Spectra and Laplacian spectra of arbitrary powers of lexicographic products of graphs

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Abstract

Consider two graphs G and H. Let $H^k[G]$ be the lexicographic product of H^k and G, where H^k is the lexicographic product of the graph H by itself k times. In this paper, we determine the spectrum of $H^k[G]$ and H^k when G and H are regular and the Laplacian spectrum of $H^k[G]$ and H^k for G and H arbitrary. Particular emphasis is given to the least eigenvalue of the adjacency matrix in the case of lexicographic powers of regular graphs, and to the algebraic connectivity and the largest Laplacian eigenvalues in the case of lexicographic powers of arbitrary graphs. This approach allows the determination of the spectrum (in case of regular graphs) and Laplacian spectrum (for arbitrary graphs) of huge graphs. As an example, the spectrum of the lexicographic power of the Petersen graph with the googol number (that is, 10^{100}) of vertices is determined. The paper finishes with the extension of some well known spectral and combinatorial invariant properties of graphs to its lexicographic powers.

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1 Introduction

The lexicographic product of a graph H with itself several times is a very special graph product, it is a kind of fractal graph which reproduces its copy in each of the positions of its vertices and connects all the vertices of each copy with another copy when they are placed in positions corresponding to adjacent vertices of H. This procedure can be repeated, reproducing a copy of the previous iterated graph in each of the positions of the vertices of H and so on. Despite the spectrum and Laplacian spectrum of the lexicographic product of two graphs (with some restrictions regarding the spectrum) expressed in terms of the two factors are well known (see [4], where a unified approach is given), it is not the case of the spectra and Laplacian spectra of graphs obtained by iterated lexicographic products, herein called lexicographic powers, of regular and arbitrary graphs, respectively. A lexicographic power H^k of a graph H can produce a graph with a huge number of vertices whose spectra and Laplacian spectra may not be determined using their adjacency and Laplacian matrices, respectively. The expressions herein deduced for the spectra and Laplacian spectra of lexicographic powers can be easily programmed, for example, in Mathematica, and the results can be obtained immediately. For instance, the spectrum of the 100-th lexicographic power of the Petersen graph, presented in Section 3, was obtained by Mathematica and the computations lasted only a few seconds. Notice that such lexicographic power has the googol number (that is, 10^{100}) of vertices.

The paper is organized as follows. In the next section, the notation is introduced and some preliminary results are given. The main results are introduced in Section 3, where the spectra (Laplacian spectra) of $H^k[G]$ and H^k, when G and H are regular (arbitrary) graphs, are deduced. Particular attention is given to the Laplacian index and algebraic connectivity of the lexicographic powers of arbitrary graphs. In Section 4, the obtained results are applied to extend some well known properties and spectral relations of combinatorial invariants of graphs H to its lexicographic powers H^k.

2 Preliminaries

In this work we deal with simple and undirected graphs. If G is such a graph of order n, its vertex set is denoted by $V(G)$ and its edge set by $E(G)$. The elements of $E(G)$ are denoted by ij, where i and j are the extreme vertices of the edge ij. The degree of $j \in V(G)$ is denoted by $d_G(j)$, the minimum and maximum degree of the vertices in G are $\delta(G)$ and $\Delta(G)$ and the set of the neighbors of a vertex j is $N_G(j)$. The *adjacency matrix* of G is the $n \times n$ matrix A_G whose (i,j)-entry is equal to 1 whether $ij \in E(G)$ and 0 otherwise. The *Laplacian matrix* of G is the matrix $L_G = D - A_G$, where D is the diagonal matrix whose diagonal elements are the degrees of the vertices of G. Since A_G and L_G are symmetric matrices, their eigenvalues are real numbers. From Geršgorin's theorem, the eigenvalues of L_G are nonnegative. The multiset (that is, the set with possible repetitions) of eigenvalues of a matrix M is called the *spectrum* of M and denoted $\sigma(M)$. Throughout the paper, we write $\sigma_A(G) = \{\lambda_1^{[g_1]}, \ldots, \lambda_s^{[g_s]}\}$ (respectively, $\sigma_L(G) = \{\mu_1^{[l_1]}, \ldots, \mu_t^{[l_t]}\}$) when $\lambda_1 > \ldots > \lambda_s$ ($\mu_1 > \ldots > \mu_t$) are the distinct eigenvalues of A_G (L_G) indexed in decreasing order - in this case, $\gamma^{[r]}$ means that the eigenvalue γ has multiplicity r. If convenient, we write $\gamma(G)$ in place of γ to indicate an eigenvalue of a matrix associated to G, and we denote the eigenvalues of A_G (respectively, L_G) indexed in non increasing order, as $\lambda_1(G) \geq \cdots \geq \lambda_n(G)$ ($\mu_1(G) \geq \cdots \geq \mu_n(G)$).

As usual, the adjacency matrix eigenvalues of a graph G are called the *eigenvalues of G*. We remember that $\mu_n(G) = 0$ (the all one vector is the associated eigenvector) and its multiplicity is equal to the number of components of G. Besides, $\mu_{n-1}(G)$ is called the *algebraic connectivity* of G [9]. Further concepts not defined in this paper can be found in [5, 7].

The *lexicographic product* (also called the *composition*) of the graphs H and G is the graph $H[G]$ (also denoted by $H \circ G$) for which the vertex set is the cartesian product $V(H) \times V(G)$ and such that a vertex (x_1, y_1) is adjacent to the vertex (x_2, y_2) whenever x_1 is adjacent to x_2 or $x_1 = x_2$ and y_1 is adjacent to y_2 (see [15] and [17] for notations and further details). This graph operation was introduced by Harary in [13] and Sabidussi in [20]. It is immediate that the lexicographic product is associative but not commutative.

The lexicographic product was generalized in [21] under the designation of *generalized composition* as follows: consider a graph H of order n and graphs G_i, $i = 1, \ldots, n$, with vertex sets $V(G_i)$s two by two disjoints is the graph such that

$$\begin{aligned}
V(H[G_1, \ldots, G_n]) &= \bigcup_{i=1}^{n} V(G_i) \quad \text{and} \\
E(H[G_1, \ldots, G_n]) &= \bigcup_{i=1}^{n} E(G_i) \;\cup \bigcup_{ij \in E(H)} E(G_i \vee G_j),
\end{aligned}$$

where $G_i \vee G_j$ denotes the join of the graphs G_i and G_j. This operation is called in [4] the *H-join* of graphs $G_1, \ldots, G_n$. In [21] and [4], the spectrum of $H[G_1, \ldots, G_n]$ is provided, where H is an arbitrary graph and $G_1, \ldots, G_n$ are regular graphs. Furthermore, in [10] and [4], using different approaches, the spectrum of the Laplacian matrix of $H[G_1, \ldots, G_n]$ for arbitrary graphs was characterized. The Laplacian spectrum of the H-join was previously obtained in [19] in the particular case of a graphs with tree structure (that is, when H is a tree).

Let H be a graph of order n and G be an arbitrary graph. If, for $1 \leq i \leq n$, G_i is isomorphic to G, it follows immediately that $H[G_1, \ldots, G_n] = H[G]$, a fact also noted in [2].

Now, let us focus on the spectrum of the adjacency and Laplacian matrix of the above generalized graph composition.

Assuming that $G_1, \ldots, G_n$ are all isomorphic to a particular graph G (which should be regular in the case of Corollary 2.1), as consequence of Theorems 5 and 8 in [4], we have the following corollaries.

Corollary 2.1. *Let H be a graph of order n with $\sigma_A(H) = \{\lambda_1^{[h_1]}(H), \ldots, \lambda_t^{[h_t]}(H)\}$, where the superscript $[h_i]$ stands for the multiplicity of the eigenvalue $\lambda_i(H)$, and let G be a p-regular graph of order m with $\sigma_A(G) = \{\lambda_1^{[g_1]}(G), \ldots, \lambda_s^{[g_s]}(G)\}$. Then*

$$\sigma_A(H[G]) = \{p^{[n(g_1-1)]}, \ldots, \lambda_s^{[ng_s]}(G)\} \cup \{(m\lambda_1(H) + p)^{[h_1]}, \ldots, (m\lambda_t(H) + p)^{[h_t]}\}.$$

Corollary 2.2. *Let H be a graph of order n with $\sigma_L(H) = \{\mu_1(H), \ldots, \mu_n(H)\}$ and let G be a graph of order m with $\sigma_L(G) = \{\mu_1(G), \ldots, \mu_m(G)\}$. Then*

$$\sigma_L(H[G]) = \left(\bigcup_{j=1}^{n} \{md_H(j) + \mu_i(G) : 1 \leq i \leq m-1\}\right) \cup \{m\mu_1(H), \ldots, m\mu_n(H)\}.$$

3 The spectra and Laplacian spectra of iterated lexicographic products of graphs

Let us consider the graphs obtained by an arbitrary number of iterations of the lexicographic product of a graph by another as follows:

$$H^0[G] = G, \;\; H^1[G] = H[G] \;\text{ and }\; H^k[G] = H[H^{k-1}[G]], \text{ for all integers } \; k \geq 2.$$

Example 3.1. *Let us consider the graph $H = C_4$ (the cycle with four vertices) and $G = K_2$ (the complete graph with two vertices). Then $H^0[G] = K_2$ and $H[G] = C_4[K_2]$ are depicted in Figure 1. Furthermore, Figure 2 depicts the graph $H^2[G] = {C_4}^2[K_2] = C_4[C_4[K_2]]$.*

In what follows, we adopt the traditional notation of the union of sets for denoting the union of multisets, where the repeated elements of the multisets A and B appear in $A \cup B$ as many times as we count them in A and B.

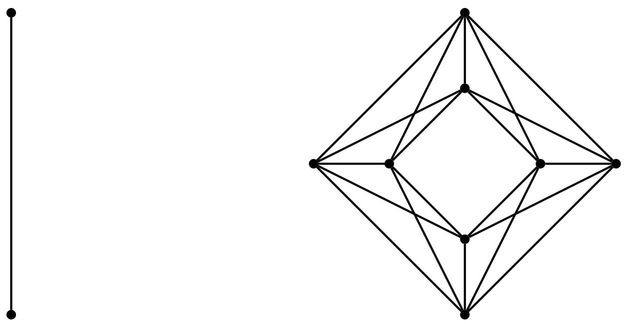

Figure 1: The graphs $H^0[G] = K_2$ and $H^1[G] = C_4[K_2]$.

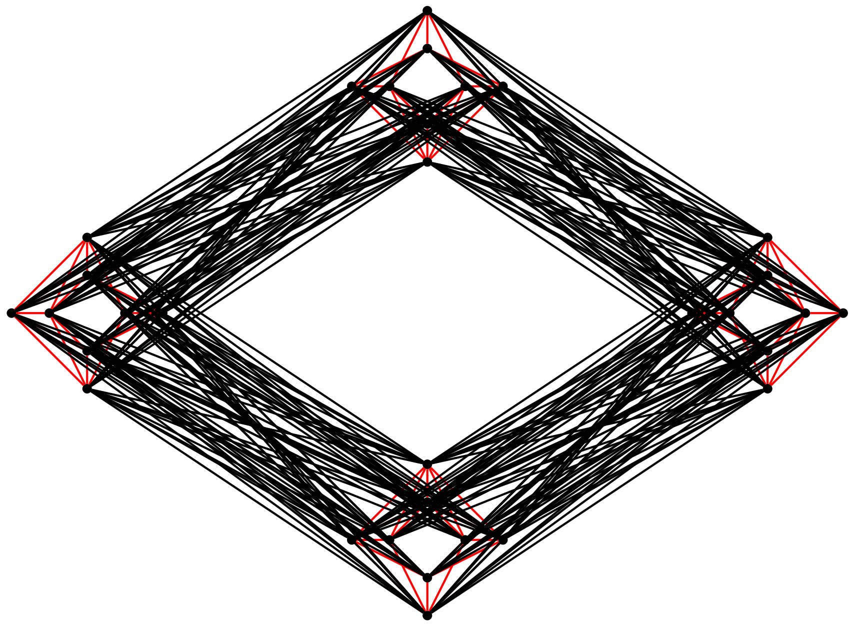

Figure 2: The graph $H^2[G] = C_4^2[K_2]$.

3.1 The spectrum in the case of a p-regular graph G and a q-regular graph H

The next theorem states the regularity degree, order and spectrum of $H^k[G]$, for $k \geq 0$, when G and H are both regular connected graphs.

Theorem 3.2. *Let H be a q-regular connected graph of order n with $\sigma_A(H) = \{q, \gamma_2^{[h_2]}(H), \ldots, \gamma_t^{[h_t]}(H)\}$ and G be a p-regular connected graph of order m with $\sigma_A(G) = \{p, \gamma_2^{[g_2]}(G), \ldots, \gamma_s^{[g_s]}(G)\}$. Then for each integer $k \geq 0$, $H^k[G]$ is a*

r_k-regular graph of order ν_k with

$$\begin{aligned}
r_k &= mq\frac{n^k-1}{n-1}+p,\\
\nu_k &= mn^k,\\
\sigma_A(H^k[G]) &= \left\{\gamma_2^{[n^k g_2]}(G),\ldots,\gamma_s^{[n^k g_s]}(G)\right\}\cup\{r_k\}\cup\Lambda_k \quad \textit{where}\\
\Lambda_k &= \bigcup_{i=0}^{k-1}\left\{(mn^i\gamma_2(H)+r_i)^{[n^{k-1-i}h_2]},\ldots,(mn^i\gamma_t(H)+r_i)^{[n^{k-1-i}h_t]}\right\},
\end{aligned}$$

assuming that $\Lambda_0=\emptyset$.

Proof. Since $H^0[G]=G$, the case $k=0$ follows. Furthermore, the case $k=1$ follows from Corollary 2.1, since $H^1[G]=H[G]$ (notice that $r_1=mq+p=m\gamma_1(H)+p$). Let us assume that the result holds for $k-1$ iterations, with $k\geq 2$. By definition of lexicographic product, we obtain

$$\begin{aligned}
r_k &= \nu_{k-1}q+r_{k-1}\\
&= mq(n^{k-1}+n^{k-2}+\cdots+n+1)+p = mq\frac{n^k-1}{n-1}+p
\end{aligned}$$

and $\nu_k=\nu(H^k[G])=\nu_{k-1}n=mn^k$. Additionally, replacing in the Corollary 2.1 the graph G by $H^{k-1}[G]$ it follows that

$$\begin{aligned}
\sigma_A(H^k[G]) &= \{\gamma_2^{[n^k g_2]}(G),\ldots,\gamma_s^{[n^k g_s]}(G)\}\cup\{r_k\}\cup\Lambda_k,\\
\text{where}\quad \Lambda_k &= \bigcup_{i=0}^{k-1}\left\{(mn^i\gamma_2(H)+r_i)^{[n^{k-1-i}h_2]},\ldots,(mn^i\gamma_t(H)+r_i)^{[n^{k-1-i}h_t]}\right\}.
\end{aligned}$$

□

Example 3.3. *For the graphs of Figure 1, we have $m=2$, $n=4$, $p=1$ and $q=2$. From Theorem 3.2, we obtain the following degree, order and spectra for $C_4^k[K_2]$ for a given $k\geq 1$ integer:*

$$\begin{aligned}
r_k &= 2\times 2\frac{4^k-1}{4-1}+1=\frac{4^{k+1}-1}{3},\\
\nu_k=\nu(H^k[G]) &= 2\times 4^k,\\
\sigma_A(H^k[G]) &= \left\{(-1)^{[4^k]}\right\}\cup\left\{\frac{4^{k+1}-1}{3}\right\}\cup\Lambda_k,
\end{aligned}$$

where $\Lambda_k=\bigcup_{i=0}^{k-1}\left\{\left(2\times 4^i\times 0+\frac{4^{i+1}-1}{3}\right)^{[4^{k-1-i}\times 2]},\left(2\times 4^i(-2)+\frac{4^{i+1}-1}{3}\right)^{[4^{k-1-i}]}\right\}$
$=\bigcup_{i=0}^{k-1}\left\{(\frac{4^{i+1}-1}{3})^{[4^{k-1-i}\times 2]},(-4^{i+1}+\frac{4^{i+1}-1}{3})^{[4^{k-1-i}]}\right\}$.

In particular, for $k = 2$ (the graph of Figure 2), it follows that

$$\begin{aligned} r_2 &= 21, \\ \nu_2 = \nu(H^2[G]) &= 32 \quad \textit{and} \\ \sigma_A(H^2[G]) &= \{21, (5)^{[2]}, (1)^{[8]}, (-1)^{[16]}, (-3)^{[4]}, -11\}. \end{aligned}$$

We may consider the graph obtained by an arbitrary number of iterations of the lexicographic product of a graph with itself. In fact, for a given q-regular graph H of order n, we assume that $H^0 = K_1$, $H^1 = H$ and that $H^k = H^{k-1}[H]$ for $k \geq 2$. Then, as an immediate consequence of Theorem 3.2, we have the following corollary.

Corollary 3.4. *Let H be a connected q-regular graph of order n with $\sigma_A(H) = \{q, \gamma_2^{[h_2]}(H), \ldots, \gamma_t^{[h_t]}(H)\}$. Then, for each integer $k \geq 1$, H^k is a r_k-regular graph of order ν_k, with*

$$\begin{aligned} r_k &= q\frac{n^k - 1}{n - 1}, \\ \nu_k &= n^k \quad \textit{and} \\ \sigma_A(H^k) &= \left(\bigcup_{i=0}^{k-1} \{(n^i\gamma_2(H) + r_i)^{[n^{k-1-i}h_2]}, \ldots, (n^i\gamma_t(H) + r_i)^{[n^{k-1-i}h_t]}\} \right) \cup \{r_k\}. \end{aligned}$$

Remark 3.5. *The least eigenvalue of H^k is $\lambda_{n^k}(H^k) = n^{k-1}\lambda_n(H) + q\frac{n^{k-1}-1}{n-1}$.*

Proof. In fact, based on the Corollary 3.4, we obtain

$$\begin{aligned} \lambda_{n^k}(H^k) &= \min_{0 \leq i \leq k-1} \{n^i\gamma_t(H) + r_i\} = \min_{0 \leq i \leq k-1} \{n^i\gamma_t(H) + q\frac{n^i - 1}{n - 1}\} \\ &= n^{k-1}\gamma_t(H) + q\frac{n^{k-1} - 1}{n - 1} = n^{k-1}\lambda_n(H) + q\frac{n^{k-1} - 1}{n - 1}. \end{aligned}$$

The third equality above is obtained taking into account that for every $i \in \{0, \ldots, k-1\}$, $n^{k-1}\gamma_t(H) + q\frac{n^{k-1}-1}{n-1} \leq n^i\gamma_t(H) + q\frac{n^i-1}{n-1} \Leftrightarrow (n^{k-1} - n^i)\gamma_t(H) \leq -q\frac{n^{k-1}-n^i}{n-1} \Leftrightarrow \gamma_t(H) \leq -\frac{q}{n-1}$ and the last inequality holds since the graph H has at least one edge and then (see [5]) $\gamma_t(H) \leq -1 \leq -\frac{q}{n-1}$. □

Remark 3.6. *Let H be a p-regular graph of order n. Then for all $k \in \mathbb{N}$ and for all nonnegative integer q, $\sigma_A(H^k) \setminus \{r_k\} \subset \sigma_A(H^{k+q})$, where r_k is the regularity of H^k and this inclusion means that all eigenvalues with the respective multiplicities of the multiset $\sigma_A(H^k) \setminus \{r_k\}$ belong to the multiset $\sigma_A(H^{k+q})$.*

Proof. This is a direct consequence of Corollary 3.4. □

Example 3.7. *Let us apply the Corollary 3.4 to the powers of the Pertersen graph P^k, with $k \in \{1, 2, 3, 100\}$.*

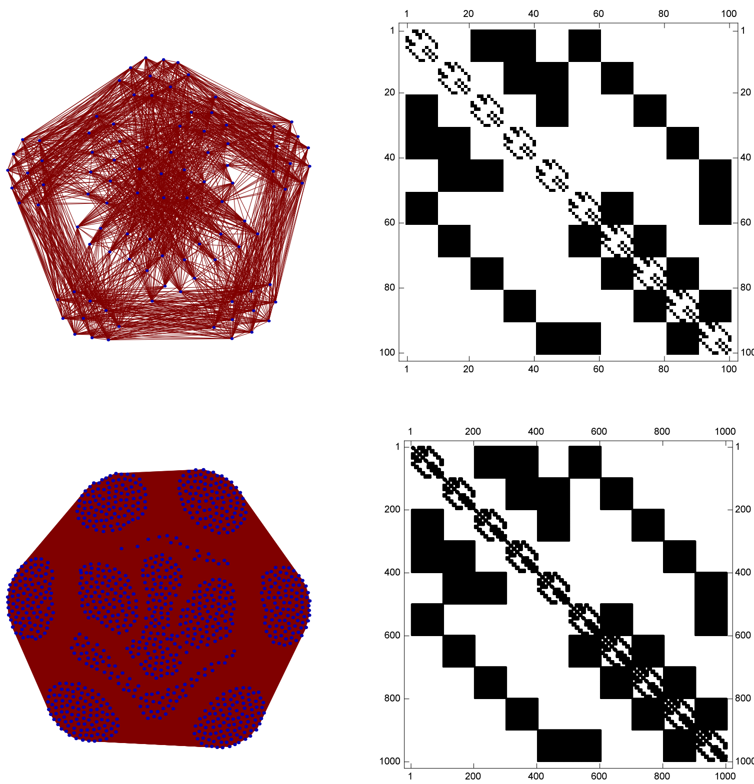

Figure 3: The second and third lexicographic powers of the Petersen graph with adjacency matrix plots.

k	*Spectrum of* P^k
$k=1$	$3,\quad 1^{[5]},\quad -2^{[4]}$
$k=2$	$33,\quad 13^{[5]},\ 1^{[50]},\quad -2^{[40]},\quad -17^{[4]}$
$k=3$	$333,\ 133^{[5]},\ 13^{[50]},\ 1^{[500]},\quad -2^{[400]},\ -17^{[40]},\ -167^{[4]}$
$k=100$	$3\times\sum_{i=0}^{99}10^i,\quad 1^{[5\times 10^{99}]},\ -2^{[4\times 10^{99}]},$ $\left(10^m+3\sum_{i=0}^{m-1}10^i\right)^{[5\times 10^{99-m}]},\ m=1,\dots,99,$ $-\left(7+10^m+6\sum_{i=1}^{m-1}10^i\right)^{[4\times 10^{99-m}]},\ m=1,\dots,99.$

Notice that the graph P^k *has* 10^k *vertices, in particular* P^{100} *has the googol number of vertices* 10^{100}*. All the computations were done by Mathematica and lasted just a few seconds.*

3.2 The Laplacian spectra

In this section we characterize the Laplacian spectrum of the iterated lexicographic product $H^k[G]$, where G and H are arbitrary graphs. The particular cases of the Laplacian spectra of these iterated lexicographic products, when H is regular and when H is arbitrary but equal to G are also presented.

Theorem 3.8. *Let G be a graph of order m with $\sigma_L(G) = \{\mu_1(G), \ldots, \mu_m(G)\}$ and let H be a graph with $V(H) = [n]$, where $[n] = \{1, 2, \cdots, n\}$, and $\sigma_L(H) = \{\mu_1(H), \ldots, \mu_n(H)\}$. Then, for each integer $k \geq 1$, $H^k[G]$ is a graph of order $\nu_k = mn^k$ with*

$$\sigma_L(H^k[G]) = \Omega_G^k \cup \Gamma_H^k$$

where

$$\begin{aligned}
\Omega_G^k &= \bigcup_{(j_1,j_2,\ldots,j_k)\in[n]^k} \left\{\mu_l(G) + m\sum_{i=1}^{k} n^{i-1}d_H(j_i) : 1 \leq l \leq m-1\right\} \text{ and} \\
\Gamma_H^k &= \bigcup_{i=2}^{k}\left(\bigcup_{(j_i,\ldots,j_k)\in[n]^{k-i+1}} \left\{mn^{i-2}\mu_l(H) + m\sum_{r=i}^{k} n^{r-1}d_H(j_r) : 1 \leq l \leq n-1\right\}\right) \cup \\
&\quad \left\{mn^{k-1}\mu_j(H) : 1 \leq j \leq n\right\}.
\end{aligned}$$

Proof. Corollary 2.2 gives us the assertion in case $k = 1$. Given an integer $k \geq 2$, let us suppose that

$$\sigma_L(H^{k-1}[G]) = \Omega_G^{k-1} \cup \Gamma_H^{k-1},$$

where

$$\begin{aligned}
\Omega_G^{k-1} &= \bigcup_{(j_1,\ldots,j_{k-1})\in[n]^{k-1}} \left\{\mu_l(G) + m\sum_{i=1}^{k-1} n^{i-1}d_H(j_i) : 1 \leq l \leq m-1\right\} \text{ and} \\
\Gamma_H^{k-1} &= \bigcup_{i=2}^{k-1}\left(\bigcup_{(j_i,\ldots,j_{k-1})\in[n]^{k-i}} \left\{mn^{i-2}\mu_l(H) + m\sum_{r=i}^{k-1} n^{r-1}d_H(j_r) : 1 \leq l \leq n-1\right\}\right) \\
&\quad \cup \left\{mn^{k-2}\mu_j(H) : 1 \leq j \leq n\right\}.
\end{aligned}$$

Then, by Corollary 2.2,

$$\begin{aligned}
\sigma_L(H^k[G]) &= \sigma_L(H[H^{k-1}[G]]) = \\
= \left(\bigcup_{j_k=1}^{n} \{mn^{k-1}d_H(j_k) + x : x \in \Omega_G^{k-1}\}\right) &\cup \left(\bigcup_{j_k=1}^{n} \{mn^{k-1}d_H(j_k) + y : y \in \Gamma_H^{k-1}\}\right),
\end{aligned}$$

where $\bigcup_{j_k=1}^{n}\left\{mn^{k-1}d_H(j_k)+x : x\in\Omega_G^{k-1}\right\}=$

$$
\begin{aligned}
&= \bigcup_{j_k=1}^{n}\left(\bigcup_{(j_1,\dots,j_{k-1})\in[n]^{k-1}}\left\{mn^{k-1}d_H(j_k)+\mu_l(G)+m\sum_{i=1}^{k-1}n^{i-1}d_H(j_i) : 1\le l\le m-1\right\}\right)\\
&= \bigcup_{(j_1,\dots,j_{k-1},j_k)\in[n]^{k}}\left\{\mu_l(G)+m\sum_{i=1}^{k}n^{i-1}d_H(j_i) : 1\le l\le m-1\right\}=\Omega_G^k \text{ and}
\end{aligned}
$$

$\bigcup_{j_k=1}^{n}\left\{mn^{k-1}d_H(j_k)+y : y\in\Gamma_H^{k-1}\right\}=$

$$
\begin{aligned}
&= \bigcup_{j_k=1}^{n}\left(\bigcup_{i=2}^{k-1}\bigcup_{(j_i,\dots,j_{k-1})\in[n]^{k-i}}\left\{mn^{k-1}d_H(j_k)+mn^{i-2}\mu_l(H)+m\sum_{r=i}^{k-1}n^{r-1}d_H(j_r) : 1\le l\le n-1\right\}\right.\\
&\quad \left.\cup\left\{mn^{k-1}d_H(j_k)+mn^{k-2}\mu_l(H) : 1\le l\le n-1\right\}\right)\cup\left\{mn^{k-1}\mu_j(H) : 1\le j\le n\right\}\\
&= \bigcup_{i=2}^{k}\bigcup_{(j_i,\dots,j_k)\in[n]^{k-i+1}}\left\{mn^{i-2}\mu_l(H)+m\sum_{r=i}^{k}n^{r-1}d_H(j_r) : 1\le l\le n-1\right\}\\
&\quad \cup\{mn^{k-1}\mu_j(H); 1\le j\le n\}=\Gamma_H^k\,.
\end{aligned}
$$

□

As an immediate consequence of the above theorem, for a regular graph H it follows

Corollary 3.9. *Let G and H as in Theorem 3.8, with H q-regular. Then, for each integer $k\ge 1$,*

$$
\begin{aligned}
\sigma_L(H^k[G]) &= \left\{\left(\mu_l(G)+mq\frac{n^k-1}{n-1}\right)^{[n^k]} : 1\le l\le m-1\right\}\cup\{0\}\cup\\
&\quad \bigcup_{i=2}^{k+1}\left\{\left(mn^{i-2}\mu_l(H)+mqn^{i-1}\frac{n^{k-i+1}-1}{n-1}\right)^{[n^{k-i+1}]} : 1\le l\le n-1\right\}.
\end{aligned}
$$

Proof. From Theorem 3.8, for all integers $k\ge 1$, it follows that

$$\sigma_L(H^k[G])=\Omega_G^k\cup\Gamma_H^k,$$

where

$$
\begin{aligned}
\Omega_G^k &= \bigcup_{(j_1,\dots,j_k)\in[n]^k}\left\{\mu_l(G)+mq\sum_{i=1}^{k}n^{i-1} : 1\le l\le m-1\right\}\\
&= \left\{\left(\mu_l(G)+mq\frac{n^k-1}{n-1}\right)^{[n^k]} : 1\le l\le m-1\right\} \qquad \text{and}
\end{aligned}
$$

$$\begin{aligned}\Gamma_H^k &= \bigcup_{i=2}^{k}\left(\bigcup_{(j_i,\dots,j_k)\in[n]^{k-i+1}}\left\{mn^{i-2}\mu_l(H)+mq\sum_{r=i}^{k}n^{r-1} : 1\le l\le n-1\right\}\right)\cup\\ &\quad \left\{mn^{k-1}\mu_j(H) : 1\le j\le n\right\}\\ &= \bigcup_{i=2}^{k+1}\left\{\left(mn^{i-2}\mu_l(H)+mqn^{i-1}\frac{n^{k-i+1}-1}{n-1}\right)^{[n^{k-i+1}]} : 1\le l\le n-1\right\}\cup\{0\}.\end{aligned}$$

□

Now, let us consider the case $G=H$.

Corollary 3.10. *Let H be a graph with $V(H)=[n]$ and $\sigma_L(H)=\{\mu_1(H),\dots,\mu_n(H)\}$. Then H^k is a graph of order $\nu_k=n^k$ and*

$$\begin{aligned}\sigma_L(H^k) &= \bigcup_{i=1}^{k-1}\left(\bigcup_{(j_i,\dots,j_{k-1})\in[n]^{k-i}}\left\{n^{i-1}\mu_l(H)+\sum_{r=i}^{k-1}n^{r}d_H(j_r) : 1\le l\le n-1\right\}\right)\cup\\ &\quad \left\{n^{k-1}\mu_j(H) : 1\le j\le n\right\},\end{aligned}$$

for all $k\ge 2$.

Proof. The first statement is obvious. Regarding the second statement, applying again Theorem 3.8 for $k\ge 2$ we obtain

$$\sigma_L(H^k)=\sigma_L(H^{k-1}[H])=$$

$$=\bigcup_{(j_1,j_2,\dots,j_{k-1})\in[n]^{k-1}}\left\{\mu_l(H)+n\sum_{i=1}^{k-1}n^{i-1}d_H(j_i) : 1\le l\le n-1\right\}\cup$$

$$\bigcup_{i=2}^{k-1}\left(\bigcup_{(j_i,\dots,j_{k-1})\in[n]^{k-i}}\left\{n^{i-1}\mu_l(H)+n\sum_{r=i}^{k-1}n^{r-1}d_H(j_r) : 1\le l\le n-1\right\}\right)\cup$$

$$\{nn^{k-2}\mu_j(H) : 1\le j\le n\}=$$

$$=\bigcup_{i=1}^{k-1}\left(\bigcup_{(j_i,\dots,j_k)\in[n]^{k-i}}\left\{n^{i-1}\mu_l(H)+\sum_{r=i}^{k-1}n^{r}d_H(j_r) : 1\le l\le n-1\right\}\right)\cup$$

$$\{n^{k-1}\mu_j(H) : 1\le j\le n\}.$$

□

Finally, the next proposition determines the algebraic connectivity and the largest Laplacian eigenvalue of H^k, for $k\ge 1$.

Theorem 3.11. *If H is a connected graph of order n with $\sigma_L(H)=\{\mu_1(H),\ldots,\mu_n(H)\}$ and $k\geq 1$, then*

$$\begin{aligned}\mu_{n^k-1}(H^k) &= n^{k-1}\mu_{n-1}(H) \qquad \textit{and} &(1)\\ \mu_1(H^k) &= n^{k-1}\mu_1(H) &(2)\end{aligned}$$

Proof. Let $k\geq 1$ be fixed. From Corollary 3.10, it follows that the second least eigenvalue of H^k is among the values $n^{k-1}\mu_{n-1}(H)$ and $n^{i-1}\mu_{n-1}(H)+\sum_{r=i}^{k-1}n^r d_H(j_r)$ for $1\leq i\leq k-1$. We may recall that $\delta(H)\geq\mu_{n-1}(H)$; then, for all $1\leq i\leq k-1$, it holds that

$$n^{i-1}\mu_{n-1}(H)+\sum_{r=i}^{k-1}n^r d_H(j_r)\geq n^{i-1}\mu_{n-1}(H)+\sum_{r=i}^{k-1}n^r\delta(H)$$

$$\geq n^{i-1}\mu_{n-1}(H)+\mu_{n-1}(H)\sum_{r=i}^{k-1}n^r=\mu_{n-1}(H)\left(n^{i-1}+\sum_{r=i}^{k-1}n^r\right)$$

$$=\ \mu_{n-1}(H)\sum_{r=i-1}^{k-1}n^r=\mu_{n-1}(H)\sum_{r=i}^{k}n^{r-1}\ \geq\ \mu_{n-1}(H)n^{k-1}.$$

Thus the equality (1) is proved. Now, let us prove the equality (2).
Applying again Corollary 3.10, it follows that the largest Laplacian eigenvalue of H^k is among the values $n^{k-1}\mu_1(H)$ and $n^{i-1}\mu_1(H)+\sum_{r=i}^{k-1}n^r d_H(j_r)$, $1\leq i\leq k-1$. Since $\mu_1(H)\geq\Delta(H)+1$, for $1\leq i\leq k-1$, it follows that

$$\begin{aligned}n^{i-1}\mu_1(H)+\sum_{r=i}^{k-1}n^r d_H(j_r) &\leq n^{i-1}\mu_1(H)+\sum_{r=i}^{k-1}n^r(\mu_1(H)-1)\\ &= \mu_1(H)\sum_{r=i-1}^{k-1}n^r-\sum_{r=i}^{k-1}n^r\\ &= \mu_1(H)n^{i-1}\frac{n^{k-i+1}-1}{n-1}-n^i\frac{n^{k-i}-1}{n-1}\\ &= \mu_1(H)n^{i-1}(\frac{n^{k-i}-1}{n-1}+n^{k-i})-n^i\frac{n^{k-i}-1}{n-1}\\ &= \mu_1(H)n^{k-1}+\frac{n^{k-i}-1}{n-1}n^{i-1}(\mu_1(H)-n)\\ &\leq n^{k-1}\mu_1(H)\end{aligned}$$

The last inequality is obtained taking into account that $\mu_1(H)-n\leq 0$. $\square$

4 Spectral and combinatorial invariant properties of lexicographic powers of graphs

In this section, a few well known spectral and combinatorial invariant properties of a graph H are extended to the lexicographic powers of H. For instance, considering that H has order $n \geq 2$, for all $k \geq 1$, we may deduce that

$$\delta(H^k) = \delta(H)\frac{n^k - 1}{n-1} \qquad \left(\Delta(H^k) = \Delta(H)\frac{n^k - 1}{n-1}\right). \tag{3}$$

Notice that since H has order n, then H^k has order n^k. The equalities (3) can be proved by induction on k, taking into account that they are obviously true for $k = 1$. Assuming that the equalities (3) are true for $k-1$, with $k \geq 2$, it is immediate that a vertex of H^k with minimum (resp. maximum) degree is a minimum (resp. maximum) degree vertex of the copy of H^{k-1} located in the position of a minimum (resp. maximum) degree vertex of H, and then its degree in H^k is equal to $\delta(H)\left(\frac{n^{k-1}-1}{n-1} + n^{k-1}\right)$ $\left(\text{resp. } \Delta(H)(\frac{n^{k-1}-1}{n-1} + n^{k-1})\right)$.

For an arbitrary graph G, let $q_1(G)$ and $q_n(G)$ be the largest and the least eigenvalue of the signless Laplacian matrix of G (that is, the matrix $A_G + D$), respectively. Taking into account the relations $2\delta(G) \leq q_1(G) \leq 2\Delta(G)$, which were proved in [6], and also the inequality $q_n(G) < \delta(G)$ [8], for the lexicographic power k of a graph H we obtain the inequalities

$$\begin{aligned} 2\delta(H)\frac{n^k-1}{n-1} \;\leq\; q_1(H^k) \;&\leq\; 2\Delta(H)\frac{n^k-1}{n-1} \qquad \text{and} \\ q_n(H^k) \;&<\; \delta(H)\frac{n^k-1}{n-1}. \end{aligned}$$

Denoting the distance between two vertices x and y in G by $d_G(x, y)$ and the diameter of G by $\mathrm{diam}(G)$, we may conclude the following interesting result concerning the diameter of the iterated lexicographic products of graphs.

Theorem 4.1. *Let H be a connected not complete graph and let G be an arbitrary graph of order m. For every $k \in \mathbb{N}$*

$$diam(H^{k+1}) = diam(H^k[G]) = diam(H).$$

Proof. Consider $V(H) = \{1, \ldots, n\}$ and $x, y \in V(H^k[G])$ (resp. $x, y \in V(H^{k+1})$). Then we have two cases (a) they are both in the same copy of $H^{k-1}[G]$ (resp. H^k) located in the position of the vertex $i \in V(H)$ or (b) they are in different copies of $H^{k-1}[G]$ (resp. H^k) located in the positions of the vertices $r, s \in V(H)$.

(a) If x and y are adjacent, then $d_{H^k[G]}(x, y) = 1$ (resp. $d_{H^{k+1}}(x, y) = 1$), otherwise there exists a vertex $j \in V(H)$ such that $ij \in E(H)$ and then there is a path x, z, y, where z is a vertex of the copy of $H^{k-1}[G]$ (resp. H^k) located in the position of the vertex $j \in V(H)$. Therefore, $d_{H^k[G]}(x, y) = 2$ (resp. $d_{H^{k+1}}(x, y) = 2$).

(b) In this case, assuming that $r, j_1, \ldots, j_t, s$ is a shortest path in H connecting the vertices r and s, there are vertices $z_1, \ldots, z_t$ in the copies of $H^{k-1}[G]$ (resp. H^k) located in the positions of the vertices $j_1, \ldots, j_t$, respectively, such that $x, z_1, \ldots, z_t, y$ is a path of length $d_H(r, s)$.

$\square$

4.1 The stability number

Regarding the stability number $\alpha(G)$ (the maximum cardinality of a vertex subset of an arbitrary graph G with pairwise nonadjacent vertices), according to [11], $\alpha(H[G]) = \alpha(H)\alpha(G)$ for an arbitrary graph H. Thus we may conclude that $\alpha(H^k) = \alpha(H)^k$ (and, denoting the complement of graph F by $\overline{F}$ and the clique number by $\omega(F)$, since $\overline{H[G]} = \overline{H}[\overline{G}]$, $\omega(H^k) = \omega(H)^k$). Furthermore, from the spectral upper bound $\alpha(G) \leq n\frac{\mu_1(G)-\delta(G)}{\Delta(G)}$, independently deduced in [18] and [12] for an arbitrary graph G, and taking into account (3) and (2), considering the k-th lexicographic power of a graph H of order n we obtain

$$\alpha(H^k) \leq n^k \frac{\mu_1(H^k) - \delta(H^k)}{\Delta(H^k)} \leq n^k \frac{\frac{n-1}{n^k-1} n^{k-1}\mu_1(H) - \delta(H)}{\Delta(H)}.$$

4.2 The vertex connectivity

Considering a graph G of order m and a graph H of order n, it is well known that the lexicographic product $H[G]$ is connected if and only if H is a connected graph [14]. On the other hand, according to [11], if both G and H are not complete, then $\upsilon(H[G]) = m\upsilon(H)$, where $\upsilon(H)$ denotes the vertex connectivity of H (that is, the minimum number of vertices whose removal yields a disconnected graph). Therefore, $\upsilon(H^k) = n^{k-1}\upsilon(H)$. Furthermore, we may conclude that when H is connected not complete (and then H^k is also connected not complete),

$$n^{k-1}\mu_{n-1}(H) \leq \upsilon(H^k) \leq \delta(H)\frac{n^k - 1}{n - 1}.$$

In fact, it should be noted that $\upsilon(G) \leq \delta(G)$ and, when G is not complete, $\mu_{n-1}(G) \leq \upsilon(G)$, see [9]. Therefore, taking into account (1) and (3) we obtain $n^{k-1}\mu_{n-1}(H) = \mu_{n-1}(H^k) \leq \upsilon(H^k) \leq \delta(H^k) = \delta(H)\frac{n^k-1}{n-1}$.

4.3 The chromatic number

Concerning the relations of the chromatic number of a graph G of order n with its spectrum, the following lower bound due to Hoffman in [16] is well known.

$$\chi(G) \quad \geq \quad 1 - \frac{\lambda_1(G)}{\lambda_n(G)}.$$

As direct consequence, if a graph H is q-regular of order n, taking into account the Remark (3.5), we may conclude the following lower bound on the chromatic number of H^k:

$$\begin{aligned}
\chi(H^k) \geq 1 - \frac{r_k}{\lambda_{n^k}(H^k)} &= 1 - q\frac{n^k - 1}{(n-1)\left(n^{k-1}\lambda_n(H) + q\frac{n^{k-1}-1}{n-1}\right)} \\
&= 1 - \frac{n^k - 1}{n^{k-1}\left((n-1)\frac{\lambda_n(H)}{q} + 1\right) - 1}.
\end{aligned}$$

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