A spanning star forest model for the diversity problem in automobile industry

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Summary

1. **Problem description.**

2. **The minimum arc cost sum spanning star forest model.**

3. **The complexity of the problem.**

4. **An algorithmic strategy.**

5. **Numerical example.**

1. Problem description

Cars are purchased with a set of active options:

- **airbags;**
- **air conditioned;**
- **radio car;**
- **etc;**

which varies from one client to another.

1. Problem description (cont.)

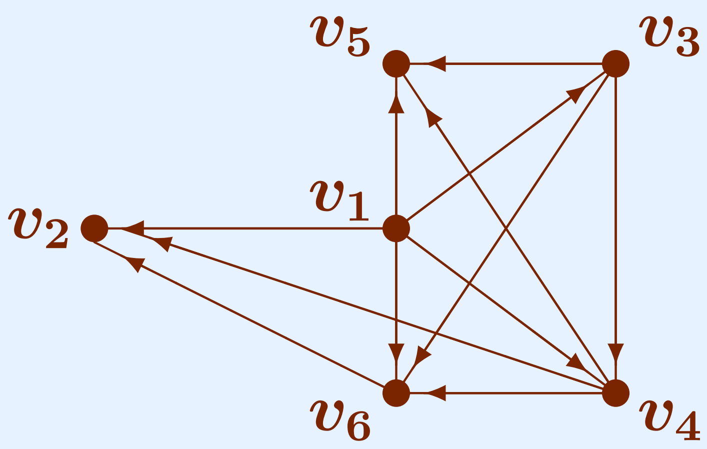
- A car has an active option connection if the car has all the necessary material for the connection.
- Each unit may be produced with more active option connections than the necessary ones.
- If a car has an active option connection which not became active, then there is cost.
- The global cost of option connections is very high in automobile industry.
- For technical reasons, it is not possible to produce a large variety of different option connection configurations.

1. Problem description (cont.)

- **Then the main goals are:**
  - **to optimize the diversity of option connection configurations;**
  - **to minimize the cost supply of produced configurations, fulfilling the market requirements;**
- **The universe of configurations is determined according to the statistic data about costumer preferences and selling expectations.**

2. The minimum arc cost sum spanning star forest

- **Consider the inclusion relation configurations digraph $DG = (V, A)$.**



DG

- **Each vertex is a configuration (a set of active option connections) and each arc $(i, j) \in A$ means that the configuration i includes configuration j.**

2. The minimum arc cost sum spanning star forest (cont.)

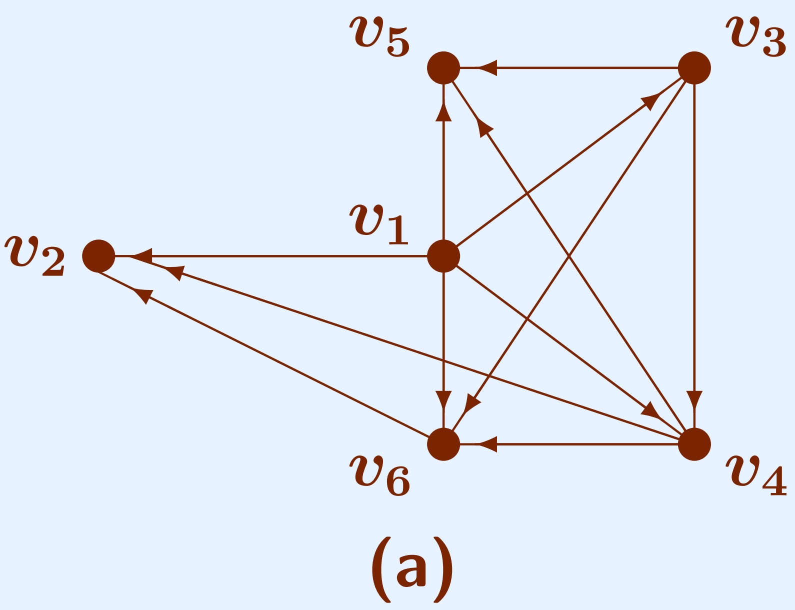
- Assuming that c_i and c_j are the unit production costs of both configurations and also that it is expected to sell n_j configurations j, each arc (i, j) has the cost

$$c_{ij} = n_j(c_i - c_j).$$

- The inclusion relation configurations digraph has the following properties: each cost c_{ij} is positive, the arcs are transitive and it is acyclic.
- The goal is to determine a minimum arc cost sum spanning star forest with no more than k star centers.

3. The minimum arc cost sum spanning star forest (cont.)

A spanning star forest with centers in $\{v_1, v_3\}$.



3. The minimum arc cost sum spanning star forest (cont.)

- **The analytic model:**

$$\min \sum_{(i,j)\in A} c_{ij}x_{ij}$$

$$\text{s. t.} \quad \sum_{i\in\delta^-(j)} x_{ij} + y_j = 1, j \in V \tag{5}$$

$$x_{ij} \le y_i, (i,j) \in A \tag{6}$$

$$\sum_{j\in V} y_j \le k \tag{7}$$

$$x_{ij}, y_p \in \mathbb{Z}_0^+,\ p \in V, (i,j) \in A, \tag{8}$$

where k is the number of allowed configurations.

3. The minimum arc cost sum spanning star forest(cont.)

- **Constraints (5) assure that each type of configuration is produced or replaced by another type of configuration.**
- **Constraints (6) assure that if configuration of type i becomes available to supply a configuration of type j, then configuration of type i must be produced.**
- **Constraint (7) assures that there are no more than k of configurations produced.**

3. The minimum arc cost sum spanning star forest (cont.)

- **Considering the model without the y variables:**

$$\min \sum_{(i,j)\in A} c_{ij}x_{ij} \tag{14}$$

$$\text{s. t.} \sum_{j\in V}\sum_{i\in\delta^-(j)} x_{ij} = n - k \tag{15}$$

$$\sum_{i\in\delta^-(j)} x_{ij} \leq 1, j \in V \tag{16}$$

$$\sum_{i\in\delta^-(j)} x_{ij} + x_{jp} \leq 1, j \in V, p \in \delta^+(j) \tag{17}$$

$$x_{ij} \in \mathbb{Z}_0^+, (i,j) \in A. \tag{18}$$

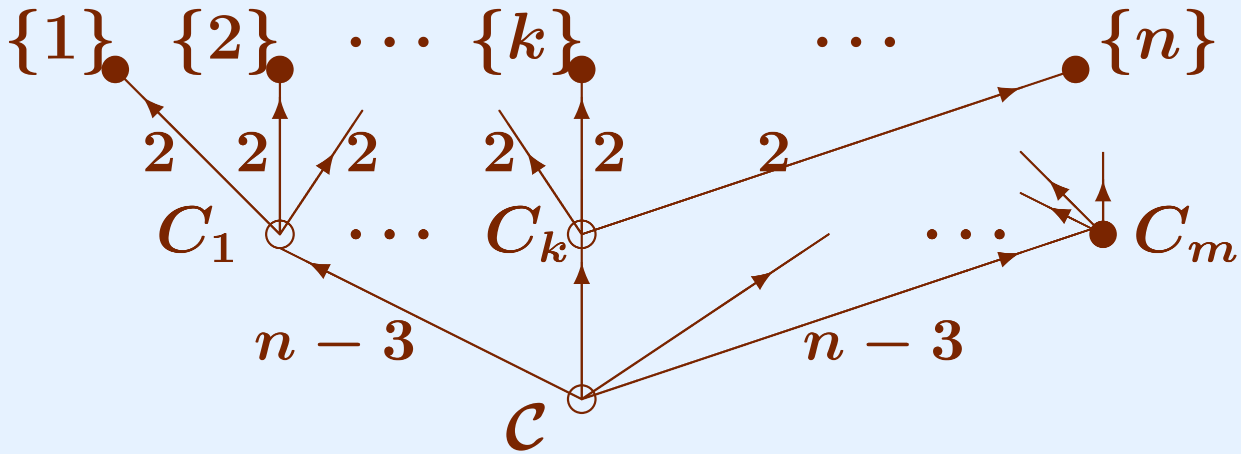
3. Complexity of the problem

- The 3-minimum cover $(C, \mathcal{F}, k)$ asks whether a collection $\mathcal{F}$ of 3-subsets of C contains a cover of C, $\mathcal{F}' \subset \mathcal{F}$, of size not greater than k.

- The 3-minimum cover $(C, \mathcal{F}, k)$ is NP-complete.

- The 3-minimum cover $(C, \mathcal{F}, k)$, with $C = \{1, \ldots, n\}$, has a solution if and only if the digraph $DG = (V, A)$ $(V = \{C\} \cup \mathcal{F} \cup \{1\} \cup \ldots \cup \{n\} \ \wedge \ (x, y) \in A \Leftrightarrow x \supseteq y)$ has a spanning star forest with $k + 1$ stars and arc cost sum not greater than $L = 2n + (|\mathcal{F}| - k)(n - 3)$.

3. Complexity of the problem (cont.)



Comparability Diagram

$$(C, \mathcal{F}, k) = (\{1, \ldots, n\}, \{C_1, \ldots, C_k, C_{k+1}, \ldots, C_m\}, k)$$

$$L = 2n + (|\mathcal{F}| - k)(n - 3)$$

4. An algorithmic strategy

- **The minimum arc cost sum spanning star forest of DG with index star centers in $I = \{i_1, \ldots, i_p\}$ is denoted by $SSF^*(I)$ and has the following tableau:**

	v_1	v_2	$\ldots$	v_{n-1}	v_n	
i_1	$c_{i_1 1}$	$c_{i_1 2}$	$\ldots$	$c_{i_1 n-1}$	$c_{i_1 n}$	$\sum_{j:i_j^*=i_1} c_{i_1 j}$
$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$	$\vdots$
i_p	$c_{i_p 1}$	$c_{i_p 2}$	$\ldots$	$c_{i_p n-1}$	$c_{i_p n}$	$\sum_{j:i_j^*=i_p} c_{i_p j}$
z	z_1	z_2	$\ldots$	z_{n-1}	z_n	$\sum_{j=1}^{n} z_j$
stc	i_1^*	i_2^*	$\ldots$	i_{n-1}^*	i_n^*	

4. An algorithmic strategy (cont.)

- **An index subset I is feasible if, fixing the vertex subset with indices in I as star centers, there is an arc from a vertex v_i, with $i \in I$, to each vertex v_j, with $j \in \{1, \dots, n\} \setminus I$.**
- **Each feasible index subset defines a spanning star forest $SSF^*(I)$.**
- **Since the inclusion relation configuration digraph DG defines the poset $(V(DG), \supseteq)$, the subset of maximal elements determines the minimum cardinality index set I_0 defining a spanning star forest $SSF^*(I_0)$.**

4. An algorithmic strategy (cont.)

- **Consider the $SSF^*(I)$ tableau in the following reduced form:**

	v_1	$\dots$	v_j	$\dots$	v_n	
z	z_1	$\dots$	z_j	$\dots$	z_n	$\sum_{j=1}^{n} z_j$
stc	i_1	$\dots$	i_j	$\dots$	i_n	

where each index i_j is equal to the index vertex center of the star to which belongs the vertex v_j.

- **Therefore, $I = \bigcup_{j=1}^{n} \{i_j\}$.**

4. An algorithmic strategy (cont.)

- Algorithm (input: DG, k);
  1. Determine $I_0 = \{i : d^-_{DG}(v_i) = 0\}$;
  2. Set $q \leftarrow 0$ and determine the $SSF^*(I_0)$ reduced tableau:

	v_1	$\dots$	v_j	$\dots$	v_n	
z^0	z_1^0	$\dots$	z_j^0	$\dots$	z_n^0	$z_0^0 = \sum_{j=1}^n z_j^0$
	i_1^0	$\dots$	i_j^0	$\dots$	i_n^0	

4. An algorithmic strategy (cont.)

3. While $|I_q| < k$ do

3.1 **Determine** $i^* \in \{1, \ldots, n\} \setminus I_q$ **such that**

$$\sum_{j=1}^{n} \min\{z_j^q, c_{i^*j}\} = \min_{i \in \{1,\ldots,n\} \setminus I_q} \sum_{j=1}^{n} \min\{z_j^q, c_{ij}\}$$

and set $z_0^{q+1} \leftarrow \sum_{j=1}^{n} \min\{z_j^q, c_{i^*j}\}$;

3.2 **Update** $SSF^*(I_q)$, **such that** for $j = 1$ **to** n **do**

- $z_j^q \leftarrow \min\{z_j^{q-1}, c_{i^*j}\}$;
- If $c_{i^*j} < z_j^{q-1}$ then $i_j^q \leftarrow i^*$;

3.3 **Set** $I_q \leftarrow \bigcup_{j=1}^{n} \{i_j^q\}$ **and** $q \leftarrow |I_q|$;

5. Numerical example

Let us apply the above algorithm to a digraph with arc cost matrix

$$C = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{pmatrix} 0 & 4 & 2 & 3 & 2 & 4 \\ \infty & 0 & \infty & \infty & \infty & \infty \\ \infty & \infty & 0 & 2 & 2 & 3 \\ \infty & 1 & \infty & 0 & 1 & 1 \\ \infty & \infty & \infty & \infty & 0 & \infty \\ \infty & 3 & \infty & \infty & \infty & 0 \end{pmatrix} \end{array},$$

considering $k = 3$.

5. Numerical example (cont.)

1. $I_0 \leftarrow \{1\}$;
2.

	v_1	v_2	v_3	v_4	v_5	v_6	
z^0	0	4	2	3	2	4	$z_0^0 = 15$
stc	1	1	1	1	1	1	

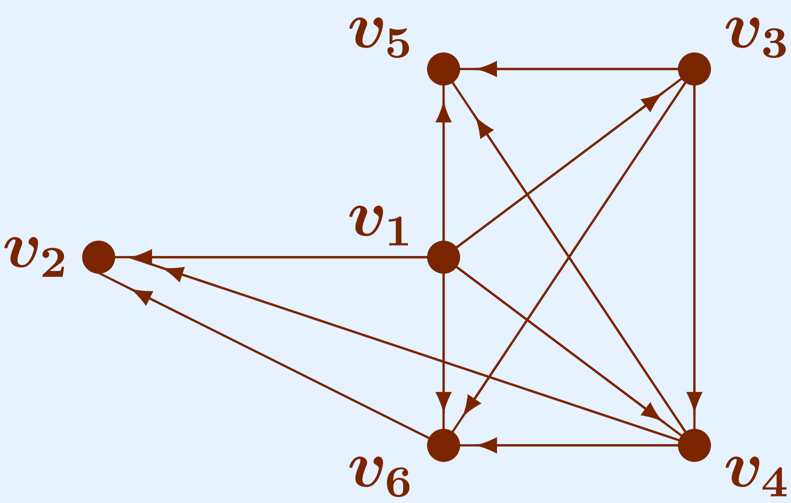
3. $|I_0| = |\{1\}| = 1 < 3$;
   $(i^* = 4)$ $\min\{0, \infty\} + \min\{4, 1\} + \min\{2, \infty\} + \min\{3, 0\} + \min\{2, 1\} + \min\{4, 1\} = 5$;

5. Numerical example (cont.)

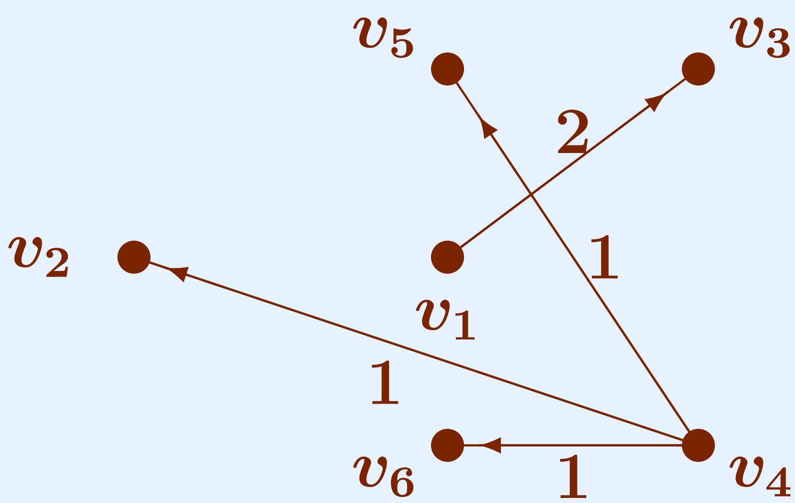
- $z_0^1 \leftarrow \sum_{j=1}^{6} \min\{z_j^1, c_{4j}\} = 5$;
- $I_1 \leftarrow \{1, 4\}$;

	v_1	v_2	v_3	v_4	v_5	v_6	
z^1	0	1	2	0	1	1	$z_0^1 = 5$
stc	1	4	1	4	4	4	

5. Numerical example (cont.)



(a)



(b)

3. $|I_1| = |\{1, 4\}| = 2 < 3$;

5. Numerical example (cont.)

$(i^* = 3)\ \min\{0, \infty\} + \min\{1, \infty\} + \min\{2, 0\} + \min\{0, 2\} + \min\{1, 2\} + \min\{1, 3\} = 3;$

- $z_0^2 \leftarrow \sum_{j=1}^{6} \min\{z_j^2, c_{3j}\} = 3;$
- $I_2 \leftarrow \{1, 3, 4\};$

	v_1	v_2	v_3	v_4	v_5	v_6	
z^1	0	1	0	0	1	1	$z_0^1 = 3$
stc	1	4	3	4	4	4	

5. Numerical example (cont.)

3. $|I_2| = |\{1, 3, 4\}| = 3.$