Second Order Conditions on the Overflow Traffic from the Erlang-B System

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Abstract. This paper presents in a unified manner mathematical properties of the second order derivatives of the overflow traffic from an Erlang loss system, assuming the number of circuits to be a nonnegative real value. It is shown that the overflow traffic function $\widehat{A}(a,x)$ is strictly convex with respect to x (number of circuits), with $x \geq 0$, taking the offered traffic, a, as a positive real parameter. The convexity (in the wider sense) has been proved by A.A. Jagers and Erik A. Van Doorn [8]. Using a similar procedure to the one used by those authors it is shown that $\widehat{A}(a,x)$ is a strictly convex function with respect to a, for all $(a,x) \in I\!R^+ \times I\!R^+$ — a well known result for the case of x being a positive integer, due to C. Palm [10, pp.180–181]. These two results are obtained by determining the sign of the second order derivatives $\widehat{A}''_{aa}(a,x)$ and $\widehat{A}''_{xx}(a,x)$ for $(a,x) \in I\!R^+ \times I\!R^+$. In the same manner it is proved that the rectangular derivatives $\widehat{A}''_{ax}(a,x)$ and $\widehat{A}''_{xa}(a,x)$ are negative for all $(a,x) \in I\!R^+ \times I\!R^+_0$. Finally, a first approach to the analysis of the strict joint convexity of $\widehat{A}(a,x)$ in some open convex subdomain of $I\!R^+ \times I\!R^+$, is discussed. Finally, based on some particular cases and extensive computational results it is conjectured that the function $\widehat{A}(a,x)$ is strictly jointly convex in areas of low blocking where the *standard offered traffic* is less than -1.

Key words. Erlang Loss System, Convexity Proprieties, Derivatives of the Overflow Traffic, Optimization Techniques.

AMS subject classifications. 33-04, 60K30, 90B22, 65G99.

1 Introduction and Main Results

The classical Erlang loss system, firstly studied by A.K. Erlang in 1917, has been subject to extensive studies regarding its mathematical properties, as a result of its importance in Teletraffic Theory and its applications. The blocking probability of the $M/M/n$ loss system in statistical equilibrium, denoted by $B(a,n)$:

$$B(a,n) \;=\; \frac{a^n/n!}{\sum_{j=0}^{n} a^j/j!}\,, \tag{1}$$

is the well known Erlang-B formula or Erlang loss formula. The positive real variable a is the *offered traffic* (in Erlangs), while the nonnegative integer variable n represents the number of servers.

Considering the usual analytic extension of $B(a,n)$, ascribed to R.Fortet [14, pag.602]:

$$B(a,x) \;=\; \left\{ a \int_0^{+\infty} e^{-az}\,(1+z)^x\;dz \right\}^{-1}\,, \tag{2}$$

the variable representing the number of servers (or circuits), x, can be taken to be real. The proof that (1) and (2) define the same function for $x \in I\!N_0$, may be found in [6, pp.531–532].

Note that the function defined by (2) is infinitely differentiable for any $(a,x) \in I\!R^+ \times I\!R$, and also the integral in (2) is uniformly convergent.

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A major known result (see *e.g.* [7]) is the following recursion obtained by partial integration of (2):

$$B(a,x)^{-1} = \frac{x}{a} B(a,x-1)^{-1} + 1, \quad \forall (a,x) \in I\!R^{+} \times I\!R\,. \tag{3}$$

Since $B(a,0) = 1$, $\forall a \in I\!R^{+}$, $B(a,x)$ may be calculated by (3) for any positive integer x.

In the following the argument (a,x) is omitted whenever there is no place of misinterpretation. The first order derivatives of $B(a,x)$ may be found in [6] and they are given by:

$$B'_a = \left(\frac{x}{a} - 1 + B\right) B\,, \tag{4}$$

$$B'_x = -a\,B^2 \int_0^{+\infty} e^{-az}\,(1+z)^x\,\ln(1+z)\,dz\,. \tag{5}$$

While $B'_x(a,x)$ is strictly negative for any $(a,x) \in I\!R^{+} \times I\!R$, $B'_a(a,x)$ is strictly positive if $x > 0$, and it vanishes if $x = 0$ (see [7, pag.1289]).

The second order derivatives of $B(a,x)$ may be obtained from the following relations:

$$B''_{aa} = \left(B'_a - \frac{x}{a^2}\right) B + \frac{(B'_a)^2}{B}\,, \tag{6}$$

$$= \left(3\,\frac{x}{a} - 3 + 2\,B\right) B^2 + \left[\,(x-a)^2 - x\,\right] \frac{B}{a^2}\,, \tag{7}$$

$$B''_{xx} = 2\,a^2\,B^3 \left[\int_0^{+\infty} e^{-az}\,(1+z)^x\,\ln(1+z)\,dz\right]^2 - \\ - a\,B^2 \int_0^{+\infty} e^{-az}\,(1+z)^x\,[\ln(1+z)]^2\,dz\,,$$

$$B''_{ax} = \left(\frac{x}{a} - 1 + 2\,B\right) B'_x + \frac{B}{a}\,. \tag{8}$$

Note that by the Schwarz theorem the rectangular derivatives are equal, that is $B''_{ax}(a,x) = B''_{xa}(a,x)$.

Now, we review two functions that play important roles in the Erlang-B system:

$$\widetilde{A}(a,x) = a\,[1 - B(a,x)]\,, \tag{9}$$

$$\widehat{A}(a,x) = a\,B(a,x) = \left\{\int_0^{+\infty} e^{-az}\,(1+z)^x\,dz\right\}^{-1}\,. \tag{10}$$

The first one, $\widetilde{A}$, is called the *carried traffic* and gives the expected number of calls in progress. The second function , $\widehat{A}$, is called the lost traffic or *overflow traffic* and gives the expected number of lost calls during the mean holding time.

In this paper the signs of the second order partial derivatives of the overflow traffic function, are derived. The first order partial derivatives of $\widehat{A}(a,x)$ may be obtained by differentiation of (10):

$$\widehat{A}'_a = B + a\,B'_a = a\,B^2 + (\,x - a + 1\,)\,B\,, \tag{11}$$

$$\widehat{A}'_x = a\,B'_x = -\left[\widehat{A}(a,x)\right]^2 \int_0^{+\infty} e^{-az}\,(1+z)^x\,\ln(1+z)\,dz\,. \tag{12}$$

After some algebraic manipulation, differentiation of (11) and (12) leads to the second order partial derivatives of $\widehat{A}(a,x)$:

$$\widehat{A}''_{xx} = a\,B''_{xx}\,, \tag{13}$$

$$\widehat{A}''_{aa} = 2\,a\,B^3 + [\,3\,(x-a) + 2\,]\,B^2 + \left[\frac{(x-a)^2}{a} + \frac{x}{a} - 2\right] B\,, \tag{14}$$

$$\widehat{A}''_{ax} = (\,x - a + 2\,a\,B + 1\,)\,B'_x + B\,. \tag{15}$$

Again, applying Schwarz theorem we have the identity $\widehat{A}''_{ax}(a,x) = \widehat{A}''_{xa}(a,x)$.

The analysis of the sign of the derivatives $\widehat{A}''_{xx}$, $\widehat{A}''_{aa}$ and $\widehat{A}''_{ax}$ is useful in the optimal design of queueing systems. Examples are some optimization problems of circuit-switched teletraffic networks (see [5]), and certain load sharing and server allocation problems (some formulations may be found in [12]). In fact, a fundamental task in order to approach the resolution of those problems is the establishment of convexity conditions of the objective functions. Indeed, the dificulties encountered in recognizing these properties are well known in nonlinear programming practice and they are an important part of problem solving.

The main result of this work is the following theorem.

Theorem 1 *For all* $(a,x) \in \mathbb{R}^+ \times \mathbb{R}_0^+$, *the following statements hold:*

$$\begin{array}{ll} 1) & \widehat{A}''_{xx}(a,x) > 0; \\ 2) & \widehat{A}''_{aa}(a,x) \geq 0 \;\wedge\; [\widehat{A}''_{aa}(a,x) = 0 \text{ iff } x = 0]; \\ 3) & \widehat{A}''_{ax}(a,x) = \widehat{A}''_{xa}(a,x) < 0. \end{array}$$

Taking a general view on the related literature, one can make the following remarks concerning the results in Theorem 1:

1. The proof of statement *1)* was established in [9], for x considered a positive integer. Later, A.A. Jagers and Erik A. Van Doorn [8] have shown that $B''_{xx}(a,x) \geq 0$ for $(a,x) \in \mathbb{R}^+ \times \mathbb{R}_0^+$ (which implies that $\widehat{A}''_{xx}(a,x) \geq 0$). More recently it has been proved that $\widehat{A}''_{xx}(a,x) > 0$, for real $x \geq 1$ (see [3]).

2. The proof of statement *2)* is well know in the case of positive integers x, and is due to C. Palm [10, pp.180–181].

3. As far as we know, statement *3)* is a new result.

Other results related to the convexity of the Erlang-B function were studied in [1]. However, that work is confined to the case where x is a positive integer. Noting that the analytic extension of the Erlang-B function, for $x \in \mathbb{R}_0^+$, becomes indispensable in many problems of analysis and optimization of teletraffic systems (see *e.g.* [11] and [15]), Theorem 1 has not only a theorical interest but also becomes relevant with respect to many applications.

As far as optimization problems are concerned, statements *1)* and *2)* of the previous theorem are specially important, since they are establish strict convexity conditions of $\widehat{A}(a,x)$ with respect to the variables x and a. In addition, statement *3)* reveals its importance in the study of the joint convexity of $\widehat{A}(a,x)$. We give a first, although incomplete, approach to this question in Section 3, where a conjecture concerning a region of strict joint convexity of function $\widehat{A}$ is proposed. The proof of Theorem 1 appears in Section 2.

2 Proof of Theorem 1

The proof of the three statements of Theorem 1 will be presented in separate subsections. The proof of statement *1)* is done using *redutio ab absurdum*. The proof of statements *2)* and *3)* follows a similar procedure to the one presented in [8, Theorem 1]. The essential observation in that procedure is the following: in order to prove the complete monotonicity of a function, it suffices to show that its Laplace transform exists and is a nonnegative function in $\mathbb{R}^+$ (see for example [4, pag.439]).

2.1 Proof of Statement *1)*

Here the objective is to prove that $\widehat{A}''_{xx}(a,x) > 0$, $\forall (a,x) \in \mathbb{R}^+ \times \mathbb{R}_0^+$.

Since $\widehat{A}''_{xx} = aB''_{xx}$, this is equivalent to prove that $B''_{xx}(a,x) > 0$, $\forall (a,x) \in \mathbb{R}^+ \times \mathbb{R}_0^+$. This result implies convexity conditions of $B(a,x)$ with respect to x. The strict convexity of $B(a,x)$ with respect to x was established by E.J. Messerli [9] in 1972 for the case where x is a nonnegative integer. In the case where x is considered to be a nonnegative real value, the convexity (in the wider sense) was established by A.A. Jagers and E. Van Doorn [8] in 1986:

$$B''_{xx}(a,x) \geq 0, \quad \forall (a,x) \in I\!R^+ \times I\!R^+_0 \,. \tag{16}$$

In [3] the authors have shown that $B''_{xx} > 0$, for all real $x \geq 1$ and for all $a > 0$. In the sequel we present a non-constructive proof for the strictness of inequality (16) in the general case.

First, let us prove the following auxiliary proposition:

$$B'_x(a,x) + \frac{1}{a} > 0, \qquad \text{for all } \; (a,x) \in I\!R^+ \times I\!R^+_0 \,. \tag{17}$$

Note that, if $x \geq 0$ and $a > 0$ then we may use the proposition (16) in order to conclude that B'_x is a non decreasing function with respect to x. Therefore:

$$B'_x(a,x) \geq B'_x(a,0), \quad \forall\, (a,x) \in I\!R^+ \times I\!R^+_0 \,.$$

Using expression (5), we have:

$$B'_x(a,x) \geq B'_x(a,0) = -a \int_0^\infty e^{-az} \ln(1+z)\, dz \,.$$

Taking into account the well known inequality $\ln(1+z) < z$, $\forall\, z \in I\!R^+$, we obtain:

$$B'_x(a,x) \;\geq\; B'_x(a,0) \;>\; -a \int_0^\infty e^{-az}\, z\, dz \;=\; -\frac{1}{a} \,.$$

So, the proposition (17) is proved.

As for the third derivatives B'''_{axx} and B'''_{xxa}, sucessive differentiation of (4) in order to x, leads to:

$$\begin{aligned} B''_{ax} &= \left(\frac{x}{a} - 1 + B\right) B'_x + \left(B'_x + \frac{1}{a}\right) B, \\ B'''_{axx} &= 2B'_x \left(B'_x + \frac{1}{a}\right) + \left(\frac{x}{a} - 1 + 2B\right) B''_{xx}. \end{aligned} \tag{18}$$

Since $B(a,x)$ is an infinitely differentiable function in $I\!R^+ \times I\!R$, by the Schwarz theorem we have:

$$\frac{\partial}{\partial\, a} B''_{xx} = B'''_{xxa} = B'''_{axx} \,. \tag{19}$$

Now, statement *1)* of Theorem 1 can be proved. Let us consider an arbitrary nonnegative real value, x_0, and define the function:

$$\begin{aligned} \varphi :\; & I\!R^+ \longrightarrow I\!R \\ & a \longmapsto \varphi(a) = B''_{xx}(a, x_0) \,. \end{aligned}$$

From proposition (16), it remains to prove that φ has no zeros in $I\!R^+$. Let us suppose, on the contrary, that there exists $a_0 \in I\!R^+$ such that $\varphi(a_0) = 0$.

By proposition (16), if φ reaches its minimum in $I\!R^+$, then a_0 is a critical point of φ:

$$\varphi'(a_0) = B'''_{xxa}(a_0, x_0) = 0 \,.$$

Taking into account equality (19), the value of $\varphi'(a_0)$ is given by equation (18). Since $B''_{xx}(a_0, x_0) = 0$, then:

$$\varphi'(a_0) = 2B'_x(a_0, x_0) \left[B'_x(a_0, x_0) + \frac{1}{a} \right] \,.$$

Now using the proposition (17) and the fact that $B'_x(a_0, x_0)$ is strictly negative, we conclude that $\varphi'(a_0) < 0$, which is a contradiction since a_0 is a critical point of φ. Therefore:

$$\varphi(a) > 0, \quad \forall\, a \in I\!R^+ \,,$$

which completes the proof.

2.2 Proof of Statement *2)*

Here the task is to prove the proposition $\widehat{A}''_{aa}(a,x) > 0$, $\forall (a,x) \in I\!R^+ \times I\!R^+$ and $\widehat{A}''_{aa}(a,0) = 0$, $\forall a \in I\!R^+$. If one restricts the domain of x to the positive integers, this is a very well known result due to C. Palm [10, pp.180–181]. In order to generalize this result to the case where $x \in I\!R_0^+$, we will use the main argument of the proof presented by A.A. Jagers and Erik A. Van Doorn in [8, Theorem 1] for establishing that $B''_{xx}(a,x) \geq 0$, $\forall (a,x) \in I\!R^+ \times I\!R_0^+$.

Let us define the functions:

$$\begin{aligned} f(a,x) &= \int_0^{+\infty} e^{-at}\,(1+t)^x\; dt\,, \qquad (20)\\ \Phi(a,x) &= 2\,[f'_a(a,x)]^2 - f(a,x)\,f''_{aa}(a,x)\,. \end{aligned}$$

From (10), we have:

$$\begin{aligned} \widehat{A}(a,x) &= \frac{1}{f(a,x)}\,,\\ \widehat{A}'_a(a,x) &= -\frac{f'_a(a,x)}{[f(a,x)]^2}\,, \qquad (21)\\ \widehat{A}''_{aa}(a,x) &= \frac{2\,[f'_a(a,x)]^2 \,-\, f(a,x)\,f''_{aa}(a,x)}{[f(a,x)]^3} = \frac{\Phi(a,x)}{[f(a,x)]^3}\,. \qquad (22) \end{aligned}$$

Note that $f(a,x)$ is a strictly positive function. Consequently, from equation (22) the functions $\widehat{A}''_{aa}(a,x)$ and $\Phi(a,x)$ have the same sign.

It remains to show the following proposition:

$$\begin{cases} \Phi(a,0) = 0, & \forall\, a \in I\!R^+\,,\\ \Phi(a,x) > 0, & \forall\,(a,x) \in I\!R^+ \times I\!R^+\,. \end{cases}$$

Suppose that $\Phi(a,x)$ is the Laplace transform of some function $g(t,x)$ (such function $g(t,x)$ exists as shown later on):

$$\begin{aligned} \Phi(a,x) &= \int_0^{+\infty} e^{-at}\,g(t,x)\,dt\,, \qquad (23)\\ g(t,x) &= \mathcal{L}^{-1}\left\{\Phi(a,x)\right\}\,,\\ g(t,x) &= 2\,\mathcal{L}^{-1}\left\{f'_a(a,x)\,f'_a(a,x)\right\} - \mathcal{L}^{-1}\left\{f(a,x)\,f''_{aa}(a,x)\right\}\,. \qquad (24) \end{aligned}$$

On the other hand, from (20):

$$\begin{aligned} f'_a(a,x) &= -\int_0^{+\infty} e^{-at}\,(1+t)^x\,t\,dt\,,\\ f''_{aa}(a,x) &= \int_0^{+\infty} e^{-at}\,(1+t)^x\,t^2\;dt\,. \end{aligned}$$

Therefore:

$$\begin{aligned} \mathcal{L}^{-1}\left\{f(a,x)\right\} &= (1+t)^x\,, \qquad (25)\\ \mathcal{L}^{-1}\left\{f'_a(a,x)\right\} &= -(1+t)^x\,t\,, \qquad (26)\\ \mathcal{L}^{-1}\left\{f''_{aa}(a,x)\right\} &= (1+t)^x\,t^2\,. \qquad (27) \end{aligned}$$

Applying the convolution theorem for Laplace transforms:

$$\begin{aligned} \mathcal{L}^{-1}\left\{f'_a(a,x)\,f'_a(a,x)\right\} &= \int_0^t \left[\,(1+u)^x\,u\,\right]\left[\,(1+t-u)^x\,(t-u)\,\right] du\,, \qquad (28)\\ \mathcal{L}^{-1}\left\{f(a,x)\,f''_{aa}(a,x)\right\} &= \int_0^t \left[\,(1+u)^x\,u^2\,\right]\left[\,(1+t-u)^x\,\right] du\,. \qquad (29) \end{aligned}$$

From equations (24), (28) and (29), we may now obtain the analytic expression for $g(t,x)$:

$$\begin{aligned} g(t,x) &= \int_0^t \left[\, 2(1+u)^x\, u\,(1+t-u)^x\,(t-u) \;-\; (1+u)^x\, u^2\,(1+t-u)^x \,\right] du\,, \\ g(t,x) &= \int_0^t [3u^2-2ut]\,[-(1+t-u)^x\,(1+u)^x]\, du\,. \end{aligned}$$

By symmetry arguments[1]:

$$g(t,x) = \int_0^{t/2} \underbrace{[6u^2-6ut+t^2]}_{q(u)}\, \underbrace{[-(1+t-u)^x\,(1+u)^x]}_{p(u)}\, du \;=\; \int_0^{t/2} q(u)\,p(u)\,du\,.$$

Note that if $x=0$, then $p(u)=-1$, $\forall u \in\ [0,t/2]$. It follows that:

$$g(t,0) = -\int_0^{t/2} q(u)\,du = -\left[2u^3-3u^2t+t^2u\right]_0^{t/2} = -\left[\frac{1}{4}t^3-\frac{3}{4}t^3+\frac{1}{2}t^3\right] = 0\,. \tag{30}$$

It remains to analyse the case $x>0$. Firstly, note that $q(0)=t^2>0$ and $q(t/2)=-t^2/2<0$. Furthermore, $q(u)$ is a strictly decreasing function in the interval $[0,t/2]$:

$$q'(u) \;=\; 12u-6t \;<\; 0\,, \quad \forall\, u \,\in\, [0,t/2[\,.$$

Thence there exists a value $u^* \in]0,t/2[$ such that:

$$q(u)>0, \forall u\in]0,u^*[\quad \wedge \quad q(u)<0, \forall u \in]u^*,t/2[\,. \tag{31}$$

On the other hand, it is easy to show that $p(u)$ is a strictly decreasing function in $[0,t/2]$:

$$x>0 \implies p'(u) = x(1+t-u)^{x-1}\,(1+u)^{x-1}\,(2u-t)<0, \quad \forall u \in\ [0,t/2[\,. \tag{32}$$

We may now use Lemma 1 (see Appendix A). In fact, propositions (30), (31) and (32) allows us to apply the condition *i)* of Lemma 1. Therefore, if $t \in I\!R^+$ and $x>0$, it may be concluded that:

$$\int_0^{t/2} q(u)\,du = 0 \implies g(t,x) = \int_0^{t/2} q(u)p(u) > 0.$$

Therefore we may state:

$$\begin{aligned} x=0 &\implies g(t,x)=0, \quad \forall t \in\ I\!R_0^+\,, \\ x>0 &\implies g(t,x)>0, \quad \forall t \in\ I\!R^+\,. \end{aligned}$$

Finally, from these propositions and relations (23) and (22), the intended result is obtained:

$$\begin{aligned} \Phi(a,0)=0, \quad \forall a \in\ I\!R^+ &\implies \widehat{A}''_{aa}(a,0)=0, \quad \forall a \in\ I\!R^+\,, \\ \Phi(a,x)>0, \quad \forall (a,x) \in\ I\!R^+ \times I\!R^+ &\implies \widehat{A}''_{aa}(a,x)>0, \quad \forall (a,x) \in\ I\!R^+ \times I\!R^+\,. \end{aligned}$$

[1] Let us define the function $p(u)=-(1+t-u)^x(1+u)^x$. Introducing the change of variable $v=t-u$: $\int_{t/2}^t (3u^2-2ut)\,p(u)\,du = -\int_{t/2}^0 [3(t-v)^2-2(t-v)t]p(v)\,dv = \int_0^{t/2}(3v^2-4vt+t^2)\,p(v)\,dv$. The previous equality leads to:

$$g(t,x) = \int_0^{t/2}(3u^2-2ut)p(u)\,du + \int_0^{t/2}(3u^2-4ut+t^2)\,p(u)\,du = \int_0^{t/2}(6u^2-6ut+t^2)\,p(u)\,du\,.$$

2.3 Proof of Statement *3)*

In order to complete the proof of Theorem 1, it remains to show that $\widehat{A}''_{ax}(a,x) = \widehat{A}''_{xa}(a,x) < 0$, $\forall(a,x) \in \mathbb{R}^+ \times \mathbb{R}_0^+$. The general idea of the proof is the same one that we have used in the previous subsection. However, the technical details are different and some tedious manipulation of the analytic expressions is needed.

As before, let us consider the function $f(a,x)$ defined by (20). Introducing the function:

$$\Psi(a,x) = 2f'_x(a,x)f'_a(a,x) - f(a,x)\, f''_{ax}(a,x)\,,$$

from expression (21) we obtain:

$$\widehat{A}''_{ax}(a,x) = \frac{2\, f'_x(a,x)\, f'_a(a,x) - f(a,x)\, f''_{ax}(a,x)}{[f(a,x)]^3} = \frac{\Psi(a,x)}{[f(a,x)]^3}\,. \tag{33}$$

Then the proof is equivalent to show the following proposition:

$$\Psi(a,x) < 0, \quad \forall(a,x) \in \mathbb{R}^+ \times \mathbb{R}_0^+\,.$$

Assume that $\Psi(a,x)$ is the Laplace transform of a function denoted by $h(t,x)$, that is:

$$\Psi(a,x) = \int_0^{+\infty} e^{-at}\, h(t,x)\, dt\,, \tag{34}$$

$$h(t,x) = \mathcal{L}^{-1}\left\{\Psi(a,x)\right\}\,,$$

$$h(t,x) = 2\,\mathcal{L}^{-1}\left\{f'_x(a,x)\, f'_a(a,x)\right\} - \mathcal{L}^{-1}\left\{f(a,x)\, f''_{ax}(a,x)\right\}\,. \tag{35}$$

From (20) and (26) we have:

$$\begin{aligned} f'_x(a,x) &= \int_0^{+\infty} e^{-at}\,(1+t)^x\, \ln(1+t)\, dt\,, \\ f''_{ax}(a,x) &= -\int_0^{+\infty} e^{-at}\,(1+t)^x\, t\, \ln(1+t)\, dt\,. \end{aligned}$$

Therefore:

$$\begin{aligned} \mathcal{L}^{-1}\left\{f'_x(a,x)\right\} &= (1+t)^x\, \ln(1+t)\,, \\ \mathcal{L}^{-1}\left\{f''_{ax}(a,x)\right\} &= -(1+t)^x\, t\, \ln(1+t)\,. \end{aligned}$$

From (25), (26) and applying the convolution theorem for Laplace transforms, it may be written:

$$\mathcal{L}^{-1}\left\{f'_x(a,x)\, f'_a(a,x)\right\} = \int_0^t \left[\,(1+u)^x\, \ln(1+u)\,\right]\left[\,-(1+t-u)^x\,(t-u)\,\right] du\,, \tag{36}$$

$$\mathcal{L}^{-1}\left\{f''_{ax}(a,x)\, f(a,x)\right\} = \int_0^t \left[\,-(1+u)^x\, u\, \ln(1+u)\,\right]\left[\,(1+t-u)^x\,\right] du\,. \tag{37}$$

Direct substitution of (36) and (37) into (35), leads to the analytic expression of the function $h(t,x)$:

$$\begin{aligned} h(t,x) &= -\int_0^t [u\ln(1+u) - 2(t-u)\ln(1+u)]\,[-(1+t-u)^x\,(1+u)^x]\, du\,, \\ h(t,x) &= -\int_0^t [(3u-2t)\ln(1+u)]\,[-(1+t-u)^x\,(1+u)^x]\, du\,. \end{aligned}$$

In order to establish the sign of the function $h(t,x)$, firstly we will analyse the case $x = 0$. To conclude that $h(t,0) < 0$, $\forall t \in \mathbb{R}^+$ some tedious manipulation is needed. Using a symbolic programming language (Mathematica), it was found:

$$h(t,0) = \int_0^t (3u-2t)\ln(1+u)\, du\,,$$

$$h(t,0) = \left(-\frac{t^2}{2} - 2t - \frac{3}{2}\right)\ln(1+t) + 5\frac{t^2}{4} + 3\frac{t}{2}\,. \tag{38}$$

Next we will use the inequality:

$$\ln(1+t) > \frac{6t+3t^2}{6+6t+t^2}, \quad \forall t \in I\!R^+ .$$

This inequality is cited in [13, pag.124] and has been established by using Padé approximation method. Direct substitution into expression (38), leads to the following inequalities (which are true for $t > 0$):

$$\begin{aligned} h(t,0) &< \left(-\frac{t^2}{2} - 2t - \frac{3}{2}\right)\frac{6t+3t^2}{6+6t+t^2} + 5\frac{t^2}{4} + 3\frac{t}{2}, \\ h(t,0) &< -\frac{t^4}{4t^2+24t+24} < 0, \quad \forall\, t \in I\!R^+ . \end{aligned} \tag{39}$$

Again, to obtain (39) we used the Mathematica program.

The following remark concludes the analysis of the case $x = 0$. Indeed, if $x = 0$, then from (39) and (34), we have $\Psi(a,0) < 0$ and consequently $\widehat{A}''_{ax}(a,0) < 0$ for all $a \in I\!R^+$.

It remains to analyse the sign of the function $h(t,x)$ in the case $x > 0$. Firstly, by symmetry arguments[2], the following expression is obtained:

$$h(t,x) = -\int_0^{t/2} \underbrace{[(3u-2t)\ln(1+u) + (t-3u)\ln(1+t-u)]}_{r(u)} \underbrace{[-(1+t-u)^x\,(1+u)^x]}_{p(u)}\, du . \tag{40}$$

From (32) $p(u)$ is a strictly decreasing function in $[0, t/2]$, for $x > 0$. Let us now analyse the sign of $r'(u)$, in $[0, t/2]$:

$$\begin{aligned} r'(u) &= 3\ln(1+u) + \frac{3u-2t}{1+u} - 3\ln(1+t-u) - \frac{t-3u}{1+t-u}, \\ r'(u) &= -3\left[\ln(1+t-u) - \ln(1+u)\right] + \frac{6u+4ut-2t^2-3t}{(1+u)(1+t-u)} . \end{aligned} \tag{41}$$

Since, for $t > 0$:

$$\begin{aligned} \ln(1+u) &< \ln(1+t-u), \quad \forall\, u \in [0, t/2[\,, \\ 6u + 4ut - 2t^2 - 3t &< 0, \quad \forall\, u \in [0, t/2[\,, \end{aligned}$$

it follows from (41):

$$r'(u) < 0, \quad \forall\, u \in [0, t/2[\,,$$

that is, $r(u)$ is a strictly decreasing function in $[0, t/2]$.

Additionnaly, for $t > 0$:

$$r(0) = t\ln(1+t) > 0 \quad \wedge \quad r(t/2) = -t\ln(1+t/2) < 0 .$$

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[2] Denoting $p(u) = -(1+t-u)^x(1+u)^x$, and after change of variable ($v = t - u$), it may be written:

$$\int_{t/2}^{t} (3u-2t)\,\ln(1+u)\; p(u)\, du = -\int_{t/2}^{0} [3(t-v)-2t]\ln(1+t-v)\; p(v)\, dv = \int_0^{t/2} (t-3v)\ln(1+t-v)\; p(v)\, dv .$$

Consequently we have:

$$\begin{aligned} h(t,x) &= -\int_0^{t/2} (3u-2t)\ln(1+u)\; p(u)\, du - \int_0^{t/2} (t-3u)\ln(1+t-u)\; p(u)\, du , \\ h(t,x) &= -\int_0^{t/2} [(3u-2t)\ln(1+u) + (t-3u)\ln(1+t-u)]\; p(u)\, du . \end{aligned}$$

Therefore it exists a real value $u^* \in]0, t/2[$, such that:

$$r(u) > 0, \quad \forall u \in]0, u^*[\quad \wedge \quad r(u) < 0, \quad \forall u \in]0, u^*[\,.$$

Taking (32) into account, the condition *iii)* of Lemma 1 (see Appendix A) may be applyed, and:

$$h(t,0) = \int_0^{t/2} r(u)\, du < 0 \quad \Longrightarrow \quad \int_0^{t/2} r(u)\, p(u)\, du \; > \; 0\,.$$

From (40) it is immediate that:

$$h(t,x) = -\int_0^{t/2} r(u)\, p(u)\, du \; < \; 0, \quad \forall t \in I\!R^+, \quad \forall x \in I\!R^+\,. \tag{42}$$

Finally, using the inequality (42) and equation (34) it is straightforward that $\Psi(a,x) < 0$, $\forall (a,x) \in I\!R^+ \times I\!R^+$. Since it has been shown that $\Psi(a,0) < 0$, $\forall a \in I\!R^+$, the proof is complete.

3 Joint Convexity Analysis

In this section the joint convexity of $\widehat{A}(a,x)$ in the two variables, in some open convex subdomain of $I\!R^+ \times I\!R_0^+$ is discussed. Indeed, it may be questioned if there exists an open convex subdomain of $I\!R^+ \times I\!R_0^+$ where $\widehat{A}(a,x)$ is convex (or strictly convex). This question raises several difficulties and the present (incomplete) discussion finishes with a conjecture.

Firtly, let us calculate the Hessian matrix of $\widehat{A}(a,x)$:

$$\nabla^2 \widehat{A}(a,x) = \begin{bmatrix} \widehat{A}''_{aa}(a,x) & \widehat{A}''_{ax}(a,x) \\ \widehat{A}''_{xa}(a,x) & \widehat{A}''_{xx}(a,x) \end{bmatrix}.$$

Note that by Schwarz theorem $\widehat{A}''_{ax}(a,x) = \widehat{A}''_{xa}(a,x)$, $\forall\,(a,x) \in I\!R^+ \times I\!R_0^+$, consequently $\nabla^2 \widehat{A}(a,x)$ is a symmetric matrix. The analytic expressions for the second order partial derivatives of $\widehat{A}(a,x)$ were presented in (13), (14) and (15).

In order to establish the strict joint convexity of $\widehat{A}(a,x)$ in some open convex subdomain of $I\!R^+ \times I\!R^+$, it suffices to show that $\nabla^2 \widehat{A}(a,x)$ is a positive definite matrix in that subdomain. To obtain an equivalent condition, we apply the Sylvester criterion. Therefore, if the two following inequalities hold,

$$\widehat{A}''_{aa} \; > \; 0\,, \tag{43}$$

$$\left|\nabla^2 \widehat{A}(a,x)\right| \; = \; \widehat{A}''_{aa} \widehat{A}''_{xx} - [\widehat{A}''_{ax}]^2 \; > \; 0\,, \tag{44}$$

then $\nabla^2 \widehat{A}(a,x)$ is a positive definite matrix. As for inequality (43) it results directly from statement *2)* of Theorem 1, in $I\!R^+ \times I\!R^+$.

Introducting the function:

$$\Delta(a,x) \; = \; \left|\nabla^2 \widehat{A}(a,x)\right| \; = \; \widehat{A}''_{aa}(a,x)\, \widehat{A}''_{xx}(a,x) - [\widehat{A}''_{ax}(a,x)]^2\,, \tag{45}$$

the problem reduces to find an open convex subdomain $\mathcal{D} \subset I\!R^+ \times I\!R^+$ such that $\Delta(a,x) \; > \; 0$.

3.1 A Counterexample

In this subsection we present a counterexample showing that $\widehat{A}(a,x)$ is not a convex function in $I\!R^+ \times I\!R^+$. Firstly note that by direct application of statement *2)* of Theorem 1, if $x = 0$ then $\widehat{A}''_{aa}(a,x) = 0$. Using (45) it is concluded that $\Delta(a,x) \; < \; 0$. Consequently, $\widehat{A}(a,x)$ is not a convex function in the domain $I\!R^+ \times I\!R_0^+$.

Now let us apply the classical recursion for calculating the Erlang-B formula:

$$\begin{array}{rclcrcl} B(a,0) & = & 1 & \Longrightarrow & \widehat{A}(a,0) & = & a\,, \\ B(a,1) & = & \dfrac{a}{a+1} & \Longrightarrow & \widehat{A}(a+1,1) & = & \dfrac{(a+1)^2}{a+2}\,, \\ B(a,2) & = & \dfrac{a^2}{a^2+2a+2} & \Longrightarrow & \widehat{A}(a+2,2) & = & \dfrac{(a+2)^3}{(a+2)^2+2(a+2)+2}\,, \\ B(a,3) & = & \dfrac{a^3}{a^3+3a^2+6a+6} & \Longrightarrow & \widehat{A}(a+3,3) & = & \dfrac{(a+3)^4}{(a+3)^3+3(a+3)^2+6(a+3)+6}\,, \end{array}$$

and consider a (fixed) arbitrary $a \in I\!R^+$. Let us consider the point $(a+2,2) \in I\!R^2$ which is the medium point of the segment whose extreme points are $(a+1,1)$ and $(a+3,3)$:

$$(a+2,2) \;=\; \frac{1}{2}\,(a+1,1) \;+\; \frac{1}{2}\,(a+3,3)\,.$$

Suppose, by hypothesis, that $\widehat{A}(a,x)$ is a convex function in the domain $I\!R^+ \times I\!R^+$. Applying the definition of convex function, the following inequality holds:

$$\widehat{A}(a+2,2) \;\leq\; \frac{1}{2}\,\widehat{A}(a+1,1) \;+\; \frac{1}{2}\,\widehat{A}(a+3,3)\,.$$

This condition is equivalent to the following ones:

$$\frac{(a+2)^3}{(a+2)^2+2(a+2)+2} - \frac{1}{2}\left[\frac{(a+1)^2}{a+2} + \frac{(a+3)^4}{(a+3)^3+3(a+3)^2+6(a+3)+6}\right] \;\leq\; 0\,,$$

$$D(a) \;=\; \frac{(a+2)^3}{(a+2)^2+2(a+2)+2} - \frac{(a+1)^2}{2(a+2)} - \frac{(a+3)^4}{2(a+3)^3+6(a+3)^2+12(a+3)+12} \;\leq\; 0\,. \qquad (46)$$

After some tedious algebric manipulation (with the aid of MATHEMATICA), the analytic expression of $D(a)$ may be simplified:

$$D(a) \;=\; \frac{2a^3+24a^2+72a+48}{a^6+20a^5+169a^4+770a^3+1986a^2+2736a+1560}\,. \qquad (47)$$

Finally, it is immediate from (47) that $D(a) > 0,\ \forall\, a \in I\!R^+$, which contradicts the condition (46). It follows that it is absurd to consider $\widehat{A}(a,x)$ as a convex function in $I\!R^+ \times I\!R^+$.

From the presented counterexample it is easy to conclude that $\widehat{A}(a,x)$ is *not* a convex function in the following subdomain:

$$\mathcal{G} \;=\; \left\{\, (a,x) \in I\!R^+ \times I\!R^+ \;:\; a \,>\, x \,\right\}\,.$$

Consequently, let us focuss our atention on the following open convex subdomain of $I\!R^+ \times I\!R^+$:

$$\mathcal{H} \;=\; \left\{\, (a,x) \in I\!R^+ \times I\!R^+ \;:\; a \,<\, x \,\right\}\,.$$

It may be questioned if $\widehat{A}(a,x)$ is a *convex* (or *strictly convex*) function in $\mathcal{H}$ (or, alternativelly, in some subdomain of $\mathcal{H}$). Note that the set $\mathcal{H}$ defines a subdomain of low blocking. In other words, it may be questioned if $\widehat{A}(a,x)$ is a joint convex (or even stictly convex) function in areas of low blocking (which are specialy important in teletraffic applications).

3.2 Computational Experiments and Conjectures

The numerical values of the function $\Delta(a,x)$, defined by (45), may be calculated using the analytical expressions of the second order partial derivatives $\widehat{A}''_{xx}$, $\widehat{A}''_{aa}$ and $\widehat{A}''_{ax}$ given, respectively, by the relations (13), (14) and (15). In this context for calculating the Erlang-B function and its derivatives we have implemented the algorithms proposed in [2] e [3].

x	a	x	a	x	a	x	a
1.5	30.4856	15.00	16.78463	70.0	69.8533	600.0	593.637
1.75	17.9625	20.0	21.47984	75.0	74.7377	700.0	692.885
2.0	13.2023	25.0	26.2363	80.0	79.626	800.0	792.185
3.0	8.46005	30.0	31.0270	85.0	84.518	900.0	891.528
4.0	8.01297	35.0	35.8417	90.0	89.4139	1E3	990.906
5.0	8.35763	40.0	40.6723	95.0	94.3124	1E4	9964.73
6.0	8.97296	45.0	45.5156	100.0	99.213	1E5	99882.0
7.0	9.71151	50.0	50.3689	200.0	197.611	1E6	999620.
8.0	10.5165	55.0	55.2304	300.0	296.389	1E7	9998793.
9.0	11.3695	60.0	60.99019	400.0	395.361	1E8	99996180.
10.0	12.2328	65.0	64.9735	500.0	494.455		

Table 1: Aproximations for the solutions of $\Delta(a, x) = 0$.

For a fixed value $x > 1$, the performed computational experiments seem to sugest that $\Delta(a, x)$ is positive for some value a sufficiently small, and negative for a sufficiently high. Additionaly, they sugests that probably the equation $\Delta(a, x) = 0$ has unique solution (for $x > 1$). Subsequently by fixing different values of x, we have solved the equation $\Delta(a, x) = 0$. The solutions were obtained by an iterative numerical method (secant method) and the results are shown in Table 1. Anyway it must be stressed that we have no proof that equation $\Delta(a, x) = 0$ has unique solution.

However, the numerical experiments presented and the discussion made in this section led us to propose a conjecture concerning a region of strict joint convexity of the function $\widehat{A}(a, x)$. The proposed region is defined by:

$$\mathcal{S} = \left\{ (a, x) \in I\!R^+ \times I\!R^+ \, : \, a < x - \sqrt{x} \right\} .$$

In [6], a *standard offered traffic* parameter is defined by:

$$c = \frac{a - x}{\sqrt{x}} .$$

Then the points of the set $\mathcal{S}$ are caracterized by the condition $c < -1$.

Finally, let us discuss a question related to the one analysed in this section: the joint convexity of the Erlang-B function itself. In fact, fixing x, for a sufficiently low we have $B''_{aa} > 0$ (see [1]). Therefore the question about the existence of some region of joint convexity of $B(a, x)$ may be posed. *A priori* it seems to be possible the existence of regions of joint convexity for both the functions $B(a, x)$ and $\widehat{A}(a, x)$.

4 Concluding Remarks

With Theorem 1 the signs of the second order derivatives of the overflow traffic function $\widehat{A}(a, x)$ from the Erlang-B system were established, thence completing and proving known results. From these results, some conditions arise on the strict convexity of $\widehat{A}(a, x)$ with respect to the variables a and x. These convexity properties have potential interest in applications, mainly concerning optimization techniques (*e.g.* load sharing and server allocation problems and optimal design of certain teletraffic networks). Also a new result concerning the sign of the rectangular second order derivatives of $\widehat{A}(a, x)$ is proved. Finally, the question of the strict convexity of $\widehat{A}(a, x)$ in some open convex subdomain of $I\!R^+ \times I\!R^+$ was discussed. This question was approached by considering some particular cases. Making use of extensive computational results, a conjecture was proposed according to which the function $\widehat{A}(a, x)$ is strictly convex in areas of low blocking, where the *standard offered traffic* is less than -1.

Appendix A Auxiliary Lemma

Lemma 1 *Let $g(z)$ and $h(z)$ be real-valued functions defined in $[\alpha, \beta[\subset I\!R$ satisfying:*

$$\{\ g(z) > 0\ \wedge\ h(z) > \kappa, \quad \forall\, z \in]\alpha, z^*[\ \}\ \ \bigwedge\ \ \{\ g(z) < 0\ \wedge\ h(z) < \kappa, \quad \forall\, z \in]z^*, \beta[\ \}\ , \tag{48}$$

for fixed values $z^* \in]\alpha, \beta[$ and $k \in I\!R$. If $U = \int_\alpha^\beta g(z)\, dz$ and $V = \int_\alpha^\beta g(z)h(z)\, dz$ exist and are finite for some $\beta \in]\alpha, +\infty]$, then $V > 0$ whenever one of the following conditions holds:

$$\textit{i)}\ U = 0\,, \qquad \textit{ii)}\ k > 0 \wedge U \geq 0\,, \qquad \textit{iii)}\ k < 0 \wedge U \leq 0\,, \qquad \textit{iv)}\ k = 0\,. \tag{49}$$

Proof. Let us consider the following decompositions:

$$V = \underbrace{\int_\alpha^{z^*} g(z)h(z)\, dz}_{V_1} + \underbrace{\int_{z^*}^\beta g(z)h(z)\, dz}_{V_2} \qquad \text{e} \qquad U = \underbrace{\int_\alpha^{z^*} g(z)\, dz}_{U_1} + \underbrace{\int_{z^*}^\beta g(z)\, dz}_{U_2}\,.$$

It is clear that V_1, V_2, U_1 and U_2 exist and are finite. From condition (48) it is true that $U_1 > 0$ and $U_2 < 0$; therefore these integrals are nonzero. Consequently, there are real numbers κ_1 and κ_2 such that $\kappa_1 = V_1/U_1$ and $\kappa_2 = V_2/U_2$. Therefore, we may write $V = \kappa_1 U_1 + \kappa_2 U_2$ and state the following:

$$g(z)h(z) > g(z)\kappa,\ \ \forall z \in]\alpha, z^*[\ \implies\ V_1 > \int_\alpha^{z^*} g(z)\kappa\, dz \implies \kappa_1 U_1 > \kappa U_1 \implies \kappa_1 > \kappa\,, \tag{50}$$

$$g(z)h(z) > g(z)\kappa,\ \ \forall z \in]z^*, \beta[\ \implies\ V_2 > \int_{z^*}^\beta g(z)\kappa\, dz \implies \kappa_2 U_2 > \kappa U_2 \implies \kappa_2 < \kappa\,. \tag{51}$$

From (50) and (51) it follows that $\kappa_1 > \kappa_2$. Finally, let us prove that each of the four conditions in (49) implies that $V > 0$:

(i) If $U = 0$, then $U_2 = -U_1$. Thus: $V = \kappa_1 U_1 + \kappa_2 U_2 = \kappa_1 U_1 - \kappa_2 U_1 = U_1[\kappa_1 - \kappa_2] > 0\,.$

(ii) If $U \geq 0$, then $U_1 \geq -U_2$. Since $\kappa > 0$, then $g(z)h(z) > g(z)\kappa > 0,\ \ \forall z \in]\alpha, z^*[$, and $V_1 > 0$. Further $U_1 > 0$, thus $\kappa_1 = V_1/U_1 > 0$. Since $\kappa_1 > 0$, then $\kappa_1 U_1 \geq -\kappa_1 U_2$. Adding $\kappa_2 U_2$ to both sides of the previous inequality we have $V \geq -\kappa_1 U_2 + \kappa_2 U_2 = U_2[\kappa_2 - \kappa_1] > 0$.

(iii) If $U \leq 0$, then $U_2 \leq -U_1$. Since $\kappa < 0$, then $g(z)h(z) > g(z)\kappa > 0, \quad \forall z \in]z^*, \beta[$, and $V_2 > 0$. Furthermore $U_2 < 0$, thus $\kappa_2 = V_2/U_2 < 0$. Since $\kappa_2 < 0$, then $\kappa_2 U_2 \geq -\kappa_2 U_1$. Adding $\kappa_1 U_1$ to both sides of the previous inequality we have $V \geq -\kappa_2 U_1 + \kappa_1 U_1 = U_1[\kappa_1 - \kappa_2] > 0$.

(iv) If $k = 0$, then $g(z)h(z) > 0, \forall z \in]\alpha, z^*[\cup]z^*, \beta[$. Clearly V is positive.

□

References

[1] Arel, A. Convexity Properties of the Erlang Loss Formula. *Operations Research* 38, 3 (1990), 499–505.

[2] J. S. Esteves, J. Craveirinha and D. Cardoso, Computing Erlang-B Function Derivatives, Research Report ET-T3, INESC-Coimbra, Nov. 1993.

[3] Esteves, J.S., Craveirinha, J., and Cardoso, D. Computing Erlang-B Function Derivatives in the Number of Servers — A Generalized Recursion. *ORSA Commun. Stat., Stochastic Models* 11, 2 (1994), 311–331.

[4] Feller, W. *An Introduction to Probability Theory and Its Applications.* Vol. II, $2.^{nd}$ Edition, Wiley, New York, 1970.

[5] Girard, A. *Routing and Dimensioning in Circuit-Switched Networks.* Addison-Wesley, Reading, Massachusetts, 1990.

[6] Jagerman, D.L. Some Properties of the Erlang Loss Function. *The Bell System Technical Journal* 53, 3 (1974), 525–551.

[7] Jagerman, D.L. Methods in Traffic Calculations. *The Bell System Technical Journal* 63, 7 (1984), 1283–1310.

[8] Jagers, A.A., and Doorn, E.A. On the Continued Erlang Loss Function. *Operations Research Letters* 5, 1 (1986), 43–46.

[9] Messerli, E.J. Proof of a Convexity Property of the Erlang-B Formula. *The Bell System Technical Journal* 51 (1972), 951–953.

[10] Palm, C. *Intensity Variations in Telephone Traffic.* North-Holland Studies in Telecomminication, Vol. 10, Amsterdam, 1988.

[11] Rapp, Y. Planning of Junction Network in a Multi-Exchange Area, Part 1. *Ericsson Technics* 20 (1964), 77–130.

[12] Reiman, M. Some Allocation Problems for Critically Loaded Loss Systems with Independent Links. *Performance Evaluation* 13 (1991), 17–25.

[13] Shenton, L.R., and Kemp, A.W. An S-fraction and $\ln^2(1+x)$. *Journal of Computational and Applied Mathematics* 25 (1989), 121-124.

[14] Sysky, R. *Congestion Theory in Telephone Systems.* Oliver and Boyd, Edinburgh, 1960.

[15] Wilkinson, R.I. Theories for Toll Traffic Engineering in the U.S.A.. *The Bell System Technical Journal* 35 (1956), 421–514.