On the multiplicity of α as eigenvalue of $A_\alpha(G)$ of graphs with pendant vertices

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Abstract

Let G be a simple undirected graph. Let $0 \leq \alpha \leq 1$. Let

$$A_\alpha(G) = \alpha D(G) + (1 - \alpha)A(G)$$

where $D(G)$ and $A(G)$ are the diagonal matrix of the vertex degrees of G and the adjacency matrix of G, respectively. Let $p(G) > 0$ and $q(G)$ be the number of pendant vertices and quasi-pendant vertices of G, respectively. Let $m_G(\alpha)$ be the multiplicity of α as eigenvalue of $A_\alpha(G)$. It is proved that

$$m_G(\alpha) \geq p(G) - q(G)$$

with equality if each internal vertex is a quasi-pendant vertex. If there is at least one internal vertex which is not a quasi-pendant vertex, the equality

$$m_G(\alpha) = p(G) - q(G) + m_N(\alpha)$$

is determined in which $m_N(\alpha)$ is the multiplicity of α as eigenvalue of the matrix N. This matrix is obtained from $A_\alpha(G)$ taking the entries corresponding to the internal vertices which are non quasi-pendant vertices. These results are applied to search for the multiplicity of α as eigenvalue of $A_\alpha(G)$ when G is a path, a caterpillar, a circular caterpillar, a generalized Bethe tree or a Bethe tree. For the Bethe tree case, a simple formula for the nullity is given.

Keywords: Adjacency matrix, signless Laplacian matrix, Laplacian matrix, convex combination of matrices, eigenvalues, nullity.
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1. Introduction

Let $G = (V(G), E(G))$ be a simple undirected graph on n vertices with vertex set $V(G)$ and edge set $E(G)$. Let $D(G)$ be the diagonal matrix of order n whose $(i,i)-$entry is the degree of the $i-th$ vertex of G and let $A(G)$ be the adjacency matrix of G. The matrices $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ are the Laplacian and signless Laplacian matrix of G, respectively. The matrices $L(G)$ and $Q(G)$ are both positive semidefinite and $(0, \mathbf{1})$ is an eigenpair of $L(G)$ where $\mathbf{1}$ is the all ones vector. For a connected graph G, the smallest eigenvalue of $Q(G)$ is positive if and only if G is non-bipartite.

In [9], the family of matrices $A_\alpha(G)$,

$$A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$$

with $\alpha \in [0,1]$, is introduced together with a number of some basic results and several open problems.

Observe that $A_0(G) = A(G)$ and $A_{1/2}(G) = \frac{1}{2}Q(G)$.

A pendant vertex is a vertex of degree 1 and a quasi-pendant vertex is a vertex adjacent to a pendant vertex. Let $p(G)$ be the number of pendant vertices and $q(G)$ be the number of quasi-pendant vertices. An internal vertex is a vertex of degree at least 2. Throughout this paper, we assume that G is a graph with pendant vertices.

The multiplicity of μ as an eigenvalue of the matrix M is denoted by $m_M(\mu)$.

The following results are due to I. Faria [6].

Lemma 1. *[6] For any graph G,*

$$m_{L(G)}(1) \geq p(G) - q(G) \tag{1}$$

and

$$m_{Q(G)}(1) \geq p(G) - q(G). \tag{2}$$

In [1], it is proved that the equalities in (1) and (2) occur if each internal vertex is a quasi-pendant vertex. Moreover, if there is at least one internal vertex which is not a quasi-pendant vertex then equalities

$$m_{L(G)}(1) = p(G) - q(G) + m_{N^-}(1)$$

and

$$m_{Q(G)}(1) = p(G) - q(G) + m_{N^+}(1)$$

are determined, where the matrices N^- and N^+ are obtained from the Laplacian matrix and signless Laplacian matrix, respectively, taking the entries corresponding to the internal vertices which are non-quasi-pendant vertices.

We already observed that $A_{1/2}(G) = \frac{1}{2}Q(G)$. Then the above mentioned results can be used to find the multiplicity of $1/2$ as eigenvalue of $A_{1/2}(G)$.

In this paper, we search for the multiplicity of $\alpha \in [0,1]$ as eigenvalue of $A_\alpha(G)$. The multiplicity of $\alpha = 0$ as eigenvalue of $A(G)$ is known as the nullity of G and it has been extensively studied (see [12, 13, 15]) and plays an important role in Chemistry. Some of its applications are described in [2, 5, 7, 8, 14, 16].

Since $A_1(G) = D(G)$, from now on, we consider $\alpha \in [0,1)$.

For simplicity, we write $m_G(\alpha)$ instead $m_{A_\alpha(G)}(\alpha)$. More precisely, we prove that

$$m_G(\alpha) \geq p(G) - q(G). \tag{3}$$

If each internal vertex of G is a quasi-pendant vertex, it is proved that the equality in (3) holds. If there is at least one internal vertex which is not a quasi-pendant vertex, the equality

$$m_G(\alpha) = p(G) - q(G) + m_N(\alpha)$$

is determined in which $m_N(\alpha)$ is the multiplicity of α as eigenvalue of the matrix N. This matrix is obtained from $A_\alpha(G)$ taking the entries corresponding to the internal vertices which are non quasi-pendant vertices.

Let $r(G)$ be the number of internal vertices of G. From now on, G is a connected graph on n vertices. Let

$$V_P = \{v \in V(G) : v \text{ is a pendant vertex}\},$$

$$V_Q = \{v \in V(G) : v \text{ is a quasi-pendant vertex}\}$$

and

$$C(G) = V(G) \setminus (V_P \cup V_Q).$$

Then $C(G)$ is the set of the internal vertices of G which are not quasi-pendant vertices.

Throughout the text, $|S|$ denotes the cardinality of the set S. Clearly $|V_P| = p(G)$, $|V_Q| = q(G)$ and $|C(G)| = n - p(G) - q(G)$.

We introduce some additional notation.

- The identity matrix is denoted by I and the zero matrix by 0.
- If M is a matrix of order $m \times m$ with $m \geq 2$ then $\widetilde{M}$ is the matrix obtained from M by deleting its last row and its last column.
- The determinant of a square matrix M is denoted by $|M|$ and the transpose of M by M^T.
- E denotes a matrix whose entries are zeros except the entry in the last row and last column which is 1.
- $G[F]$ denotes the subgraph of G induced by $F \subseteq V(G)$.

The orders of the matrices I, 0 and E will be clear from the context.

We recall Lemma 2.2 of [11] that will play an important role in this paper.

Lemma 2. *[11] For $i = 1, 2, \ldots, m$, let B_i be a matrix of order $k_i \times k_i$ and $\mu_{i,j}$ be arbitrary scalars. Then*

$$\left| \begin{array}{ccccc} B_1 & \mu_{1,2}E & \cdots & \mu_{1,m-1}E & \mu_{1,m}E \\ \mu_{2,1}E^T & B_2 & \cdots & \cdots & \mu_{2,m}E \\ \mu_{3,1}E^T & \mu_{3,2}E^T & \ddots & \cdots & \vdots \\ \vdots & \vdots & \vdots & B_{m-1} & \mu_{m-1,m}E \\ \mu_{m,1}E^T & \mu_{m,2}E^T & \cdots & \mu_{m,m-1}E^T & B_m \end{array} \right|$$

$$= \left| \begin{array}{ccccc} |B_1| & \mu_{1,2}\left|\widetilde{B_2}\right| & \cdots & \mu_{1,m-1}\left|\widetilde{B_{m-1}}\right| & \mu_{1,m}\left|\widetilde{B_m}\right| \\ \mu_{2,1}\left|\widetilde{B_1}\right| & |B_2| & \cdots & \cdots & \mu_{2,m}\left|\widetilde{B_m}\right| \\ \mu_{3,1}\left|\widetilde{B_1}\right| & \mu_{3,2}\left|\widetilde{B_2}\right| & \ddots & \cdots & \vdots \\ \vdots & \vdots & \vdots & |B_{m-1}| & \mu_{m-1,m}\left|\widetilde{B_m}\right| \\ \mu_{m,1}\left|\widetilde{B_1}\right| & \mu_{m,2}\left|\widetilde{B_2}\right| & \cdots & \mu_{m,m-1}\left|\widetilde{B_{m-1}}\right| & |B_m| \end{array} \right|.$$

The following result is Corollary 1.3 in [1].

Corollary 1. *If $k_i = 1$, for some $1 \leq i \leq m$, then defining $\widetilde{B_i} = 1$ the equality in Lemma 2 also holds.*

Another previous result that we need is the following.

Lemma 3. *Consider the square matrix of order $s+1$*

$$S(\alpha) = \begin{bmatrix} \alpha & 0 & \dots & 0 & 1-\alpha \\ 0 & \alpha & \dots & 0 & 1-\alpha \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \alpha & 1-\alpha \\ 1-\alpha & 1-\alpha & \dots & 1-\alpha & \alpha d \end{bmatrix}.$$

Then the characteristic polynomial of $S(\alpha)$ is

$$|xI - S(\alpha)| = (x-\alpha)^{s-1}((x-\alpha d)(x-\alpha) - s(1-\alpha)^2).$$

Throughout this paper, unless otherwise stated, $v_1, v_2, \dots, v_{r(G)}$ are the internal vertices of G. As usual, $u \sim v$ means that the vertices u and v are adjacent. For the internal vertices, d_i is the degree of the vertex v_i as a vertex of G and $\varepsilon_{i,j} = 1$ if $v_i \sim v_j$ and $\varepsilon_{i,j} = 0$, otherwise. For instance, $d_7 = 6$, $\varepsilon_{3,7} = 1$ and $\varepsilon_{7,10} = 0$ (see Figure 1 and Figure 2).

We label the vertices of G with the numbers $1, 2, \dots, n$, starting with the vertices of the stars $K_{1,s_1}, \dots, K_{1,s_{q(G)}}$, at each star the vertices are labeled beginning with the pendant vertices, and finishing with the internal vertices which are not quasi-pendants. This labeling of the vertices of G is herein called the global labeling of the vertices of G. Notice that, since the internal vertices of G are also denoted by $v_1, \dots, v_{q(G)}, \dots, v_{r(G)}$, for $j = 1, \dots, q(G)$, each of these vertices v_j corresponds, in the global labeling of the vertices of G, to the vertex $\sum_{i=1}^{j} s_i + j$ and the vertices $v_{q(G)+1}, \dots, v_{r(G)}$ correspond to the vertices $q(G) + p(G) + 1, \dots, n$, respectively.

2. Each internal vertex is a quasi-pendant vertex

In this section we consider the case in which each internal vertex of G is a quasi-pendant vertex. Then $r(G) = q(G)$ and there are stars $K_{1,s_1}, \dots, K_{1,s_{q(G)}}$ such that G is obtained by identifying the root of K_{1,s_i} with the $i-$th vertex of the graph induced by the quasi-pendant vertices. Moreover, $s_1 + s_2 + \dots + s_{q(G)} + q(G) = n$ and $s_i \geq 1$ for all i. We denote this graph G by $G(s_1, s_2, \dots, s_{q(G)})$. In Figure 1, we have an example of such a graph.

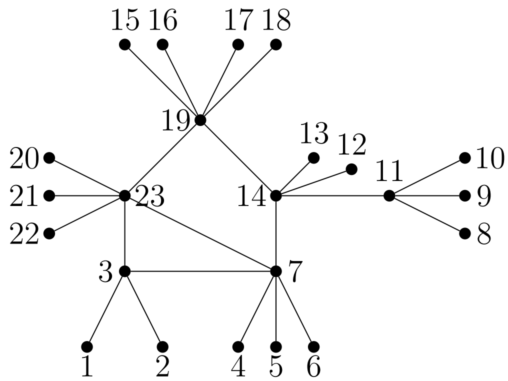

Figure 1: A example where each internal vertex is quasi-pendant.

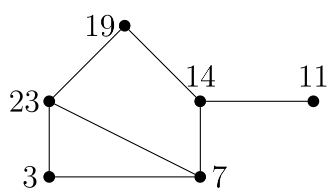

Figure 2: The subgraph induced by the internal vertices $v_1 = 3, v_2 = 7, v_3 = 11, v_4 = 14, v_5 = 19$ and $v_6 = 23$ of the graph in Figure 1.

For $i = 1, 2, \ldots, q(G)$, let

$$S_i(\alpha) = \begin{bmatrix} \alpha & 0 & \ldots & 0 & 1-\alpha \\ 0 & \alpha & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \alpha & 1-\alpha \\ 1-\alpha & \ldots & \ldots & 1-\alpha & \alpha d_{v_i} \end{bmatrix}$$

of order $(s_i + 1) \times (s_i + 1)$ and

$$C_i(\alpha) = \begin{bmatrix} \alpha & (1-\alpha)\sqrt{s_i} \\ (1-\alpha)\sqrt{s_i} & \alpha d_{v_i} \end{bmatrix}.$$

From the definition of the matrices S_i and C_i, and Lemma 3, we have the following corollary.

Corollary 2. *For* $i = 1, 2, \ldots, q(G)$,

1. $$|xI - S_i(\alpha)| = (x-\alpha)^{s_i - 1}|xI - C_i(\alpha)|.$$

   and

2. $$|\widetilde{xI - S_i}(\alpha)| = (x-\alpha)^{s_i}.$$

From now on, let $1 - \alpha = \beta$. Using the above mentioned labeling the matrix $A_\alpha(G)$, $G = G(s_1, s_2, \ldots, s_{q(G)})$, becomes

$$A_\alpha(G) = \begin{bmatrix} S_1(\alpha) & \varepsilon_{1,2}\beta E & \ldots & \varepsilon_{1,q(G)}\beta E \\ \varepsilon_{1,2}\beta E^T & S_2(\alpha) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \varepsilon_{q(G)-1,q(G)}\beta E \\ \varepsilon_{1,q(G)}\beta E^T & \ldots & \varepsilon_{q(G)-1,q(G)}\beta E^T & S_{q(G)}(\alpha) \end{bmatrix}$$

The next theorem gives the spectrum of $A_\alpha(G)$ if $G = G\left(s_1, s_2, \ldots, s_{q(G)}\right)$.

Theorem 1. *If* $G = G\left(s_1, s_2, \ldots, s_{q(G)}\right)$, *the eigenvalues of* $A_\alpha(G)$ *are* α *with multiplicity at least* $p(G) - q(G)$ *and the eigenvalues of the* $2q(G) \times 2q(G)$ *matrix*

$$X = \begin{bmatrix} C_1(\alpha) & \varepsilon_{1,2}\beta E & \cdots & \cdots & \varepsilon_{1,q(G)}\beta E \\ \varepsilon_{1,2}\beta E & C_2(\alpha) & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & C_{q(G)-1}(\alpha) & \varepsilon_{q(G)-1,q(G)}\beta E \\ \varepsilon_{1,q(G)}\beta E & \cdots & \cdots & \varepsilon_{q(G)-1,q(G)}\beta E & C_{q(G)}(\alpha) \end{bmatrix}.$$

Proof. Applying Lemma 2 and Corollary 2, together with a factoring in each column, we have $|xI - A_\alpha(G)| =$

$$\begin{vmatrix} (xI - S_1(\alpha)) & -\varepsilon_{1,2}\beta E & \ldots & -\varepsilon_{1,q(G)}\beta E \\ -\varepsilon_{1,2}\beta E^T & (xI - S_2(\alpha)) & \ddots & \vdots \\ \vdots & \ddots & \ddots & -\varepsilon_{q(G)-1,q(G)}\beta E \\ -\varepsilon_{1,q(G)}\beta E^T & \ldots & -\varepsilon_{q(G)-1,q(G)}\beta E^T & (xI - S_{q(G)}(\alpha)) \end{vmatrix} =$$

$$\prod_{j=1}^{q(G)}(x-\alpha)^{s_j-1}\begin{vmatrix} |xI-C_1(\alpha)| & -\varepsilon_{1,2}\beta(x-\alpha) & \dots & -\varepsilon_{1,q(G)}\beta(x-\alpha) \\ -\varepsilon_{1,2}\beta(x-\alpha) & |xI-C_2(\alpha)| & \dots & \vdots \\ \vdots & \vdots & \ddots & -\varepsilon_{q(G)-1,q(G)}\beta(x-\alpha) \\ -\varepsilon_{1,q(G)}\beta(x-\alpha) & -\varepsilon_{2,q(G)}\beta(x-\alpha) & \dots & |xI-C_{q(G)}(\alpha)| \end{vmatrix}.$$

Applying again Lemma 2, it follows that $|xI-X|=$

$$\begin{vmatrix} |xI-C_1(\alpha)| & -\varepsilon_{1,2}\beta(x-\alpha) & \dots & -\varepsilon_{1,q(G)-1}\beta(x-\alpha) & -\varepsilon_{1,q(G)}\beta(x-\alpha) \\ -\varepsilon_{1,2}\beta(x-\alpha) & |xI-C_2(\alpha)| & \dots & -\varepsilon_{2,q(G)-1}\beta(x-\alpha) & -\varepsilon_{2,q(G)}\beta(x-\alpha) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\varepsilon_{1,q(G)-1}\beta(x-\alpha) & -\varepsilon_{2,q(G)-1}\beta(x-\alpha) & \dots & |xI-C_{q(G)-1}(\alpha)| & -\varepsilon_{q(G)-1,q(G)}\beta(x-\alpha) \\ -\varepsilon_{1,q(G)}\beta(x-\alpha) & -\varepsilon_{2,q(G)}\beta(x-\alpha) & \dots & -\varepsilon_{q(G)-1,q(G)}\beta(x-\alpha) & |xI-C_{q(G)}(\alpha)| \end{vmatrix}.$$

Finally, observe that $\prod_{j=1}^{q(G)}(x-\alpha)^{s_j-1}=(x-\alpha)^{p(G)-q(G)}$. ■

Corollary 3. *The multiplicity of α as an eigenvalue of $A_\alpha(G)$, where $G=G\left(s_1,s_2,\dots,s_{q(G)}\right)$, is exactly $p(G)-q(G)$.*

Proof. It is sufficient to prove that $|\alpha I-X|\neq 0$. From the above expression for $|xI-X|$, we have $|\alpha I-X|=$

$$\begin{vmatrix} |\alpha I-C_1(\alpha)| & 0 & \dots & 0 \\ 0 & |\alpha I-C_2(\alpha)| & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & |\alpha I-C_{q(G)}(\alpha)| \end{vmatrix}$$

and for $i=1,2,\dots,q(G)$, $|\alpha I-C_i(\alpha)|=-(1-\alpha)^2 s_i\neq 0$. Hence $|\alpha I-X|\neq 0$.
■

3. Graphs having internal vertices which are non quasi-pendant

Let $\mathbf{e}$ be a column vector of zeros except its last entry which is 1. As before, the dimension of $\mathbf{e}$ will be clear from the context.

Suppose $r(G)>q(G)$. Then there are $r(G)-q(G)$ internal vertices which are non quasi-pendant vertices and $q(G)$ internal vertices which are the roots of the stars $K_{1,s_1},K_{1,s_2},\dots,K_{1,s_{q(G)}}$. Let us denote such a graph G by $G(s_1,\dots,s_{q(G)},\mathbf{0})$, where $\mathbf{0}$ indicates a vector of zeros with $r(G)-q(G)$ entries. Without loss of generality, we assume that $V_Q=\{v_1,v_2,\dots,v_{q(G)}\}$ and $C(G)=\{v_{q(G)+1},v_{q(G)+2},\dots,v_r(G)\}$. We recall that the global labeling for the vertices of $G(s_1, \ldots, s_{q(G)}, \mathbf{0})$ is such that the labels $1, 2, \ldots, p(G) + q(G)$ are used for the vertices of the stars $K_{1,s_1}, K_{1,s_2}, \ldots, K_{1,s_{q(G)}}$, and the labels $p(G) + q(G) + 1, \ldots, n$ are used for the internal vertices which are non quasi-pendant, as illustrated in Figure 3. Using this global labeling jointly with the labels $v_1, \ldots, v_{r(G)}$ for the internal vertices (as before) the matrix $A_\alpha(G)$, where $G = G(s_1, \ldots, s_{q(G)}, \mathbf{0})$, is

$$A_\alpha(G) = \begin{bmatrix} U & V \\ V^T & N \end{bmatrix}$$

where

$$U = \begin{bmatrix} S_1(\alpha) & \varepsilon_{1,2}\beta E & \ldots & \varepsilon_{1,q(G)-1}\beta E & \varepsilon_{1,q(G)}\beta E \\ \varepsilon_{1,2}\beta E^T & S_2(\alpha) & \ldots & \varepsilon_{2,q(G)-1}\beta E & \varepsilon_{2,q(G)}\beta E \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \varepsilon_{1,q(G)-1}\beta E^T & \varepsilon_{2,q(G)-1}\beta E & \ldots & S_{q(G)-1}(\alpha) & \varepsilon_{q(G)-1,q(G)}\beta E \\ \varepsilon_{1,q(G)}\beta E^T & \varepsilon_{2,q(G)}\beta E & \ldots & \varepsilon_{q(G)-1,q(G)}\beta E^T & S_{q(G)}(\alpha) \end{bmatrix},$$

$$V = \beta \begin{bmatrix} \varepsilon_{1,q(G)+1}\mathbf{e} & \varepsilon_{1,q(G)+2}\mathbf{e} & \ldots & \varepsilon_{1,r(G)-1}\mathbf{e} & \varepsilon_{1,r(G)}\mathbf{e} \\ \varepsilon_{2,q(G)+1}\mathbf{e} & \varepsilon_{2,q(G)+2}\mathbf{e} & \ldots & \varepsilon_{1,r(G)-1}\mathbf{e} & \varepsilon_{2,r(G)}\mathbf{e} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \varepsilon_{q(G)-1,q(G)+1}\mathbf{e} & \varepsilon_{q(G)-1,q(G)+2}\mathbf{e} & \ldots & \varepsilon_{q(G)-1,r(G)-1}\mathbf{e} & \varepsilon_{q(G)-1,r(G)}\mathbf{e} \\ \varepsilon_{q(G),q(G)+1}\mathbf{e} & \varepsilon_{q(G),q(G)+2}\mathbf{e} & \ldots & \varepsilon_{q(G),r(G)-1}\mathbf{e} & \varepsilon_{q(G),r(G)}\mathbf{e} \end{bmatrix}$$

and

$$N = \begin{bmatrix} \alpha d_{q(G)+1} & \varepsilon_{q(G)+1,q(G)+2}\beta & \ldots & \varepsilon_{q(G)+1,r(G)-1}\beta & \varepsilon_{q(G)+1,r(G)}\beta \\ \varepsilon_{q(G)+1,q(G)+2}\beta & \alpha d_{q(G)+2} & \ldots & \varepsilon_{q(G)+2,r(G)-1}\beta & \varepsilon_{q(G)+2,r(G)}\beta \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \varepsilon_{q(G)+1,r(G)-1}\beta & \varepsilon_{q(G)+2,r(G)-1}\beta & \ldots & \alpha d_{r(G)-1} & \varepsilon_{r(G)-1,r(G)}\beta \\ \varepsilon_{q(G)+1,r(G)}\beta & \varepsilon_{q(G)+2,r(G)}\beta & \ldots & \varepsilon_{r(G)-1,r(G)}\beta & \alpha d_{r(G)} \end{bmatrix}$$

where $d_{q(G)+1}, d_{q(G)+2}, \ldots, d_{r(G)-1}, d_{r(G)}$ are the degrees of the vertices $v_{q(G)+1}, v_{q(G)+2}, \ldots, v_{r(G)-1}, v_{r(G)}$, respectively.

Applying Lemma 2, Corollary 1 and Corollary 2, together with a factoring in each of the first $q(G)$ columns of the resulting determinant, one can prove the following theorem.

Theorem 2. *If $G = G\left(s_1, s_2, \ldots, s_{q(G)}, \boldsymbol{0}\right)$, the eigenvalues of $A_\alpha(G)$ are α with multiplicity at least $p(G) - q(G)$ and the eigenvalues of the $(n + q(G) - p(G)) \times (n + q(G) - p(G))$ matrix*

$$X = \begin{bmatrix} Q & R \\ R^T & N \end{bmatrix}$$

where

$$Q = \begin{bmatrix} C_1(\alpha) & \varepsilon_{1,2}\beta E & \ldots & \varepsilon_{1,q(G)-1}\beta E & \varepsilon_{1,q(G)}\beta E \\ \varepsilon_{1,2}\beta E & C_2(\alpha) & \ldots & \varepsilon_{2,q(G)-1}\beta E & \varepsilon_{2,q(G)}\beta E \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \varepsilon_{1,q(G)-1}\beta E & \varepsilon_{2,q(G)-1}\beta E & \ldots & C_{q(G)-1}(\alpha) & \varepsilon_{q(G)-1,q(G)}\beta E \\ \varepsilon_{1,q(G)}\beta E & \varepsilon_{2,q(G)}\beta E & \ldots & \varepsilon_{q(G)-1,q(G)}\beta E & C_{q(G)}(\alpha) \end{bmatrix},$$

$$N = \begin{bmatrix} \alpha d_{q(G)+1} & \varepsilon_{q(G)+1,q(G)+2}\beta & \ldots & \varepsilon_{q(G)+1,r(G)-1}\beta & \varepsilon_{q(G)+1,r(G)}\beta \\ \varepsilon_{q(G)+1,q(G)+2}\beta & \alpha d_{q(G)+2} & \ldots & \varepsilon_{q(G)+2,r(G)-1}\beta & \varepsilon_{q(G)+2,r(G)}\beta \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \varepsilon_{q(G)+1,r(G)-1}\beta & \varepsilon_{q(G)+2,r(G)-1}\beta & \ldots & \alpha d_{r(G)-1} & \varepsilon_{r(G)-1,r(G)}\beta \\ \varepsilon_{q(G)+1,r(G)}\beta & \varepsilon_{q(G)+2,r(G)}\beta & \ldots & \varepsilon_{r(G)-1,r(G)}\beta & \alpha d_{r(G)} \end{bmatrix}$$

and

$$R = \beta \begin{bmatrix} \varepsilon_{1,q(G)+1}\mathbf{e} & \varepsilon_{1,q(G)+2}\mathbf{e} & \ldots & \varepsilon_{1,r(G)-1}\mathbf{e} & \varepsilon_{1,r(G)}\mathbf{e} \\ \varepsilon_{2,q(G)+1}\mathbf{e} & \varepsilon_{2,q(G)+2}\mathbf{e} & \ldots & \varepsilon_{2,r(G)-1}\mathbf{e} & \varepsilon_{2,r(G)}\mathbf{e} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \varepsilon_{q(G)-1,q(G)+1}\mathbf{e} & \varepsilon_{q(G)-1,q(G)+2}\mathbf{e} & \ldots & \varepsilon_{q(G)-1,r(G)-1}\mathbf{e} & \varepsilon_{q(G)-1,r(G)}\mathbf{e} \\ \varepsilon_{q(G),q(G)+1}\mathbf{e} & \varepsilon_{q(G),q(G)+2}\mathbf{e} & \ldots & \varepsilon_{q(G),r(G)-1}\mathbf{e} & \varepsilon_{q(G),r(G)}\mathbf{e} \end{bmatrix}.$$

Theorem 3. *Let $G = G\left(s_1, s_2, \ldots, s_{q(G)}, \boldsymbol{0}\right)$. Let X, Q and N be the matrices in Theorem 2. Then*

1. $m_X(\alpha) = m_N(\alpha)$.
2. $m_G(\alpha) = p(G) - q(G) + m_N(\alpha)$.

Proof.
1. We have $|\alpha I - X| = \begin{vmatrix} \alpha I - Q & -R \\ -R^T & \alpha I - N \end{vmatrix}$. Hence, applying Corollary 1,

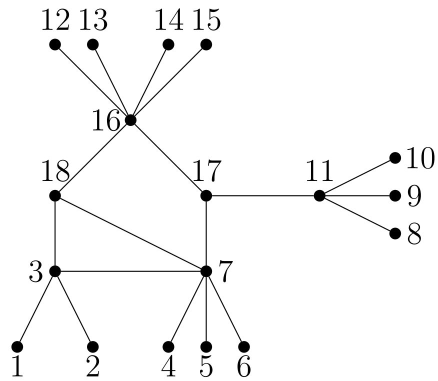

Figure 3: The graph $G(2, 3, 3, 4, \mathbf{0})$.

we obtain $|\alpha I - X| =$

$$\begin{vmatrix} -(1-\alpha)^2 s_1 & 0 & \cdots & 0 & * \\ 0 & -(1-\alpha)^2 s_2 & \vdots & \vdots & \vdots \\ \vdots & 0 & \ddots & 0 & \vdots \\ \vdots & \cdots & \cdots & -(1-\alpha)^2 s_{q(G)} & * \\ 0 & 0 & \cdots & 0 & |\alpha I - N| \end{vmatrix} =$$

$$(-1)^{q(G)} (1-\alpha)^{2q(G)} s_1 s_2 \ldots s_{q(G)} \left| \alpha I - N \right|.$$

Since $1 - \alpha \neq 0$ and $s_i \geq 1$, for $1 \leq i \leq q(G)$, we obtain $|\alpha I - X| = 0$ if and only if $|\alpha I - N| = 0$. Hence $m_X(\alpha) = m_N(\alpha)$.

2. From Theorem 2, $m_G(\alpha) = p(G) - q(G) + m_X(\alpha)$. Since $m_X(\alpha) = m_N(\alpha)$, the result follows. ■

We see that the matrix N in Theorem 2 is obtained from $A_\alpha(G)$ by taking the entries corresponding to the internal vertices which are not quasi-pendant vertices.

We recall that $G[F]$ denotes the subgraph of G induced by $F \subseteq V(G)$.

Theorem 4. *Let $G = G\left(s_1, s_2, \ldots, s_{q(G)}, \mathbf{0}\right)$. Let X, Q and N be the matrices in Theorem 3. Let $C_1, \ldots, C_t$ be the components of the subgraph induced by $C(G)$. Then*

$$m_G(\alpha) = p(G) - q(G) + \textstyle\sum_{i=1}^{t} m_{N_i}(\alpha),$$

where, for $1 \leq i \leq t$, $N_i = (1-\alpha)A(G[C_i]) + \alpha D_i$, where D_i is a diagonal matrix of order $|C_i|$ in which each diagonal entry is the degree in G of the corresponding vertex.

Proof. From the hypothesis, there is a labeling of the vertices of $C(G)$ such that $N = \bigoplus_{i=1}^{t} N_i$ (the direct sum of the matrices $N_i, i = 1, \ldots, t$). Therefore $m_N(\alpha) = \sum_{i=1}^{t} m_{N_i}(\alpha)$. Now, the result is immediate by Theorem 3. ■

Example 1. *For the graph G in Figure 3, we have $p(G) = 12$, $q(G) = 4$. We see that $N = \begin{bmatrix} 3\alpha & 0 \\ 0 & 3\alpha \end{bmatrix}$. Hence $|\alpha I - N| = \begin{vmatrix} \alpha - 3\alpha & 0 \\ 0 & \alpha - 3\alpha \end{vmatrix} = 0$ if and only if $\alpha = 0$. Therefore, α is not an eigenvalue of N, when $\alpha \neq 0$. Then, from Theorem 3, the multiplicity of α as an eigenvalue of $A_\alpha(G)$ is*

$$\begin{cases} p(G) - q(G) = 8, & \textit{when } \alpha \neq 0; \\ p(G) - q(G) + m_N(0) = 10, & \textit{otherwise.} \end{cases}$$

We recall that the nullity of a graph G, denoted by $\eta(G)$, is the multiplicity of 0 as eigenvalue of $A(G)$. From Theorem 4, we obtain

Corollary 4. *Let $G = G\left(s_1, s_2, \ldots, s_{q(G)}, \boldsymbol{0}\right)$. Let $C_1, \ldots, C_t$ be the components of the subgraph induced by $C(G)$. Then*

$$\eta(G) = p(G) - q(G) + \sum_{i=1}^{t} \eta(N_i), \tag{4}$$

where, for $1 \leq i \leq t$, $N_i = A(G[C_i])$ and $\eta(N_i)$ is the multiplicity of 0 as eigenvalue of N_i.

Corollary 5. *Let $G = G\left(s_1, s_2, \ldots, s_{q(G)}, \boldsymbol{0}\right)$. Let $C_1, \ldots, C_t$ be the components of the subgraph induced by $C(G)$. Let H be the induced subgraph of G obtained by deleting one pendant vertex together with the vertex adjacent to it. Then $\eta(G) = \eta(H)$.*

Proof. Clearly $C(H) = C(G)$ and the components of the subgraphs induced by $C(H)$ and $C(G)$ are the same. In addition, $p(H) - q(H) = p(G) - 1 - (q(G) - 1) = p(G) - q(G)$. Applying (4), we obtain $\eta(H) = \eta(G)$. ■

A version of Corollary 5 is proved in [3] assuming that G is a bipartite graph with at least one pendant vertex.

4. Applications on some particular graphs

In this section, the above results are applied to search for the multiplicity of $\alpha \in [0,1)$ as an eigenvalue of $A_\alpha(G)$ when G is a path, a caterpillar, a circular caterpillar, a generalized Bethe tree or a Bethe tree.

4.1. The multiplicity of α as an eigenvalue of $A_\alpha(P_n)$

Let P_n be the path of n vertices. It is well known that 0 is an eigenvalue of $A(P_n)$ if and only if n is odd. Moreover, if n is odd then 0 is a simple eigenvalue of $A(P_n)$. We begin considering the cases $2 \leq n \leq 4$.

- If $n = 2$ then $A_\alpha(P_2) = \begin{bmatrix} \alpha & 1-\alpha \\ 1-\alpha & \alpha \end{bmatrix}$. We have $|A_\alpha(P_2) - \alpha I_2| = \begin{vmatrix} 0 & 1-\alpha \\ 1-\alpha & 0 \end{vmatrix} = 0$ if and only if $\alpha = 1$. Hence $\alpha \in (0,1)$ is not an eigenvalue of $A_\alpha(P_2)$.
- If $n = 3$ then P_3 is a graph with 2 pendant vertices and 1 quasi-pendant vertex. From Corollary 3, $m_{P_3}(\alpha) = 2 - 1 = 1$.
- If $n = 4$ then P_4 is a graph with 2 pendant vertices and 2 quasi-pendant vertices. From Corollary 3, $m_{P_4}(\alpha) = 2 - 2 = 0$. Hence $\alpha \in (0,1)$ is not an eigenvalue of $A_\alpha(P_4)$.

From now on let us assume that P_n is such that $n \geq 5$. Since $\alpha \neq 1$, the vertices of P_n can be labeled such that $A_\alpha(P_n)$ is a symmetric tridiagonal matrix with nonzero codiagonal entries. Hence the eigenvalues of $A_\alpha(P_n)$ are simple and, in particular, if $\alpha \neq 1$ is an eigenvalue of $A_\alpha(P_n)$ then it will be a simple eigenvalue. We recall the following lemma (see [4]).

Lemma 4. *The eigenvalues of the symmetric tridiagonal matrix*

$$N(\alpha) = \begin{bmatrix} 2\alpha & 1-\alpha & & & \\ 1-\alpha & 2\alpha & 1-\alpha & & \\ & \ddots & \ddots & \ddots & \\ & & 1-\alpha & 2\alpha & 1-\alpha \\ & & & 1-\alpha & 2\alpha \end{bmatrix} \qquad (5)$$

of order $s \times s$ are

$$2\alpha + 2(1-\alpha)\cos(\frac{\pi j}{s+1})$$

for $j = 1, 2, \ldots, s$.

Corollary 6. *Let $N(\alpha)$ as in (5).*

1. *If $2/3 \leq \alpha < 1$ then α is not an eigenvalue of $N(\alpha)$.*
2. *γ is an eigenvalue of $N(\gamma)$ if and only if*

$$\gamma = \frac{-2\cos(\frac{\pi j}{s+1})}{1 - 2\cos(\frac{\pi j}{s+1})}$$

   for some $j = 1, \ldots, s$.
3. *Let*

$$\alpha_j = \frac{-2\cos(\frac{\pi j}{s+1})}{1 - 2\cos(\frac{\pi j}{s+1})}. \tag{6}$$

   Then α_j is an eigenvalue of $N(\alpha_j)$ and $0 < \alpha_j < 1$ if and only if $j \in \{\lfloor \frac{s+3}{2} \rfloor, \ldots, s\}$.

Proof.

1. Assume $2/3 \leq \alpha < 1$. Using this hypothesis, the matrix $N(\alpha) - \alpha I$ is irreducible and diagonally dominant with strict inequality in at least one row. Then, by Theorem 1.21 in [17], $N(\alpha) - \alpha I$ is an invertible matrix. Hence α is not an eigenvalue of $N(\alpha)$.
2. It is immediate from Lemma 4.
3. Let α_j as in (6). Then α_j is an eigenvalue of $N(\alpha_j)$. The following fact is immediate:

$$0 < \frac{x}{1+x} < 1 \text{ if and only if } x > 0.$$

   Let $j \in \{\lfloor \frac{s+3}{2} \rfloor, \ldots, s\}$. Then $-2\cos(\frac{\pi j}{s+1}) > 0$. From the above fact, $0 < \alpha_j = \frac{-2\cos(\frac{\pi j}{s+1})}{1-2\cos(\frac{\pi j}{s+1})} < 1$. Conversely, suppose that $0 < \alpha_j < 1$. We use again the above mentioned fact, to obtain that $-2\cos(\frac{\pi j}{s+1}) > 0$. Hence $j \in \{\lfloor \frac{s+3}{2} \rfloor, \ldots, s\}$.

■

Since $n \geq 5$, P_n has 2 pendant vertices, 2 quasi-pendant vertices and $n-4$ internal vertices which are not quasi-pendants. By Theorem 3, $m_{P_n}(\alpha) =$

$2 - 2 + m_N(\alpha) = m_N(\alpha)$, where

$$N = N(\alpha) = \begin{bmatrix} 2\alpha & 1-\alpha & & & \\ 1-\alpha & 2\alpha & 1-\alpha & & \\ & \ddots & \ddots & \ddots & \\ & & 1-\alpha & 2\alpha & 1-\alpha \\ & & & 1-\alpha & 2\alpha \end{bmatrix}$$

of order $(n-4) \times (n-4)$.

In particular, for $n = 5$ and $n = 6$, we have

- $m_{P_5}(\alpha) = m_N(\alpha)$ where $N = N(\alpha) = [2\alpha]$. Since $\alpha \neq 0$ is not an eigenvalue of $[2\alpha]$, we have that $\alpha \neq 0$ is not an eigenvalue of $A_\alpha(P_5)$.
- $m_{P_6}(\alpha) = m_N(\alpha)$ where $N = N(\alpha) = \begin{bmatrix} 2\alpha & 1-\alpha \\ 1-\alpha & 2\alpha \end{bmatrix}$. We have $|N(\alpha) - \alpha I_2| = \begin{vmatrix} \alpha & 1-\alpha \\ 1-\alpha & \alpha \end{vmatrix} = 0$ if and only if $\alpha = 1/2$. Hence $\alpha \in (0,1)$ is an eigenvalue of $A_\alpha(P_6)$ if and only if $\alpha = 1/2$.

Applying Theorem 4 and Corollary 6, we get

Corollary 7. *Let $n \geq 7$. Then $\alpha \in (0,1)$ is an eigenvalue of $A_\alpha(P_n)$ if and only if*

$$\alpha = \frac{-2\cos(\frac{\pi j}{n-3})}{1 - 2\cos(\frac{\pi j}{n-3})}$$

for some $j \in \{\lfloor \frac{n-1}{2} \rfloor, \ldots, n-4\}$.

4.2. Multiplicity of α as eigenvalue of $A_\alpha(G)$ when G is a caterpillar or a circular caterpillar

We recall that a graph G is a caterpillar (respectively, a circular caterpillar) if its internal vertices induce a path (respectively, a cycle). We say that a caterpillar or a circular caterpillar is complete if each internal vertex is a quasi-pendant vertex. From Corollary 3, we get

Corollary 8. *If G is a complete caterpillar or a complete circular caterpillar then $m_G(\alpha) = p(G) - q(G)$.*

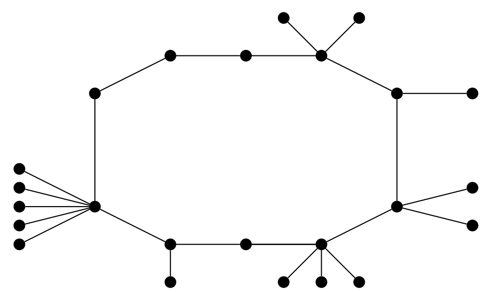

Figure 4: An example of a circular caterpillar.

If G is a caterpillar or a circular caterpillar having internal vertices which are not quasi-pendant vertices, we may use Theorem 4 together with equation 6 to find $m_G(\alpha)$.

Example 2. *Let us find the exact multiplicity of* $\alpha \in [0, 1)$ *as eigenvalue of* $A_\alpha(G)$ *where* G *is the circular caterpillar in Figure 4. Applying Theorem 4, we obtain*

$$m_G(\alpha) = p(G) - q(G) + m_{N_1}(\alpha) + m_{N_2}(\alpha) = 14 - 6 + m_{N_1}(\alpha) + m_{N_2}(\alpha)$$

where $N_1 = N_1(\alpha) = \begin{bmatrix} 2\alpha & 1-\alpha & 0 \\ 1-\alpha & 2\alpha & 1-\alpha \\ 0 & 1-\alpha & 2\alpha \end{bmatrix}$ *and* $N_2 = N_2(\alpha) = [2\alpha]$. *By equation (6) in Corollary 6,* $\alpha \in (0, 1)$ *is an eigenvalue of* $N_1(\alpha)$ *if and only if* $\alpha = \alpha_3 = \frac{-2\cos(\frac{3\pi}{4})}{1-2\cos(\frac{3\pi}{4})} = \frac{\sqrt{2}}{1+\sqrt{2}}$.

Moreover, $\alpha \in [0, 1)$ *is an eigenvalue of* N_2 *if and only if* $\alpha = 0$. *Then, from Theorem 4, the multiplicity of* α *as an eigenvalue of* $A_\alpha(G)$ *is*

$$\begin{cases} 8 + m_{N_1}(\alpha) + m_{N_2}(\alpha) = 8 + 1 + 0 = 9, & \text{when } \alpha = \alpha_3; \\ 8 + m_{N_1}(0) + m_{N_2}(0) = 8 + 0 + 1 = 9, & \text{when } \alpha = 0; \\ 8 + m_{N_1}(\alpha) + m_{N_2}(\alpha) = 8, & \text{when } \alpha \in (0, 1)\setminus\{\alpha_3\}. \end{cases}$$

4.3. Multiplicity of α *as eigenvalue of* $A_\alpha(T)$ *when* T *is a generalized Bethe tree*

Given a rooted graph, the level of a vertex is equal to its distance to the root vertex increased by one. A generalized Bethe tree is a rooted tree in which vertices at the same level have the same degree.

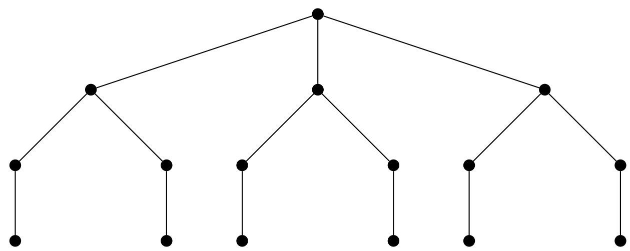

Figure 5: An example of a generalized Bethe tree of 4 levels.

In this section, B_k is a generalized Bethe tree of $k > 1$ levels. Given a B_k and an integer $1 \leq j \leq k$, n_{k-j+1} is the number of vertices at level j and d_{k-j+1} is the degree of them. In particular, $d_1 = 1$, $n_k = 1$, $n_1 = p(B_k)$ and $n_2 = q(B_k)$.

Definition 1. *For $j = 1, 2, \ldots, k-1$, let T_j be the $j \times j$ leading principal submatrix of the $k \times k$ symmetric tridiagonal matrix*

$$T_k = \begin{bmatrix} \alpha & \beta\sqrt{d_2 - 1} & 0 & \cdots & 0 & 0 & 0 \\ \beta\sqrt{d_2 - 1} & \alpha d_2 & \beta\sqrt{d_3 - 1} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha d_{k-2} & \beta\sqrt{d_{k-1} - 1} & 0 \\ 0 & 0 & 0 & \cdots & \beta\sqrt{d_{k-1} - 1} & \alpha d_{k-1} & \beta\sqrt{d_k} \\ 0 & 0 & 0 & \cdots & 0 & \beta\sqrt{d_k} & \alpha d_k \end{bmatrix}$$

where $\beta = 1 - \alpha$.

Let $\sigma(M)$ be the multiset of eigenvalues of the matrix M.

Since $d_s > 1$ for all $s = 2, 3, ..., j$, each matrix T_j has nonzero codiagonal entries. Then the eigenvalues of each T_j are simple and $\sigma(T_j) \cap \sigma(T_{j+1}) = \phi$ for $j = 1, \ldots, k-1$.

The following theorem was proved in [10].

Theorem 5. *Let B_k be a generalized Bethe tree.*

1. *The multiset of the eigenvalues of $A_\alpha(B_k)$ is*

$$\sigma(A_\alpha(B_k)) = \sigma(T_1) \cup \cdots \cup \sigma(T_k). \tag{7}$$

2. *The multiplicity of each eigenvalue of T_j as an eigenvalue of $A_\alpha(B_k)$ is $n_j - n_{j+1}$ if $1 \leq j \leq k-1$, and is 1 if $j = k$. If some eigenvalues obtained in different matrices are equal, their multiplicities are added together.*
3. *The largest eigenvalue of T_k is the largest eigenvalue of $A_\alpha(B_k)$.*

We observe that for $k \geq 3$,

$$T_2 = \begin{bmatrix} \alpha & (1-\alpha)\sqrt{d_2 - 1} \\ (1-\alpha)\sqrt{d_2 - 1} & \alpha d_2 \end{bmatrix}.$$

Hence $|T_2 - \alpha I_2| = \begin{vmatrix} 0 & (1-\alpha)\sqrt{d_2 - 1} \\ (1-\alpha)\sqrt{d_2 - 1} & \alpha(d_2 - 1) \end{vmatrix} = -(1-\alpha)^2(d_2 - 1) \neq 0$ because $\alpha \neq 1$. The next corollary follows easily from Theorem 5.

Corollary 9. *Let $\alpha \in [0, 1)$. Then*

1. $m_{B_2}(\alpha) = n_1 - 1$.
2. $m_{B_3}(\alpha) = n_1 - n_2$ *if* $\alpha \neq 0$ *and* $m_{B_3}(0) = n_1 - n_2 + 1$.
3. *For $k \geq 4$,*

$$m_{B_k}(\alpha) = n_1 - n_2 + \sum_{j=3}^{k-1} (n_j - n_{j+1}) m_{T_j}(\alpha) + m_{T_k}(\alpha) \tag{8}$$

in which, for $j = 3, \ldots, k$, $m_{T_j}(\alpha) = 1$ if $\alpha \in \sigma(T_j)$ and $m_{T_j}(\alpha) = 0$ otherwise.

We already observed that $\sigma(T_j) \cap \sigma(T_{j+1}) = \phi$ for $j = 1, \ldots, k-1$.

Example 3. *Let us find $m_{B_4}(\alpha)$. Using (8),*

$$m_{B_4}(\alpha) = n_1 - n_2 + (n_3 - n_4) m_{T_3}(\alpha) + m_{T_4}(\alpha)$$

where $n_4 = 1$, $T_3 = \begin{bmatrix} \alpha & (1-\alpha)\sqrt{d_2 - 1} & 0 \\ (1-\alpha)\sqrt{d_2 - 1} & \alpha d_2 & (1-\alpha)\sqrt{d_3 - 1} \\ 0 & (1-\alpha)\sqrt{d_3 - 1} & \alpha d_3 \end{bmatrix}$

and

$$T_4 = \begin{bmatrix} \alpha & (1-\alpha)\sqrt{d_2 - 1} & 0 & 0 \\ (1-\alpha)\sqrt{d_2 - 1} & \alpha d_2 & (1-\alpha)\sqrt{d_3 - 1} & 0 \\ 0 & (1-\alpha)\sqrt{d_3 - 1} & \alpha d_3 & (1-\alpha)\sqrt{d_4} \\ 0 & 0 & (1-\alpha)\sqrt{d_4} & \alpha d_4 \end{bmatrix}.$$

It is easily to find that

$$|T_3 - \alpha I_3| = -\alpha(1-\alpha)^2(d_2-1)(d_3-1)$$

and

$$|T_4 - \alpha I_4| = -(1-\alpha)^2(d_2-1)(\alpha^2(d_3-1)(d_4-1)-(1-\alpha)^2 d_4).$$

Hence $\alpha \in [0,1)$ *is an eigenvalue of* T_3 *if and only if* $\alpha = 0$ *and* $\alpha \in [0,1)$ *is an eigenvalue of* T_4 *if and only if* $\alpha = \alpha_0 = \frac{\sqrt{d_4}}{\sqrt{d_4}+\sqrt{(d_3-1)(d_4-1)}}$. *Therefore the multiplicity of* α *as an eigenvalue of* $A_\alpha(B_4)$ *is*

$$\begin{cases} n_1 - n_2 + n_3 - 1, & \text{when } \alpha = 0; \\ n_1 - n_2 + 1, & \text{when } \alpha = \alpha_0; \\ n_1 - n_2, & \text{when } \alpha \in (0,1)\backslash\{\alpha_0\}. \end{cases}$$

4.4. Multiplicity of α *as eigenvalue of* $A_\alpha(T)$ *and the nullity of* T *when* T *is a Bethe tree*

A Bethe tree $B(d,k)$ is a rooted tree of k levels in which the root has degree d, the vertices in level j $(2 \le j \le k-1)$ have degree equal to $d+1$ and the vertices in level k have degree equal to 1 (pendant vertices). Clearly, any Bethe tree $B(d,k)$ is a generalized Bethe tree in which the matrix T_j is the $j \times j$ leading principal submatrix of the $k \times k$ symmetric tridiagonal matrix

$$T_k = \begin{bmatrix} \alpha & \beta\sqrt{d} & 0 & \cdots & 0 & 0 & 0 \\ \beta\sqrt{d} & \alpha d & \beta\sqrt{d} & \cdots & 0 & 0 & 0 \\ 0 & \beta\sqrt{d} & \alpha d & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha d & \beta\sqrt{d} & 0 \\ 0 & 0 & 0 & \cdots & \beta\sqrt{d} & \alpha d & \beta\sqrt{d} \\ 0 & 0 & 0 & \cdots & 0 & \beta\sqrt{d} & \alpha d \end{bmatrix} \tag{9}$$

where $\beta = 1-\alpha$. Moreover, $n_j = d^{k-j}$ for $j = 1, \ldots, k$. From Corollary 9, we get

Corollary 10. *Let* $\alpha \in [0,1)$. *Then*

1. $m_{B(d,2)}(\alpha) = d-1$.
2. $m_{B(d,3)}(\alpha) = (d-1)d$ *if* $\alpha \neq 0$ *and* $\eta(B(d,3)) = m_{B(d,3)}(0) = (d-1)d+1$.

3. *For $k \geq 4$,*

$$m_{B(d,k)}(\alpha) = (d-1)(d^{k-2} + \sum_{j=3}^{k-1} d^{k-j-1} m_{T_j}(\alpha)) + m_{T_k}(\alpha) \tag{10}$$

in which the matrix T_j, for $j = 3, \ldots, k$, is the $j \times j$ principal submatrix of T_k as in (9) and $m_{T_j}(\alpha) = 1$ if $\alpha \in \sigma(T_j)$ and $m_{T_j}(\alpha) = 0$ otherwise.

Finally, we find the nullity of $B(d,k)$. Let $\alpha = 0$. Then, $\beta = 1$ and the matrix T_k becomes

$$T_k = \begin{bmatrix} 0 & \sqrt{d} & 0 & \cdots & 0 & 0 & 0 \\ \sqrt{d} & 0 & \sqrt{d} & \cdots & 0 & 0 & 0 \\ 0 & \sqrt{d} & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \sqrt{d} & 0 \\ 0 & 0 & 0 & \cdots & \sqrt{d} & 0 & \sqrt{d} \\ 0 & 0 & 0 & \cdots & 0 & \sqrt{d} & 0 \end{bmatrix} \tag{11}$$

Corollary 11. *The nullity of $B(d,k)$ is given by*

$$\eta(B(d,k)) = \begin{cases} \frac{d^k-1}{d+1}, & \text{if } k \text{ is even;} \\ \\ \frac{d^k+1}{d+1}, & \text{otherwise.} \end{cases}$$

Proof. We recall that, for $j = 1, \ldots, k-1$, T_j is the $j \times j$ principal submatrix of T_k as in (11). One can easily see that 0 is an eigenvalue of T_j if and only if j is odd.

1. Let k be an even integer.
   For $k = 2$, we have $m_{B(d,2)}(0) = \eta(B(d,2)) = d - 1 = \frac{d^2-1}{d+1}$. Let $k \geq 4$. From (10), we obtain

   $$\eta(B(d,k) = (d-1)(d^{k-2} + d^{k-4} + \ldots + d^2 + 1) = \frac{d^k - 1}{d+1}.$$

2. Let k be an odd integer.
   For $k = 3$, we have $m_{B(d,3)}(0) = \eta(B(d,3)) = (d-1)d + 1 = \frac{d^3+1}{d+1}$. Let $k \geq 5$. From (10), we obtain

   $$\eta(B(d,k)) = (d-1)(d^{k-2} + d^{k-4} + \ldots + d^3 + d) + 1 = \frac{d^k + 1}{d+1}.$$

■

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