Eigenvectors and eigenvalues of graphs with regularity constraints

Domingos M. Cardoso[(1)], Charles Delorme[(2)]
and Paula Rama[(1)]

[(1)] Dep. de Matemática, Univ. Aveiro, 3810-193 Aveiro, Portugal

[(2)] Lab. de Recherche en Informatique, Univ. Paris-Sud, 91405 Orsay, France

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Abstract

Relations between Laplacian eigenvectors and eigenvalues and the existence of almost equitable partitions (which are generalizations of equitable partitions) are presented. Based in some properties of the adjacency eigenvectors of a graph, a necessary and sufficient condition for the graph to be primitive strongly regular is introduced. Furthermore, results about the intersection of (k, τ)-regular sets of regular graphs and a necessary and sufficient condition for a particular subset of vertices of a primitive strongly regular graph be (k, τ)-regular are also introduced.

Keywords: graph eigenvectors and eigenvalues, equitable partitions.

1 Introduction

Along this paper we consider a simple graph G of order n with a set of vertices $V(G)$ and a set of edges $E(G)$. An element of $E(G)$, which has the vertices i and j as endvertices, is denoted by ij. If $v \in V(G)$, then we denote the neighborhood of v by $N_G(v)$, that is, $N_G(v) = \{w : vw \in E(G)\}$. The number of neighbors of $v \in V(G)$ will be denoted by $d_G(v)$ and called, as usually, degree of v. Given a subset of vertices S of a graph G, the vector $x \in \mathbb{R}^V$ with $x_v = 1$ if $v \in S$ and $x_v = 0$ if $v \notin S$ is called the characteristic vector of S. Throughout this paper, A_G will denote the adjacency matrix of the graph G of order $n > 1$, that is, $A_G = (a_{ij})_{n \times n}$ is such that

$$a_{ij} = \begin{cases} 1, & \text{if } ij \in E(G) \\ 0, & \text{otherwise.} \end{cases}$$

Note that A_G is a symmetric matrix and then has n real eigenvalues. The spectrum of a matrix A (that is, the set of its eigenvalues) will be denoted by $\sigma(A)$, and the null space by $Ker(A)$. Throughout the text $\hat{e}$ will denote the all-ones vector with n components and I will denote the identity matrix.

Given a k-partition of $V(G)$, $\pi = (V_1, ..., V_k)$, the matrix P^π whose columns are the characteristic vectors of the subsets $V_1, \ldots, V_k$, will be called the characteristic matrix of π. It is known that a k-partition $\pi = (V_1, ..., V_k)$ of $V(G)$ is *equitable* if for any pair $i, j \in \{1, ..., k\}$ and $\forall v \in V_i$ the number

$$m_{ij} \quad = \quad |N_G(v) \cap V_j| \tag{1}$$

depends only on i and j, i.e, the number of neighbors which a vertex in V_i has in V_j is independent of the choice of the vertex in V_i.
Equitable partitions were introduced in [9], [10] and [11]. In the later one equitable bipartitions are used to obtain information about eigenvalues and eigenvectors of graphs. Equitable partitions appear related with automorphism groups of graphs [6], walk partitions and colorations [8] distance-regular graphs and covering graphs (see [4], for details and further references). Recent applications of equitable partitions may be found in [7] applied to the study of graphs with three eigenvalues and in [12] applied to the study of correlation structure of landscapes.
According to [2], a (k, τ)*-regular set* S of a graph G is a subset of vertices which induces in G a k-regular subgraph such that every vertex out of S has τ neighbors in it. Therefore, if G has a (k, τ)-regular set S then in its complement, $\bar{G}$, S is $(|S| - k - 1, |S| - \tau)$-regular. From the above definition it is immediate that for all $\tau > 0$ a $(0, \tau)$-regular set of a graph is a maximal stable set.
The *quotient graph* G/π of G with respect to the equitable k-partition π is a multi-digraph with the subsets of π, $V_1, \ldots, V_k$, as its vertices and with d_{ij} arcs going from V_i to V_j. From the above definitions we may conclude that a bipartition $\pi = (S, V(G) \setminus S)$ is equitable with

$$A_{G/\pi} \quad = \quad \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}, \tag{2}$$

iff S is (m_{11}, m_{21})-regular and $V(G) \setminus S$ is (m_{22}, m_{12})-regular. If G is a p-regular graph and $S \subset V(G)$ is (k, τ)-regular then $\pi = (S, V(G) \setminus S)$ is an equitable bipartition such that $A_{G/\pi} = \begin{bmatrix} k & p-k \\ \tau & p-\tau \end{bmatrix}$.

We define now an *almost equitable k-partition* of the vertices of G, a k-partition of $V(G)$ $\pi = (V_1, \ldots, V_k)$, such that $\forall i, j \in \{1, \ldots, k\}$ with $i \neq j$

$$\forall x \in V_i \;\; |N_G(x) \cap V_j| = d_{ij}.$$

According to this definition, every equitable partition is almost equitable but the converse, in general, is not true. However, when the graph is regular, the concepts of almost equitable partition and equitable partition are both the same.
Related with an almost equitable k-partition π, let us define also the *generalized Laplacian matrix* L^π as the $k \times k$ matrix such that

$$(L^\pi)_{ij} = \begin{cases} -d_{ij} & \text{if } i \neq j, \\ s_i & \text{otherwise,} \end{cases}$$

where $s_i = \sum_{j \neq i} d_{ij}$. When π is an equitable partition then L^π is the Laplacian matrix of the quotient graph G/π.
It must be noted that every graph admits an almost equitable partition. In fact, every graph of order n admits the almost equitable n-partition $\pi = (V_1, \ldots, V_n)$, where each V_i is a singleton. Another almost equitable partition which every graph admits is the 1-partition $\pi = (V_1)$, where $V_1 = V(G)$. These almost

equitable partitions will be called *trivial almost equitable partitions.*
Whether π is the trivial almost equitable n-partition then $L^\pi = L$, where L is the Laplacian matrix of G. Whether π is the trivial almost equitable 1-partition then $L^\pi = 0$.
As an example, the graph C_5, depicted in Figure 1, admits the almost equitable 3-partition $\pi = (V_1, V_2, V_3)$, where $V_1 = \{1\}$, $V_2 = \{2, 5\}$ and $V_3 = \{3, 4\}$. Therefore, the Laplacian matrix L^π of this almost equitable k-partition is the 3×3 matrix

$$L^\pi = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \end{array} \begin{array}{c} \begin{array}{ccc} V_1 & V_2 & V_3 \end{array} \\ \begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \end{array}.$$

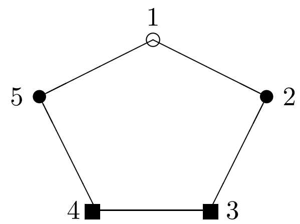

Figure 1: Example of an almost equitable 3-partition of C_5.

2 Basic results

From now on, it is assumed that every (k, τ)-regular set is such that $\tau > 0$.
The next two propositions follows from Lemmas 2.1 and 2.2 of chapter 5 of [4], which give similar results considering equitable partitions and adjacency matrices.

Proposition 1 *Let G be a graph, L its Laplacian matrix, $\pi = (V_1, \ldots, V_k)$ a k-partition of $V(G)$ and P^π the characteristic matrix of π. Then π is an almost equitable k-partition if and only if there is a $k \times k$ matrix B such that*

$$LP^\pi = P^\pi B. \tag{3}$$

If π is an almost equitable k-partition then B is the generalized Laplacian matrix L^π.

Proof. Suppose that π is an almost equitable k-partition. Let us start by considering the matrix product $P^\pi L^\pi$. Suppose that $i \in V_j$. Then the entries of the line $(P^\pi L^\pi)_i$ are given by

$$-d_{j1} \quad \ldots \quad -d_{j,j-1} \quad \sum_{r \neq j} d_{jr} \quad -d_{j,j+1} \quad \ldots \quad -d_{j,k}.$$

Now, let us consider the matrix product LP^π. Suppose again that $i \in V_j$. Then the entries of the line $(LP^\pi)_i$ are given by

$$-|V_1 \cap N_G(i)| \quad \ldots \quad -|V_{j-1} \cap N_G(i)| \quad d_G(i) - |V_j \cap N_G(i)| \quad -|V_{j+1} \cap N_G(i)| \quad \ldots -|V_k \cap N_G(i)|$$

where $d_G(i)$ is the degree of vertex i in G. Taking into account that $d_G(i) - |V_j \cap N_G(i)| = \sum_{r \neq j} d_{jr}$ and $|V_r \cap N_G(i)| = d_{jr}$, for $r \neq j$, then this line is equal to

$$-d_{j1} \quad \dots \quad -d_{j,j-1} \quad \sum_{r \neq j} d_{jr} \quad -d_{j,j+1} \quad \dots \quad -d_{j,k}$$

and thus $LP^\pi = P^\pi L^\pi$.

Conversely, suppose that π is a k-partition of $V(G)$ that satisfies (3). Hence each column of LP^π is the linear combination of the columns of P^π. Consequently, each column of LP^π is constant on the indices correspondent to vertices in the same subsets of π. Finally, for $i \neq j$,

$$(LP^\pi)_{ij} = \sum_{r=1}^{n} L_{ir} P^\pi_{rj} = - \sum_{r \in N_G(i)} P^\pi_{rj} = -|N_G(i) \cap V_j|$$

and $|N_G(i) \cap V_j| = -(LP^\pi)_{ij} = -(LP^\pi)_{tj} = |N_G(t) \cap V_j|$ for every t in the same subset as i, concluding that π is an almost equitable k-partition. ■

Proposition 2 *Let G, L^π, P^π and L as defined in Proposition 1. Then*

$$u \in Ker(L^\pi - \lambda I) \setminus \{0\} \quad \Leftrightarrow \quad P^\pi u \in Ker(L - \lambda I) \setminus \{0\}.$$

Proof.

$$\begin{aligned} u \in Ker(L^\pi - \lambda I) \setminus \{0\} \quad &\Leftrightarrow \quad L^\pi u = \lambda u && (4)\\ &\Leftrightarrow \quad P^\pi L^\pi u = \lambda P^\pi u && (5)\\ &\Leftrightarrow \quad LP^\pi u = \lambda P^\pi u && (6)\\ &\Leftrightarrow \quad P^\pi u \in Ker(L - \lambda I) \setminus \{0\}. && (7) \end{aligned}$$

It must be noted that the equivalence between (4) and (5) follows taking into account that the columns of P^π are linearly independent and the equivalence between (5) and (6) follows from the equality (3). Finally, the equivalence between (6) and (7) follows taking into account that $u \neq 0$ and, again, from the linearly independence of the columns of P^π. ■

The next proposition is a direct consequence of the above ones and can help to find out if there exists an almost equitable partition (which is equitable when the graph is regular) for particular graphs.

Proposition 3 *Let L be the Laplacian matrix of the graph G. If $\pi = (V_1, \dots, V_k)$ is an almost equitable k-partition of G, then the spectra $\sigma(L^\pi)$ and $\sigma(L)$ satisfy $\sigma(L^\pi) \subset \sigma(L)$ and $\forall \lambda \in \sigma(L^\pi)$ $\exists u \in Ker(L - \lambda I) \setminus \{0\}$ such that $\forall j \in \{1, \dots, k\}$*

$$\forall r, s \in V_j \qquad u_r = u_s. \tag{8}$$

Proof. Assuming that L is the Laplacian matrix of the graph G and $\pi = (V_1, \dots, V_k)$ is an almost equitable k-partition of G, then from proposition 2 it follows that $\forall \lambda \in \sigma(L^\pi)$ and $\forall u \in Ker(L^\pi - \lambda I) \setminus \{0\}$

$$LP^\pi u = \lambda P^\pi u.$$

Therefore, $P^\pi u \in Ker(L - \lambda I) \setminus \{0\}$ and, from the definition of the characteristic matrix P^π, it follows that $\forall j \in \{1, \dots, k\}$ and

$$\forall r, s \in V_j \qquad (P^\pi u)_r = (P^\pi u)_s.$$

■

Therefore, if there exists an almost equitable bipartition (that is, a 2-partition) for a graph with Laplacian matrix L, then L has at least two eigenvectors defining (by their subsets of components with constant values) the almost equitable bipartition. More generally, if there exists an almost equitable k-partition then L has k eigenvectors defining such almost equitable k-partition. However, since the Laplacian matrix has always an eigenvector with eigenvalue equal to zero and components with equal value, then the definition of the nontrivial almost equitable k-partition must be obtained from the remaining $k-1$ eigenvectors. As a direct consequence of Proposition 3, we may conclude that there exists graphs without any nontrivial almost equitable partition, as it is the case of the graph of Figure 2. In fact, from the spectrum of the Laplacian matrix of

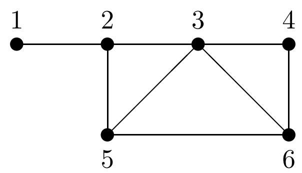

Figure 2: Graph without any nontrivial almost equitable partition.

the graph of Figure 2, $\sigma(L) = \{5.1183, 4.5494, 3.4659, 2.1353, 0.7312, 0\}$, we may conclude that each eigenspace has dimension 1. Therefore if there exists an almost equitable k-partition, with $1 < k < 6$, then it will be defined by at least two of the eigenvectors (that is, there exists two Laplacian eigenvectors with constant values for the same subsets of components). Analyzing the Laplacian eigenvectors

u^1	u^2	u^3	u^4	u^5	u^6
0.0995	0.1040	−0.2376	0.3286	0.8051	1.0000
−0.4097	−0.3692	0.5858	−0.3730	0.2164	1.0000
0.8410	−0.2308	0.1549	−0.1147	−0.1886	1.0000
−0.1808	0.2799	0.3392	0.6718	−0.3949	1.0000
−0.0727	0.6988	−0.1902	−0.5365	−0.1255	1.0000
−0.2774	−0.4827	−0.6521	0.0238	−0.3125	1.0000,

the eigenvector u^6 is the unique eigenvector that has two components with equal value. Therefore, there exists only one almost equitable k-partition, with $k > 1$, which is the trivial almost equitable 6-partition.

Proposition 4 *Let G be a graph with Laplacian matrix L. If G has an almost equitable bipartition (that is, a 2-partition) $\pi = (V_1, V_2)$ such that $\forall x \in V_1$, $|N_G(x) \cap V_2| = \tau_2$ and $\forall x \in V_2$, $|N_G(x) \cap V_1| = \tau_1$, then*

$$\tau_1 + \tau_2 \in \sigma(L) \cap \sigma(L^\pi).$$

Proof. According to the definition of π it follows that $L^\pi = \begin{bmatrix} \tau_2 & -\tau_2 \\ -\tau_1 & \tau_1 \end{bmatrix}$ and then $\sigma(L^\pi) = \{0, \tau_2 + \tau_1\}$. Therefore, by proposition 2, the result follows. ■

For instance, for the graph of Figure 3, $\pi = (\{1,5\},\{2,3,4,6\})$ is an almost equitable bipartition and the eigenvalues of L^π are 0 and $\tau_2 + \tau_1 = 2 + 1 = 3$.

As immediate consequence of the above proposition we have the following corollary.

Corollary 1 *If a graph G of order $n > 2$ has no integer Laplacian eigenvalues different from zero then there are no almost equitable bipartitions in G.*

Therefore, since the graph of Figure 1 has the Laplacian spectrum $\sigma(L) = \{[3.6180]^2, [1.3820]^2, [0.0000]^1\}$ [1] this graph has no almost equitable bipartitions. Nevertheless, there are graphs with an integer Laplacian eigenvalue $\lambda \neq 0$ and equitable bipartitions π, none of them is such that $\lambda \in \sigma(L^\pi)$. This is true for the graph of Figure 3, for which $2 \in \sigma(L) = \{[5.2361]^1, [3]^2, [2]^1, [0.7639]^1, [0]^1\}$ but $2 \notin \sigma(L^\pi) = \sigma(L^{\pi'})$.

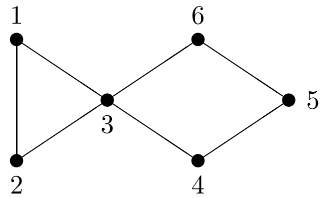

Figure 3: Graph with only two almost equitable bipartitions $\pi = \{\{1,5\},\{2,3,4,6\}\}$ and $\pi' = \{\{2,5\},\{1,3,4,6\}\}$.

3 Intersection of (k,τ)-regular sets in regular graphs

As a consequence of Propositions 3 and 4 we have the following result, which was previously obtained in [13].

Proposition 5 *Let G be a p-regular graph, $S \subset V(G)$. There are $k \in \mathbb{N} \cup \{0\}$ and $\tau \in \mathbb{N}$, with $k - \tau = \lambda$, such that S is (k,τ)-regular if and only if $\exists u \in Ker(A_G - \lambda I)$ such that*

$$u_i = \begin{cases} (1 - \frac{n}{|S|}) & \text{if} \quad i \in S, \\ 1 & \text{otherwise.} \end{cases} \tag{9}$$

Proof.

($\Rightarrow$) Let $S \subset V(G)$ be (k,τ)-regular and $\lambda = k - \tau$. Since G is a p-regular graph, then $\pi = (S, V(G) \setminus S)$ is an equitable bipartition such that $\forall x \in S$, $|N_G(x) \cap (V(G) \setminus S)| = p - k$ and $\forall x \in V(G) \setminus S$, $|N_G(x) \cap S| = \tau$. Therefore, according to Proposition 4, $p-k+\tau \in \sigma(L) \cap \sigma(L^\pi)$ and, according to Proposition 3, $\exists u \in Ker(L-(p-k+\tau)I) \setminus \{0\}$ such that $u_i = \gamma\ \forall i \in S$ and $u_j = \beta\ \forall j \in V(G) \setminus S$. On the other hand, the p-regularity of G implies that $p-(p-k+\tau) = k-\tau \in \sigma(A_G)$, $u \in Ker(A_G - (k-\tau)I) \setminus \{0\}$ and also that

$$0 = \sum_{j \in V(G)} u_j = \sum_{j \in S} \gamma + \sum_{j \in V(G)\setminus S} \beta = \gamma|S| + \beta(n - |S|).$$

Therefore, $\gamma = (1 - \frac{n}{|S|})\beta$ and, without loss of generality, setting $\beta = 1$ the result follows.

[1] The exponent denotes the multiplicity of the eigenvalue.

($\Leftarrow$) Consider $u \in Ker(A_G - \lambda I)$ such that (9) holds. Then

$$A_G u = \lambda u$$

$$\Updownarrow$$

$$\sum_{j \in N_G(i)} u_j = \lambda u_i \ \forall i \in V(G)$$

$$\Updownarrow$$

$$(1 - \frac{n}{|S|})|N_G(i) \cap S| + |N_G(i) \cap (V(G) \setminus S)| = \begin{cases} \lambda(1 - \frac{n}{|S|}), & \text{if } i \in S \\ \lambda, & \text{if } i \notin S \end{cases}$$

$$\Updownarrow$$

$$|N_G(i) \cap S| + |N_G(i) \cap (V(G) \setminus S)| - \frac{n}{|S|}|N_G(i) \cap S| = \begin{cases} \lambda(1 - \frac{n}{|S|}), & \text{if } i \in S \\ \lambda, & \text{if } i \notin S \end{cases}$$

$$\Updownarrow$$

$$|N_G(i) \cap S| = \begin{cases} \frac{|S|}{n}(p - \lambda(1 - \frac{n}{|S|})), & \text{if } i \in S \\ \frac{|S|}{n}(p - \lambda), & \text{if } i \notin S \end{cases}$$

$$\Updownarrow$$

$$|N_G(i) \cap S| = \begin{cases} \frac{|S|}{n}(p - \lambda) + \lambda, & \text{if } i \in S \\ \frac{|S|}{n}(p - \lambda), & \text{if } i \notin S. \end{cases}$$

Therefore, setting $k = \frac{|S|}{n}(p-\lambda)+\lambda$ and $\tau = \frac{|S|}{n}(p-\lambda)$, it follows that $k-\tau = \lambda$.

■

As a consequence of Proposition 5 and taking into account that according to [5], section 9.3, if G has an equitable partition $\pi = (V_1, ..., V_k)$ then for every adjacency eigenvector $v \in Ker(A_G - \lambda I)$, with $\lambda \in \sigma(A_G) \backslash \sigma(A_{G/\pi})$, $\sum_{j \in V_i} v_j = 0 \ \ \forall i \in \{1, ..., k\}$, we have the following result (obtained also in [13], using a different approach).

Proposition 6 *If G is a p-regular graph with the (k_1, τ_1)-regular set S_1 and the (k_2, τ_2)-regular set S_2, with $k_1 - \tau_1 \neq k_2 - \tau_2$, then*

$$|S_1 \cap S_2| = \frac{n\tau_1\tau_2}{(p - (k_1 - \tau_1))(p - (k_2 - \tau_2))}. \tag{10}$$

Proof. According to [5], if $u \in Ker(A_G - \lambda I) \setminus \{0\}$, with $\lambda \neq k_1 - \tau_1$, then $\sum_{v \in S_1} u_v = 0$. Therefore, choosing the eigenvector u, defined in Proposition 5 by S_2, that is, such that

$$u_v = \begin{cases} (1 - \frac{n}{|S_2|}) & \text{if} \quad v \in S_2, \\ 1 & \text{otherwise,} \end{cases}$$

it follows that
$0 = \sum_{j \in S_1} u_j = \sum_{j \in S_1 \cap S_2} u_j + \sum_{j \in S_1 \setminus S_2} u_j = |S_1 \cap S_2| \frac{|S_2| - n}{|S_2|} + |S_1 \setminus S_2| = \frac{|S_2||S_1| - n|S_1 \cap S_2|}{|S_2|}$
Therefore, $|S_1 \cap S_2| = \frac{|S_1||S_2|}{n}$ and, taking into account that from [13], if S_i is a (k_i, τ_i)-regular set of a p-regular graph of order n then

$$|S_i| = \frac{n\tau_i}{p - (k_i - \tau_i)}, \tag{11}$$

we get (10). ■

According to [1], a graph $G \neq P_2$ has a perfect matching if and only if its line graph $L(G)$ has a $(0,2)$-regular set. Furthermore, we may conclude the following result about perfect induced matchings, that is, induced matchings covering all edges of the graph.

Proposition 7 *A graph G has the perfect induced matching $IM \subset E(G)$ if and only if $L(IM)$ is a $(0,1)$-regular set of $L(G)$.*

Proof. Supposing that $IM \subset E(G)$ is a perfect induced matching of G then it is immediate that $L(IM) \subset V(L(G))$ is a $(0,1)$-regular set of $L(G)$. Conversely, let $L(X) \subset V(L(G))$ be a $(0,1)$-regular set, then $X \subset E(G)$ is a matching and $\forall v \notin L(X)$, $|N_{L(G)}(v) \cap L(X)| = 1$, that is, every edge not in X is adjacent to one and only one edge of X and then X is a perfect induced matching. ■

As a consequence of the above results and taking into account (11) we have the following corollary.

Corollary 2 *If a p-regular graph G, with $|E(G)| = m$, has a perfect matching M and a perfect induced matching IM then*

- $|IM| = |L(IM)| = \frac{m}{2p-1}$;
- $|M \cap IM| = \frac{m}{p(2p-1)}$.

A strongly regular graph with parameters $(n, p; a, c)$ is a non-null p-regular graph $(0 < p < n-1)$ such that each pair of adjacent vertices has a common neighbors and each pair of nonadjacent vertices has c common neighbors. It is well known that a connected strongly regular graph has diameter two and three distinct adjacency eigenvalues (noting that any graph with diameter d has at least $d+1$ distinct eigenvalues). One of these adjacency eigenvalues is p and the other two, usually called *restricted adjacency eigenvalues*, are such that (see, for instance, [4])

$$\lambda = \frac{a - c \pm \sqrt{\Delta}}{2}, \tag{12}$$

with $\Delta = (a-c)^2 + 4(p-c)$.
It follows some strongly regular graphs which are examples of graphs with a (k_1, τ_1)-regular set and a (k_2, τ_2)-regular set such that $k_1 - \tau_1 \neq k_2 - \tau_2$.

- The Petersen Graph, P, that is, the strongly regular graph with parameters $(10, 3; 0, 1)$, has a $(0,2)$-regular set S_1 of size 4 and a $(2,1)$-regular set S_2 of size 5. Therefore, according to proposition 6, $|S_1 \cap S_2| = \frac{20}{(3+2)(3-1)} = 2$.
- The Hoffman-Singleton graph, that is, the strongly regular graph with parameters $(50, 7; 0, 1)$, has a $(0,3)$-regular set S_0 of size 15 and the vertex set can be partitioned into five $(3,1)$-regular sets S_j of size 10 (inducing Petersen graphs). Therefore, $|S_0 \cap S_j| = \frac{150}{(7+3)(7-2)} = 3$, for $j = 1, \ldots, 5$, that is, the maximum stable set of a Hoffman-Singleton graph has three vertices in each of the five induced Petersen graphs in which the vertex set can be partitioned.

The line graph of P has a $(0,2)$-regular set $L(M_1)$ and a $(0,1)$-regular set $L(M_2)$. Therefore M_1 is a perfect matching and, according to Proposition 7, M_2 is a perfect induced matching of P. Furthermore, according to Corollary 2, $|M_2| = \frac{15}{6-1} = 3$ and $|M_1 \cap M_2| = \frac{15}{3(6-1)} = 1$.
For the fourth graph of Moore, F, that is, the strongly regular graph with parameters $(3250, 57; 0, 1)$ and spectrum $\sigma(A_F) = \{[-8]^{1520}, [7]^{1729}, [57]^1\}$, if it contains a $(0,8)$-regular set S_1 (and then its size is 400) and a (k_2, τ_2)-regular set S_2 with $k_2 - \tau_2 = 7$ then, from Proposition 6,

$$|S_1 \cap S_2| = \frac{26\ 000\tau_2}{(57+8)(57-(k_2-\tau_2))} = \frac{50\tau_2}{57-(k_2-\tau_2)} = \tau_2.$$

On the other hand, from (11)

$$|S_2| = 65\tau_2 \tag{13}$$

and, from [2], if G is a graph with a (k_1, τ_1)-regular set S_1 and a (k_2, τ_2)-regular set S_2, then

$$\tau_2|S_1 \setminus S_2| - \tau_1|S_2 \setminus S_1| = |S_1 \cap S_2|(k_1 - k_2)$$

which is equivalent to

$$\begin{aligned} |S_1 \cap S_2|k_2 = |S_2 \setminus S_1|8 - |S_1 \setminus S_2|\tau_2 \quad &\Leftrightarrow \quad \tau_2 k_2 = (|S_2| - \tau_2)8 - (|S_1| - \tau_2)\tau_2 \\ &\Leftrightarrow \quad \tau_2(k_2 - \tau_2 + 8 + 400) = |S_2|8 \\ &\Leftrightarrow \quad |S_2| = \frac{415}{8}\tau_2. \end{aligned} \tag{14}$$

Therefore, comparing (13) with (14), if there exists a strongly regular graph with parameters $(3250, 57; 0, 1)$ with a stable set of size 400, then it has no (k_2, τ_2)-regular set with $k_2 - \tau_2 \neq -8$.

4 Eigenvectors of strongly regular graphs

According to [4], a strongly regular graph G is *primitive* if both G and its complement $\bar{G}$ are connected, otherwise it is called *imprimitive*. A strongly regular graph with parameters $(n, p; a, c)$ is imprimitive if and only if $c = p$ or $c = 0$ (see [4], pag. 178).
Denoting the distance between the vertices x and y in a graph G by $d_G(x, y)$, it is known [4] that G is distance-regular if and only if the distance partition $V_0, V_1, \ldots, V_d$ is equitable, where fixing $v \in V(G)$, $V_i = \{x \in V(G) : d_G(v, x) = i\}$, for $i = 1, \ldots, d$, and d is the diameter of G. Therefore, since a strongly regular graph is a distance-regular graph with diameter 2, we may conclude the following proposition.

Proposition 8 *A graph G is strongly regular with parameters $(n, p; a, c)$ if and only if $\forall k \in V(G)$, denoting $N_G[k] = N_G(k) \cup \{k\}$, there exists the equitable 3-partition $\pi_k = (V_0, V_1, V_2)$, where $V_0 = \{k\}$, $V_1 = N_G(k)$ and $V_2 = V(G) \setminus N_G[k]$, for which*

$$L^{\pi_k} = \begin{array}{c c} & \begin{array}{ccc} V_0 & V_1 & V_2 \end{array} \\ \begin{array}{c} V_0 \\ V_1 \\ V_2 \end{array} & \left(\begin{array}{ccc} p & -p & 0 \\ -1 & p-a & -(p-a-1) \\ 0 & -c & c \end{array} \right) \end{array}.$$

Let G be a p-regular graph, with Laplacian matrix L, then

$$A_G v = \lambda v \quad \Leftrightarrow \quad Lv = (pI - A_G)v = (p - \lambda)v. \tag{15}$$

If G is strongly regular, with parameters $(n, p; a, c)$, then, from Proposition 8, $\forall k \in V(G)$ there exists the equitable 3-partition $\pi = (\{k\}, N_G(k), V(G) \setminus N_G[k])$ and, by Proposition 2,

$$\sigma(L^\pi) \subseteq \sigma(L).$$

Therefore, since from (15) $\lambda \in \sigma(L)$ iff $p - \lambda \in \sigma(A_G)$, we may conclude that

$$\sigma(pI - L^\pi) \subseteq \sigma(A_G),$$

with

$$pI - L^\pi = \begin{bmatrix} 0 & p & 0 \\ 1 & a & p-a-1 \\ 0 & c & p-c \end{bmatrix}.$$

In fact, since the characteristic polynomial of $pI - L^\pi$ has the same roots of the ones obtained from the parameters of the strongly regular graph G with parameters $(n, p; a, c)$, we may conclude that

$$\sigma(pI - L^\pi) \quad = \quad \sigma(A_G). \tag{16}$$

Now we have the following proposition.

Proposition 9 *Let G be a connected graph of order n with at least one edge, $p \in \mathbb{N}$ and $\lambda \in \mathbb{N} \setminus \{p\}$ an adjacency eigenvalue of G. Then G is a primitive strongly regular graph with parameters $(n, p; a, c)$ if and only if $\forall k \in V(G)$ $\exists v^k \in Ker(A_G - \lambda I) \setminus \{0\}$ such that*

$$v_i^k \quad = \quad \begin{cases} 1 & \text{if} \quad i = k, \\ \frac{\lambda}{p} & \text{if} \quad i \in N_G(k), \\ \frac{c\lambda}{p(\lambda-(p-c))} & \text{if} \quad i \in V(G) \setminus N_G[k], \end{cases} \tag{17}$$

with $p \neq c \neq 0$.

Proof.

($\Rightarrow$) Let G be a primitive strongly regular graph with parameters $(n, p; a, c)$. Then $p \neq c \neq 0$ and, according to Proposition 8, there exists an equitable 3-partition $\pi = (\{k\}, N_G(k), V(G) \setminus N_G[k])$, where $k \in V(G)$, such that

$$L^\pi = \begin{bmatrix} p & -p & 0 \\ -1 & p-a & -(p-a-1) \\ 0 & -c & c \end{bmatrix}.$$

Let $\lambda \neq p$ be an eigenvalue of A_G. Therefore, denoting by L the Laplacian matrix of G, taking into account (15), it follows that $p - \lambda = \beta \in \sigma(L)$ and, by (16), $\beta \in \sigma(L^\pi)$ with $\beta \neq 0$. If u is the eigenvector of L^π corresponding to β then

$$L^\pi u = \beta u \quad \Leftrightarrow \quad \left\{ \begin{array}{rcrcrcl} pu_1 & - & pu_2 & & & = & \beta u_1, \\ -u_1 & + & (p-a)u_2 & - & (p-a-1)u_3 & = & \beta u_2, \\ & - & cu_2 & + & cu_3 & = & \beta u_3. \end{array} \right. \tag{18}$$

If $u_1 = 0$ then, according to (18), $u = 0$. Thus, we may conclude that $u_1 \neq 0$ and, without loss of generality, we may assume that $u_1 = 1$. As a consequence, $u_2 = \frac{p-\beta}{p}$ and $u_3 = \frac{c(p-\beta)}{p(c-\beta)}$. Note that, $c - \beta = 0 \Rightarrow p - \beta = 0$ and then $p = c$, which implies that G is imprimitive, contradicting the hypothesis.
Being P^π the characteristic matrix of π, from Proposition 2, it follows that $v^k = P^\pi u$ is an eigenvector of L, corresponding to the eigenvalue β and such that $v^k_k = 1$, $v^k_i = \frac{p-\beta}{p}$ $\forall i \in N_G(k)$ and $v^k_j = \frac{c(p-\beta)}{p(c-\beta)}$ $\forall j \in V(G) \setminus N_G[k]$.
Finally, according to (15), the vector v^k is also an eigenvector of A_G corresponding to the eigenvalue $\lambda = p - \beta$ and then $v^k_i = \frac{\lambda}{p}$ $\forall i \in N_G(k)$ and $v^k_j = \frac{c\lambda}{p(\lambda-(p-c))}$ $\forall j \in V(G) \setminus N_G[k]$.

($\Leftarrow$) Suppose that $\forall k \in V(G)$ $\exists v^k \in Ker(A_G - \lambda I) \setminus \{0\}$ as described in (17). First, note that $\lambda \neq 0$ otherwise v^k will be 0 in all its components except for $v^k_k = 1$ and $\forall i \in \{1, \ldots, n\}$ $(A_G v^k)_i = 0$ which leads to k being an isolated vertex contradicting the hypothesis that G is connected.
Now, let us prove that G is strongly regular.

1. From $(A_G v^k)_k = \lambda v^k_k \Leftrightarrow \sum\limits_{j \in N_G(k)} v^k_j = \lambda \Leftrightarrow |N_G(k)| \frac{\lambda}{p} = \lambda \Leftrightarrow |N_G(k)| = p$ we conclude that G is p-regular.
2. Let $k, j \in V(G)$ with $kj \in E(G)$. Hence $j \in N_G(k)$ and
$$(A_G v^k)_j = \lambda v^k_j \Leftrightarrow 1 + \sum_{i \in N_G(k) \cap N_G(j)} \frac{\lambda}{p} + \sum_{i \in (V(G) \setminus N_G[k]) \cap N_G(j)} \frac{c\lambda}{p(\lambda - (p-c))} = \frac{\lambda^2}{p}.$$
Let $a = |N_G(k) \cap N_G(j)|$. Then the last equality is equivalent to
$$1 + a\frac{\lambda}{p} + (p - a - 1)\frac{c\lambda}{p(\lambda - p + c)} = \frac{\lambda^2}{p}$$
$$\Updownarrow$$
$$\left(\frac{c\lambda}{p(\lambda - p + c)} - \frac{\lambda}{p}\right) a = 1 + (p-1)\frac{c\lambda}{p(\lambda - p + c)} - \frac{\lambda^2}{p}$$
$$\Updownarrow$$
$$a\lambda(c - (\lambda - p + c)) = p(\lambda - p + c) + (p-1)c\lambda - \lambda^2(\lambda - p + c)$$
$$\Updownarrow$$
$$a = \frac{(p - \lambda^2)(\lambda - p + c) + (p-1)c}{(p - \lambda)}$$
concluding that for $kj \in E(G)$ $a = |N_G(k) \cap N_G(j)|$ does not depend on the chosen vertices k and j.
3. Let $k, j \in V(G)$ with $k \neq j$ and $kj \notin E(G)$. Hence $j \in V(G) \setminus N_G[k]$ and
$$(A_G v^k)_j = \lambda v^k_j \Leftrightarrow \sum_{i \in N_G(k) \cap N_G(j)} \frac{\lambda}{p} + \sum_{i \in (V(G) \setminus N_G[k]) \cap N_G(j)} \frac{c\lambda}{p(\lambda - p + c)} = \frac{c\lambda^2}{p(\lambda - p + c)}.$$
Let $q = |N_G(k) \cap N_G(j)|$. Then the last equality is equivalent to
$$q\frac{\lambda}{p} + (p - q)\frac{c\lambda}{p(\lambda - p + c)} = \frac{c\lambda^2}{p(\lambda - p + c)}$$
$$\Updownarrow$$
$$q = \frac{c(\lambda - p)}{(\lambda - p)} = c$$
concluding that for $kj \notin E(G)$ $|N_G(k) \cap N_G(j)| = c$ does not depend on the chosen vertices k and j.

Therefore, according to 1, 2 and 3, G is a strongly regular graph with parameters $(n, p; a, c)$. Finally, the primitivity comes immediately from the hypothesis ($0 \neq c \neq p$).

■

As immediate consequence of Proposition 9 we have the following Corollary.

Corollary 3 *Let S be a maximum stable set of a primitive strongly regular graph G, with parameters $(n, p; a, c)$. Then*

$$\forall k \in S \qquad \alpha(G) = \sum_{j \in N_G(k)} \frac{|N_G(j) \cap S|}{c} - \frac{p-c}{c}.$$

Proof. Consider that $k \in S$ and that v is the eigenvector of A_G corresponding to the restricted eigenvalue λ which, according to the Proposition 9, is such that $v_k = 1$, $v_i = \frac{\lambda}{p}$ $\forall i \in N_G(k)$ and $v_j = \frac{c\lambda}{p(\lambda - p + c)}$ $\forall j \in V(G) \setminus N_G[k]$. Let us suppose also that x is the characteristic vector of S. Then

$$\begin{aligned}
x^T A_G v = \lambda x^T v \quad &\Leftrightarrow \quad \sum_{j \notin S} |N_G(j) \cap S| v_j = \lambda \sum_{j \in S} v_j \\
&\Leftrightarrow \quad \sum_{j \in N_G(k)} |N_G(j) \cap S| \frac{\lambda}{p} + \sum_{j \notin S \cup N_G(k)} |N_G(j) \cap S| \frac{c\lambda}{p(\lambda - (p-c))} \\
&\qquad = \lambda + (\alpha(G) - 1) \frac{c\lambda^2}{p(\lambda - (p-c))} \\
&\Leftrightarrow \quad \sum_{j \in N_G(k)} |N_G(j) \cap S| + \frac{c}{\lambda - (p-c)} \sum_{j \notin S \cup N_G(k)} |N_G(j) \cap S| \\
&\qquad = p + (\alpha(G) - 1) \frac{c\lambda}{\lambda - (p-c)} \\
&\Leftrightarrow \quad \sum_{j \in N_G(k)} |N_G(j) \cap S| \left(1 - \frac{c}{\lambda - (p-c)}\right) + \frac{c}{\lambda - (p-c)} \sum_{j \notin S} |N_G(j) \cap S| \\
&\qquad = p + (\alpha(G) - 1) \frac{c\lambda}{\lambda - (p-c)} \\
&\Leftrightarrow \quad \frac{\lambda - p}{\lambda - (p-c)} \sum_{j \in N_G(k)} |N_G(j) \cap S| + \frac{cp}{\lambda - (p-c)} \alpha(G) \\
&\qquad = \frac{p(\lambda - (p-c)) + c\lambda(\alpha(G) - 1)}{\lambda - (p-c)} \\
&\Leftrightarrow \quad (\lambda - p) \sum_{j \in N_G(k)} |N_G(j) \cap S| + cp\alpha(G) \\
&\qquad = c\lambda\alpha(G) + \lambda(p-c) - p(p-c) \\
&\Leftrightarrow \quad \alpha(G)c(p - \lambda) = (p - \lambda) \sum_{j \in N_G(k)} |N_G(j) \cap S| - (p-\lambda)(p-c) \\
&\Leftrightarrow \quad \alpha(G) = \sum_{j \in N_G(k)} \frac{|N_G(j) \cap S|}{c} - \frac{p-c}{c}.
\end{aligned}$$

■

Taking into account the Corollary 3, if a maximum stable set S of a primitive strongly regular graph G, with parameters $(n, p; a, c)$, is $(0, \tau)$-regular then

$$\alpha(G) \quad = \quad \frac{p(\tau - 1) + c}{c}. \tag{19}$$

On the other hand, if $T \subset V(G)$ is $(0,\tau)$-regular and λ_1 and λ_2 are the restricted eigenvalues of G, with $\lambda_2 < 0 < \lambda_1$, then it is known [2] that $\lambda_2 = -\tau$ and also that $c - p = \lambda_1\lambda_2$. Therefore, according to Proposition 2.4 in [3], T is a maximum stable set and thus (19) implies

$$|T| = \frac{p(\tau-1)+c}{c} \Leftrightarrow \alpha(G) = -\lambda_2\frac{p-\lambda_1}{c}.$$

Proposition 10 *Let G be a primitive strongly regular graph with parameters $(n,p;a,c)$ and $\lambda \in \sigma(A_G)\setminus\{p\}$. If there exists $S \subset V(G)$ such that*

$$\forall v \in Ker(A_G - \lambda I) \;\; \sum_{j\in S} v_j = 0,$$

then S is a (k,τ)-regular set, with

$$k = \frac{c\lambda(|S|-1)+pc}{\lambda(p-\lambda)} - \frac{p}{\lambda} \text{ and } \tau = \frac{c}{p-\lambda}|S|.$$

Proof. Let $S \subset V(G)$ be such that $\forall v \in Ker(A_G - \lambda I)$ $\sum_{i\in S} v_i = 0$. Consider an eigenvector v of A_G, as defined in (17), that is, such that

$$v_i = \begin{cases} 1 & \text{if} \quad i = x, \\ \frac{\lambda}{p} & \text{if} \quad i \in N_G(x), \\ \frac{c\lambda}{p(\lambda-(p-c))} & \text{if} \quad i \in V(G)\setminus N_G[x], \end{cases}$$

- If $x \in S$ then

$$\begin{aligned} 0 &= \sum_{j\in S} v_j \\ &= 1 + \sum_{j\in N_G(x)\cap S}\frac{\lambda}{p} + \sum_{j\in S\setminus N_G[x]}\frac{c\lambda}{p(\lambda-(p-c))} \\ &= \frac{p}{\lambda} + |N_G(x)\cap S| + (|S| - |N_G[x]\cap S|)\frac{c}{\lambda-p+c} \\ &= \frac{p}{\lambda} - \frac{c}{\lambda-p+c} + |N_G(x)\cap S|\frac{\lambda-p}{\lambda-p+c} + |S|\frac{c}{\lambda-p+c} \\ &= |N_G(x)\cap S|(\lambda-p) + \frac{c\lambda(|S|-1)+p(\lambda-p+c)}{\lambda} \\ &\Updownarrow \\ |N_G(x)\cap S| &= \frac{c\lambda(|S|-1)+pc-p(p-\lambda)}{\lambda(p-\lambda)} \\ &= \frac{c\lambda(|S|-1)+pc}{\lambda(p-\lambda)} - \frac{p}{\lambda}. \end{aligned}$$

- If $x \in V(G)\setminus S$ then

$$\begin{aligned} 0 &= \sum_{j\in S} v_j \\ &= \sum_{j\in N_G(x)\cap S}\frac{\lambda}{p} + \sum_{j\in S\setminus N_G(k)}\frac{c\lambda}{p(\lambda-(p-c))} \\ &= |N_G(x)\cap S| + (|S| - |N_G(x)\cap S|)\frac{c}{\lambda-(p-c)} \\ &= |N_G(x)\cap S|\frac{\lambda-p}{\lambda-p+c} + |S|\frac{c}{\lambda-p+c} \end{aligned}$$

$$
\begin{aligned}
&= |N_G(x) \cap S|(\lambda - p) + |S|c. \\
&\Updownarrow \\
|N_G(x) \cap S| &= \frac{c}{p-\lambda}|S|.
\end{aligned}
$$

■

In [5], section 9.3, considering a graph G with an equitable partition $\pi = (V_1, \ldots, V_k)$ and an adjacency eigenvector $v \in Ker(A_G - \lambda I)$, with $\lambda \in \sigma(A_G) \setminus \sigma(A_{G/\pi})$, then $\sum_{j \in V_i} v_j = 0 \;\; \forall i \in \{1, ..., k\}$. This result, combined with Proposition 10, allow us to conclude a necessary and sufficient condition for a subset of vertices of a primitive strongly regular graph to be (k, τ)-regular.

Corollary 4 *Let G be a primitive strongly regular graph with parameters $(n, p; a, c)$ and $\lambda \in \sigma(A_G) \setminus \{k - \tau, p\}$. Then $S \subset V(G)$ is (k, τ)-regular if and only if $\forall v \in Ker(A_G - \lambda I)$,*

$$\sum_{j \in S} v_j = 0.$$

Furthermore, if S is (k, τ)-regular then $k = \frac{c\lambda(|S|-1)+pc}{\lambda(p-\lambda)} - \frac{p}{\lambda}$ and $\tau = \frac{c}{p-\lambda}|S|$.

References

[1] Barbosa. R., Cardoso, D. M. *On regular-stable graphs.* To appear in Ars Combinatoria.

[2] Cardoso, D. M. and P. Rama, *Spectral results on regular graphs with (k, τ)-regular sets.* Universidade de Aveiro. Cadernos de Matemática CM02/I22 (2002): 14 p.

[3] Cardoso, D. M. and P. Rama, *Equitable bipartions of graphs and related results.* Journal of Mathematical Sciences, special volume - Aveiro Seminar on Control, Optimization and Graph Theory, 120 (2004), 869-880.

[4] Godsil, C. D., *Algebraic Combinatorics.* Chapman & Hall Mathematics Series, New York (1993).

[5] Godsil, C. D. and G. Royle, *Algebraic Graph Theory.* Springer-Verlag, New York (2001).

[6] McKay, G. J., *Backtrack programming and the graph isomorphism problem.* M. Sc. Thesis, University of Melbourne, (1976).

[7] Muzychuk, M. and M. Klin, *On graphs with three eigenvalues.* Discrete Mathematics, 189 (1998): 191–207.

[8] Powers, D. L. and M. M. Sulaiman, *The walk partition and colorations of a graph.* Linear Algebra and Its Applications, 48 (1982): 145–159.

[9] Sachs, H., *Über Teiler, Faktoren und Charakteristische Polynome von Graphen.* Teil I. Wiss. Z. TH Ilmenau 12 (1966): 7–12.

[10] Sachs, H., *Über Teiler, Faktoren und Charakteristische Polynome von Graphen.* Teil II. Wiss. Z. TH Ilmenau 13 (1967): 405–412.

[11] Schwenk, A. J., *Computing the characteristic polynomial of a graph*. In R. Bari and Frank Harary, editors. Graphs and Combinatorics, Springer, Berlin, (1974): 153–172.

[12] Stadler, P. F. and G. Tinhofer, *Equitable partitions, coherent algebras and random walks: Applications to the correlation structure of landscapes.* MATCH - Communications in Mathematics and in Computer Chemistry, 40 (1999): 215–261.

[13] Thompson, D. M., *Eigengraphs: constructing strongly regular graphs with block designs*. Utilitas Math., 20 (1981): 83-115.