On the maximum cardinality of k-regular induced subgraphs

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Outline

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2. **Maximum size regular induced subgraphs.**

3. **Some applications.**

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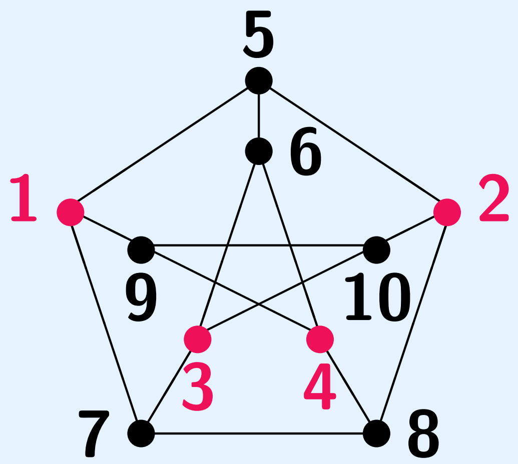
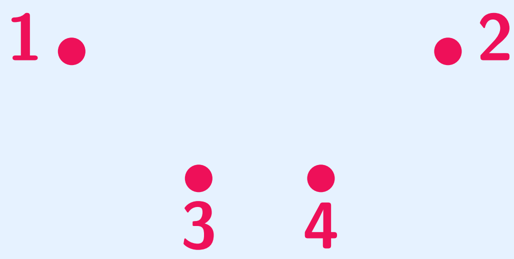
1. Introduction

Many combinatorial problems can be formulated as the determination of maximum size k-regular induced subgraphs.

- **A maximum stable set is 0-regular;**
- **A maximum induced matching is 1-regular;**
- **A maximum clique is $(\omega(G) - 1)$-regular;**
- **An Hamiltonian cycle of a graph G is a maximum cardinality 2-regular induced connected subgraph of $L(G)$;**
- **etc.**

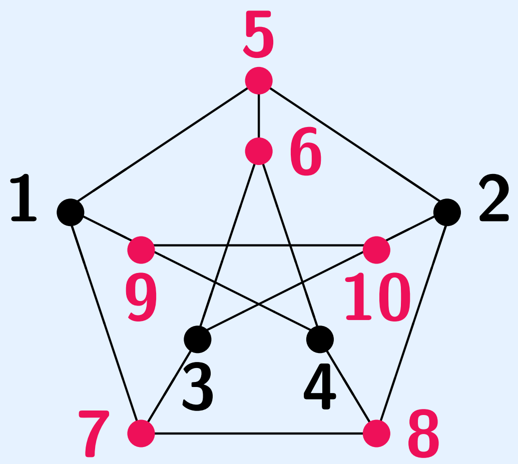
1. Introduction (cont.)

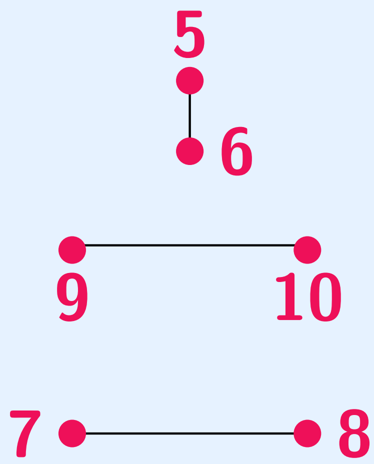




The vertex subset $S_0 = \{1, 2, 3, 4\}$ induces a 0-regular subgraph.

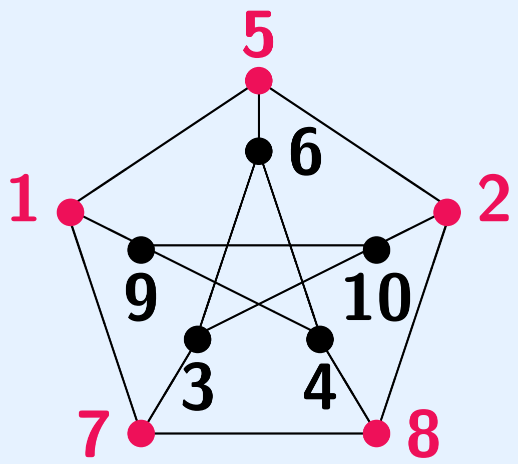
1. Introduction (cont.)

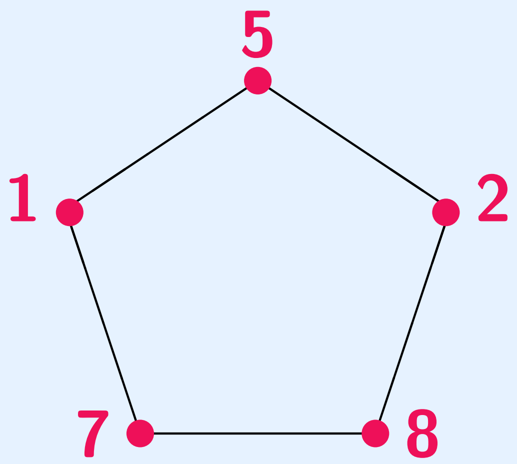




The vertex subset $S_1 = \{5, 6, 7, 8, 9, 10\}$ induces a 1-regular subgraph.

1. Introduction (cont.)





The vertex-subsets $S_2 = \{1, 2, 5, 7, 8\}$ induces a 2-regular subgraph.

1. Introduction (cont.)

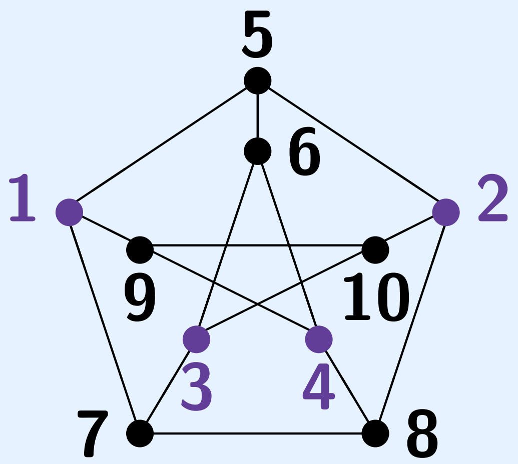
- **As it is well known the determination of a maximum cardinality k-regular induced subgraph is NP-hard;**

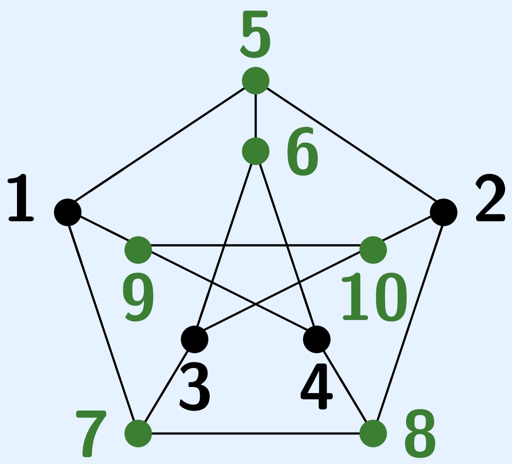
- **It is crucial to obtain polynomial-time upper bounds;**

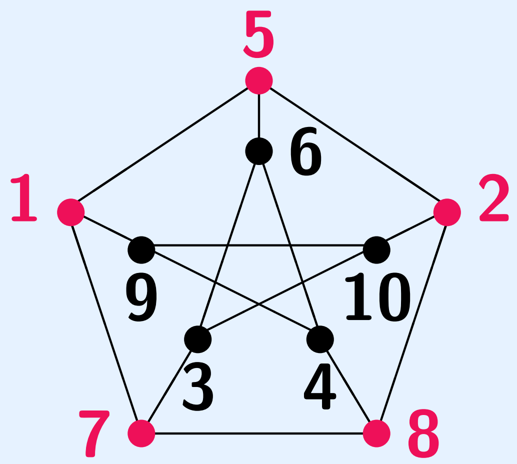
- **A vertex subset $S \subseteq V(G)$ is (k,τ)-regular if induces a k-regular subgraph and**

$$\forall v \notin S \qquad |N_G(v) \cap S| = \tau.$$

1. Introduction (cont.)







$S_0 = \{1, 2, 3, 4\}$ is $(0, 2)$-regular;

$S_1 = \{5, 6, 7, 8, 9, 10\}$ is $(1, 3)$-regular;

$S_2 = \{1, 2, 5, 7, 8\}$ is $(2, 1)$-regular.

2. Maximum size regular induced subgraphs

Let G be a graph of order n with at least one edge, then

$$\upsilon_k(G) = \max_{x \geq 0} 2\hat{e}^T x - \frac{\tau}{k+\tau} x^T \left(\frac{A_G}{\tau} + I_n\right) x,$$

- **where $\hat{e}$ is the all-ones vector;**
- **A_G is the adjacency matrix of G and $\tau = -\lambda_{min}(A_G)$;**
- **I_n is the identity matrix and k is a scalar;**

is a convex quadratic program.

2. Maximum size regular induced subgraphs(cont.)

Let G be a simple graph of order n with at least one edge and $-\lambda_{min}(A_G) = \tau$. If $S \subseteq V(G)$ induces a subgraph of G such that $\bar{d}_{G[S]} = k$ (where $\bar{d}_H = \frac{1}{|V(H)|}\sum_{v \in V(H)} d_H(v)$), then the following properties hold.

- $|S| \leq \upsilon_k(G)$;
- $|S| = \upsilon_k(G)$ if and only if S induces a k-regular subgraph and

$$\tau + k \leq |N_G(v) \cap S| \;\; \forall v \notin S.$$

2. Maximum size regular induced subgraphs(cont.)

Assuming that G is regular, then

- $v_k(G) = n\frac{k-\lambda_{min}(A_G)}{\lambda_{max}(A_G)-\lambda_{min}(A_G)}$;
- $|S| = v_k(G)$ **if and only if** S **is** $(k, k+\tau)$**-regular.**

For the Petersen graph G (where $\lambda_{min}(A_G) = -2$),

$$v_1(G) = 10\frac{1-(-2)}{3-(-2)} = 6$$

Note that $S_1 = \{5, 6, 7, 8, 9, 10\}$ is $(1, 3)$-regular and induces the maximum induced matching $M = \{56, 9(10), 78\}$.

3. Some applications

As immediate consequence of the previous results, considering a regular graph G with $\tau = -\lambda_{min}(A_G)$, we have the following extension of the Hoffman bound: If $S \subseteq V(G)$ induces a k-regular subgraph, then

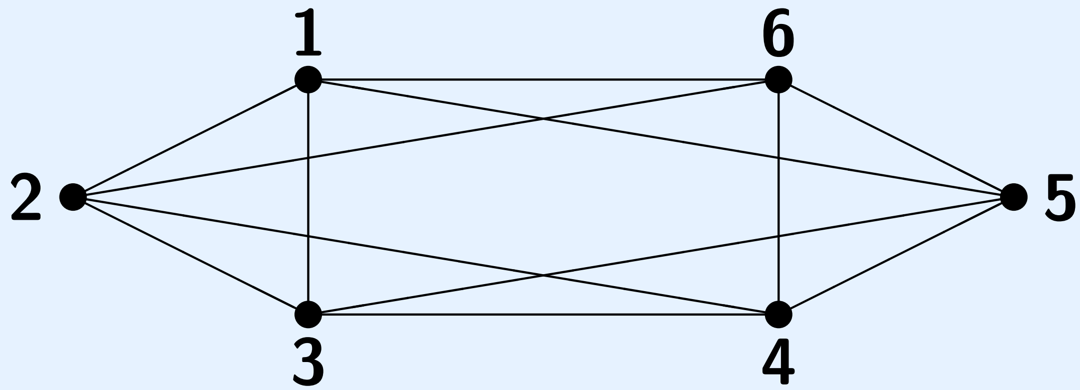
$$|S| \leq n \frac{k - \lambda_{min}(A_G)}{\lambda_{max}(A_G) - \lambda_{min}(A_G)}.$$

Furthermore,

$$|S| = n \frac{k - \lambda_{min}(A_G)}{\lambda_{max}(A_G) - \lambda_{min}(A_G)}$$

iff S is $(k, k + \tau)$-regular.

3. Some applications(cont.)



For this 4-regular graph G (where $\lambda_{min}(A_G) = -2$)

$$v_2(G) = 6\frac{2-(-2)}{4-(-2)} = 4.$$

Note that $S = \{1,3,4,6\}$ is $(2,4)$-regular.

3. Some applications(cont.)

- If $g(G)$ is the girth of a p-regular graph G, with $p > 1$,

then

$$g(G) \leq n \frac{2 - \lambda_{min}(A_G)}{p - \lambda_{min}(A_G)}.$$

- Another consequence:

Let G be an arbitrary graph (regular or nonregular) of order n with at least one edge and $\tau = -\lambda_{min}(A_G)$.

If $S \subset V(G)$ is $(k, k + \tau)$-regular then S is a maximum cardinality set inducing a k-regular subgraph of G.

3. Some applications(cont.)

The Hoffman-Singleton graph G is a strongly regular graph with parameters $(50, 7; 0, 1)$ (for which $\lambda_{min}(A_G) = -3$). If $S \subset V(G)$ is a stable set and $M \subset E(G)$ is an induced matching, then

$|S| \le 50\frac{0-(-3)}{7-(-3)} = 15.$

$2|M| \le 50\frac{1-(-3)}{7-(-3)} = 20 \Rightarrow |M| \le 10$. The equality holds iff $V(M)$ is $(1, 4)$-regular.

4. Final remarks

The Moore graph G of valency 57 if there exists (so far it is open) is a strongly regular graph with parameters $(3250, 57; 0, 1)$ (for which $\lambda_{min}(A_G) = -8$).

If $S \subset V(G)$ is a stable set and $M \subset E(G)$ is an induced matching, then

$$|S| \leq 3250\frac{0-(-8)}{57-(-8)} = 400.$$

$2|M| \leq 3250\frac{1-(-8)}{57-(-8)} = 450 \Rightarrow |M| \leq 225$. **The equality holds iff $V(M)$ is $(1, 9)$-regular.**

4. Final remarks(cont.)

The recognition of (k, τ)-regular sets in regular graphs can be done using the following result obtained in (Thompson, 1981).

- A p-regular graph has a (k, τ)-regular set S, with $k < p$, iff $k - \tau$ is an adjacency eigenvalue and

$$(p - k + \tau)x(S) - \tau\hat{e},$$

is a $(k - \tau)$-eigenvector.

- D. M. Thompson, Eigengraphs: constructing strongly regular graphs with block designs. Utilitas Math., 20 (1981): 83-115.