# Distance matrices on the $H - join$ of graphs: a general result and applications

Domingos M. Cardoso*
Roberto C. Díaz and Oscar Rojo†

**Abstract**

Given a graph $H$ with vertices $1, \ldots, s$ and a set of pairwise vertex disjoint graphs $G_1, \ldots, G_s$, the vertex $i$ of $H$ is assigned to $G_i$. Let $G$ be the graph obtained from the graphs $G_1, \ldots, G_s$ and the edges connecting each vertex of $G_i$ with all the vertices of $G_j$ for all edge $ij$ of $H$. The graph $G$ is called the $H - join$ of $G_1, \ldots, G_s$. Let $M(G)$ be a matrix on a graph $G$. A general result on the eigenvalues of $M(G)$, when the all ones vector is an eigenvector of $M(G_i)$ for $i = 1, 2, \ldots, s$, is given. This result is applied to obtain the distance eigenvalues, the distance Laplacian eigenvalues and as well as the distance signless Laplacian eigenvalues of $G$ when $G_1, \ldots, G_s$ are regular graphs. Finally, we introduce the notions of the distance incidence energy and distance Laplacian-energy like of a graph and we derive sharp lower bounds on these two distance energies among all the connected graphs of prescribed order in terms of the vertex connectivity. The graphs for which those bounds are attained are characterized.

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## 1 Introduction

The *distance matrix* of a graph $G$ of order $n$ is the $n \times n$ matrix $\mathcal{D}(G) = (d_{i,j})$, indexed by the vertices of $G$, where $d_{i,j}$ is the distance (number of edges of a shortest path) between vertices $v_i$ and $v_j$. The very beginning of the distance matrix research goes back to the thirties of the twentieth century [22] and [33]. The main motivation from graph theory point of view was the problem of realizability of distance matrices, first presented in [8] and investigated in [23, 24, 25, 29], [5], [30] and [4], among many other papers. However, it was proven in [1] and

*Department of Mathematics, Universidade de Aveiro, Aveiro, Portugal.
†Department of Mathematics, Universidad Católica del Norte, Antofagasta, Chile.

[31] that the problem of determining the minimum weight graph realization of a distance matrix with integer entries is **NP**-complete. The distance matrices have also deserved the attention of the spectral graph theory community since the paper published by Graham and Pollak in 1971 [14], where a relationship between the number of negative eigenvalues of the distance matrix and the addressing problem in data communication systems is established. In the same paper the authors also proved the following very impressive result, which drew the attention of many researchers, transforming the spectral properties of distance matrices in a hot topic of spectral graph theory. Assuming that $T$ is a tree of order $n \geq 2$ with distance matrix $\mathcal{D}(T)$, then

$$\det\mathcal{D}(T) = (-1)^{n-1}(n-1)\,2^{n-2}. \tag{1}$$

Thus, the determinant of the distance matrix of a tree depends only from its number of vertices. The concepts of *distance Laplacian* and *distance signless Laplacian* were introduced in [3], where the authors proven the equivalence between the distance signless Laplacian, distance Laplacian and the distance spectra for the class of transmission regular graphs.
A very complete survey of the state of the art on distance matrices up to 2014 appears in [2] (see also [27]). More recently, extremal graphs were characterized in terms of the eigenvalues of distance signed Laplacian and distance Laplacian matrices. Namely, the graphs of order $n$ with least distance signed Laplacian eigenvalue equal to $n-2$ and the graphs of order $n \geq 11$ with second least distance signed Laplacian eigenvalue in the interval $[n-2, n]$ [19]; the graphs of order $n$ with largest distance Laplacian eigenvalue equal to $n-2$ [18] (in this paper the authors proven the conjecture proposed in [2] that the multiplicity of the largest eigenvalue of the distance Laplacian matrix of a non complete graph is not greater than $n-2$ with equality if and only if either the graph is isomorphic to a star or to a regular complete bipartite graph).

This paper is devoted to the determination of distance, distance Laplacian and distance signless Laplacian eigenvalues of graphs obtained by the $H$-joint operation [7] over a family of regular graphs. Furthermore, sharp lower bounds on distance incidence energy and distance Laplacian-energy like of a graph are deduced for graphs of prescribed order in terms of the vertex connectivity and the graphs for which these bounds are attained are characterized.

In the remaining part of this section, we introduce the notation and basic definitions of the concepts used throughout the paper.

Let $G = (V(G), E(G))$ be a simple undirected graph of order $n$, that is, on $n$ vertices, with vertex set $V(G)$ and edge set $E(G)$. The cardinality of $V(G)$ is called the order of $G$. If $e \in E(G)$ has end vertices $u$ and $v$, then we say that $u$ and $v$ are adjacent and this edge is denoted by $uv$. If $u \in V(G)$, then $N_G(u)$ is the set of neighbors of $u$ in $G$, that is, $N_G(u) = \{v \in V(G) : uv \in E(G)\}$. The cardinality of $N_G(u)$ is said to be the *degree* of $u$. A graph $G$ is called $d$-degree regular when every vertex has the same degree equal to $d$.
The vertex connectivity (or just connectivity) of a graph $G$, denoted by $\kappa(G)$, is the minimum number of vertices of $G$ whose deletion disconnects $G$. It is conventional to define $\kappa(K_n) = n-1$.
The adjacency matrix of a graph $G$ of order $n$, $A(G)$, is a $0-1$-matrix of order $n$ with entries

$a_{ij}$ such that $a_{ij} = 1$ if $ij \in E(G)$ and $a_{ij} = 0$ otherwise. Other matrices on $G$ are the Laplacian matrix $L(G) = D(G) - A(G)$ and the signless Laplacian matrix $Q(G) = D(G) + L(G)$, where $D(G)$ is the diagonal matrix of vertex degrees. It is well known that $L(G)$ and $Q(G)$ are positive semidefinite matrices and that $(0, \mathbf{1})$ is an eigenpair of $L(G)$ where $\mathbf{1}$ is the all ones vector. Let us denote by $\sigma(M)$ the spectrum (the multiset of eigenvalues) of a square matrix $M$. An eigenvalue of a matrix $M$ will be denoted by $\lambda(M)$ and throughout the paper, assuming that $M$ has order $n$, the eigenvalues $M$ are indexed in non increasing order, that is, $\lambda_1(M) \geq \cdots \geq \lambda_n(M)$. Given an eigenvalue $\lambda_i(M)$ its eigenspace will be denoted by $\Lambda_{\lambda_i}(M)$. If $M$ is a nonnegative matrix then, by the Perron-Frobenius Theorem, $M$ has an eigenvalue equal to its spectral radius, called the Perron root of $M$. In addition, if $M$ is irreducible then the Perron root of $M$ is a simple eigenvalue with a corresponding positive eigenvector, called the Perron vector of $M$.

Given two vertex disjoint graphs $G_1$ and $G_2$, the join of $G_1$ and $G_2$ is the graph $G = G_1 \vee G_2$ such that $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{xy : x \in V(G_1), y \in V(G_2)\}$. This join operation can be generalized as follows [6, 7]: Let $H$ be a graph of order $s$. Let $V(H) = \{1, \ldots, s\}$. Let $\{G_1, \ldots, G_s\}$ be a set of pairwise vertex disjoint graphs. For $1 \leq i \leq s$, the vertex $i \in V(H)$ is assigned to the graph $G_i$. Let $G$ be the graph obtained from the graphs $G_1, \ldots, G_s$ and the edges connecting each vertex of $G_i$ with all the vertices of $G_j$ if and only if $ij \in E(H)$. That is, $G$ is the graph with vertex set $V(G) = \bigcup_{i=1}^{s} V(G_i)$ and edge set

$$E(G) = \left(\bigcup_{i=1}^{s} E(G_i)\right) \cup \left(\bigcup_{ij \in E(H)} \{uv : u \in V(G_i), v \in V(G_j)\}\right).$$

This graph operation introduced in [6] under the designation of $H-join$ of the graphs $G_1, \ldots, G_s$ it is denoted by

$$G = \bigvee_{H} \{G_i : 1 \leq i \leq s\}.$$

The same graph operation was introduced in [11] as a generalization of the lexicographic product [9, 10]. In [11] this graph operation was designated *generalized composition and denoted by* $H[G_1, G_2, \ldots, G_s]$. Clearly if each $G_i$ is a graph of order $n_i$, then the $H-join$ of $G_1, \ldots, G_s$ (generalized composition $H[G_1, G_2, \ldots, G_s]$) is a graph of order $n = n_1 + n_2 + \ldots + n_s$. The same operation appear in [26] under the designation of joined union. In [26] the distance spectrum of the joined union of regular graphs is determined and this technique is applied to the construction of distance equienergetic graphs with diameter greater than two.

As usual, let $P_n, C_n, S_n$ and $K_n$ be the path, cycle, star and the complete graph on $n$ vertices, respectively.

**Example 1** *Let* $H = P_3$, $G_1 = P_2, G_2 = K_4$ *and* $G_3 = P_3$. *Then the graph* $\bigvee_H \{G_1, G_2, G_3\}$ *is depicted in Figure 1.*

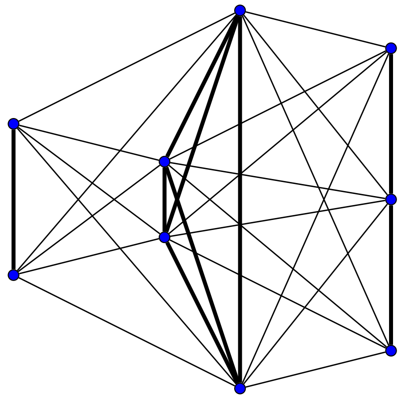

**Figure 1**. *The graph* $\bigvee_{P_3}\{P_2, K_4, P_3\}$.

The Wiener index $W(G)$ of a connected graph $G$ is $W(G) = \frac{1}{2}\sum_{u,v\in V(G)} d(u,v)$,

$$W(G) = \frac{1}{2}\sum_{u,v\in V(G)} d(u,v),$$

where $d(u,v)$ is the distance between $u,v \in V(G)$, that is, the length of the shortest path connecting $u$ and $v$. The *transmission* $Tr(v)$ of a vertex $v \in V(G)$ is the sum of the distances from $v$ to all other vertices of $G$, that is,

$$Tr(v) = \sum_{u\in V(G)} d(v,u).$$

A graph $H$ is said to be $r-$ *transmission regular* if $Tr(v) = r$ for each vertex $v \in V(H)$. The eigenvalues of the distance matrix $\mathcal{D}(G)$ of the graph $G$ are called the *distance eigenvalues* of $G$ and they are denoted by

$$\partial_1(G) \geq \partial_2(G) \geq \ldots \geq \partial_n(G).$$

In [3] Aouchiche and Hansen introduce, for a connected graph $G$, the distance Laplacian matrix $\mathcal{L}(G)$ and the signless Laplacian matrix $\mathcal{Q}(G)$, respectively, as follows

$$\mathcal{L}(G) = Tr(G) - \mathcal{D}(G) \text{ and } \mathcal{Q}(G) = Tr(G) + \mathcal{D}(G),$$

where $Tr(G) = diag[Tr(v_1), Tr(v_2), \ldots, Tr(v_n)]$ is the diagonal matrix of the vertex transmissions in $G$. The eigenvalues of $\mathcal{L}(G)$ and $\mathcal{Q}(G)$ are called the *distance Laplacian eigenvalues* and the *distance signless Laplacian eigenvalues* of G and they are denoted by

$$\partial_1^L(G) \geq \partial_2^L(G) \geq \ldots \geq \partial_n^L(G) \text{ and } \partial_1^Q(G) \geq \partial_2^Q(G) \geq \ldots \geq \partial_n^Q(G),$$

respectively. Notice that $\mathcal{L}(G)$ and $\mathcal{Q}(G)$ are both real symmetric matrices and then, from Geršgorin's Theorem, it follows that their eigenvalues are nonnegative real numbers. Let **1** be the all one vector. Clearly each row sum of $\mathcal{L}(G)$ is 0. Then $(0, \mathbf{1})$ is an eigenpair of $\mathcal{L}(G)$ and, when $G$ is connected graph, 0 is a simple eigenvalue.

It is clear that if $G$ is a $r-$ transmission regular graph, then

$$\mathcal{L}(G) = rI_n - \mathcal{D}(G) \text{ and } \mathcal{Q}(G) = rI_n + \mathcal{D}(G),$$

where $I_n$ is the identity matrix of order $n$ and, for $i = 1, \ldots, n$, $\partial_i^L(G) = r - \partial_{n-i+1}(G)$ and $\partial_i^Q(G) = r + \partial_i(G)$.

A basic result on $\partial_1^L(G)$ is the following:

**Theorem 1** *[3, Cor. 3.6] Let $G$ be a connected graph of order $n \geq 3$. Then*

$$\partial_i^L(G) \geq \partial_i^L(K_n) = n, \qquad \text{for } i = 1, 2, \ldots, n-1,$$

*and $\partial_n^L(G) = \partial_n^L(K_n) = 0$.*

Throughout this paper, $G = \bigvee_H \{G_i : 1 \leq i \leq s\}$ where, for $1 \leq i \leq s$, it is assumed that $G_i$ is a graph order $n_i$ such that the all one vector $\mathbf{1}_{n_i}$ is an eigenvector for the eigenvalue $\mu_i$ of $G_i$. Moreover, $I$ and 0 are the identity and the zero matrices of the appropriate order, respectively.

Let $M(G)$ be a matrix on a graph $G$. In this paper, a general result on the eigenvalues of $M(G)$, when the all one vector is an eigenvector of $M(G_i)$ for $i = 1, 2, \ldots, s$, is given. This result is applied to obtain the distance eigenvalues, the distance Laplacian eigenvalues and as well as the distance signless Laplacian eigenvalues of $G$ when $G_1, \ldots, G_s$ are regular graphs.

Finally, we introduce the notions of the distance incidence energy and distance Laplacian-energy like of a graph and derive sharp lower bounds on these two distance energies among all the connected graphs $G$ of order $n$, with $m \geq n$ edges and $\kappa(G) \leq k$. The graphs for which those bounds are attained are characterized.

## 2 A general result on the *H-join* of graphs

Consider the vertices of $G$ with the labels $1, \ldots, \sum_{i=1}^{s} n_i$ starting with the vertices of $G_1$, continuing with the vertices of $G_2, G_3, \ldots, G_{s-1}$ and finally with the vertices of $G_s$.

**Example 2** *For the graph in Example 1, our labeling is*

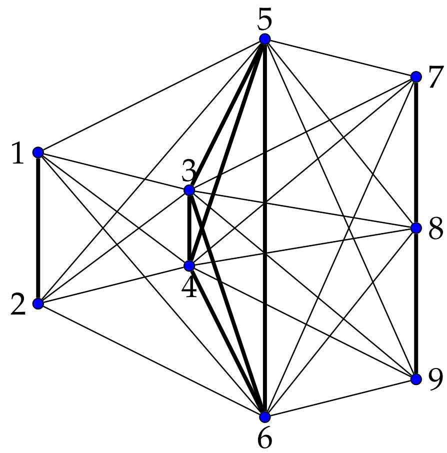


**Figure 2**. *The graph of Fig. 1 with vertex labels as described.*

With the above mentioned labeling, we get

$$M\left(G\right)=\left[\begin{array}{cccc} M_1 & \delta_{12}\mathbf{1}_{n_1}\mathbf{1}_{n_2}^T & \cdots & \delta_{1s}\mathbf{1}_{n_1}\mathbf{1}_{n_s}^T \\ \delta_{12}\mathbf{1}_{n_2}\mathbf{1}_{n_1}^T & M_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \delta_{(s-1)s}\mathbf{1}_{n_{s-1}}\mathbf{1}_{n_s}^T \\ \delta_{1s}\mathbf{1}_{n_s}\mathbf{1}_{n_1}^T & \cdots & \delta_{(s-1)s}\mathbf{1}_{n_s}\mathbf{1}_{n_{s-1}}^T & M_s \end{array}\right] \tag{2}$$

where the diagonal blocks $M_i$ are symmetric matrices such that

$$M_i\mathbf{1}_{n_i}=\mu_i\mathbf{1}_{n_i} \tag{3}$$

for $i=1,\ldots,s$ and $\delta_{i,j}$ are scalars for $1\leq i<j\leq s$.

Using a strategy similar to the one used in [6], the following lemmas are proven and the spectrum of the matrix $M(G)$ is deduced as it is stated by Theorem 2.

**Lemma 1** *Consider the block matrices $M_i$ which appear in the expression of $M(G)$ in (2) and the eigenvalues $\mu_i$ in (3), both for $i=1,\ldots,s$. Then*

$$\cup_{i=1}^{s}\left(\sigma(M_i)-\{\mu_i\}\right)\subseteq\sigma(M\left(G\right)).$$

**Proof** Let $\lambda_i\neq\mu_i$ be an eigenvalue of $M_i$ with multiplicity $m_i$ and let $(\lambda_i,\mathbf{u}_i)$ be an eigenpair of $M_i$, where $\mathbf{u}_i$ is an arbitrary vector in $\Lambda_{\lambda_i}(M_i)\setminus\{\mathbf{0}\}$, for $i=1,\ldots,s$. Then, for $i=1$, we have

$$M\left(G\right)\left[\begin{array}{c}\mathbf{u}_1\\0\\\vdots\\0\end{array}\right]=\left[\begin{array}{c}M_1\mathbf{u}_1\\0\\\vdots\\0\end{array}\right]=\left[\begin{array}{c}\lambda_1\mathbf{u}_1\\0\\\vdots\\0\end{array}\right]=\lambda_1\left[\begin{array}{c}\mathbf{u}_1\\0\\\vdots\\0\end{array}\right]$$

and thus $\lambda_1 \in \sigma(M(G))$ with multiplicity $m_1$. Similarly,

$$M(G)\begin{bmatrix} 0 \\ \mathbf{u}_2 \\ \vdots \\ 0 \end{bmatrix} = \lambda_2 \begin{bmatrix} 0 \\ \mathbf{u}_2 \\ \vdots \\ 0 \end{bmatrix}, \qquad \cdots \qquad M(G)\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \mathbf{u}_s \end{bmatrix} = \lambda_s \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \mathbf{u}_s \end{bmatrix}$$

and thus $\lambda_2, \ldots, \lambda_s \in \sigma(M(G))$, with multiplicity $m_2, \ldots, m_s$, respectively. Furthermore, for each $i \in \{1, \ldots, s\}$, if $\mu_i$ has multiplicity $p_i > 1$, we can consider $p_i - 1$ linear independent vectors from $\Lambda_{\mu_i}(M_i) \setminus \{\mathbf{0}\}$ orthogonal to $\mathbf{1}_i$ and using each of them in a similar way as above, we obtain $p_i - 1$ linear independent eigenvectors of $M(G)$ associated to $\mu_i$. Therefore, $\mu_i \in \sigma(M(G))$ with multiplicity $p_i - 1$, for $i = 1, \ldots, s$. □

**Lemma 2** *Considering the matrix $M(G)$ in (2) and the $s \times s$ symmetric matrix*

$$F_s = \begin{bmatrix} \mu_1 & \delta_{12}\sqrt{n_1 n_2} & \ldots & \delta_{1(s-1)}\sqrt{n_1 n_{s-1}} & \delta_{1s}\sqrt{n_1 n_s} \\ \delta_{12}\sqrt{n_1 n_2} & \mu_2 & \ldots & \delta_{2(s-1)}\sqrt{n_2 n_{s-1}} & \delta_{2s}\sqrt{n_2 n_s} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \delta_{1(s-1)}\sqrt{n_2 n_{s-1}} & \delta_{2(s-1)}\sqrt{n_2 n_{s-1}} & \ldots & \mu_{s-1} & \delta_{(s-1)s}\sqrt{n_{s-1} n_s} \\ \delta_{1s}\sqrt{n_1 n_s} & \delta_{2s}\sqrt{n_2 n_s} & \ldots & \delta_{(s-1)s}\sqrt{n_{s-1} n_s} & \mu_s \end{bmatrix}, \tag{4}$$

*it follows that*

$$\sigma(F_s) \subseteq \sigma(M(G)).$$

**Proof** Let $\lambda \in \sigma(F_s)$. There exists $\begin{bmatrix} x_1 & x_2 & \ldots & x_s \end{bmatrix}^T \neq \begin{bmatrix} 0 & 0 & \ldots & 0 \end{bmatrix}^T$ such that

$$\begin{bmatrix} \mu_1 & \delta_{12}\sqrt{n_1 n_2} & \ldots & \delta_{1s}\sqrt{n_1 n_s} \\ \delta_{12}\sqrt{n_1 n_2} & \mu_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \delta_{(s-1)s}\sqrt{n_{s-1} n_s} \\ \delta_{1s}\sqrt{n_1 n_s} & \ldots & \delta_{(s-1)s}\sqrt{n_{s-1} n_s} & \mu_s \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_s \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_s \end{bmatrix}. \tag{5}$$

We claim that $\lambda \in \sigma(M(G))$ with an associated eigenvector

$$\mathbf{y}^{\mathbf{T}} = \begin{bmatrix} x_1 \mathbf{1}_{n_1}^T & x_2\sqrt{\frac{n_1}{n_2}}\mathbf{1}_{n_2}^T & \ldots & x_{s-1}\sqrt{\frac{n_1}{n_{s-1}}}\mathbf{1^T}_{n_{s-1}} & x_s\sqrt{\frac{n_1}{n_s}}\mathbf{1}_{n_s}^T \end{bmatrix}.$$

Indeed

$$M(G)\,\mathbf{y} = \begin{bmatrix} M_1 & \delta_{12}\mathbf{1}_{n_1}\mathbf{1}_{n_2}^T & \cdots & \delta_{1s}\mathbf{1}_{n_1}\mathbf{1}_{n_s}^T \\ \delta_{12}\mathbf{1}_{n_2}\mathbf{1}_{n_1}^T & M_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \delta_{(s-1)s}\mathbf{1}_{n_{s-1}}\mathbf{1}_{n_s}^T \\ \delta_{1s}\mathbf{1}_{n_s}\mathbf{1}_{n_1}^T & \cdots & \delta_{(s-1)s}\mathbf{1}_{n_s}\mathbf{1}_{n_{s-1}}^T & M_s \end{bmatrix} \begin{bmatrix} x_1\mathbf{1}_{n_1} \\ x_2\sqrt{\frac{n_1}{n_2}}\mathbf{1}_{n_2} \\ \vdots \\ x_s\sqrt{\frac{n_1}{n_s}}\mathbf{1}_{n_s} \end{bmatrix}$$

$$= \begin{bmatrix} \left(\mu_1 x_1 + \delta_{12}\sqrt{n_1 n_2}x_2 + \cdots + \delta_{1s}\sqrt{n_1 n_s}x_s\right)\mathbf{1}_{n_1} \\ \left(\delta_{12}n_1 x_1 + \mu_2 x_2\sqrt{\frac{n_1}{n_2}} + \cdots + \delta_{1s}\sqrt{n_2 n_s}x_s\right)\mathbf{1}_{n_2} \\ \vdots \\ \left(\delta_{1s}n_1 x_1 + \delta_{2s}\sqrt{n_1 n_2}x_2 + \cdots + \sqrt{\frac{n_1}{n_s}}\mu_s x_s\right)\mathbf{1}_{n_s} \end{bmatrix}$$

$$= \begin{bmatrix} \left(\mu_1 x_1 + \delta_{12}\sqrt{n_1 n_2}x_2 + \cdots + \delta_{1s}\sqrt{n_1 n_s}x_s\right)\mathbf{1}_{n_1} \\ \sqrt{\frac{n_1}{n_2}}\left(\delta_{12}\sqrt{n_1 n_2}x_1 + \mu_2 x_2 + \cdots + \delta_{1s}\sqrt{n_2 n_s}x_s\right)\mathbf{1}_{n_2} \\ \vdots \\ \sqrt{\frac{n_1}{n_s}}\left(\delta_{1s}\sqrt{n_1 n_s}x_1 + \delta_{2s}x_2\sqrt{n_2 n_s} + \cdots + \mu_s x_s\right)\mathbf{1}_{n_s} \end{bmatrix} = \lambda \begin{bmatrix} x_1\mathbf{1}_{n_1} \\ \sqrt{\frac{n_1}{n_2}}x_2\mathbf{1}_{n_2} \\ \vdots \\ \sqrt{\frac{n_1}{n_s}}x_s\mathbf{1}_{n_s} \end{bmatrix}.$$

The last equality is a consequence of (5). Hence $M(G)\,\mathbf{y} = \lambda\mathbf{y}$ and then $\sigma(F_s) \subseteq \sigma(M(G))$. □

From Lemma 1 and Lemma 2, we get the following more strong result on the spectrum of $M(G)$, where $G = \bigvee_H \{G_i : 1 \le i \le s\}$.

**Theorem 2** *Consider the matrix $M(G)$ (2) and the matrix $F_s$ (4). Then the spectrum of $M(G)$ is*

$$\sigma(M(G)) = \cup_{i=1}^{s}\left(\sigma(M_i) - \{\mu_i\}\right) \cup \sigma(F_s),$$

## 3 Determination of the eigenvalues of *H-join* distance matrices

In this section, we assume that $H$ is a connected graph of order $s$ and $d_{i,j}$ denotes the distance between $i, j \in V(H)$. Here, we apply Theorem 2 to determine the eigenvalues of the distance matrix, distance Laplacian matrix and distance signless Laplacian matrix of the *H-join* $G = \bigvee_H \{G_i : 1 \le i \le s\}$, when $G_1, \ldots, G_s$ are regular graphs. Therefore, throughout this section we deal with the *H-join* of a family of regular graphs $G_1, \ldots, G_s$.

### 3.1 Distance eigenvalues

Taking into account (2), we get that the distance matrix of the *H-join* $G = \bigvee_H \{G_i : 1 \leq i \leq s\}$, where $G_i$ is a $d_i$-degree regular graph of order $n_i$, for each $i = 1, \ldots, s$, it follows that

$$\mathcal{D}(G) = \begin{bmatrix} M_1 & d_{1,2}\mathbf{1}_{n_1}\mathbf{1}_{n_2}^T & \ldots & d_{1,s}\mathbf{1}_{n_1}\mathbf{1}_{n_s}^T \\ d_{1,2}\mathbf{1}_{n_2}\mathbf{1}_{n_1}^T & M_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & d_{s-1,s}\mathbf{1}_{n_{s-1}}\mathbf{1}_{n_s}^T \\ d_{1,s}\mathbf{1}_{n_s}\mathbf{1}_{n_1}^T & \ldots & d_{s-1,s}\mathbf{1}_{n_s}\mathbf{1}_{n_{s-1}}^T & M_s \end{bmatrix}, \quad (6)$$

and from (4)

$$F_s = \begin{bmatrix} \lambda_1(M_1) & d_{1,2}\sqrt{n_1 n_2} & \ldots & d_{1,s}\sqrt{n_1 n_s} \\ d_{1,2}\sqrt{n_1 n_2} & \lambda_1(M_2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & d_{s-1,s}\sqrt{n_{s-1}n_s} \\ d_{1,s}\sqrt{n_1 n_s} & \ldots & d_{s-1,s}\sqrt{n_{s-1}n_s} & \lambda_1(M_s) \end{bmatrix}. \quad (7)$$

**Remark 1** *Since $H$ is connected, then $M_i = 2(J_{n_i} - I_{n_i}) - A(G_i)$ (in (6)), where $J_{n_i}$ is the all one square matrix of order $n_i$, $I_{n_i}$ is the identity matrix of order $n_i$ and $A(G_i)$ is the adjacency matrix of $G_i$, for each $i = 1, \ldots, n$. Further, as $G_i$ is a $d_i$-degree regular graph it follows that $M_i\mathbf{1}_{n_i} = \lambda_1(M_i)\mathbf{1}_{n_i}$, where the eigenvalues $\lambda_1(M_i)$ (in (7)) are given by*

$$\lambda_1(M_i) = 2(n_i - 1) - d_i,$$

*for $i = 1, \ldots, s$. Moreover, since for each $i \in \{1, \ldots, s\}$, the matrices $J_{n_i} - I_{n_i}$ and $A(G_i)$ commute, then the spectrum of $M_i$ is completely determined by the spectrum of $A(G_i)$. That is,*

$$\sigma(M_i) = \{2(n_i - 1) - d_i\,,\ -2 - \lambda_{n_i}(A(G_i))\,,\ \ldots\,,\ -2 - \lambda_2(A(G_i))\}, \quad (8)$$

*for $i = 1, \ldots, s$.*

Taking into account the Remark 1, applying Theorem 2, we obtain the following corollary.

**Corollary 1** *Let $H$ a connected graph of order $s$. If for each $i \in \{1, \ldots, s\}$, $G_i$ is a $d_i$-degree regular graph, then the spectrum of $\mathcal{D}(G)$, where $G = \bigvee_H \{G_i : 1 \leq i \leq s\}$, is*

$$\sigma(\mathcal{D}(G)) = \cup_{i=1}^{s}\Big(\sigma(M_i) - \{2(n_i - 1) - d_i\}\Big) \cup \sigma(F_s)$$

*where $F_s$ is the $s \times s$ matrix given in (7).*

Notice that the matrices $M_i$ in the Theorem 2 are not necessarily the distance matrices $D(G_i)$. This can be observed with the following example.

**Example 3** *Let* $H = P_3$, $G_1 = C_6$, $G_2 = K_2$, $G_3 = K_3$ *and consider the graph* $G = \bigvee_H \{G_1, G_2, G_3\}$.

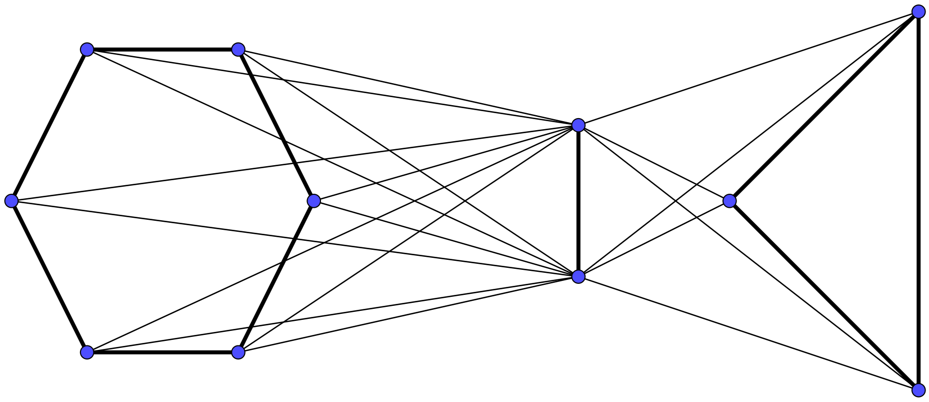

**Figure 3**. *The graph* $G = \bigvee_{P_3} \{C_6, K_2, K_3\}$.

*In this example, we have* $\sigma(M_1) = \{8,\ 0,\ -1^{[2]},\ -3^{[2]}\}$, $\sigma(M_2) = \{1,\ -1\}$ *and* $\sigma(M_3) = \{2,\ -1^{[2]}\}$. *Notice that* $M_1 \neq \mathcal{D}(C_6)$. *Further,*

$$F_3 = \begin{bmatrix} 8 & \sqrt{12} & 2\sqrt{18} \\ \sqrt{12} & 1 & \sqrt{6} \\ 2\sqrt{18} & \sqrt{6} & 2 \end{bmatrix}.$$

*Therefore, to four decimal places* $\sigma(F_3) = \{15.2621,\ -0.2621,\ -4\}$ *and from Corollary 1, we get*

$$\sigma(\mathcal{D}(G)) = \{15.2621,\ 0,\ -0.2621,\ -1^{[5]},\ -3^{[2]},\ -4\}.$$

### 3.2 Distance Laplacian eigenvalues

For $i = 1, \dots, s$, let us consider the matrices $L_i = k_i I_{n_i} - M_i$, where $k_i = \lambda_1(M_i) + \sum_{j \neq i} d_{i,j} n_j$, with $M_i$, $\lambda_1(M_i)$ as in the Remark 1. Notice that in this case $L_i \mathbf{1}_{n_i} = \lambda_{n_i}(L_i)\mathbf{1}_{n_i}$, where

$$\lambda_{n_i}(L_i) \;=\; \sum_{j \neq i} d_{i,j} n_j, \tag{9}$$

for each $i = 1, \dots, s$ and therefore, we can write

$$k_i = \lambda_1(M_i) + \lambda_{n_i}(L_i), \tag{10}$$

for each $i = 1, \dots, s$. Taking into account (2), we get that the distance Laplacian matrix of the H-Join $G = \bigvee_H \{G_i : 1 \leq i \leq s\}$ is

$$\mathcal{L}(G) = \begin{bmatrix} L_1 & -d_{1,2}\mathbf{1}_{n_1}\mathbf{1}_{n_2}^T & \dots & -d_{1,s}\mathbf{1}_{n_1}\mathbf{1}_{n_s}^T \\ -d_{12}\mathbf{1}_{n_2}\mathbf{1}_{n_1}^T & L_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -d_{s-1,s}\mathbf{1}_{n_{s-1}}\mathbf{1}_{n_s}^T \\ -d_{1,s}\mathbf{1}_{n_s}\mathbf{1}_{n_1}^T & \dots & -d_{s-1,s}\mathbf{1}_{n_s}\mathbf{1}_{n_{s-1}}^T & L_s \end{bmatrix} = \bigoplus_{i=1}^{s} k_i I_{n_i} - \mathcal{D}(G),$$

where $\mathcal{D}(G)$ is as in (6). Notice also that $\mathcal{L}(G)\mathbf{1}_n = \mathbf{0}_n$.
On the other hand, from (4) we obtain

$$F_s = \begin{bmatrix} \lambda_{n_1}(L_1) & -d_{1,2}\sqrt{n_1 n_2} & \dots & -d_{1,s}\sqrt{n_1 n_s} \\ -d_{1,2}\sqrt{n_1 n_2} & \lambda_{n_2}(L_2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & -d_{s-1,s}\sqrt{n_{s-1}n_s} \\ -d_{1,s}\sqrt{n_1 n_s} & \dots & -d_{s-1,s}\sqrt{n_{s-1}n_s} & \lambda_{n_s}(L_s) \end{bmatrix}, \tag{11}$$

where for each $i = 1, \dots, s$, $\lambda_{n_i}(L_i)$ is as in (9). Therefore, applying Theorem 2, we obtain the following corollary.

**Corollary 2** *Let $H$ a connected graph of order $s$. If for each $i = 1, \dots, s$, $G_i$ is a regular graph, the spectrum of $\mathcal{L}(G)$ where $G = \bigvee_H \{G_i : 1 \leq i \leq s\}$ is*

$$\sigma(\mathcal{L}(G)) = \cup_{i=1}^{s} \left(\sigma(L_i) - \{\lambda_{n_i}(L_i)\}\right) \cup \sigma(F_s)$$

*where $F_s$ is the $s \times s$ matrix in (11) and $\lambda_{n_i}(L_i)$ is as in (9), for each $i = 1, \dots, s$.*

**Example 4** *Consider the same* H-join *graph of Example 3. Making some computation we have that $k_1 = 16$, $k_2 = 10$ and $k_3 = 16$. Thus, $L_1 = 16I_6 - M_1$, $L_2 = 10I_2 - M_2$, $L_3 = 16I_3 - M_3$, $\sigma(L_1) = \{19^{[2]},\ 17^{[2]},\ 16,\ 8\}$, $\sigma(L_2) = \{11,\ 9\}$, $\sigma(L_3) = \{17^{[2]},\ 14\}$, and*

$$F_3 = \begin{bmatrix} 8 & -\sqrt{12} & -2\sqrt{18} \\ -\sqrt{12} & 9 & -\sqrt{6} \\ -2\sqrt{18} & -\sqrt{6} & 14 \end{bmatrix}.$$

*Therefore, $\sigma(F_3) = \{20,\ 11,\ 0\}$ and, from Corollary 2, we get*

$$\sigma(\mathcal{L}(G)) = \{20,\ 19^{[2]},\ 17^{[4]},\ 16,\ 11^{[2]},\ 0\}.$$

### 3.3 Distance signless Laplacian eigenvalues

Finally, we consider the distance signless Laplacian matrix of the *H-join* of regular graphs. For $i = 1, \dots, s$, let us consider the matrices $Q_i = k_i I_{n_i} + M_i$, where $k_i$ is again as in (10) and $M_i$, $\lambda_1(M_i)$ are as in the Remark 1, for each $i = 1, \dots, s$. Moreover, notice that $Q_i \mathbf{1}_{n_i} = \lambda_1(Q_i)\mathbf{1}_{n_i}$, where

$$\lambda_1(Q_i) \;=\; k_i + \lambda_1(M_i) \;=\; 2(2(n_i - 1) - d_i) + \lambda_{n_i}(L_i), \tag{12}$$

for each $i = 1, \dots, s$. Taking into account (2), we get that the distance signless Laplacian matrix of $G = \bigvee_H \{G_i : 1 \leq i \leq s\}$ is

$$\mathcal{Q}(G) = \begin{bmatrix} Q_1 & d_{1,2}\mathbf{1}_{n_1}\mathbf{1}_{n_2}^T & \dots & d_{1,s}\mathbf{1}_{n_1}\mathbf{1}_{n_s}^T \\ d_{12}\mathbf{1}_{n_2}\mathbf{1}_{n_1}^T & Q_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & d_{s-1,s}\mathbf{1}_{n_{s-1}}\mathbf{1}_{n_s}^T \\ d_{1,s}\mathbf{1}_{n_s}\mathbf{1}_{n_1}^T & \dots & d_{s-1,s}\mathbf{1}_{n_s}\mathbf{1}_{n_{s-1}}^T & Q_s \end{bmatrix} = \bigoplus_{i=1}^{s} k_i I_{n_i} + \mathcal{D}(G),$$

where $\mathcal{D}(G)$ is as in (6). Also, from (4) we obtain

$$F_s = \begin{bmatrix} \lambda_1(Q_1) & d_{1,2}\sqrt{n_1 n_2} & \cdots & d_{1,s}\sqrt{n_1 n_s} \\ d_{1,2}\sqrt{n_1 n_2} & \lambda_1(Q_2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & d_{s-1,s}\sqrt{n_{s-1} n_s} \\ d_{1,s}\sqrt{n_1 n_s} & \cdots & d_{s-1,s}\sqrt{n_{s-1} n_s} & \lambda_1(Q_s) \end{bmatrix}, \tag{13}$$

where for each $i \in \{1, \ldots, s\}$, $\lambda_1(Q_i)$ is as in (12). Therefore, applying Theorem 2, we obtain the following corollary.

**Corollary 3** *Let $H$ be a connected graph of order $s$. If for each $i \in \{1, \ldots, s\}$, $G_i$ is a regular graph, the spectrum of $\mathcal{Q}(G)$, with $G = \bigvee_H \{G_i : 1 \leq i \leq s\}$, is*

$$\sigma(\mathcal{Q}(G)) = \cup_{i=1}^{s} \left(\sigma(Q_i) - \{\lambda_1(Q_i)\}\right) \cup \sigma(F_s)$$

*where $F_s$ is the $s \times s$ matrix in (13) and $\lambda_1(Q_i)$ is as in (12) for each $i = 1, \ldots, s$.*

**Example 5** *Consider the same* H-join *as in Example 3. Similarly to Example 4, we get that $Q_1 = 16I_6 + M_1$, $Q_2 = 10I_2 + M_2$, and $Q_3 = 16I_3 + M_3$, $\sigma(Q_1) = \{24,\ 16,\ 15^{[2]},\ 13^{[2]}\}$, $\sigma(Q_2) = \{11,\ 9\}$ and $\sigma(Q_3) = \{18,\ 15^{[2]}\}$ and*

$$F_3 = \begin{bmatrix} 24 & \sqrt{12} & 2\sqrt{18} \\ \sqrt{12} & 11 & \sqrt{6} \\ 2\sqrt{18} & \sqrt{6} & 18 \end{bmatrix}.$$

*Therefore, to four decimal places $\sigma(F_3) = \{30.9043,\ 12,\ 10.0957\}$ and thus, from Corollary 3, we get*

$$\sigma(\mathcal{Q}(G)) = \{30.9043,\ 16,\ 15^{[4]},\ 13^{[2]},\ 12,\ 10.0957,\ 9\}.$$

# 4 Sharp lower bounds on the distance incidence energy and distance Laplacian-energy like

In this section, we define the distance incidence energy and the distance Laplacian-energy like of connected graphs. Then we derive sharp lower bounds on these two energies among the graphs $G$ on $n$ vertices in terms of connectivity. These lower bounds are attained if and only if $G = K_k \vee (K_1 \cup K_{n-k-1})$.

Three well known energies on a graph are the (adjacency) energy, introduced by Gutman in 1978 as the sum of the absolute values of the eigenvalues of the adjacency matrix, the Laplacian energy introduced by Gutman and Zhou in [15] and the signless Laplacian energy which was defined in analogy with the Laplacian energy. The distance energy [28], the distance Laplacian energy [32] and the distance signless Laplacian energy [13] have been introduced as follows

$$E^{\mathcal{D}}(G) = \sum_{j=1}^{n} |\partial_j(G)|, \quad E^{\mathcal{L}}(G) = \sum_{j=1}^{n} \left|\partial_j^{\mathcal{L}}(G) - \frac{2W(G)}{n}\right| \quad \text{and} \quad E^{\mathcal{Q}}(G) = \sum_{j=1}^{n} \left|\partial_j^{\mathcal{Q}}(G) - \frac{2W(G)}{n}\right|,$$

respectively.

Other energies on a graph are the incidence energy [17] and the Laplacian-energy like [34]. For a graph $G$ of order $n$, its incidence energy, denoted by $IE(G)$, becomes

$$IE\left(G\right)=\sum_{i=1}^{n}\sqrt{q_i(G)}$$

where $q_1(G)\geq q_2(G)\geq\ldots\geq q_n(G)$, are the signless Laplacian eigenvalues of $G$; and its Laplacian-energy like energy, denoted by $LEL(G)$, is

$$LEL\left(G\right)=\sum_{i=1}^{n}\sqrt{\mu_i}$$

where $\mu_1(G)\geq\mu_2(G)\geq\ldots\geq\mu_n(G)=0$, are the Laplacian eigenvalues of $G$.

From now on, $\mathcal{V}(n,k)$ is the family of connected graphs $G$ of order $n$ such that $\kappa(G)\leq k$.

In [21] and in [34], sharp upper bounds on the incidence energy and on the Laplacian-energy like, respectively, for the graphs in $\mathcal{V}(n,k)$ are obtained. These upper bounds are attained if and only if $G=K_k\vee(K_1\cup K_{n-k-1})$.

In this section we extend the above concepts of incidence energy and Laplacian-energy like to the distance matrix context, introducing the notions of distance Laplacian energy like, denoted by $DLEL(G)$, and distance incidence energy, denoted by $DIE(G)$, as follows

$$DLEL(G)=\sum_{i=1}^{n}\sqrt{\partial_i^L(G)}\qquad\text{and}\qquad DIE(G)=\sum_{i=1}^{n}\sqrt{\partial_i^Q(G)}.$$

In this study the following theorem plays a crucial role.

**Theorem 3** *([3], Theorem 3.5) Let $G$ be a connected graph on $n$ vertices and $m\geq n$ edges. Consider the connected graph $\widetilde{G}$ obtained from $G$ by the deletion of an edge.*

- *Let $\partial_1^L(G),\ldots,\partial_n^L(G)$ and $\partial_1^L(\widetilde{G}),\ldots,\partial_n^L(\widetilde{G})$ be the distance Laplacian eigenvalues of $G$ and $\widetilde{G}$, respectively. Then $\partial_i^L(\widetilde{G})\geq\partial_i^L(G)$ for $i=1,\ldots,n$.*
- *Let $\partial_1^Q(G),\ldots,\partial_n^Q(G)$ and $\partial_1^Q(\widetilde{G}),\ldots,\partial_n^Q(\widetilde{G})$ be the distance signless Laplacian eigenvalues of $G$ and $\widetilde{G}$, respectively. Then $\partial_i^Q(\widetilde{G})\geq\partial_i^Q(G)$ for $i=1,\ldots,n$.*

Clearly $trace(\mathcal{L}(\widetilde{G}))>trace(\mathcal{L}(G))$ and $trace(\mathcal{Q}(\widetilde{G}))>trace(\mathcal{Q}(G))$. Therefore, from Theorem 3, the following corollaries are immediate.

**Corollary 4** *If $G$ and $\widetilde{G}$ are connected graphs such that $\widetilde{G}$ is obtained from $G$ by the deletion of an edge, then $DLEL(\widetilde{G})>DLEL(G)$ and $DIE(\widetilde{G})>DIE(G)$.*

**Corollary 5** *Among the all connected graphs on $n$ vertices and $m \geq n$ edges, the complete graph $K_n$ has the smallest distance Laplacian-energy like and the smallest distance incidence energy.*

For $i = 1, 2, 3$, let $G_i$ be a $d_i-$regular graph of order $n_i$. Then $G = G_1 \vee (G_2 \cup G_3)$ is a graph of order $n = n_1 + n_2 + n_3$. Observe that $G = G_1 \vee (G_2 \cup G_3)$ is a $P_3 - join$ graph in which the central vertex of $P_3$ is assigned to $G_1$, one pendent vertex of $P_3$ is assigned to $G_2$ and the other to $G_3$.

Labelling the vertices of $G = G_1 \vee (G_2 \cup G_3)$ starting with the vertices of $G_1$, continuing with the vertices of $G_2$ and finishing with the vertices of $G_3$, and using the results obtained in the Subsection 3.3, the distance signless Laplacian matrix $\mathcal{Q}(G)$ becomes

$$\mathcal{Q}(G) = \begin{bmatrix} Q_1 & \mathbf{1}_{n_1}\mathbf{1}_{n_2}^T & \mathbf{1}_{n_1}\mathbf{1}_{n_3}^T \\ \mathbf{1}_{n_2}\mathbf{1}_{n_1}^T & Q_2 & 2\mathbf{1}_{n_2}\mathbf{1}_{n_3}^T \\ \mathbf{1}_{n_3}\mathbf{1}_{n_1}^T & 2\mathbf{1}_{n_3}\mathbf{1}_{n_2}^T & Q_3 \end{bmatrix} \tag{14}$$

where, for $i = 1, 2, 3$,

$$Q_i = k_i I_{n_i} + 2(J_{n_i} - I_{n_i}) - A(G_i) \tag{15}$$

and

$$\begin{aligned} k_1 &= 2(n_1 - 1) - d_1 + n_2 + n_3 \\ k_2 &= 2(n_2 - 1) - d_2 + n_1 + 2n_3 \\ k_3 &= 2(n_3 - 1) - d_3 + n_1 + 2n_2 \end{aligned} \tag{16}$$

Clearly, the largest eigenvalues of $Q_1, Q_2$ and $Q_3$ are

$$\begin{aligned} \lambda_1(Q_1) &= 4(n_1 - 1) - 2d_1 + n_2 + n_3 \\ \lambda_1(Q_2) &= 4(n_2 - 1) - 2d_2 + n_1 + 2n_3 \\ \lambda_1(Q_3) &= 4(n_3 - 1) - 2d_3 + n_1 + 2n_2 \end{aligned} \tag{17}$$

with eigenvectors $\mathbf{1}_{n_1}$, $\mathbf{1}_{n_2}$ and $\mathbf{1}_{n_3}$, respectively.

Applying Corollary 3, we get the following result.

**Theorem 4** *If $G = G_1 \vee (G_2 \cup G_3)$ and, for $i = 1, 2, 3$, $G_i$ is a $d_i-$regular graph then*

$$\sigma(\mathcal{Q}(G)) = (\sigma(Q_1) \cup \sigma(Q_2) \cup \sigma(Q_3) - \{\lambda_1(Q_1), \lambda_1(Q_2), \lambda_1(Q_3)\}) \cup \sigma(F_3)$$

*where $Q_1, Q_2, Q_3$ are as in (15), $\lambda_1(Q_1), \lambda_1(Q_2), \lambda_1(Q_3)$ are as in (17) and*

$$F_3(G) = \begin{bmatrix} \lambda_1(Q_1) & \sqrt{n_1 n_2} & \sqrt{n_1 n_3} \\ \sqrt{n_1 n_2} & \lambda_1(Q_2) & 2\sqrt{n_2 n_3} \\ \sqrt{n_1 n_3} & 2\sqrt{n_2 n_3} & \lambda_1(Q_3) \end{bmatrix}. \tag{18}$$

Let $n$ and $k$ be positive integers, with $k \leq n - 1$ and consider the graph

$$G(i) = K_k \vee (K_i \cup K_{n-k-i}), \tag{19}$$

where, without loss of generality, we assume $1 \leq i \leq \left\lfloor \frac{n-k}{2} \right\rfloor$. Then, for the graph $G(i)$, the matrices $Q_1, Q_2$ and $Q_3$ in (15) are

$$\begin{array}{rcl} Q_1 & = & (n-1)I_k + A(K_k) \\ Q_2(i) & = & (2n-k-i-1)I_i + A(K_i) \\ Q_3(i) & = & (n+i-1)I_{n-k-i} + A(K_{n-k-i}), \end{array} \tag{20}$$

respectively, and the matrix $F_3(G(i))$ in (18) becomes

$$F_3\left(G\left(i\right)\right) = \left[\begin{array}{ccc} n+k-2 & \sqrt{ki} & \sqrt{k\left(n-k-i\right)} \\ \sqrt{ki} & 2n-k-2 & 2\sqrt{i(n-k-i)} \\ \sqrt{k\left(n-k-i\right)} & 2\sqrt{i(n-k-i)} & 2n-k-2 \end{array}\right].$$

Taking into account that the adjacency eigenvalues of $K_s$ are $s-1$ and $-1$ with multiplicity $s-1$, the spectra of $Q_1, Q_2\left(i\right)$ and $Q_3\left(i\right)$ in (20) are

$$\begin{array}{rcl} \sigma\left(Q_1\right) & = & \left\{n+k-2, (n-2)^{[k-1]}\right\}, \\ \sigma\left(Q_2\left(i\right)\right) & = & \left\{2n-k-2, (2n-k-i-2)^{[i-1]}\right\} \text{ and} \\ \sigma\left(Q_3\left(i\right)\right) & = & \left\{2n-k-2, (n+i-2)^{[n-k-i-1]}\right\}, \end{array}$$

where $\lambda^{[t]}$ denotes that $\lambda$ is an eigenvalue with multiplicity $t$. One can obtain that the spectrum of $F_3\left(G\left(i\right)\right)$ which is

$$\sigma\left(F_3\left(G\left(i\right)\right)\right) = \left\{f_1\left(i\right), f_2, f_3\left(i\right)\right\},$$

where $f_{1,3}(i) = 2(n-1) - \frac{k}{2} \pm \sqrt{(\frac{k}{2})^2 + 4i(n-k-i)}$ and $f_2 = n-2$.
Now, applying Theorem 4 to $\mathcal{Q}(G(i))$ the next corollary follows.

**Corollary 6** *Let $G(i)$ be the graph defined in* (19). *Then*

$$\sigma\left(\mathcal{Q}\left(G\left(i\right)\right)\right) = \left\{f_1\left(i\right), f_2, f_3\left(i\right), (n-2)^{[k-1]}, (2n-k-i-2)^{[i-1]}, (n+i-2)^{[n-k-i-1]}\right\}.$$

Let $|S|$ be the cardinality of a finite set $S$ and let us define $\mathcal{W}(n,k)$ as follows:

$$\mathcal{W}(n,k) = \{G \in \mathcal{V}(n,k) : |E(G)| \geq n\}.$$

Furthermore, consider

$$\begin{array}{rcl} b(n,k) & = & k\sqrt{n-2} + (n-k-2)\sqrt{n-1} \;+\; \sqrt{2(n-1) - \frac{k}{2} + \sqrt{(\frac{k}{2})^2 + 4(n-k-1)}} \\ & & +\; \sqrt{2(n-1) - \frac{k}{2} - \sqrt{(\frac{k}{2})^2 + 4(n-k-1)}}. \end{array}$$

Then, denoting by $K_0$ the empty graph (that is, the graph without edges and without vertices) we have the following theorem.

**Theorem 5** *If $G \in \mathcal{W}(n,k)$, then*

$$DIE\,(G) \geq b\,(n,k)\,, \;\; for\; k = 1, \ldots, n-1. \tag{21}$$

*Additionally, the inequalities (21) hold as equalities if and only if $G = K_k \vee (K_1 \cup K_{n-k-1})$.*

**Proof** Let $G \in \mathcal{W}(n,k)$. We first consider $k = n-1$. From Corollary 5, $DIE\,(G) \geq DIE\,(K_n)$ with equality if and only if $G = K_n$. Moreover

$$b\,(n, n-1) = \sqrt{2(n-1)} + (n-1)\sqrt{n-2} = DIE\,(K_n)\,.$$

Then the result is true for $k = n-1$. Now let $1 \leq k \leq n-2$ and let $G \in \mathcal{W}(n,k)$ such that $DIE\,(G)$ is a minimum. Let $S \subseteq V\,(G)$ such that $G - S$ is a disconnected graph. Let $C_1, C_2, \ldots, C_r$ be the connected components of $G - S$. We claim that $r = 2$. If $r > 2$ then we can construct a new graph $H = G \cup \{e\}$ where $e$ is an edge connecting a vertex in $C_1$ with a vertex in $C_2$. Clearly, $H \in \mathcal{W}(n,k)$ and $G = H - e$. By Corollary 4, $DIE\,(G) > DIE\,(H)$, which is a contradiction. Therefore $r = 2$, that is, $G - S = C_1 \cup C_2$. By hypothesis $|S| \leq k$. Now, we claim that $|S| = k$. Suppose $|S| < k$. Since $G - S = C_1 \cup C_2$, we may construct a graph $H = G + e$ where $e$ is an edge joining a vertex $u \in V\,(C_1)$ with a vertex $v \in V\,(C_2)$. We see that $H - S$ is a connected graph and the deletion of the vertex $u$ disconnected it. This tell us that $H \in \mathcal{W}(n,k)$. By Corollary 4, $DIE\,(G) > DIE\,(H)$, which is also a contradiction. Hence $G - S = C_1 \cup C_2$ and $|S| = k$. Let $|C_1| = i$. Then $|C_2| = n-k-i$. Repeated application of Corollary 4 enables to conclude that

$$G = K_k \vee (K_i \cup K_{n-k-i}) = G\,(i)$$

for some $1 \leq i \leq \left\lfloor \frac{n-k}{2} \right\rfloor$. We have proved $DIE\,(G) \geq DIE\,(G\,(i))$ for all $G \in \mathcal{W}_n^k$. We now search for the value of $i$ for which $DIE\,(G\,(i))$ is minimum. From Corollary 6,

$$\begin{aligned} DIE\,(G\,(i)) &= k\sqrt{n-2} + (i-1)\sqrt{2n-k-i-2} + (n-k-i-1)\sqrt{n+i-2} \\ &+ \sqrt{2n-2-\frac{k}{2}+\sqrt{(\frac{k}{2})^2+4i(n-k-i)}} + \sqrt{2n-2-\frac{k}{2}-\sqrt{(\frac{k}{2})^2+4i(n-k-i)}}. \end{aligned}$$

Defining the function

$$\begin{aligned} f(x) &= (x-1)\sqrt{2n-k-x-2} + (n-k-x-1)\sqrt{n+x-2} \\ &+ \sqrt{2n-2-\frac{k}{2}+\sqrt{(\frac{k}{2})^2+4x(n-k-x)}} + \sqrt{2n-2-\frac{k}{2}-\sqrt{(\frac{k}{2})^2+4x(n-k-x)}} \end{aligned}$$

and observing that $f\,(x) = f\,(n-k-x)$, for $x \in [0, n-k]$, after some algebraic manipulation, we may conclude that $f$ is a strictly increasing function in the interval $[0, \frac{n-k}{2}]$. Hence $DIE\,(G) \geq DIE\,(G\,(1))$ for all $G \in \mathcal{W}(n,k)$. Moreover, since $G\,(1) = K_k \vee (K_1 \cup K_{n-k-1})$ and $DIE\,(G\,(1)) = b\,(n,k)$, the inequality (21) holds as equality if and only if $G = K_k \vee (K_1 \cup K_{n-k-1})$. □

To find a sharp lower bound on the distance Laplacian-energy like among the graphs in $\mathcal{W}(n,k)$ is easier. Using the above mentioned labelling for the vertices of $G = G_1 \vee (G_2 \cup G_3)$ and the results obtained in the subsection 3.2, we obtain

$$\mathcal{L}(G) = \left[ \begin{array}{ccc} L_1 & -\mathbf{1}_{n_1}\mathbf{1}_{n_2}^T & -\mathbf{1}_{n_1}\mathbf{1}_{n_3}^T \\ -\mathbf{1}_{n_2}\mathbf{1}_{n_1}^T & L_2 & -2\mathbf{1}_{n_2}\mathbf{1}_{n_3}^T \\ -\mathbf{1}_{n_3}\mathbf{1}_{n_1}^T & -2\mathbf{1}_{n_3}\mathbf{1}_{n_2}^T & L_3 \end{array} \right] \tag{22}$$

where, for $i = 1, 2, 3$,

$$L_i = k_i I_{n_i} - 2(J_{n_i} - I_{n_i}) + A(G_i) \tag{23}$$

and $k_i$ are as in (16).

The smallest eigenvalues of $L_1, L_2$ and $L_3$ are

$$\lambda_{n_1}(L_1) = n_2 + n_3, \;\; \lambda_{n_2}(L_2) = n_1 + 2n_3 \;\text{ and }\; \lambda_{n_3}(L_3) = n_1 + 2n_2, \tag{24}$$

with eigenvectors $\mathbf{1}_{n_1}$, $\mathbf{1}_{n_2}$ and $\mathbf{1}_{n_3}$, respectively.

Applying Corollary 2, the following theorem is obtained.

**Theorem 6** *If $G = G_1 \vee (G_2 \cup G_3)$ and, for $i = 1, 2, 3$, $G_i$ is a $d_i-$regular graph then*

$$\sigma(\mathcal{L}(G)) = (\sigma(L_1) \cup \sigma(L_2) \cup \sigma(L_3) - \{\lambda_{n_1}(L_1), \lambda_{n_2}(L_2), \lambda_{n_3}(L_3)\}) \cup \sigma(F_3)$$

*where $L_1, L_2, L_3$ are as in (23), $\lambda_{n_1}(L_1), \lambda_{n_2}(L_2), \lambda_{n_3}(L_3)$ are as in (24) and*

$$F_3(G) = \left[ \begin{array}{ccc} \lambda_{n_1}(L_1) & -\sqrt{n_1 n_2} & -\sqrt{n_1 n_3} \\ -\sqrt{n_1 n_2} & \lambda_{n_2}(L_2) & -2\sqrt{n_2 n_3} \\ -\sqrt{n_1 n_3} & -2\sqrt{n_2 n_3} & \lambda_{n_3}(L_3) \end{array} \right]. \tag{25}$$

For the graph $G(i)$, the matrices $L_1, L_2$ and $L_3$ in (23) are

$$\begin{array}{rcl} L_1 & = & (n-1)I_k - A(K_k), \\ L_2(i) & = & (2n-k-i-1)I_i - A(K_i), \\ L_3(i) & = & (n+i-1)I_{n-k-i} - A(K_{n-k-i}), \end{array} \tag{26}$$

respectively, and the matrix $F_3(G(i))$ in (25) becomes

$$F_3\left(G\left(i\right)\right) = \left[ \begin{array}{ccc} n-k & -\sqrt{ki} & -\sqrt{k\left(n-k-i\right)} \\ -\sqrt{ki} & 2n-k-2i & -2\sqrt{i(n-k-i)} \\ -\sqrt{k\left(n-k-i\right)} & -2\sqrt{i(n-k-i)} & k+2i \end{array} \right].$$

The spectra of $L_1, L_2\left(i\right)$ and $L_3\left(i\right)$ in (26) are

$$\begin{array}{rcl} \sigma\left(L_1\right) & = & \left\{n^{[k-1]}, n-k\right\}, \\ \sigma\left(L_2\left(i\right)\right) & = & \left\{\left(2n-k-i\right)^{[i-1]}, 2n-k-2i\right\} \text{ and} \\ \sigma\left(L_3\left(i\right)\right) & = & \left\{\left(n+i\right)^{[n-k-i-1]},\; k+2i\right\}. \end{array}$$

In this case, the spectrum of $F_3(G(i))$ is

$$\sigma(F_3(G(i))) = \{2n-k,\ n,\ 0\}.$$

Applying Theorem 6 to $\mathcal{L}(G(i))$, we obtain the next corollary.

**Corollary 7** *Let $G(i)$ be the graph defined in (19). Then*

$$\sigma(\mathcal{L}(G(i))) = \left\{2n-k,\ n^{[k]},\ (2n-k-i)^{[i-1]},\ (n+i)^{[n-k-i-1]},\ 0\right\}.$$

By similar arguments to the ones used in the proof of Theorem 5, we get the following result.

**Theorem 7** *If $G \in \mathcal{W}(n,k)$, then*

$$DLEL(G) \geq c(n,k) \tag{27}$$

*where $c(n,k) = \sqrt{2n-k} + k\sqrt{n} + (n-k-2)\sqrt{n+1}$, for $k = 1,\ldots,n-1$. The inequality (27) holds as equality if and only if $G = K_k \vee (K_1 \cup K_{n-k-1})$.*

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