A SHORT NOTE ON COPS AND ROBBERS PLAYING ON TOTAL GRAPHS

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Abstract

Cop Robber game is a two player game played on an undirected graph. In this game, the cops try to capture a robber moving on the vertices of the graph. The cop number of a graph is the least number of cops needed to guarantee the robber will be caught. The total graph $T(G)$ of a graph G has a vertex for each edge and vertex of G and an edge in $T(G)$ for every edge-edge, vertex-edge, and vertex-vertex adjacency in G. In this paper, we play the game on the total graph $T(G)$, showing in particular that for planar graphs $c(T(G)) \leq 3$.

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1 Introduction

The Game of Cops and Robbers is a vertex pursuit game played on a graph G introduced by Nowakowski and Winkler [8]. In this game we consider two players, a set of k cops (or searchers) C, where $k > 0$ is a fixed integer, and the robber R. Cops initiate the game placing themselves on a set of k vertices and they move in alternate rounds with the robber. A move is considered as placing itself on adjacent vertex to currently occupied or staying in the same place. Many cops are allowed to occupy a single vertex. The players know each others actual position (that is, the game is played with complete information). The cops win and the game ends if one of the cop move to the vertex occupied by the robber; if robber has a strategy to avoid cops, then R wins. The minimum number of cops needed to catch the robber (regardless of robber's strategy) is called the cop number of G, and is denoted by $c(G)$. This parameter is well studied in literature (see [1, 5]).

Throughout this paper we consider only simple graphs that is graphs without loops and parallel edges. For additional definitions on graphs we refer the book [6] and for the game we refer the book [5].

The total graph $T(G)$ of a graph G is the graph obtained by taking as the vertex the set $V(T(G)) = V(G) \cup E(G)$ and two distinct vertices x and y are adjacent in $T(G)$ if x and y are adjacent vertices of G, adjacent edges of G,or an incident vertex and edge of G (see [2, 3, 4]). In this paper, we play the game of Cops and Robbers on the total graph $T(G)$ of G. We define the *total cop number* as $c^t(G) = c(T(G))$.

The total graph can also be defined as follows: $T(G)$ is the total graph of G if it is obtained by taking a copy of G and a copy of its line graph $L(G)$ and one vertex in G is joined to one vertex in $L(G)$ if the corresponding elements are incident in the original graph. If G is a simple graph of order n, then the following properties are immediate:

1. If each connected component of G is K_2, then $c^t(G) = n/2$.
2. If each connected component of G is K_3, then $c^t(G) = 2n/3$.
3. If G is the path P_n of order $n \geq 4$, then $c^t(P_n) = 1$.
4. If G is the cycle C_n of order $n \geq 3$, then $c^t(C_n) = 2$.
5. If G is the star $K_{1,n}$, then $c^t(K_{1,n}) = 1$.

The *edge cop number* was defined by Dudek, Gordinowicz and Prałat [7] as $\overline{c}(G) = c(L(G))$, where $L(G)$ is the line graph of G. See their paper for more results on this topic.

Theorem 1. *Let G be any connected simple graph of order n. Then $c(G) \leq c^t(G) \leq c(G) + 1$.*

Proof. At first assume that the robber moves along graph $T(G)$ only on the vertices corresponding to $V(G)$. Notice that if we put a cop at a vertex corresponding to $e \in E(G)$, then it is adjacent to exactly two vertices in $V(G)$ part, which are endpoints of the edge e. Thus if we move the cop from $E(G)$ part of $T(G)$ to $V(G)$ part of $T(G)$, then the cop will cover more vertices in $V(G)$ than before. (Since in $E(G)$ part each vertex is adjacent to exactly two vertices in $V(G)$ part). In addition notice that to attack the robber we can not make a shortcut using vertices corresponding to $E(G)$, i.e. reach a vertex u which is at distance k from the vertex occupied by the cop in a graph G is not less than k steps in the graph $T(G)$ (since in $T(G)$ and in G $d_G(v, v) = d_T(u, v)$ for any two arbitrary vertices u and v in G, where $d_G(u, v)$-denote the distance between the vertices u and v in G). Therefore less than $c(G)$ cops will never be suffice to catch the robber on $T(G)$. This follows the lower bound.

The strategy for catching the robber with $c(G) + 1$ is similar to the one presented in [7]. Whenever the robber moves through the graph $T(G)$ on vertices corresponding to the $V(G)$ we follow optimal strategy to catch the robber in graph G. Whenever it moves to a vertex corresponding to the $e \in E(G)$ we treat it as if he would occupy one endpoint vertex of the edge e. The only thing we have to keep trace of is that when robber moves to a vertex corresponding to the edge $e' \in E(G)$ adjacent to e, then we continue the game as if it would move to an adjacent vertex in G. At some point we reach the state in which we "catch" the robber (meaning the vertex that we treat as the one occupied by robber is surrounded by cops).

Let us show that despite the next move of the robber we can put one cop at a vertex adjacent to the vertex occupied by the robber. First assume that robber was at a vertex corresponding to $V(G)$, then in the final state there has to be a cop which is at a vertex adjacent to it. This cop can now follow the robber staying always at vertex adjacent to his (and at the same time preventing robber from getting back to the vertices from the $V(G)$ part). In the other case we assume that robber was at a vertex corresponding to $E(G)$. In this case it might happen that none of the cops stays at vertex adjacent to it. Robber has two option, either in the next round he keep his position, which allows us to move a cop onto a vertex adjacent to him, or he moves along the vertices corresponding to $E(G)$. He has to choose a vertex which is adjacent to some vertex from the neighbourhood of the vertex that we focus on in the graph G. Since we assumed that we reach the state in which robber lose the game, all such vertices have a cop staying on them. Thus we can move a cop on a vertex adjacent to the one that robber chose. Again this cop can now follow the robber staying always at vertex adjacent to robber position. See Figure 1 for details.

Thus now robber can only move along vertices corresponding to $E(G)$ having a cop chasing him on adjacent vertex corresponding to one endpoint of $e \in E(G)$. We continue the game using optimal strategy for cop-robber game on graph G assuming that robber occupy other vertex of e. At some point remaining $c(G)$ cops will catch robber in such game. At this point we would have cops occupying both ends of the edge e, obtaining a state in which robber can not run away from the cops and he lose the game.

□

Lemma 2. *If $\overline{c}(G) < c(G)$, then $c^t(G) = c(G)$.*

Proof. As showed in the proof of Theorem 1 with $c(G)$ cops we can always force

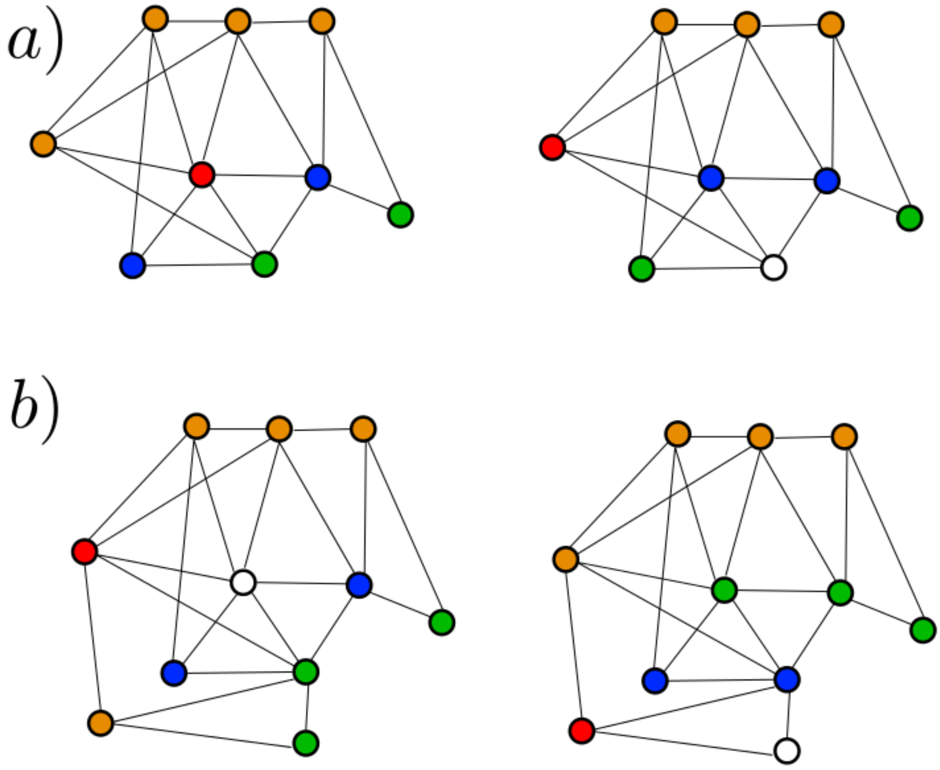

Figure 1: Figure present game on total graph $T(G)$. Green vertices represent vertices corresponding to $V(G)$, orange vertices represent vertices corresponding to $E(G)$, blue vertices are position of the cops, red vertex is a robber position, white vertex is a position which cops treats as robber position in their pursuit on vertices of $V(G)$. In the case a) cops caught robber on vertex of $V(G)$, his only option now is to move to vertex from part corresponding $E(G)$, namely vertex corresponding to e which is not adjacent to cops positions. This allows cop to move to a vertex adjacent to robber position. In the case b) cops caught robber on vertex of $V(G)$, while he is moving along the part corresponding to $E(G)$. Again in the next move one cop can place himself at a vertex adjacent to robber position. In both cases game can be continued as if robber was at the other vertex adjacent to e.

the robber to move into the part of the graph corresponding to the $E(G)$ and place a cop on a vertex adjacent to the robber. This cop can prevent robber from moving into the part of the graph corresponding to the $V(G)$, as $c(G) - 1$ cops are enough to catch robber on the line graph of G, they will eventually win the game. ◻

As proven in [7] Petersen graph has edge cop number equal to 2 having cop number equal to 3. The line graph of the dodecahedron is the icosidodecahedral graph. Again as proven in [7] dodecahedron has edge cop number equal to 2 having cop number equal to 3. In general it is not known for which graphs the inequality $\overline{c}(G) < c(G)$ holds.

Corollary 3. *If G is the Petersen graph or the dodecahedron graph, then $c(G) = c^t(G) = 3$.*

A graph G is said to be cop win if $c(G) = 1$ and is said to be total cop win if $c^t(G) = 1$. For a vertex x in a graph G, the neighborhood of x is denoted by $N(x)$ and is defined as $N(x) = \{x \in V \mid (x, y) \in E\}$. The closed neighborhood of x is defined as $N[x] = \{x\} \cup N(x)$. A vertex x is said to be the *corner*, if there exist a vertex y such that $N[x] \subseteq N[y]$. Corner vertices plays an important role in characterizing cop win graphs (see [5]). A graph G is *dismantlable* if we can reduce G to a single vertex after the successive removal of all corner vertices. In case of total graphs we can prove following statement.

Theorem 4. *G is total cop win if and only if G is a tree.*

Proof. Notice that vertices corresponding to vertices of degree one in G are corners in total graph. After deleting those vertices, vertices which correspond to edges incident to them are corners. Thus we can successively remove vertices of degree one and their neighbours until we get a single vertex, therefore total graph of a tree is dismantlable. Assume that graph G is not a tree, as mentioned above we can delete one by one all vertices corresponding to vertices of degree one and to edges incident to them. What we are left with is a graph with minimal degree two. We can show that none of its vertices is a corner. Vertices corresponding to edges in G are not corners since two such vertices can not dominate neighbourhood of one another as they are connected to different pairs of vertices. Vertex which correspond to a vertex in G can not dominate the neighbourhood of such vertex since it is not connected to all edges adjacent to given edge (as minimal degree in a graph is two). On the other hand vertices corresponding to vertices in G can not dominate each other as we always find an edge which is incident with one of them and not incident with the other. It is clear that a vertex corresponding to an edge can not dominate vertex coming from a vertex of G. ◻

Corollary 5. *If G is the complete graph K_n, then* $c^t(K_n) = \begin{cases} 1, & \text{if } n = 1,2 \\ 2, & \text{if } n \geq 3 \end{cases}$.

The cop-win graphs was completely characterized by Nowakowski and Winkler [8].

Theorem 6 ([8])**.** *A graph G is cop win if and only if it is dismantlable.*

Based on the above results we have the following:

Corollary 7. *Let G be a dismantlable graph which is not a tree, then $c^t(G) = 2$.*

Proof. Let G be a dismantlable graph which is not a tree. Since G is dismantlable, we have $c(G) = 1$. Also G is not a tree, by Theorem 4 $c^t(G) \neq 1$. This implies

$$c^t(G) \geq 2. \tag{1}$$

Again, by Theorem 1 $c^t(G) \leq c(G) + 1$. This yields

$$c^t(G) \leq 2. \tag{2}$$

From (1) and (2) the result follows □

Theorem 8. *Let G be any connected planar graph. Then*

$$c^t(G) \leq 3$$

Proof. By Theorem 1 there exist planar graphs for which total cop number is at least 3. We now show that 3 cops are always enough to catch the robber. Proof follows the strategy invented by Aigner and Fromme [1]. We say that a cop c controls the Path if whenever the robber tries to cross P he is caught by c in the next round. Crucial in the proof of Aigner and Fromme is the following lemma

Lemma 9 ([1])**.** *Let G be any graph, $u, v \in V(G)$ and P be the shortest path between u and v. Then a single cop c on G can, after a finite number of moves, prevent the robber from entering P. That is, R will be immediately caught if he moves onto P.*

We need a little bit different version of this lemma

Lemma 10. *Let G be a graph and $T(G)$ be its total graph, $u, v \in V(G)$ and P be the shortest path between u and v in G. Then a single cop c on $T(G)$ can, after a finite number of moves, prevent the robber from entering or crossing the P. That is, given the embedding of P in the plane, R will be immediately caught if he moves onto vertices corresponding to $V(P)$ or $E(G)$ or cross the path.*

Proof. We take any planar embedding of G and put vertices corresponding to $E(G)$ in the middle of the edges of this embedding. Denote such embedding of $T(G)$ as R. Denote by $d(x, y)$ the distance (=lenght of a shortest path) between x and y. In [1] Aigner and Fromme proved that cop c placed on path P once obtained, can preserve a property

$$d(r, z) \geq d(c, z) \text{ for all } z \in V(P)$$

where r stands for the position of the robber. This property is true due to distance in a graph preserving the triangle inequality. Strategy is as follows: If robber stays put, so does the cop. If robber moves toward the path P then cop moves accordingly preserving the distances.

Notice that above property is true for any graph, in particular also for total graph. It remains to prove that the stated lemma is true when robber moves onto vertices corresponding to $E(G)$.

First assume that robber moves onto $v = (x, y) \in E(G)$ from vertices corresponding to $V(G)$. This can only happen if he was at either the vertex x or y, as both of them are vertices of the path P he would be caught in the previous step.

In the second case robber moves onto v from a vertex corresponding to some edge u adjacent to (x, y) in G. Without loss of generality we can assume that vertex x is adjacent to v in G. In order to preserve the distance property when robber moved onto u cop had to move onto vertex x. Thus if robber moves onto v he will be caught by the cop in the next move. The only case for robber to cross the path P in embedding R without moving onto vertices corresponding to $V(P)$ or $E(G)$ is to move from u to another vertex corresponding to edge $q = (x, p)$ in G, such that p and y are on different sides of P in R. As in the previous case if robber moves onto q he will be caught by the cop in the next move.

□

The idea of the proof of Theorem 10 is to secure by two cops two shortest paths between two vertices u and v in G. Both those paths create a cycle and robber is placed outside of it. Proof use the embedding of the planar graph G and the property robber can not enter the area inside the circle between u and v without being caught. Property that as we showed in lemma also holds in the case of total graphs of planar graphs (once in embedding we put vertices corresponding to edges on the edges of the graph G). Then in next rounds we can extend the cycle by other vertices (using third cop) reducing the subgraph outside of it. In general in stage i, we assign to the robber a certain subgraph G_i, the robber's territory, which contains all edges which r may still safely enter, and we show that, after a finite number of moves, G_i is reduced to $G_{i+1} \subset G_i$. Game ends as eventually there is no safe place for the robber to move. See [1] for the details of the proof.

□

2 Open Problems

There are graphs with $c^t(G) = c(L(G))$, for example consider the paths, cycles, trees and complete graphs. Characterization of all families of graphs with this property is an open problem. Another challenging question which we left unsolved is to characterize the graphs having $c(G) = c^t(G) + 1$.

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