Extensions of the Motzkin-Straus Result on the Stability Number of Graphs

Domingos M. Cardoso (dcardoso@mat.ua.pt)
Universidade de Aveiro, Departamento de Matemática, 3810-193 Aveiro, Portugal

Carlos J. Luz (cluz@est.ips.pt)
Escola Superior de Tecnologia de Setúbal, Departamento de Matemática, 2914-508 Setúbal, Portugal

Abstract. In 1965 Motzkin and Straus showed that the stability number of a graph can be determined by minimizing an indefinite quadratic form on the simplex. This result is now extended to a family of quadratic programming problems, $(P_G(\tau))$, which are convex for certain values of the parameter τ. A necessary and sufficient condition for the absence of spurious optimal solutions of $(P_G(\tau))$ is introduced and several properties of the optimal values of these quadratic programming problems are given. The recognition of graphs for which the stability number is equal to the optimal value of the problems $(P_G(\tau))$ (called graphs with convex-QP stability number) is analyzed and the polynomial-time recognition of this type of graphs, when they are (C_4, P_5)-free or (claw, P_5)-free, is proved. Furthermore, a quadratic programming finite sequence algorithm for the recognition of claw-free graphs with convex-QP stability number, when they have no particular subgraphs, is presented.

Keywords: Stable sets (05C69); Programming involving graphs (90C35).

1. Introduction

In this paper we deal with undirected simple graphs, $G = (V(G), E(G))$, where $V(G)$ denotes the nonempty set of vertices and $E(G)$ the set of edges. It is also assumed that G is of order n, i.e., $|V(G)| = n$. An element of $E(G)$, which the endpoints are the vertices i and j, is denoted by ij and, in such case, we say that the vertex i is adjacent to the vertex j. If $v \in V(G)$, then we call neighborhood of v the vertex set denoted by $N_G(v) = \{w : vw \in E(G)\}$. Given a graph G and a set of vertices $U \subseteq V(G)$, the subgraph of G induced by U, $G[U]$, is such that $V(G[U]) = U$ and $E(G[U]) = \{ij : i, j \in U \wedge ij \in E(G)\}$. Consider the set of vertices of the graph G, $X = \{x_1, x_2, \ldots, x_k\}$, and suppose that $x_i x_{i+1} \in E(G)\ \forall i \in \{1, \ldots, k-1\}$. If the vertices of X are all distinct then we say that X induces a path (and if they are all distinct but x_1 and x_k, which are equal, then we say that X induces a cycle). If there is no $i \in \{1, \ldots, k-2\}$ and $j > 1$ such that $x_i x_{i+j} \in E(G)$, then X is denoted by P_k (C_k) and we say that it is a chorthless path (cycle). A graph of order n in which all pairs of vertices are adjacent is called a complete graph and denoted by K_n. A star is a graph of order $p+1$, denoted by $K_{1,p}$, for which $\exists v \in V(K_{1,p})$ such that $V(K_{1,p}) = N_{K_{1,p}}(v) \cup \{v\}$ and there is no pair of adjacent vertices in the subgraph induced by $N_{K_{1,p}}(v)$. In particular a star of order 4, i.e., $K_{1,3}$, is called claw. A graph G is connected if $\forall i, j \in V(G)$ exists a path between i

and j and is disconnected if it is not connected. A component of a graph G is a connected subgraph of G, H, such that $\forall v \in V(G) \setminus V(H)$ the induced subgraph $G[V(H) \cup \{v\}]$ is disconnected. Given a graph H, the graph G is called H-free if G has no copy of H as an induced subgraph. For instance, a claw-free graph is a $K_{1,3}$-free graph. Extending this definition to more than one forbidden subgraph, given the graphs $H_1, \ldots, H_k$, with $k \geq 2$, we say that the graph G is $(H_1, \ldots, H_k)$-free if G contains no copy of any of the graphs $H_1, \ldots, H_k$, as induced subgraphs. Given a graph G, the line graph of G, which is denoted by $L(G)$ is constructed by taking the edges of G as vertices of $L(G)$, and joining two vertices in $L(G)$, by an edge, whenever the corresponding edges in G have a common vertex. A set S of vertices is called stable (clique) set if no (every) two vertices are adjacent. A stable (clique) set S is called maximum stable (clique) set if there is no other stable (clique) set with greater number of vertices. The number of vertices, in a maximum stable (clique) set of a graph G, is called the stability (clique) number of G and is denoted by $\alpha(G)$ $(\omega(G))$. The complement of G, denoted by $\bar{G}$, is such that $V(\bar{G}) = V(G)$ and $E(\bar{G}) = \{ij : i,j \in V(G) \wedge ij \notin E(G)\}$. It is well known that $\alpha(G) = \omega(\bar{G})$ and then the determination of the stability number is equivalent to the determination of the clique number. As proved in (Karp, 1972), given a nonnegative integer k, to determine if G has a stable set of size k is NP-complete and therefore, in general, the determination of $\alpha(G)$ (as well as $\omega(G)$) is a hard problem. However there are several particular classes of graphs for which the stability number can be determined in polynomial-time as is the case of perfect graphs (Lovász, 1986), the claw-free graphs (Minty, 1980) and (Sbihi, 1980), and the $(P_5, banner)$-free graphs (Lozin, 2000), among many others. A matching in a graph G is a subset of edges, $M \subseteq E(G)$, no two of which have a common vertex. A matching with maximum cardinality is designated maximum matching. Furthermore, if for each vertex $v \in V(G)$ there is one edge of the matching M incident with v, then M is called a perfect matching. Note that the determination of a maximum stable set of a line graph $L(G)$ is equivalent to the determination of a maximum matching of G. Based on the Edmonds perfect matching algorithm (Edmonds, 1965), the are several polynomial-time algorithms for the determination of a maximum matching of a graph. Given a subset of vertices S of a graph G, the vector $x \in \mathbb{R}^V$ with $x_v = 1$ if $v \in S$ and $x_v = 0$ if $v \notin S$ is called the characteristic vector of S. Throughout this paper, A_G will denote the adjacency matrix of the graph G of order n, that is, $A_G = (a_{ij})_{n \times n}$ is such that

$$a_{ij} = \begin{cases} 1 & \text{,if } ij \in E(G) \\ 0 & \text{,otherwise} \end{cases}$$

and $\lambda_{min}(A_G)$ will denote the minimum eigenvalue of A_G. It is well known that if G has at least one edge then $\lambda_{min}(A_G) \leq -1$. Actually, $\lambda_{min}(A_G) = 0$ iff G has no edges, $\lambda_{min}(A_G) = -1$ iff G has at least one edge and every component complete and $\lambda_{min}(A_G) \leq -\sqrt{2}$ otherwise, (Doob, 1982). The basic properties of line graphs, $L(G)$, are that they are claw-free and $\lambda_{min}(A_{L(G)}) \geq -2$. Given a set of vertices (edges) U (F) included in $V(G)$ $(E(G))$, $G-U$ $(G-F)$ will denote the graph with $V(G-U) = V(G) \setminus U$ $(V(G-F) = V(G))$ and $E(G-U) = \{ij : i, j \in V(G-U), ij \in E(G)\}$ $(E(G-F) = E(G) \setminus F)$.

The remainder of the paper is organized in four parts. The next section contains the properties of a family of quadratic programming problems which includes the Motzkin-Straus one (Motzkin and Straus, 1965) as a particular case. A necessary and sufficient condition for the absence of spurious optimal solutions are also included. In section 3, results related with the recognition of graphs with stability number equal to the optimal values of quadratic programs of the above family are analyzed. Also adverse subgraphs, which are graphs that do not allow the recognition of graphs with stability number equal to the optimal value of a convex quadratic program (called graphs with convex-QP stability number) are characterized. Furthermore, some results which imply the absence of the above referred adverse subgraphs are presented. Such results allow the polynomial-time recognition of (C_4, P_5)-free and (claw, P_5)-free graphs with convex-QP stability number. Section 4, includes a quadratic programming finite sequence algorithm for the recognition of claw-free graphs with convex-QP stability number, when they have no particular subgraphs. In section 5, some conclusions and open problems are presented.

2. Continuous Optimization Approach to the Stability Number of Graphs

The continuous optimization approach to the approximate resolution of combinatorial NP-hard problems has been deeply researched, and several readable texts with reviews on the subject have been published, among which we may refer, for instance (Bomze et al, 1999). Several parameterizations of the indefinite Motzkin-Straus quadratic program (Motzkin and Straus, 1965) have been also studied, with different results from the ones herein introduced, as is the case of (Gibbons et al, 1997).

Let us define the quadratic programming problem

$$(P_G(\tau)) \;\; \upsilon_G(\tau) = \max\{2\hat{e}^T x - x^T(\frac{1}{\tau}A_G + I_n)x : x \geq 0\},$$

with $\tau > 0$, where $\hat{e}$ is the all ones vector and I_n is the identity matrix of order n (these notations will be used throughout the paper). Then we may conclude the following result.

THEOREM 2.1. *$\forall \tau > 0$, if $x^*(\tau)$ is an optimal solution for $(P_G(\tau))$ then*

$$\forall i \in V(G) \;\; [x^*(\tau)]_i = \max\{0, 1 - \frac{a_G^i x^*}{\tau}\},$$

where $[x^*(\tau)]_i$ *is the* i*-th component of* $x^*(\tau)$ *and* a^i_G *be the* i*-th row of the matrix* A_G.

Proof. The proof is similar to the one obtained in (Cardoso, 2001). ■

As immediate consequences of this theorem we have the following.

- If $x^*(\tau)$ is an optimal solution for $(P_G(\tau))$ then
$$\forall i \in V(G) \;\; 0 \leq [x^*(\tau)]_i \leq 1.$$
- On the other hand, since $\forall \tau > 0$
$$0 \leq x \leq \hat{e} \;\Rightarrow\; 2\hat{e}^T x - x^T(\frac{A_G}{\tau} + I_n)x \leq 2\hat{e}^T x - ||x||^2 \leq n$$
and since for $2\hat{e}^T\hat{e}_j - \hat{e}_j^T(\frac{A_G}{\tau} + I_n)\hat{e}_j = 1$ (where $\hat{e}_j$ denotes the j-th vector of the canonical basis of $I\!R^n$) we may conclude that
$$1 \leq \upsilon_G(\tau) \leq n.$$
- Furthermore, the above lower and upper bounds for $\upsilon_G(\tau)$ are the best possible, since $\upsilon_G(\tau) = 1$ if G is a clique and $\upsilon_G(\tau) = n$ if G has no edges.

The next theorem introduces the main properties of $\upsilon_G(\tau)$ and $(P_G(\tau))$. Before that, however, it is convenient to remind that an optimal solution, x^*, of $(P_G(\tau))$ is called spurious when $\upsilon_G(\tau) = \alpha(G)$ but x^* not defines the characteristic vector of a maximum stable set. The presence of spurious solutions among the optimal ones of the Motzkin-Straus quadratic programming problem (Motzkin and Straus, 1965) first observed in (Pardalos and Phillips, 1990), has been deeply studied (see, (Pelillo and Jagota, 1995), (Bomze, 1997) and (Bomze, 1998)).

THEOREM 2.2. *Given a graph,* G*, the function* $\upsilon_G :]0, +\infty[\mapsto [1, n]$ *has the following properties:*

2.2.1. $\forall \tau > 0 \;\; \alpha(G) \leq \upsilon_G(\tau)$.

2.2.2. $0 < \tau_1 < \tau_2 \Rightarrow \upsilon_G(\tau_1) \leq \upsilon_G(\tau_2)$.

2.2.3. $\forall \tau \geq -\lambda_{min}(A_G) \;\; (P_G(\tau))$ *is a convex program.*

2.2.4. Assuming $\tau^* > 0$*, then the following statements are equivalent.*

a) $\exists \bar{\tau} \in]0, \tau^*[$ *such that* $\upsilon_G(\bar{\tau}) = \upsilon_G(\tau^*)$*;*

b) $\upsilon_G(\tau^*) = \alpha(G)$ *and* $\forall \tau \in]0, \tau^*[\;\; (P_G(\tau))$ *has no spurious optimal solutions;*

c) $\forall \tau \in]0, \tau^*] \;\; \upsilon_G(\tau) = \alpha(G)$.

2.2.5. $\forall U \subset V(G)\ \ \forall \tau > 0\ \ \upsilon_{G-U}(\tau) \leq \upsilon_G(\tau)$.

Proof.

2.2.1. For any characteristic vector, $\bar{x}$, of a stable set of G, $\bar{x}^T A_G \bar{x} = 0$. Therefore, if $\bar{x}$, is the characteristic vector of a maximum stable set then

$$\forall \tau > 0\ \ 2\hat{e}^T\bar{x} - \bar{x}^T(\frac{A_G}{\tau} + I_n)\bar{x} = 2\hat{e}^T\bar{x} - ||\bar{x}||^2 = \alpha(G) \leq \upsilon_G(\tau).$$

2.2.2. Let us supouse that $0 < \tau_1 < \tau_2$, then $\forall x \geq 0$

$$\frac{x^T A_G x}{\tau_2} \leq \frac{x^T A_G x}{\tau_1}$$
$$\Updownarrow$$
$$2\hat{e}^T x - x^T(\frac{A_G}{\tau_1} + I_n)x \leq 2\hat{e}^T x - x(\frac{A_G}{\tau_2} + I_n)x.$$

Therefore

$$\upsilon_G(\tau_1) = \max_{x\geq 0}\ 2\hat{e}^T x - x^T(\frac{A_G}{\tau_1} + I_n)x \leq \max_{x \geq 0}\ 2\hat{e}^T x - x^T(\frac{A_G}{\tau_2} + I_n)x = \upsilon_G(\tau_2).$$

2.2.3. $\forall \tau \geq -\lambda_{min}(A_G)$, the minimum eigenvalue of $\frac{A_G}{\tau} + I_n$ is not less than zero, and then the Hessian matrix of the objective function of $(P_G(\tau))$ is negative semidefinite.

2.2.4. • Let us prove that $a) \Rightarrow b)$.

Assuming that $\bar{\tau} \in]0, \tau^*[$ and $x(\bar{\tau})$ is an optimal solution for $(P_G(\bar{\tau}))$, then

$$\begin{aligned} \upsilon_G(\bar{\tau}) &= 2\hat{e}^T x(\bar{\tau}) - x(\bar{\tau})^T(\frac{A_G}{\bar{\tau}} + I_n)x(\bar{\tau}) \\ &\leq 2\hat{e}^T x(\bar{\tau}) - x(\bar{\tau})^T(\frac{A_G}{\tau^*} + I_n)x(\bar{\tau}) \\ &\leq \upsilon_G(\tau^*). \end{aligned}$$

As a consequence, $\upsilon_G(\bar{\tau}) = \upsilon_G(\tau^*)$ implies that $x(\bar{\tau})$ is an optimal solution for $(P_G(\tau^*))$ and also that

$$\frac{x(\bar{\tau})^T A_G x(\bar{\tau})}{\bar{\tau}} = \frac{x(\bar{\tau})^T A_G x(\bar{\tau})}{\tau^*}.$$

Since $\bar{\tau} < \tau^*$, from the above equality, we may conclude that

$$x(\bar{\tau})^T A_G x(\bar{\tau}) = 0.$$

This is equivalent to say that the support of $x(\bar{\tau})$ (that is, the subset of indices $\{j : x(\bar{\tau})_j > 0\}$) defines the the characteristic vector of a stable set of G and then (since $x(\bar{\tau})$ is an optimal solution for $(P_G(\bar{\tau}))$) is the characteristic vector of a maximum stable set of G. Therefore $\upsilon_G(\tau^*) = \alpha(G)$ and $\forall \tau \in]0, \tau^*[$ $(P_G(\tau))$ has no spurious optimal solutions.

- Now let us prove that $b) \Rightarrow c)$
  Assuming that $\upsilon_G(\tau^*) = \alpha(G)$, according to 2.2.2,

$$\forall \tau \in]0, \tau^*] \qquad \alpha(G) \leq \upsilon_G(\tau) \leq \upsilon_G(\tau^*) = \alpha(G).$$

- The implication $c) \Rightarrow a)$ is obvious.

2.2.5. Let $U \subset V(G)$ and $\bar{x}^*$ an optimal solution for $(P_{G-U}(\tau))$, with $\tau > 0$. Let $x^* \in I\!R^n$ such that

$$x_i^* = \begin{cases} \bar{x}_i^*, & \text{if } i \notin U \\ 0, & \text{otherwise.} \end{cases}$$

Then $\upsilon_{G-U}(\tau) = 2\hat{e}^T x^* - x^{*T}(\frac{A_G}{\tau} + I_n)x^* \leq \upsilon_G(\tau)$. ■

Given an arbitrary graph, G, from the above theorem, we conclude that $\upsilon_G(\tau)$ is a monotone upper bound on the stability number of G. Being $\tau^* = \max\{\tau : \upsilon_G(\tau) = \alpha(G)\}$ (and then, as will be seen later, $\tau^* \geq 1$), we may conclude that $\forall \tau \in]0, \tau^*[$ $(P_G(\tau))$ has no spurious optimal solutions, and $\forall U \subset V(G)$ $\forall \tau > 0$ $\upsilon_{G-U}(\tau) \leq \upsilon_G(\tau)$.
The figure 1 illustrates the graph of the function $\upsilon_G(\tau)$, for the graph G depicted in figure 2.

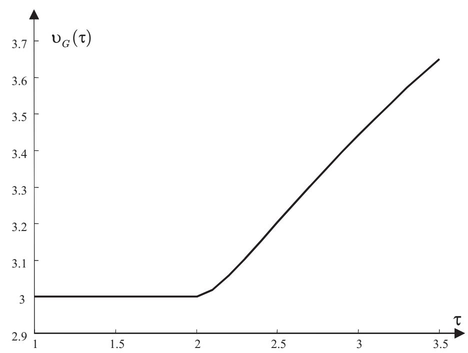

Figure 1. Graph of the function $\upsilon_G(\tau)$ defined for the graph of figure 2.

The next theorem allows to conclude that the indefinite quadratic program of Motzkin-Straus (Motzkin and Straus, 1965) is a particular case of the family of quadratic programming problems $(P_G(\tau))$.

THEOREM 2.3. *Consider the quadratic programming problem*

$$(Q_G(\tau)) \qquad \nu_G(\tau) = \min\{z^T(\frac{A_G}{\tau} + I_n)z : \hat{e}^T z = 1, z \geq 0\},$$

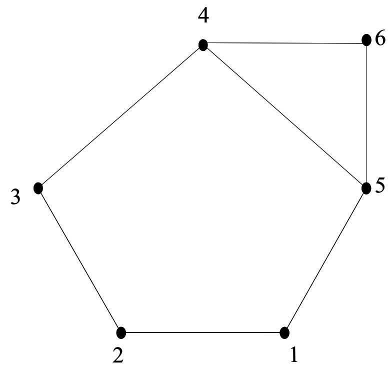

Figure 2. Graph G, with $\lambda_{min}(A_G) = -1.7566$, for which $\upsilon_G(2) = 3 = \alpha(G)$ and $S = \{1, 3, 6\}$ is the only maximum stable set.

with $\tau > 0$, *and let* x^* *and* z^* *be optimal solutions for* $P_G(\tau)$ *and* $Q_G(\tau)$, *respectively. Then* $\frac{z^*}{\nu_G(\tau)}$ *and* $\frac{x^*}{\upsilon_G(\tau)}$ *are optimal solutions for* $P_G(\tau)$ *and* $Q_G(\tau)$, *respectively, and* $\upsilon_G(\tau) = \frac{1}{\nu_G(\tau)}$.

Proof. From optimality of x^* for $P_G(\tau)$, by Karush-Khun-Tucker conditions (Bazaraa et al, 1993), $\exists y^* \geq 0$ such that

$$A_G x^* = \tau(\hat{e} - x^*) + y^*, \tag{1}$$
$$x^{*T} y^* = 0, \tag{2}$$

Then $x^{*T}(\frac{A_G}{\tau} + I_n)x^* = \hat{e}^T x^* = \upsilon_G(\tau)$, and therefore

$$\frac{1}{\upsilon_G(\tau)} = \frac{x^{*T}}{\upsilon_G(\tau)}(\frac{A_G}{\tau} + I_n)\frac{x^*}{\upsilon_G(\tau)}.$$

On the other hand, taking into account that $\frac{x^*}{\upsilon_G(\tau)}$ is feasible for $(Q_G(\tau))$, from the optimality of z^* for $(Q_G(\tau))$ we have

$$z^{*T}(\frac{A_G}{\tau} + I_n)z^* \leq \frac{x^{*T}}{\upsilon_G(\tau)}(\frac{A_G}{\tau} + I_n)\frac{x^*}{\upsilon_G(\tau)}.$$

Therefore

$$\upsilon_G(\tau)z^{*T}(\frac{A_G}{\tau} + I_n)\upsilon_G(\tau)z^* \leq x^{*T}(\frac{A_G}{\tau} + I_n)x^*$$

$$\Updownarrow$$

$$\upsilon_G(\tau) = 2\hat{e}^T x^* - x^{*T}(\frac{A_G}{\tau} + I_n)x^* \leq 2\hat{e}^T(\upsilon_G(\tau)z^*) - (\upsilon_G(\tau)z^*)^T(\frac{A_G}{\tau} + I_n)(\upsilon_G(\tau)z^*),$$

since $\hat{e}^T x^* = \upsilon_G(\tau)$ and $\hat{e}^T z^* = 1$. Thus $\upsilon_G(\tau)z^*$ is an optimal solution for $(P_G(\tau))$ and $\upsilon_G(\tau) = (\upsilon_G(\tau))^2 z^{*T}(\frac{A_G}{\tau} + I_n)z^* \Leftrightarrow \upsilon_G(\tau) = \frac{1}{\nu_G(\tau)}$. From the last

equality we also conclude that $\frac{z^*}{\nu_G(\tau)}$ is an optimal solution for $(P_G(\tau))$.
On the other hand, the optimality of x^* for $(P_G(\tau))$ implies

$$\upsilon_G(\tau) = 2\hat{e}^T x^* - x^{*T}(\frac{A_G}{\tau} + I_n)x^* \geq 2\hat{e}^T(\upsilon_G(\tau)z^*) - (\upsilon_G(\tau)z^*)^T(\frac{A_G}{\tau} + I_n)(\upsilon_G(\tau)z^*).$$

Then, since $\hat{e}^T x^* = \hat{e}^T(\upsilon_G(\tau)z^*)$, we have

$$(\upsilon_G(\tau)z^*)^T(\frac{A_G}{\tau} + I_n)(\upsilon_G(\tau)z^*) \geq x^{*T}(\frac{A_G}{\tau} + I_n)x^* = \upsilon_G(\tau)$$

$$\Updownarrow$$

$$\nu_G(\tau) = z^{*T}(\frac{A_G}{\tau} + I_n)z^* \geq \frac{x^{*T}}{\upsilon_G(\tau)}(\frac{A_G}{\tau} + I_n)\frac{x^*}{\upsilon_G} = \frac{1}{\upsilon_G(\tau)}.$$

■

The above proof could be obtained applying theorem 5 of (Bomze, 1998) to the quadratic programming problems $(P_G(\tau))$ and $(Q_G(\tau))$.

Considering the Motzkin-Straus result on the stability number of graphs (Motzkin and Straus, 1965), which is equivalent to the equality

$$\frac{1}{\min\{z^T(A_G + I_n)z : \hat{e}^T z = 1, z \geq 0\}} = \alpha(G), \tag{3}$$

(see proposition 2 in (Gibbons et al, 1997)) and, combining this result with theorem 2.3, we obtain the equalities

$$\upsilon_G(1) = \frac{1}{\nu_G(1)} = \alpha(G).$$

Then theorem 2.2-2.2.4 implies that $\forall \tau \in]0,1[$ $(P_G(\tau))$ has no spurious optimal solutions. On the other hand, if x^* is an optimal solution of $(P_G(\tau))$, with $\tau \geq 1$, then $2\hat{e}^T x^* - x^{*T}(A_G + I_n)x^* \leq \upsilon_G(1) = \alpha(G) \leq \upsilon_G(\tau)$. Thus, since $\upsilon_G(\tau) = \hat{e}^T x^* = x^*(\frac{A_G}{\tau} + I_n)x^*$, we obtain the lower and upper bounds on the stability number of G, $\upsilon_G(\tau) - \frac{\tau-1}{\tau}x^{*T}A_G x^* \leq \alpha(G) \leq \upsilon_G(\tau)$. These inequalities are equivalent to

$$||x^*||^2 - (\tau - 2)(\upsilon_G(\tau) - ||x^*||^2) \leq \alpha(G) \leq \upsilon_G(\tau).$$

The upper bound on the stability number of G, $\upsilon_G(\tau)$, with $\tau = -\lambda_{min}(A_G)$, was introduced in (Luz, 1995), where the following necessary and sufficient condition to obtain the equality is given.

- If G has at least one edge then $\alpha(G) = \upsilon_G(-\lambda_{min}(A_G))$ if and only if for a maximum stable set S of G (and then for all),

$$-\lambda_{\min}(A_G) \leq \min\{|N_G(i) \cap S| : i \notin S\}. \tag{4}$$

Now, we assert the following extension, which can be deduced applying the Karush-Khun-Tucker conditions, in a way similar to the proof of (4) in (Luz, 1995).

- If G has at least one edge then $\forall \tau \geq -\lambda_{min}(A_G)$, $\alpha(G) = \upsilon_G(\tau)$ if and only if for a maximum stable set S of G (and then for all),

$$\tau \leq \min\{|N_G(i) \cap S| : i \notin S\}. \tag{5}$$

There are infinite-many graphs, G, for which the Motzkin-Straus quadratic program $(Q_G(1))$ has spurious optimal solutions (as was proved in (Pelillo and Jagota, 1995)), and then the same for $(P_G(1))$. Therefore, according to theorem 2.1, for such graphs there is no $\tau > 1$ such that $\upsilon_G(\tau) = \alpha(G)$. Given a graph G and $\tau \in]1, -\lambda_{min}(A_G)[$, it remains as an open problem to know if the condition (5), when fulfilled for every maximum stable set of G, is sufficient to obtain the equality $\upsilon_G(\tau) = \alpha(G)$. The graph G, depicted in figure 3, with $\lambda_{min}(A_G) = -2.1774$, show that the existence of a maximum stable set S such that $\tau \leq \min\{|N_G(j) \cap S| : j \notin S\}$, with $\tau \in]1, -\lambda_{min}(A_G)[$, is not sufficient to obtain the equality $\upsilon_G(\tau) = \alpha(G)$. Note that $S = \{1, 4\}$ is a maximum stable set for the graph G, depicted in figure 3, such that $2 \leq \min\{|N_G(i) \cap S| : i \notin S\}$ and $\alpha(G) = 2 < \frac{7}{3} \leq \upsilon_G(2)$ (it must be noted, however, that there are also other maximum stable sets, $S' = \{2, 3\}$ and $S'' = \{5, 2\}$, for which the inequality (5), with $\tau = 2$, is not fulfilled).
Taking into account the theorem 2.2-2.2.4., we may conclude that if S is a

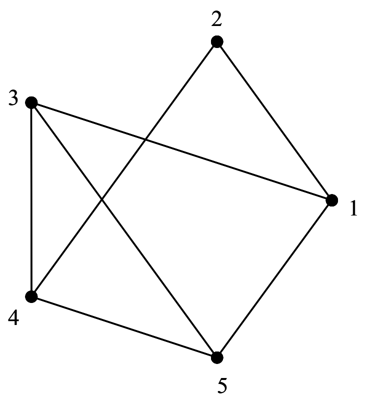

Figure 3. Graph, G, with $\lambda_{min}(A_G) = -2.1774$, where exists a maximum stable set, $S = \{1, 4\}$, for which the condition (5) is fulfilled for $\tau = 2$, but $\upsilon_G(2) > \alpha(G)$.

maximum stable set of G, $\lambda_{min}(A_G)$ is not an integer, and $|N_G(v) \cap S| \geq -\lambda_{min}(A_G)$ $\forall v \notin S$ (as is the case of the graph depicted in figure 2), then S is the only maximum stable set of G. In fact, in such case, for $\tau \geq -\lambda_{min}(A_G)$, the quadratic program $(P_G(\tau))$ is convex. Therefore the convex combination of optimal solutions of $(P_G(\tau))$ are also optimal solutions. On the other hand, setting $\tau_1 = -\lambda_{min}(A_G)$ and $\tau_2 = \lceil -\lambda_{min}(A_G) \rceil$ (where $\lceil \delta \rceil$ denotes the least integer not less than δ), since

$$\tau_1 \leq \min\{|N_G(v) \cap S| : v \notin S\} \Rightarrow \tau_2 \leq \min\{|N_G(v) \cap S| : v \notin S\},$$

by (5), we obtain $\upsilon_G(\tau_1) = \alpha(G) = \upsilon_G(\tau_2)$. Therefore, assuming that there are two maximum stable sets in G, the corresponding characteristic vectors are optimal solutions for $(P_G(\tau_1))$ and for $(P_G(\tau_2))$, and the convex combinations of them are also optimal solutions for both programs. Thus there are spurious optimal solutions for $(P_G(\tau_1))$, contradicting the theorem 2.2-2.2.4.
When $\tau \in]1, -\lambda_{min}(A_G)[$, we may state the following necessary condition for the equality $\upsilon_G(\tau) = \alpha(G)$.

THEOREM 2.4. *If $\upsilon_G(\tau) = \alpha(G)$, then for any maximum stable set S of G,*

$$|N_G(v) \cap S| \geq \tau \qquad \forall v \in V(G) \setminus S.$$

Proof. If $\upsilon_G(\tau) = \alpha(G)$, then the characteristic vector of every maximum stable set of G is an optimal solution for $(P_G(\tau))$. Thus, being $\bar{x}$ the characteristic vector of a maximum stable set of G, S, by Karush-Khun-Tucker conditions (Bazaraa et al, 1993), $\exists \bar{y} \geq 0$ such that $\bar{y}^T \bar{x} = 0$, and $A_G \bar{x} = \tau(\hat{e} - \bar{x}) + \bar{y}$. Therefore, for each vertex $v \notin S$, we get the following equality

$$|N_G(v) \cap S| = \tau(1 - \bar{x}_v) + \bar{y}_v \Leftrightarrow |N_G(v) \cap S| = \tau + \bar{y}_v,$$

and then $|N_G(v) \cap S| \geq \tau$. ■

Despite the condition (5) be not sufficient to obtain the equality $\upsilon_G(\tau) = \alpha(G)$ when $\tau \in]1, -\lambda_{min}(A_G)[$, assuming that certain components of the complementary vector of an optimal solution of $(P_G(\tau))$ are zero valued, this equality holds.

THEOREM 2.5. *Let x^* be an optimal solution for $(P_G(\tau))$ and y^* the corresponding complementary vector. If there is a maximum stable set S of G, such that $y^*_i = 0$ $\forall i \in S$ and $\tau \leq \min\{|N_G(i) \cap S| : i \notin S\}$ then $\upsilon_G(\tau) = \alpha(G)$.*

Proof. According to the hypothesis $A_G x^* = \tau(\hat{e} - x^*) + y^*$. Then, being $\bar{x}$ the characteristic vector of S, we have

$$\bar{x}^T A_G x^* = \tau(\alpha(G) - \sum_{j \in S} x^*_j) + \sum_{j \in S} y^*_j \Leftrightarrow \sum_{j \in S} x^*_j + \sum_{j \notin S} \frac{|N_G(j) \cap S|}{\tau} x^*_j = \alpha(G),$$

since $\sum_{j \in S} y^*_j = 0$. Therefore, if $\frac{|N_G(j) \cap S|}{\tau} \geq 1$ $\forall j \notin S$ then

$$\upsilon_G(\tau) \leq \sum_{j \in S} x^*_j + \sum_{j \notin S} \frac{|N_G(j) \cap S|}{\tau} x^*_j = \alpha(G) \leq \upsilon(G).$$

■

Note that this proof also implies that if $\exists j \notin S$ such that $|N_G(j) \cap S| > \tau$, then $x_j^* = 0$.

Combining theorems 2.5 and 2.4, it follows that given an arbitrary graph G, $\forall \tau > 0$, if there is an optimal solution for $(P_G(\tau))$ which is a critical point for its objective function (that is, the corresponding complementary vector is zero) then $\alpha(G) = \upsilon_G(\tau)$ if and only if $\tau \leq \min\{|N_G(i) \cap S| : i \notin S\}$.

3. Recognition of Graphs, G, with $\upsilon_G(\tau)$ Stability Number

The recognition of graphs G with $\upsilon_G(\tau)$ stability number (that is, such that $\upsilon_G(\tau) = \alpha(G)$) can be done determining $\upsilon_G(\tau')$, with $\tau' < \tau$, and thus, applying theorem 2.2-2.2.4. (from which we may conclude that $\alpha(G) = \upsilon_G(\tau)$ if and only if $\upsilon_G(\tau) = \upsilon_G(\tau')$). Thus, if $\lambda_{min}(A_G)$ is not an integer, taking into account the necessary and sufficient condition (5), $\upsilon_G(-\lambda_{min}(A_G)) = \alpha(G)$ if and only if $\upsilon_G(\lceil -\lambda_{min}(A_G) \rceil) = \upsilon_G(-\lambda_{min}(A_G))$ or, again, by theorem 2.2-2.2.4., if and only if the optimal solution of the convex quadratic programming problem $(P_G(-\lambda_{min}(A_G))$ is integer (and then the characteristic vector of the unique maximum stable set of G). Therefore, when $\lambda_{min}(A_G)$ is not integer it is easy to recognize if G has or not $\upsilon_G(-\lambda_{min}(A_G))$ stability number. However, if $\lambda_{min}(A_G)$ is integer, from theorem 2.2 such recognition may be hard to obtain. In fact, if $\lambda_{min}(A_G)$ is integer, since $(P_G(\tau))$ is an indefinite quadratic programming problem for $\tau \in]0, -\lambda_{min}(A_G)[$, the results of theorem 2.2-2.2.4. are difficult to be applied.
It must be noted that given an arbitrary graph, G, $\upsilon_G(\tau) = \alpha(G)$ if and only if $(P_G(\tau))$ has an integer optimal solution. However, to recognize if such solution exists, among the optimal ones, is an NP-hard problem (Garey and Johnson, 1979), even for convex quadratic programming problems.
Let us denote by $\mathcal{Q}_\tau$ the class of graphs G such that $\alpha(G) = \upsilon_G(\tau)$, (that is, with $\upsilon_G(\tau)$ stability number), with $\tau > 0$. This class of graphs is not hereditary (that is, is not closed for induced subgraphs) (Lozin and Cardoso, 1999). However, by theorem 3.2, $\mathcal{Q}_\tau$ is closed under deletion of α-redundant subsets of vertices (defining an α-redundant subset of vertices as being a subset $U \subseteq V(G)$ such that $\alpha(G) = \alpha(G - U)$).
When $\tau = -\lambda_{min}(A_G)$, the graphs belonging to $\mathcal{Q}_\tau$ will be called, as in (Cardoso, 2001), *graphs with convex-QP stability number* (where QP means quadratic programming). There is an infinite number of graphs with convex-QP stability number. Actually, a connected graph G with at least one edge, which is neither a star nor a triangle has a perfect matching if and only if its line graph has convex-QP stability number (Cardoso, 2001).
The polynomial-time recognition of graphs with convex-QP stability number remains, as an open problem. However, as it will be seen later, for several particular cases, it is not hard to recognize if $G \in \mathcal{Q}_\tau$, when $\tau = -\lambda_{min}(A_G)$.
A strategy for the recognition of graphs, $G \in \mathcal{Q}_\tau$, when $\tau = -\lambda_{min}(A_G)$ is

integer, can be implemented with the study of the optimal values of convex quadratic programming problems $(P_{G'}(\tau))$ for induced subgraphs G' of G.

THEOREM 3.1. *Let G be a graph for which $\exists X \subset V(G)$ such that*

$$\upsilon_G(-\lambda_{min}(A_G)) = \upsilon_{G-X}(-\lambda_{min}(A_{G-X})).$$

If $\lambda_{min}(A_G) < \lambda_{min}(A_{G-X})$, then $\alpha(G) = \upsilon_G(\tau)$, with $\tau = -\lambda_{min}(A_G)$, and the unique optimal solution of $(P_{G-X}(\tau))$ is the characteristic vector of the unique maximum stable set of G.

Proof. Part of this proof is similar to the proof of theorem 3.4 presented in (Luz and Cardoso, 1998), taking into account that, by interlacing properties, $\forall U \subseteq V(G)\ \ \lambda_{min}(A_G) \le \lambda_{min}(A_{G-U})$. Then, supposing that

$$\exists X \subset V(G) \text{ such that } \upsilon_G(-\lambda_{min}(A_G)) = \upsilon_{G-X}(-\lambda_{min}(A_{G-X})),$$

and setting $\tau = -\lambda_{min}(A_G)$, from the inequalities $\upsilon_{G-X}(-\lambda_{min}(A_{G-X})) \le \upsilon_{G-X}(\tau) \le \upsilon_G(\tau)$, we obtain the equalities

$$\upsilon_{G-X}(-\lambda_{min}(A_{G-X})) = \upsilon_{G-X}(\tau), \tag{6}$$

$$\upsilon_{G-X}(\tau) = \upsilon_G(\tau). \tag{7}$$

These equalities mean that any optimal solution of $(P_{G-X}(-\lambda_{min}(A_{G-X}))$, $\bar{x}^*$, is such that $\bar{x}^{*T} A_{G-X} \bar{x}^* = 0$, and then is the characteristic vector of a maximum stable set of $G-X$. On the other hand, since $(P_{G-X}(\tau))$ has a single set of optimal solutions which includes all optimal solutions of $(P_{G-X}(-\lambda_{min}(A_{G-X}))$, then the latter program has also only one optimal solution with a support (i.e., the indices of the non null components) which also defines an optimal solution for $(P_G(\tau))$. ■

The following theorems 3.2, 3.3 and 3.4, with proofs similar to the ones presented in (Cardoso, 2001), provide an algorithmic strategy for the recognition of several particular cases of graphs with $\upsilon_G(\tau)$ stability number.

Unless otherwise stated, throughout the remaining part of this section, it is assumed that τ has an arbitrary positive value.

THEOREM 3.2. *If $\upsilon_G(\tau) = \alpha(G)$ and $\exists U \subseteq V(G)$ such that $\alpha(G) = \alpha(G-U)$ then $\upsilon_{G-U}(\tau) = \alpha(G-U)$.*

THEOREM 3.3. *If $\exists v \in V(G)$ such that*

$$\upsilon_G(\tau) \ne \max\{\upsilon_{G-\{v\}}(\tau), \upsilon_{G-N_G(v)}(\tau)\}$$

then $\upsilon_G(\tau) > \alpha(G)$.

As immediate consequence of theorem 3.3, if $\upsilon_G(\tau) = \alpha(G)$ then

$$\forall v \in V(G),\ \upsilon_G(\tau) = \max\{\upsilon_{G-\{v\}}(\tau), \upsilon_{G-N_G(v)}(\tau)\}.$$

THEOREM 3.4. *Consider that* $\upsilon_G(\tau) = \max\{\upsilon_{G-\{v\}}(\tau), \upsilon_{G-N_G(v)}(\tau)\}$ *and* $\upsilon_{G-\{v\}}(\tau) \neq \upsilon_{G-N_G(v)}(\tau)$.

a) If $\upsilon_G(\tau) = \upsilon_{G-\{v\}}(\tau)$ *then*

$$\upsilon_G(\tau) = \alpha(G) \text{ iff } \upsilon_{G-\{v\}}(\tau) = \alpha(G - \{v\}).$$

b) If $\upsilon_G(\tau) = \upsilon_{G-N_G(v)}(\tau)$ *then*

$$\upsilon_G(\tau) = \alpha(G) \text{ iff } \upsilon_{G-N_G(v)}(\tau) = \alpha(G - N_G(v)).$$

COROLLARY 3.1. *Let G be a graph such that $\upsilon_G(\tau) = \alpha(G)$. Then, for any vertex $v \in V(G)$, the following implications hold.*

3.1.1. $\upsilon_G(\tau) > \upsilon_{G-N_G(v)}(\tau) \Rightarrow \alpha(G) = \alpha(G - \{v\})$;

3.1.2. $\upsilon_G(\tau) > \upsilon_{G-\{v\}}(\tau) \Rightarrow \alpha(G) = \alpha(G - N_G(v))$.

Proof. The proof follows immediately from theorems 3.3 and 3.4. ■

None of the theorems 3.3 and 3.4 and corollary 3.1 can be applied to graphs G and values of τ for which $\forall v \in V(G)\ \ \upsilon_G(\tau) = \upsilon_{G-\{v\}}(\tau) = \upsilon_{G-N_G(v)}(\tau)$. Supposing that G has no isolated vertices (that is, there is no vertex $j \in V(G)$ such that $N_G(j) = \emptyset$) then

$$\forall v \in V(G)\ \ \upsilon_G(\tau) = \upsilon_{G-N_G(v)}(\tau) \Rightarrow \forall u \in V(G)\ \ \upsilon_G(\tau) = \upsilon_{G-\{u\}}(\tau).$$

In fact, assuming that $\forall v \in V(G)\ \upsilon_G(\tau) = \upsilon_{G-N_G(v)}(\tau)$, since G has no isolated vertices, $\forall u \in V(G)\ \ \exists w \in N_G(u)$ and then, from the inequalities $\upsilon_{G-N_G(w)}(\tau) \leq \upsilon_{G-\{u\}}(\tau) \leq \upsilon_G(\tau)$, the equality $\upsilon_{G-\{u\}}(\tau) = \upsilon_G(\tau)$ is obtained. By similar arguments we may conclude that

$$\forall v \in V(G)\ \ \lambda_{min}(A_G) = \lambda_{min}(A_{G-N_G(v)}) \Rightarrow \forall u \in V(G)\ \ \lambda_{min}(A_G) = \lambda_{min}(A_{G-\{u\}}).$$

From now on, assuming that $\lambda_{min}(A_G), \upsilon_G(\tau) \in \mathbb{Z}$, we say that the subgraph G' of G is *adverse*, if the above results can not be applied to recognize, in polynomial-time, if G has or not convex-QP stability number. Then, for a while, according to the above analysis, considering a graph G such that $\tau = -\lambda_{min}(A_G)$ and $\upsilon_G(\tau)$ are both integers, a subgraph G' of G is adverse if and only if the following equalities hold.

$$\upsilon_{G'}(\tau) = \upsilon_G(\tau), \tag{8}$$
$$\lambda_{min}(A_{G'}) = \lambda_{min}(A_G), \tag{9}$$
$$\forall v \in V(G')\ \ \upsilon_{G'}(\tau) = \upsilon_{G-N_{G'}(v)}(\tau), \tag{10}$$
$$\forall u \in V(G')\ \ \lambda_{min}(A_{G'}) = \lambda_{min}(A_{G'-N_{G'}(u)}). \tag{11}$$

The figure 4 gives an example of an adverse graph, that is, a graph for which all the equalities (8)-(11) hold with $\tau = 2$.

Denoting the closed neighborhood of a vertex v by $N_G[v]$ (that is, $N_G[v] = N_G(v) \cup \{v\}$), the next theorem provides some conditions under which the equalities (10) could not be fulfilled.

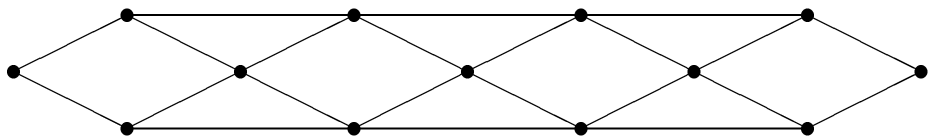

Figure 4. An adverse graph, G, with $\lambda_{min}(A_G) = -2$ and $\upsilon_G(2) = \alpha(G) = 5$.

THEOREM 3.5. *Given a graph G and $\tau > 1$, if $\exists p, q \in V(G)$ such that $N_G[q] \subseteq N_G[p]$ then $\upsilon_G(\tau) > \upsilon_{G-N_G(p)}(\tau)$.*

Proof. Let x^* be an optimal solution of $(P_G(\tau))$, with $\tau > 1$, and y^* the corresponding complementary vector. Then, from Karush-Kuhn-Tucker conditions, subtracting the q-th row to the p-th row of system (1) we get the equality

$$\tau(x_p^* - x_q^*) + \sum_{j \in N_G(p)} x_j^* - \sum_{j \in N_G(q)} x_j^* \;=\; y_p^* - y_q^*. \tag{12}$$

$$\Updownarrow$$

$$(\tau - 1)(x_p^* - x_q^*) + \sum_{j \in N_G[p]} x_j^* - \sum_{j \in N_G[q]} x_j^* \;=\; y_p^* - y_q^*. \tag{13}$$

$$\Updownarrow$$

$$(\tau - 1)(x_p^* - x_q^*) + \sum_{j \in N_G[p] \setminus N_G[q]} x_j^* - \sum_{j \in N_G[q] \setminus N_G[p]} x_j^* \;=\; y_p^* - y_q^*. \tag{14}$$

Since (by hypothesis) $N_G[q] \subseteq N_G[p]$, then $N_G[q] \setminus N_G[p] = \emptyset$ and (14) can be written as $(\tau - 1)(x_p^* - x_q^*) + \sum_{j \in N_G[p] \setminus N_G[q]} x_j^* = y_p^* - y_q^*$, and implies

$$(\tau - 1)(x_p^* - x_q^*) \;\leq\; y_p^* - y_q^*. \tag{15}$$

If $\upsilon_G(\tau) = \upsilon_{G-N_G(p)}(\tau)$ then exists an optimal solution x^{*p} of $(P_G(\tau))$, and the corresponding complementary vector y^{*p} such that $x_p^{*p} = 1$. As a consequence $y_p^{*p} = 0$ and $x_j^{*p} = 0 \; \forall j \in N_G(p)$ (in particular, $x_q^{*p} = 0$). Therefore, considering $x^* = x^{*p}$, the inequality (15) becomes equivalent to the inequality

$$\tau - 1 \leq -y_q^* \Leftrightarrow \tau \leq 1,$$

which is a contradiction. ∎

From now on, supposing that the chordless path, P_4, is induced by the vertices x_1, x_2, x_3, x_4, the vertices x_2 and x_3 will be called (as usual) the midpoints of P_4. Theorem 3.5 implies the following corollary.

COROLLARY 3.2. *Let G be a graph and $\tau > 1$. If $\exists pq \in E(G)$ such that $\upsilon_G(\tau) = \upsilon_{G-N_G(p)}(\tau) = \upsilon_{G-N_G(q)}(\tau)$ then pq belongs to a C_4 or p and q are the midpoints of a P_4.*

Proof. By theorem 3.5, $N_G[q] \not\subseteq N_G[p] \wedge N_G[p] \not\subseteq N_G[q]$ and hence $\exists x \in N_G[p] \setminus N_G[q]$ and $\exists y \in N_G[q] \setminus N_G[p]$. Therefore if $xy \in E(G)$ then pq belongs to a C_4 else p and q are the midpoints of a P_4. ■

Reminding that a *banner* is a graph isomorphic to the graph with vertices x_1, x_2, x_3, x_4, x_5 and edges x_1x_2, x_2x_3, x_3x_4, x_4x_5 and x_5x_2, from corollary 3.2 of this section and corollary 1 in (Brandstädt and Lozin, 2001) (where it is stated that if G is a $(banner, P_5)$-free graph, then any midpoint of a P_4, v, is such that $\alpha(G) = \alpha(G - \{v\})$), we have the following.

COROLLARY 3.3. *Let G be a graph without isolated vertices, for which the equalities (10) hold, with $\tau > 1$. If G is (C_4, P_5)-free, then $\forall v \in V(G)$ $\alpha(G) = \alpha(G - \{v\})$.*

Proof. Since G is a C_4-free graph, for which the equalities (10) hold with $\tau > 1$, and has no isolated vertices, by corollary 3.2, any vertex is the midpoint of a P_4. Therefore, since any C_4-free graph is *banner*-free, from corollary 1 in (Brandstädt and Lozin, 2001), we may conclude that $\forall v \in V(G)$ $\alpha(G) = \alpha(G - \{v\})$. ■

Given two vertices $x, y \in V(G)$, the vertex x dominates vertex y if $N_G(y) \subseteq N_G[x]$ and then we say that the vertices x and y are comparable. This binary relation is a *preorder* (that is, is reflexive and transitive) which is called, in (Földes and Hammer, 1978), *vicinal preorder*. Therefore, the graph $D(G)$ such that $V(D(G)) = V(G)$ and

$$E(D(G) = \{ij : i, j \in V(G) \wedge N_G(i) \subseteq N_G[j] \text{ or } N_G(j) \subseteq N_G[i]\},$$

is the comparability graph of the vicinal preorder of G.
Considering the Dilworth number of a graph G, $\text{dilw}(G)$, introduced in (Földes and Hammer, 1978) which is the largest number of pairwise incomparable vertices of G, then $\text{dilw}G) = \alpha(D(G))$ and we have the following theorem.

THEOREM 3.6. *Let G be a graph and $\tau > 1$. If $dilw(G) < \omega(G)$ then $\exists pq \in E(G)$ such that $\upsilon_G(\tau) > \upsilon_{G-N_G(p)}(\tau)$ or $\upsilon_G(\tau) > \upsilon_{G-N_G(q)}(\tau)$.*

Proof. Suppose that $\exists pq \in E(G)$ such that $\upsilon_G(\tau) = \upsilon_{G-N_G(p)}(\tau) = \upsilon_{G-N_G(q)}(\tau)$. Then, theorem 3.5 implies that $N_G[q] \not\subseteq N_G[p]$ and $N_G[p] \not\subseteq N_G[q]$ that is, p and q are incomparable. Therefore, if $\forall pq \in E(G)$ $\upsilon_G(\tau) = \upsilon_{G-N_G(p)}(\tau) = \upsilon_{G-N_G(q)}(\tau)$, then any set of pairwise adjacent vertices is a set of incomparable vertices, and then

$$\forall pq \in E(G) \;\; \upsilon_G(\tau) = \upsilon_{G-N_G(p)}(\tau) = \upsilon_{G-N_G(q)}(\tau) \;\; \Rightarrow \;\; dilw(G) \geq \omega(G).$$

■

4. Convex Quadratic Programming Determination of Maximum Stable Sets of Particular claw-free Graphs

Taking into account that a claw-free graph remains claw-free for induced subgraphs, the next theorem helps to recognize claw-free graphs which have $\upsilon_G(\tau)$ stability number, when $\tau > 1$. Before that it is convenient to remind that $v \in V(G)$ ($pq \in E(G)$) is called α-critical if $\alpha(G) > \alpha(G - \{v\})$ ($\alpha(G) < \alpha(G - \{pq\})$).

THEOREM 4.1. *Let G be a claw-free graph and $\tau > 1$. If $\exists pq \in E(G)$ such that $\upsilon_G(\tau) = \upsilon_{G-N_G(p)}(\tau) = \upsilon_{G-N_G(q)}(\tau)$ and p and q are not the midpoints of a P_4, then neither p nor q are α-critical.*

Proof. According to corollary 3.2, $N_G[p] \setminus N_G[q] \neq \emptyset$ and $N_G[q] \setminus N_G[p] \neq \emptyset$. Suppose that $p \in S$, where S is a maximum stable set of G.

- If the edge pq is α-critical, then it is well known that neither p nor q are α-critical and hence the result is proved.

- Suppose that pq is not α-critical. Then (since G is claw-free) $\exists! v \in (N_G(q) \cap S) \setminus \{p\}$ and, according to the hypothesis, $\exists u \in N_G[p] \setminus N_G[q]$. Since p and q are not the midpoints of a P_4, $uv \in E(G)$ and then the set of vertices $X = \{u, p, q, v\}$ induces a C_4 and is such that $p, v \in S$, and $q, u \notin S$. Furthermore, the neighbors of q and u in S are only p and v. Therefore $S' = (S \setminus X) \cup (X \setminus S)$ is a maximum stable set of G, and thus neither p nor q are α-critical.

■

Considering the chordless path P_5 defined by the sequence of vertices $x_1x_2x_3x_4x_5$, the vertex x_3 is called the mid point of P_5.

THEOREM 4.2. *Let G be a claw-free graph without isolated vertices such that $\forall ij \in E(G)$, ij belongs to a C_4 or i and j are the midpoints of a P_4. Then $\forall w \in V(G)$, w is the midpoint of a P_5, $uvwxy$, such that $u, w, y \in S$, where S is a maximum stable set of G, or $\alpha(G) = \alpha(G - \{w\})$.*

Proof. Let S be a maximum stable set of G and w an arbitrary vertex. If $w \notin S$ then, obviously, $\alpha(G) = \alpha(G - \{w\})$. Let us suppose that $w \in S$. Then, since w is not isolated, $\exists x \in V(G)$ such that $wx \in E(G)$.

- If wx is α-critical then $\alpha(G) = \alpha(G - \{w\})$.

- Let us assume that wx is not α-critical. Then, $\exists y \in N_G(x) \cap (S \setminus \{w\})$. On the other hand, since wx belongs to a C_4 or w and x are the midpoints of a P_4, $\exists v \in N_G(w) \setminus N_G[x]$.

  - If $vy \in E(G)$ then, since G is claw-free, $(S \setminus \{w, y\}) \cup \{v, x\}$ is a maximum stable set of G and then $\alpha(G) = \alpha(G - \{w\})$.

- If $vy \notin E(G)$ then or vw is α-critical (and thus $\alpha(G) = \alpha(G - \{w\})$) or $\exists u \in N_G(v) \cap (S \setminus \{w\})$.
  - $*$ Suppose that $\exists u \in N_G(v) \cap (S \setminus \{w\})$. If $ux \in E(G)$ then $\alpha(G) = \alpha(G - \{w\})$ else $uvwxy$ is a P_5 with $u, w, y \in S$. ■

An immediate consequence of combining this theorem with theorem 4.1 is the following corollary.

COROLLARY 4.1. *If G is a $(claw, P_5)$-free graph without isolated vertices, for which the equalities (10) hold, with $\tau > 1$, then $\forall v \in V(G)$ $\alpha(G) = \alpha(G - \{v\})$.*

The next theorem allows the recognition of α-redundant subsets of vertices in claw-free graphs with comparable non adjacent vertices, in the sense of the vicinal preorder relation (Földes and Hammer, 1978)).

THEOREM 4.3. *Let G be a claw-free graph and $p, q \in V(G)$ such that $pq \notin E(G)$. If $N_G(p) \subseteq N_G(q)$ then $N_G(p)$ is an α-redundant subset of vertices.*

Proof. Assume that $\exists u \in N_G(p)$ such that $u \in S$, where S is a maximum stable set of G.

– If $S \cap N_G(p) = \{u\}$ then up is α-critical and hence u is not. Therefore the result is proved.

– Suppose $|S \cap N_G(p)| > 1$. Then, since G is claw-free, $\exists v \neq u$ such that $S \cap N_G(p) = \{u, v\}$. On the other hand, since $N_G(p) \subseteq N_G(q)$, we may conclude that the set of vertices $X = \{p, u, v, q\}$ induces a C_4 such that $X \cap S = \{u, v\}$. Therefore $S' = (S \setminus X) \cup (X \setminus S)$ is also a maximum stable set of G and thus $N_G(p)$ is an α-redundant subset of vertices.

■

From corollaries 3.3 and 4.1, it is immediate to conclude that we may recognize, in polynomial-time, if a (C_4, P_5)-free or (claw, P_5)-free graph has or not convex-QP stability number. On the other hand, applying theorems 3.2, 3.3, 3.4, 4.2 and 4.3 and corollary 3.1, we are able to introduce a finite sequence convex quadratic programming algorithm for the polynomial-time recognition of claw-free graphs, G, with convex-QP stability number, unless we obtain an induced adverse subgraph. It must be noted that if G is *claw*-free and $\lambda_{min}(A_G) < -2$ then G does not have convex-QP stability number.
In order to recognize if a claw-free graph, G, has or not convex-QP stability number, assuming that $\upsilon_G(\tau) = \alpha(G)$, we may proceed as follows. If $dilw(G) < |V(G)|$ then $\exists u, v \in V(G)$ such that $N_G(u) \subseteq N_G[v]$ and hence if $uv \in E(G)$ then, by theorem 3.5 and corollary 3.1, we may delete the vertex v. Otherwise, by theorem 4.3 we may delete the neighbors of the vertex u. On the other hand, taking into account the corollary 3.2 and theorem 4.2, we may define by a *claw-free-adverse* subgraph of a claw-free graph G, any of its

subgraphs G' without isolated vertices, for which all the equalities (8)-(11) hold with $\tau = 2$, and additionally, such that the following conditions are satisfied:

$$\mathrm{dilw}(G') = |V(G')|, \tag{16}$$

$$\forall v \in V(G') \qquad v \text{ is the midpoint of a } P_5. \tag{17}$$

Note that, according to the above definition, despite the graph G of figure 4 be adverse and claw-free, it is not claw-free-adverse since $\mathrm{dilw}(G) < |V(G)|$. The following algorithm determines, by a convex quadratic programming finite sequence, a maximum stable set of a claw-free graph, G, with $\lambda_{min}(A_G) = -2$ or recognizes that $\upsilon_G(2) > \alpha(G)$ (and then $\alpha(G) \leq \lceil \upsilon_G(2) \rceil - 1$) or determines a claw-free-adverse subgraph. If $\lambda_{min}(A_G) > -2$ and $G \in \mathcal{Q}_2$ then the optimal solution of $(P_G(2))$ is the characteristic vector of the unique maximum stable set of G.

In the algorithm 4.1, $Iso(H)$ will denote the set of isolated vertices of H.

ALGORITHM 4.1. *[***Input***: A connected claw-free graph, G, with $\lambda_{min}(A_G) = -2$.* **Output***: A maximum stable set S of G, or the conclusion that $G \notin \mathcal{Q}_2$ or a claw-free-adverse subgraph G'.]*

– **BEGIN**

1. **If** $\upsilon_G(2) \notin \mathbb{Z}$ **then STOP** *($G \notin \mathcal{Q}_2$)* [1] **else set** $k = 0$, $S_0 = \emptyset$ *and* $G_0 = G$*;*
2. **If** *the optimal solution of* $(P_{G_k}(2))$, x^*, *is such that* $x^* \in \{0,1\}^{|V(G_k)|}$ **then go to** *step 5.;* [2]
3. **while** $\exists q \in V(G_k)$ *such that* $N_{G_k}(q) \subseteq N_{G_k}[p]$ *for some* $p \in V(G_k)$ **do**
   - **begin**
     - **if** $pq \in E(G_k)$ **then set** $S = Iso(G_k - \{p\})$ *and* $G_{k+1} = G_k - (S \cup \{p\})$ [3] **else set** $S = Iso(G^k - N_{G_k}(q))$ *and* $G_{k+1} = G_k - (S \cup N_{G_k}(q))$ [4]*;*
     - **set** $S_{k+1} = S_k \cup S$ *and* $k = k + 1$*;*
   - **end;**
4. **if** $\exists v \in V(G_k)$ *such that* $\upsilon_{G_k}(2) > \upsilon_{G_k - N_{G_k}(v)}(2)$
   - **then**
     - **If** $\upsilon_{G_k}(2) = \upsilon_{G_k - \{v\}}(2)$ **then set** $S = Iso(G^k - \{v\})$, $G_{k+1} = G_k - (S \cup \{v\})$ [5], $S_{k+1} = S_k \cup S$, $k = k + 1$ *and* **go to** *step 2* **else STOP** *($G \notin \mathcal{Q}_2$)* [6]*;*
   - **else**

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[1] It is clear that if $\upsilon_G(2)$ is not integer then $\upsilon_G(2) > \alpha(G)$.

[2] If x^* is integer then x^* is the characteristic vector of a maximum stable set of G_k.

[3] Assuming that $\upsilon_{G_k}(2) = \alpha(G_k)$, according to theorem 3.5 and corollary 3.1, $\alpha(G_k) = \alpha(G_k - \{p\})$.

[4] By theorem 4.3 $N_{G_k}(q)$ is α-redundant.

[5] Since $\upsilon_{G_k}(2) = \upsilon_{G_k - \{v\}}(2) > \upsilon_{G_k - N_{G_k}(v)}(2)$, by theorem 3.4 $\upsilon_{G_k}(2) = \alpha(G_k)$ iff $\upsilon_{G_k - \{v\}}(2) = \alpha(G_k - \{v\})$.

[6] Since $\upsilon_{G_k}(2) > \max\{\upsilon_{G_k - \{v\}}(2), \upsilon_{G_k - N_{G_k}(v)}(2)\}$, by theorem 3.3, $\upsilon_{G_k}(2) > \alpha(G_k)$.

* **if** $\exists w \in V(G_k)$ *such that* w *is not the midpoint of a* P_5 **then set** $S = Iso(G^k - \{w\})$, $G_{k+1} = G_k - (S \cup \{w\})$, [7] $S_{k+1} = S_k \cup S$, $k = k+1$ *and* **go to** *step 2* **else set** $G' = G_k$ *and* **STOP** *(G_k is a claw-free-adverse graph)* [8].

5. **Set** $S = S_k$ *and add to* S *all the vertices which correspond to the support of* x^**;*

6. **If** $\upsilon_G(2) = |S|$ **then** $G \in \mathcal{Q}_2$ *and* S *is a maximum stable set of* G **else** $G \notin \mathcal{Q}_2$*;*

– **END.**

According to (Cardoso, 2001), a connected graph, G, of order greater than one, such that $L(G)$ is not complete, has a perfect matching if and only if $\upsilon_{L(G)}(2) = \alpha(L(G))$. Therefore, since any line graph $L(G)$ is claw-free and $\lambda_{min}(A_{L(G)}) \geq -2$, with the help of algorithm 4.1 we are able to recognize in polynomial-time if a connected graph with at least one edge and without claw-free-adverse subgraphs, has or not a perfect matching.

5. Conclusions and Open Problems

In this paper we have studied a family of quadratic programming problems $(P_G(\tau))$ and the connections of their optimal values with the stability number of G. We proved that this family includes the Motzkin-Straus quadratic program (Motzkin and Straus, 1965). Several necessary and sufficient conditions for the equality $\upsilon_G(\tau) = \alpha(G)$ be fulfilled were analyzed. Despite remaining, as an open problem, the polynomial-time recognition of graphs with convex-QP stability number, we introduce an algorithm, based on solving a finite sequence of quadratic programming problems (assigned to induced subgraphs) which gives such recognition for claw-free graphs or determines a claw-free-adverse subgraph. As a consequence, the recognition of graphs with a perfect matching by a finite sequence convex quadratic programming algorithm becomes available under the absence of claw-free-adverse subgraphs in the corresponding line graphs. It is an open problem to know if there exists an adverse or claw-free-adverse graph without convex-QP stability number. Corollaries 3.3 and 4.1 imply the polynomial-time recognition of (C_4, P_5)-free and (claw, P_5)-free graphs with convex-QP stability number. By theorem 3.6 there are several other types of graphs without induced subgraphs, G', with $dilw(G') \geq \omega(G')$, for which we also may recognize, in polynomial-time, if they have or not convex-QP stability number. This is the case of threshold graphs which are graphs with Dilworth number equal to one (for details about graphs with Dilworth number not greater than two see (Brandstädt et al, 1999)).

[7] Since the equalities (10) hold for G_k, by Corollary 3.2, $\forall ij \in E(G_k)$ ij belongs to a C_4 or i and j are the midpoints of a P_4. Therefore, by theorem 4.2, since w is not the midpoint of a P_5, $\alpha(G_k) = \alpha(G_k - \{w\})$.

[8] In such case, $\upsilon_{G_k}(2) = \upsilon_G(2)$, $\lambda_{min}(A_{G_k}) = \lambda_{min}(A_G)$ and the equalities (10)-(11) and (16)-(17) hold for G_k.

If $\tau \in]1, -\lambda_{min}(A_G)[$, the counterexample presented in figure 3 shows that the existence of a maximum stable set S of G for which (5) is fulfilled is not sufficient to obtain the equality $\upsilon_G(\tau) = \alpha(G)$. However, given an arbitrary graph G and $\tau \in]1, -\lambda_{min}(A_G)[$, it remains as an open problem to know if the condition (5), when fulfilled for every maximum stable set of G, is sufficient for the upper bound $\upsilon_G(\tau)$ on $\alpha(G)$ be attained.

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