On Optimal Control Problems which Admit an Infinite Continuous Group of Transformations

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Fundamental Problem of the Calculus of Variations

$$J[x(\cdot)] = \int_a^b L\left(t, x(t), \dot{x}(t)\right) \mathrm{d}t \longrightarrow \min$$

$$(x(a), x(b)) \in \mathcal{F}$$

$$x(\cdot) \in W_{1,\infty}\left([a,b]\,;\,\mathbb{R}^n\right)$$

$$C^1 \ni L : [a,b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

★ If $x(\cdot)$ is a minimizer of the FPCV, then $x(\cdot)$ is an Euler-Lagrange extremal (ELE):

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}}\left(t, x(t), \dot{x}(t)\right) = \frac{\partial L}{\partial x}\left(t, x(t), \dot{x}(t)\right)$$

★ A quantity $C\left(t, x(t), \dot{x}(t)\right)$ which is constant along every ELE,

$$C\left(t, x(t), \dot{x}(t)\right) = \text{constant}\,, \tag{1}$$

is called a *First Integral* or a Conserved Current (CC);
(1) is the corresponding *Conservation Law*.

Noether's First Theorem $\rightsquigarrow$ Finite Symmetries

Let $G_\rho\left(T, X\right)$ be a ρ parameter group of transformations $t \to T\left(t, x, \varepsilon_s\right)$, $x \to X\left(t, x, \varepsilon_s\right)$, where ε_s, $s = 1, \ldots, \rho$, denote the ρ independent parameters of the group.

Definition. The FPCV is invariant under $G_\rho\left(T, X\right)$ if, and only if,

$$L\left(T\left(t, x(t), \varepsilon_s\right), X\left(t, x(t), \varepsilon_s\right), \frac{\frac{\mathrm{d}X(t,x(t),\varepsilon_s)}{\mathrm{d}t}}{\frac{\mathrm{d}T(t,x(t),\varepsilon_s)}{\mathrm{d}t}}\right) \frac{\mathrm{d}T\left(t, x(t), \varepsilon_s\right)}{\mathrm{d}t} = L\left(t, x(t), \dot{x}(t)\right)$$

Noether's First Theorem. If the FPCV is invariant under a finite continuous group $G_\rho\left(T, X\right)$, then there result ρ CCs:

$$\left(L\left(t, x(t), \dot{x}(t)\right) - \frac{\partial L}{\partial \dot{x}}\left(t, x(t), \dot{x}(t)\right) \cdot \dot{x}(t)\right) \frac{\partial}{\partial \varepsilon_s} T\left(t, x(t), \varepsilon_s\right)\Big|_0 + \frac{\partial L}{\partial \dot{x}}\left(t, x(t), \dot{x}(t)\right) \cdot \frac{\partial}{\partial \varepsilon_s} X\left(t, x(t), \varepsilon_s\right)\Big|_0$$

Noether's Second Theorem $\leftrightsquigarrow$ Infinite Symmetries

Let $G_{\rho\infty}(T,X)$ be an infinite continuous group of transformations depending upon ρ arbitrary and independent functions $p_s(\cdot) \in C^m([a,b];\mathbb{R})$ together with their derivatives ($s = 1,\ldots,\rho$):
$t \to T\left(t,x,p_s(t),\ldots,p_s^{(m)}(t)\right)$, $x \to X\left(t,x,p_s(t),\ldots,p_s^{(m)}(t)\right)$

Definition. The FPCV is invariant under $G_{\rho\infty}(T,X)$ iff

$$L\left(T(\alpha(t)),X(\alpha(t)),\frac{\frac{\mathrm{d}X(\alpha(t))}{\mathrm{d}t}}{\frac{\mathrm{d}T(\alpha(t))}{\mathrm{d}t}}\right)\frac{\mathrm{d}T(\alpha(t))}{\mathrm{d}t} = L(t,x(t),\dot{x}(t))$$

where $\alpha(t) = \left(t,x(t),p_s(t),\dot{p}_s(t),\ldots,p_s^{(m)}(t)\right)$

Noether's Second Theorem. If the FPCV is invariant under an infinite group $G_{\rho\infty}(T,X)$, then there result $\rho(m+1)$ CCs:

$$\left(L(t,x(t),\dot{x}(t)) - \frac{\partial L}{\partial \dot{x}}(t,x(t),\dot{x}(t))\cdot\dot{x}(t)\right)\frac{\partial}{\partial p_s^{(i)}}T(\alpha(t))\Bigg|_0 + \frac{\partial L}{\partial \dot{x}}(t,x(t),\dot{x}(t))\cdot\frac{\partial}{\partial p_s^{(i)}}X(\alpha(t))\Bigg|_0$$

The Source be with You

- ➜ Noether's Second Theorem has been widely forgotten
- ➜ Failure to appreciate that Emmy Noether offered not one but two theorems in her 1918 paper is source of widespread confusion (cf. K. Brading, *Noether's Theorem and Gauge Symmetries*)
- ➜ The few mentions to Noether's Second Theorem usually assert, wrongly, that the result is a particular case of the First Theorem! (e.g. Giaquinta & Hildebrandt, Calculus of variations I)

★ There is an insight in original Noether's fundamental paper which I claim to be important enough to be called the *Third Noether Theorem*: if one looks at G_ρ has a rigid subgroup of G_{ρ^∞}, in the sense that G_ρ arises from G_{ρ^∞} by fixing $p_s(t) = \varepsilon_s = \text{const}$, then the First Theorem is a consequence of the Second.

The Facts of Life

- ★ The Calculus of Variations is part of Optimal Control
- ★ The concepts of symmetry, conserved current, and reduction are a very useful tool in Optimal Control (e.g. stability, regularity)
- ★ The optimal control literature on First Noether-Type theorems is now vast: e.g.
  - ➜ D. S. Đukić, 1973
  - ➜ A. van der Schaft, 1981, 1987
  - ➜ H. Sussmann, 1995
  - ➜ V. Jurdjevic, 1997
  - ➜ G. Blankenstein & A. van der Schaft, 2001
  - ➜ D. F. M. Torres, 2002

No Second Noether-type theorem is available for Optimal Control

The Optimal Control Problem (OCP)

$$J[x(\cdot),u(\cdot)] = \int_a^b L\left(t, x(t), u(t)\right) \mathrm{d}t \longrightarrow \min$$

$$\dot{x}\left(t\right) = \varphi\left(t, x\left(t\right), u\left(t\right)\right)$$

$$(x(a), x(b)) \in \mathcal{F}$$

$$x\left(\cdot\right) \in W_{1,1}\left(\left[a,b\right]; \mathbb{R}^n\right), \quad u\left(\cdot\right) \in L_\infty\left(\left[a,b\right]; \Omega \subseteq \mathbb{R}^r\right)$$

$$C^1 \ni L : [a,b] \times \mathbb{R}^n \times \Omega \to \mathbb{R}, \quad C^1 \ni \varphi : [a,b] \times \mathbb{R}^n \times \Omega \to \mathbb{R}^n$$

★ If $(x(\cdot), u(\cdot))$ is a minimizer, then $\exists\ (\psi_0, \psi(\cdot)) \neq 0$, $\psi_0 \leq 0$, $\psi(\cdot) \in W_{1,1}^n$, s.t. $(x(\cdot), u(\cdot), \psi_0, \psi(\cdot))$ is a Pontryagin extremal:

$$\dot{\psi} = -\frac{\partial H}{\partial x}, \quad H\left(t, x(t), u(t), \psi_0, \psi(t)\right) = \sup_{u \in \Omega} H\left(t, x(t), u, \psi_0, \psi(t)\right)$$

with the Hamiltonian $H(t, x, u, \psi_0, \psi) = \psi_0 L\left(t, x, u\right) + \psi \cdot \varphi\left(t, x, u\right)$.

Definition. A function $C\left(t, x, u, \psi_0, \psi\right)$ which is constant in t along every Pontryagin extremal, will be called a Noether current

The Semi-Invariance Definition $\leftrightsquigarrow$ Gauge Symmetry

Let $C^m \ni p : [a, b] \to \mathbb{R}^\rho$ be an arbitrary function, and let $\alpha(t)$ to denote $\left(t, x(t), u(t), p(t), \dot{p}(t), \ldots, p^{(m)}(t)\right)$. We say that the OCP is *semi-invariant* under the C^1 transformation group $G_{\rho^\infty}(T, X, U)$ iff

★ $$\left(\lambda^0 \cdot p(t) + \lambda^1 \cdot \dot{p}(t) + \cdots + \lambda^m \cdot p^{(m)}(t)\right) \frac{\mathrm{d}}{\mathrm{dt}} L\left(t, x(t), u(t)\right) + L\left(t, x(t), u(t)\right) + \frac{\mathrm{d}}{\mathrm{dt}} F\left(\alpha(t)\right) = L\left(T\left(\alpha(t)\right), X\left(\alpha(t)\right), U\left(\alpha(t)\right)\right) \frac{\mathrm{d}}{\mathrm{d}t} T\left(\alpha(t)\right) ,$$

★ $$\frac{\mathrm{d}}{\mathrm{d}t} X\left(\alpha(t)\right) = \varphi\left(T\left(\alpha(t)\right), X\left(\alpha(t)\right), U\left(\alpha(t)\right)\right) \frac{\mathrm{d}}{\mathrm{d}t} T\left(\alpha(t)\right) ,$$

for some function F of class C^1 and for some $\lambda^0, \ldots, \lambda^m \in \mathbb{R}^\rho$.

➜ We are assuming that to $p(t) = \dot{p}(t) = \cdots = p^{(m)}(t) = 0$ there corresponds the identity transformation: $T(t, x, u, 0, 0, \ldots, 0) = t$, $X(t, x, u, 0, 0, \ldots, 0) = x$, $U(t, x, u, 0, 0, \ldots, 0) = u$

Generalized Noether-type Identities

Theorem. Suppose that the OCP is semi-invariant under the infinite group $G_{\rho^\infty}(T, X, U)$. Then, the following conditions hold:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\lambda_s^i L + \left.\frac{\partial F(\alpha(t))}{\partial p_s^{(i)}}\right|_0\right) = \frac{\partial L}{\partial t} \left.\frac{\partial T(\alpha(t))}{\partial p_s^{(i)}}\right|_0 + \frac{\partial L}{\partial x} \cdot \left.\frac{\partial X(\alpha(t))}{\partial p_s^{(i)}}\right|_0 + \frac{\partial L}{\partial u} \cdot \left.\frac{\partial U(\alpha(t))}{\partial p_s^{(i)}}\right|_0 + L \frac{\mathrm{d}}{\mathrm{d}t} \left.\frac{\partial T(\alpha(t))}{\partial p_s^{(i)}}\right|_0$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \left.\frac{\partial X(\alpha(t))}{\partial p_s^{(i)}}\right|_0 = \frac{\partial \varphi}{\partial t} \left.\frac{\partial T(\alpha(t))}{\partial p_s^{(i)}}\right|_0 + \frac{\partial \varphi}{\partial x} \cdot \left.\frac{\partial X(\alpha(t))}{\partial p_s^{(i)}}\right|_0 + \frac{\partial \varphi}{\partial u} \cdot \left.\frac{\partial U(\alpha(t))}{\partial p_s^{(i)}}\right|_0 + \varphi \frac{\mathrm{d}}{\mathrm{d}t} \left.\frac{\partial T(\alpha(t))}{\partial p_s^{(i)}}\right|_0$$

✔ The proof follows from the previous definition of semi-invariance

➜ The conditions hold for all functions $x(\cdot) \in W_{1,1}^n$ and $u(\cdot) \in L_\infty^r$, whether they are Pontryagin extremals or not

Second Noether Theorem for Optimal Control

Theorem. If the OCP is semi-invariant under $G_{\rho\infty}(T,X,U)$, then there exist $\rho\,(m+1)$ Noether currents of the form

$$\psi_0\left(\lambda_s^i L\left(t,x(t),u(t)\right)+\left.\frac{\partial F\left(\alpha(t)\right)}{\partial p_s^{(i)}}\right|_0\right)+\psi(t)\cdot\left.\frac{\partial X\left(\alpha(t)\right)}{\partial p_s^{(i)}}\right|_0$$

$$-H\left(t,x(t),u(t),\psi_0,\psi(t)\right)\left.\frac{\partial T\left(\alpha(t)\right)}{\partial p_s^{(i)}}\right|_0$$

Example

$L\equiv 1\,,\ \varphi=u$	$G_{1\infty}\left(T,X,U\right)$	Noether Currents
$T\longrightarrow \min$	$T=p(t)+t\,,$	ψ_0-H
$\dot{x}(t)=u(t)$	$X=(\dot{p}(t)+1)^2 x(t)\,,$	$2\psi(t)x(t)$
$u(t)\in(-1,1)$	$U=(\dot{p}(t)+1)u(t)+2\ddot{p}(t)x(t)\,,$	
$x(0)=\alpha\,,\ x(T)=\beta$	$F=p(t)\,,\quad \lambda_1^i=0\,(i=0,1,2)$	

Proof of the Second Noether-type Theorem

★ Let $(x(\cdot), u(\cdot), \psi_0, \psi(\cdot))$ be any extremal. Maximality Condition $\Rightarrow$

$$\psi_0 \frac{\partial L}{\partial u} \cdot \frac{\partial U(\alpha(t))}{\partial p_s^{(i)}}\bigg|_0 + \psi(t) \cdot \frac{\partial \varphi}{\partial u} \cdot \frac{\partial U(\alpha(t))}{\partial p_s^{(i)}}\bigg|_0 = 0 \tag{2}$$

★ (2) + Generalized Noether Identities $\Rightarrow$

$$\psi_0 \left(\frac{\partial L}{\partial t} \frac{\partial T}{\partial p_s^{(i)}}\bigg|_0 + \frac{\partial L}{\partial x} \cdot \frac{\partial X}{\partial p_s^{(i)}}\bigg|_0 + L \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial T}{\partial p_s^{(i)}}\bigg|_0 - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial F}{\partial p_s^{(i)}}\bigg|_0 - \lambda_s^i \frac{\mathrm{d}}{\mathrm{d}t} L \right)$$
$$+\psi(t) \cdot \left(\frac{\partial \varphi}{\partial t} \frac{\partial T}{\partial p_s^{(i)}}\bigg|_0 + \frac{\partial \varphi}{\partial x} \cdot \frac{\partial X}{\partial p_s^{(i)}}\bigg|_0 + \varphi \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial T}{\partial p_s^{(i)}}\bigg|_0 - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial X}{\partial p_s^{(i)}}\bigg|_0 \right) = 0 \tag{3}$$

★ (3) + Adjoint system + property $\frac{dH}{dt} = \frac{\partial H}{\partial t}$ of the extremals $\Rightarrow$

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\psi_0 \frac{\partial F}{\partial p_s^{(i)}}\bigg|_0 + \psi_0 \lambda_s^i L + \psi(t) \cdot \frac{\partial X}{\partial p_s^{(i)}}\bigg|_0 - H \frac{\partial T}{\partial p_s^{(i)}}\bigg|_0 \right) = 0$$

Conclusions

The genuine Emmy Noether's theorem is the second, not the first!

- ✔ In this paper we provide an extension of second Noether's theorem from the calculus of variations to the more general optimal control framework.
- ✔ As corollaries one can obtain, using Noether's remark, the previous optimal control versions of the first Noether theorem.

Our Main Result admits several extensions. E.g.

- ➜ More general Symmetries can be considered (cf. the quasi-invariance notion introduced by Torres in Proc. 10th Mediterranean Conference on Control and Automation, MED2002)
- ➜ A broad class of optimal control problems involving holonomic or nonholonomic constraints can be considered. A version for problems with mixed constrains is under development.