Conservation Laws in Optimal Control

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Lagrange Problem of Optimal Control

Problem (P)

$$J[x(\cdot),\, u(\cdot)] = \int_a^b L\left(t,\, x(t),\, u(t)\right)\, \mathrm{d}t \longrightarrow \min$$

$$\dot{x}\,(t) = \varphi\left(t,\, x\,(t)\,,\, u\,(t)\right)$$

$$u\,(\cdot) \in L_\infty\left([a,\, b]\,;\, \Omega \subseteq \mathbb{R}^m\right), \qquad x\,(\cdot) \in W_{1,1}\left([a,\, b]\,;\, \mathbb{R}^n\right)$$

$$\begin{aligned} C^1 \quad &\ni \quad L : [a,\, b] \times \mathbb{R}^n \times \Omega \longrightarrow \mathbb{R}, \\ C^1 \quad &\ni \quad \varphi : [a,\, b] \times \mathbb{R}^n \times \Omega \longrightarrow \mathbb{R}^n \end{aligned}$$

Necessary Optimality Condition for (P)

Pontryagin Maximum Principle (PMP)

If $(x(\cdot),\, u(\cdot))$ is a minimizer of (P), then $\exists\ (\psi_0,\, \psi(\cdot)) \neq 0,\ \psi_0 \leq 0,$ $\psi(\cdot) \in W^n_{1,1}$, such that $(x(\cdot),\, u(\cdot),\, \psi_0,\, \psi(\cdot))$ satisfies:

★ the **Hamiltonian system** $\dot{x} = \dfrac{\partial H}{\partial \psi}, \quad \dot{\psi} = -\dfrac{\partial H}{\partial x};$

★ The **maximality condition**

$$H\left(t,\, x(t),\, u(t),\, \psi_0,\, \psi(t)\right) = \sup_{u \in \Omega} H\left(t,\, x(t),\, u,\, \psi_0,\, \psi(t)\right);$$

with the Hamiltonian $H = \psi_0\, L\,(t,\, x,\, u) + \psi \cdot \varphi\,(t,\, x,\, u)$.

Definition. A quadruple $(x(\cdot),\, u(\cdot),\, \psi_0,\, \psi(\cdot))$ satisfying the PMP is called a (Pontryagin) extremal.

A Particular Case: $\varphi = u$, $\Omega = \mathbb{R}^n$

Fundamental Problem of the Calculus of Variations (CV)

- ★ Hamiltonian: $H = -L + \psi \cdot u$;
- ★ Hamiltonian system: $\dot{x} = u$, $\dot{\psi} = \dfrac{\partial L}{\partial x}$;
- ★ maximality condition: $\psi = \dfrac{\partial L}{\partial u}$

If $x(\cdot)$ is a minimizer, it satisfies the Euler–Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial u}\left(t, x(t), \dot{x}(t)\right) = \frac{\partial L}{\partial x}\left(t, x(t), \dot{x}(t)\right). \tag{1}$$

Definition. A solution $x(\cdot)$ of (1) is called an (Euler–Lagrange) extremal.

Definition. A quantity $C\left(t, x(t), \dot{x}(t)\right)$ that is constant in t along every extremal $x(\cdot)$, is called a Conservation Law (CL).

General Method of Constructing CLs in the CV

Def. If $C^1 \ni h^s(t, x) = (\mathcal{T}(t, s), \mathcal{X}(x, s)) : [a, b] \times \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n$, $s \in (-\varepsilon, \varepsilon)$; $h^0(t, x) = (t, x)$ for all $(t, x) \in [a, b] \times \mathbb{R}^n$;

$$\int_{\mathcal{T}(a,0)}^{\mathcal{T}(b,0)} L\left(t^s, \mathcal{X}\left(x(t^s)\right), \frac{\mathrm{d}}{\mathrm{d}t^s}\mathcal{X}\left(x(t^s)\right)\right) \mathrm{d}t^s = \int_a^b L\left(t, x(t), \dot{x}(t)\right) \mathrm{d}t,$$

for $t^s = \mathcal{T}(t, s)$, all $s \in (-\varepsilon, \varepsilon)$ and all $x(\cdot)$; then the problem of the CV is said invariant under h^s.

Emmy Noether's Theorem (1918)

If the problem is invariant under h^s then

$$\psi(t) \cdot \frac{\partial}{\partial s} \mathcal{X}\left(x(t), s\right)\Big|_{s=0} - H\left(t, x(t), \dot{x}(t), \psi(t)\right) \frac{\partial}{\partial s} \mathcal{T}\left(t, s\right)\Big|_{s=0}$$

is a conservation law.

Example. Time invariance $\Rightarrow$ 2nd Erdmann condition

The Universal Principle of Emmy Noether

Invariance $\Rightarrow$ Existence of a CL

Classical Formulations

Well Known:

★ Invariant problems of the CV defined on a manifold M;

★ Problems of the CV with multiple integrals;

★ Invariance with respect to r-parameter families of maps from finite– or infinite–dimensional Lie group;

★ Invariance of the Lagrangian up to addition of an exact differential $\mathrm{d}\mathcal{F}(t, x, s)$, with $\mathcal{F}$ linear on the parameter s.

Not Well Known:

➜ Noether's theorem is still valid for maps h^s depending on $\dot{x}$.

(I. M. Gelfand, S. V. Fomin. *Calculus of Variations*. Prentice-Hall, 1963.)

The Universal Principle of Emmy Noether

Invariance $\Rightarrow$ Existence of a CL

Recent Formulations

★ Hamiltonian control systems (van der Schaft, '81)

★ (Higher-order) Supermechanics (Cariñena & Figueroa, '94)

★ Discrete systems (e.g. cellular automata) (Baez & Gilliam, '94)

Optimal Control:

★ "Biextremals" $(x(\cdot), \psi(\cdot))$; $x \mapsto h^s(x)$ (Sussmann, '95)

★ Pontryagin extremals; $x \mapsto h^s(x)$ (Jurdjevic, '97)

Our Contribution

✍ Pontryagin extremals; Invariance up to addition of an exact differential $\mathrm{d}\mathcal{F}(t, x, u, s)$; $(t, x, u) \mapsto h^s(t, x, u)$

Main Result

Definition. A quantity $C(t, x(t), u(t), \psi_0, \psi(t))$ constant in t along every Pontryagin extremal, will be called a conservation law (CL).

Noether Theorem for Optimal Control

Definition. Problem (P) is invariant under $h^s \in C^1$, $h^s(t, x, u) = (\mathcal{T}, \mathcal{X}, \mathcal{U})|_{(t,x,u,s)}$, $|s| < \varepsilon$, if $h^0(t, x, u) = (t, x, u)$ and

★ $\dfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{X}\left(t, x(t), u(t), s\right) = \varphi \circ h^s\left(t, x(t), u(t)\right) \dfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{T}\left(t, x(t), u(t), s\right)$;

★ $L\left(t, x(t), u(t)\right) + \dfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}\left(t, x(t), u(t), s\right) = L \circ h^s\left(t, x(t), u(t)\right) \dfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{T}$.

Theorem. If (P) is invariant under h^s then the following CL holds:

$$\psi_0 \, \frac{\partial}{\partial s}\mathcal{F}(t, x(t), u(t), s)\Big|_{s=0} + \psi(t) \cdot \frac{\partial}{\partial s}\mathcal{X}(t, x(t), u(t), s)\Big|_{s=0} - H(t, x(t), u(t), \psi_0, \psi(t)) \, \frac{\partial}{\partial s}\mathcal{T}(t, x(t), u(t), s)\Big|_{s=0} .$$

A Necessary and Sufficient Condition

$\psi_0 F(t,x,u) + \psi \cdot X(t,x,u) - H(t,x,u,\psi_0,\psi) T(t,x,u)$ is a CL iff

$$\psi_0 \left(\frac{\partial L}{\partial t} T + \frac{\partial L}{\partial x} \cdot X + L \frac{\mathrm{d}}{\mathrm{d}t} T - \frac{\mathrm{d}}{\mathrm{d}t} F \right) \\ + \psi \cdot \left(\frac{\partial \varphi}{\partial t} T + \frac{\partial \varphi}{\partial x} \cdot X + \varphi \frac{\mathrm{d}}{\mathrm{d}t} T - \frac{\mathrm{d}}{\mathrm{d}t} X \right) = 0$$

holds along every Pontryagin extremal $(x(\cdot), u(\cdot), \psi_0, \psi(\cdot))$.

Proof. By definition, C is a CL iff $\frac{\mathrm{d}}{\mathrm{d}t} C = 0$ along every extremal. Having in mind the adjoint system $\dot{\psi} = -\frac{\partial H}{\partial x}$ and the property

$$\frac{\mathrm{d}H\left(t, x(t), u(t), \psi_0, \psi(t)\right)}{\mathrm{d}t} = \frac{\partial H\left(t, x(t), u(t), \psi_0, \psi(t)\right)}{\partial t},$$

the required conclusion follows by direct computations. □

Proof of the Main Result

Let

$$T(t,x,u) = \left.\frac{\partial \mathcal{T}(t,x,u,s)}{\partial s}\right|_{s=0}, \quad X(t,x,u) = \left.\frac{\partial \mathcal{X}(t,x,u,s)}{\partial s}\right|_{s=0},$$

$$U(t,x,u) = \left.\frac{\partial \mathcal{U}(t,x,u,s)}{\partial s}\right|_{s=0}, \quad F(t,x,u) = \left.\frac{\partial \mathcal{F}(t,x,u,s)}{\partial s}\right|_{s=0}.$$

★ From the definition of invariance:

$$\frac{\mathrm{d}}{\mathrm{d}t}F = \frac{\partial L}{\partial t}T + \frac{\partial L}{\partial x}X + \frac{\partial L}{\partial u}U + L\frac{\mathrm{d}}{\mathrm{d}t}T\,; \tag{2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}X = \frac{\partial \varphi}{\partial t}T + \frac{\partial \varphi}{\partial x}\cdot X + \frac{\partial \varphi}{\partial u}\cdot U + \varphi\frac{\mathrm{d}}{\mathrm{d}t}T\,; \tag{3}$$

★ From the maximality condition:

$$\psi_0\frac{\partial L}{\partial u}U + \psi(t)\cdot\frac{\partial \varphi}{\partial u}U = 0\,; \tag{4}$$

★ Multiplying (2) and (3) respectively by ψ_0 and $\psi(t)$; using (4) for simplification; proof follows from condition of previous slide.

An Example

$$n = r = 1$$

$$J\left[x(\cdot), u(\cdot)\right] = \int_a^b \mathrm{e}^{tx(t)} u(t)\,\mathrm{d}t \longrightarrow \min\,,$$

$$\dot{x}(t) = tx(t)u(t)^2\,, \quad u \in \Omega\,.$$

Proposition. The following expression holds along any extremal:

$$\psi_0 t\mathrm{e}^{tx(t)} u(t) + \psi(t)x(t)\left((tu(t))^2 + 1\right) \equiv \textit{constant}\,. \tag{5}$$

Proof. Setting $\mathcal{F} \equiv 0$ and $h^s(t,x,u) = (\mathrm{e}^{-s}t, \mathrm{e}^s x, \mathrm{e}^s u)$, it follows from our result that $\psi x + Ht$ is a conservation law. $\square$

➜ Equality (5) holds for a given arbitrary control set Ω.

➜ Equality (5) is a necessary optimality condition not predictable from previous known results.

Final Remarks
Conservation Laws are a Useful Tool

Calculus of Variations

- ✔ Lipschitzian regularity of the minimizers (Clarke & Vinter, '85)
- ✔ Lavrentiev phenomenon (Heinricher & Mizel, '88)
- ✔ Existence of minimizers (Clarke, '93)

Control Theory

- ✔ System decomposition in terms of lower dimensional subsystems (Respondek, '82; Grizzle & Marcus, '85)

Optimal Control

- ✔ Lipschitzian regularity of the minimizers (Sarychev & DT, 2000)