Graphs whose stability number is easily determined

Domingos M. Cardoso
Universidade de Aveiro, Portugal

Conference on Graph Theory
80th Birthday of Professor Horst Sachs,
Ilmenau, March 27-30, 2007

1. **Introduction**.

2. **A Motzkin-Straus like approach**.

3. **$\mathcal{Q}$-graphs and its recognition**.

4. **Related properties and extensions**.

5. **References**.

1. Introduction

In this presentation we deal with simple graphs (just called graphs) G and the main subject is the **stability number** ($\alpha(G)$) and the **maximum stable set problem** (MSSP).

- Given a nonnegative integer k, to determine if a graph G has a stable set of size k is NP-hard (Karp, 1972).

- Furthermore, considering H-free graphs, if H contains **a)** a cycle, or **b)** a vertex of degree more than three, or **c)** two vertices of degree three in the same connected component, then the MSSP is NP-hard in the class of H-free graphs (Alekseev, 1982).

There are several classes of graphs for which the maximum stable set problem can be solved in polynomial time, for example:

- Claw-free graphs, which includes the line-graphs [(Berge, 1957), (Minty, 1980), (Sbihi, 1980)].

- Particular subclasses of P_5-free graphs [(Mosca, 1997), (Mosca, 1999)], including :
  - $(P_5, K_{1,m})$-free graphs;
  - $(P_5, K_{2,3})$-free graphs;
  - (P_6, C_4)-free graphs.

- etc.

The focus is the class of graphs whose stability number is determined by solving a convex quadratic programming problem ($\mathcal{Q}$-graphs). The results will be presented crossing the following topics:

- Connections of the above convex quadratic program with the Motzkin-Straus quadratic formulation of the stability number.

- Characterization of $\mathcal{Q}$-graphs and analysis of its recognition.

- Graph eigenvalue properties of particular $\mathcal{Q}$-graphs.

- Extensions to the more general case of the maximum size k-regular induced subgraph problem.

By graph eigenvalues we mean (here) adjacency eigenvalues. Where, as usually, the adjacency matrix of a graph G of order n is a $n \times n$ matrix $A_G = \left(a_{ij}\right)$ such that

$$a_{ij} = \begin{cases} 1, & \text{if } ij \in E(G) \\ 0, & \text{otherwise.} \end{cases}$$

Thus A_G is symmetric and it has n real eigenvalues

$$\lambda_{max}(A_G) = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = \lambda_{min}(A_G).$$

If G has at least one edge, then

$$\lambda_{min}(A_G) \leq -1.$$

In fact,

$$\lambda_{min}(A_G) = -1$$

if and only if each component of G is complete.

2. A Motzkin-Straus like approach

Consider a graph G and the quadratic program

$$f(G) = \max\{\frac{1}{2}x^T A_G x : x \in \Delta\},$$

where $\Delta = \{x \geq 0 : \hat{e}^T x = 1\}$.

Theorem 1 (Motzkin-Straus, 1965) *If G is a graph with clique number $\omega(G)$, then*

$$f(G) = \frac{1}{2}(1 - \frac{1}{\omega(G)}). \qquad (1)$$

Therefore, from (1) and after some algebraic manipulation,

$$\frac{1}{\alpha(G)} = \min_{x \in \Delta} x^T (A_G + I)x. \qquad (2)$$

Now, let us consider the families of quadratic programs (with $\tau > 0$):

$$\nu_G(\tau) = \min_{x\in\Delta} x^T(\frac{A_G}{\tau} + I)x, \quad (3)$$

$$\upsilon_G(\tau) = \max_{y\geq 0} 2\hat{e}^T y - y^T(\frac{A_G}{\tau} + I)y \quad (4)$$

Then $\nu_G(1)$ is the modified quadratic formulation of Motzkin-Straus (2).

Theorem 2 (C, 2003) *If* x^* *and* y^* *are optimal solutions for* (3) *and* (4), *respectively, then*

$$\frac{x^*}{\nu_G(\tau)} \text{ and } \frac{y^*}{\upsilon_G(\tau)}$$

are optimal solutions of (4) *and* (3), *respectively. Furthermore,* $\upsilon_G(\tau) = \frac{1}{\nu_G(\tau)}$.

As consequence of this theorem, $\upsilon_G(1) = \alpha(G)$.

The family of quadratic programs

$$v_G(\tau) = \max_{y \geq 0} 2\hat{e}^T y - y^T (\frac{A_G}{\tau} + I) y,$$

has the following properties (for all $\tau > 0$):

- $\alpha(G) \leq v_G(\tau)$.
- $1 \leq v_G(\tau) \leq n$.
- $v_G(\tau) = 1$ if and only if G is complete, and $v_G(\tau) = n$ if and only if G has no edges.

Furthermore, assuming that $E(G) \neq \emptyset$, the quadratic programs are convex for $\tau \geq -\lambda_{min}(A_G)$ (the convex quadratic program, obtained with $\tau = -\lambda_{min}(A_G)$, was firstly introduced as an upper bound for $\alpha(G)$ in (Luz, 1995)).

The function $\upsilon_G :]0, +\infty[\mapsto [1, n]$ verifies:

- $0 < \tau_1 < \tau_2 \Rightarrow \upsilon_G(\tau_1) \leq \upsilon_G(\tau_2)$.
- $\exists \tau^* \geq 1$ such that $\upsilon_G(\tau) = \alpha(G)\ \forall \tau \in]0, \tau^*]$.
- $\forall U \subset V(G)\ \upsilon_{G-U}(\tau) \leq \upsilon_G(\tau)$.

Theorem 3 (Luz, 1995) *Let G be a graph with at least one edge. Then $\upsilon_G(-\lambda_{min}(A_G)) = \alpha(G)$ if and only if for a stable set $S \subset V(G)$ (and then for all)*

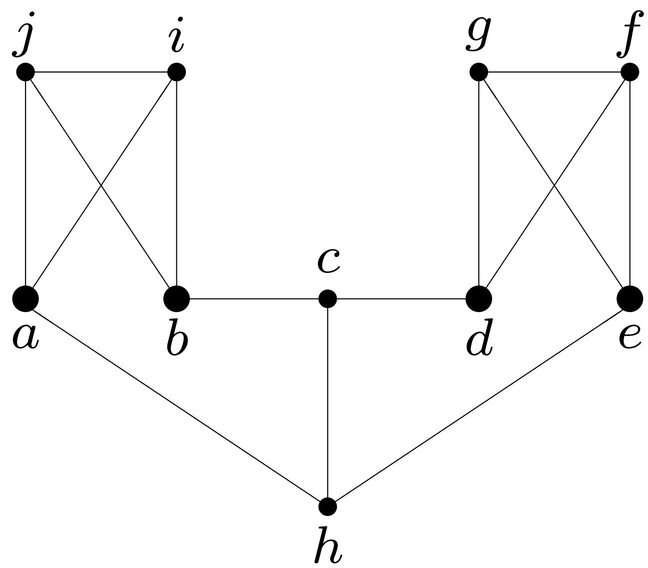
$$-\lambda_{min}(A_G) \leq |N_G(v) \cap S|\ \ \forall v \in V(G) \setminus S.$$

A graph G with at least one edge such that $\upsilon(-\lambda_{min}(A_G)) = \alpha(G)$ is designated graph with **convex QP-stability number**, where QP means quadratic program.

For instance, the cubic graph G depicted in the next figure is such that $\lambda_{min}(A_G) = -2$ and

$$\upsilon_G(2) = 4 = \alpha(G).$$

Therefore, it has convex-QP stability number.



From now on the graphs with **convex** QP-**stability number** are denoted $\mathcal{Q}$-**graphs** and $\upsilon_G(\tau)$, with $\tau = -\lambda_{min}(A_G)$, is simple denoted $\upsilon(G)$.

3. $\mathcal{Q}$-graphs and its recognition

The class of $\mathcal{Q}$-graphs is not hereditary (it is not closed under vertex deletion) (Lozin and C, 2001). However, if G is a $\mathcal{Q}$-graph and $\exists U \subseteq V(G)$ such that $\alpha(G) = \alpha(G - U)$, then $G - U$ is a $\mathcal{Q}$-graph.

There exists an infinite number of $\mathcal{Q}$-graphs (C, 2001):

- A connected graph with at least one edge, which is nor a star neither a triangle, has a perfect matching if and only if its line graph is a $\mathcal{Q}$-graph.

- If each component of G has a nonzero even number of edges then $L(L(G))$ is a $\mathcal{Q}$-graph.

Among several famous $\mathcal{Q}$-graphs we have the **Petersen** graph and the **Hoffman-Singleton** graph.

The following results (C, 2001) can be used on the recognition of $\mathcal{Q}$-graphs.

- Every graph G has an induced $\mathcal{Q}$-subgraph H such that $\alpha(H) = \alpha(G)$.

- A graph G is a $\mathcal{Q}$-graph if and only if each of its components is a $\mathcal{Q}$-graph.

- If $\exists U \subseteq V(G)$ such that $\upsilon(G) = \upsilon(G - U)$ and $\lambda_{min}(A_G) < \lambda_{min}(A_{G-U})$, then G is a $\mathcal{Q}$-graph.

- If $\exists v \in V(G)$ such that

$$\upsilon(G) \neq \max\{\upsilon(G-v), \upsilon(G-N_G(v))\},$$

then G is not a $\mathcal{Q}$-graph.

- Consider that $\exists v \in V(G)$ such that

$$\upsilon(G-v) \neq \upsilon(G-N_G(v)).$$

1. If $\upsilon(G) = \upsilon(G-v)$ then G is a $\mathcal{Q}$-graph if and only if $G-v$ is a $\mathcal{Q}$-graph.

2. If $\upsilon(G) = \upsilon(G-N_G(v))$ then G is a $\mathcal{Q}$-graph if and only if $G-N_G(v)$ is a $\mathcal{Q}$-graph.

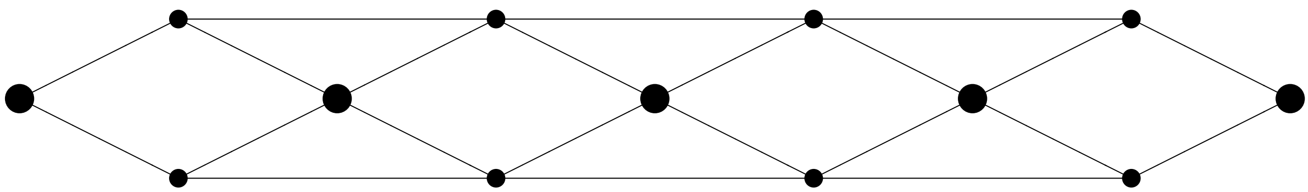
Thus, we have problems when $\forall v \in V(G)$ $\upsilon(G) = \upsilon(G-v) = \upsilon(G-N_G(v))$ and $\lambda_{min}(A_G) = \lambda_{min}(A_{G-v}) = \lambda_{min}(A_{G-N_G(v)})$.

The above results allow the recognition of $\mathcal{Q}$-graphs, except for **adverse graphs**, which are graphs having an induced subgraph G without isolated vertices such that $\upsilon(G)$ is integer and $\forall v \in V(G)$ the following conditions hold:

1. $\upsilon(G) = \upsilon(G - N_G(v))$.

2. $\lambda_{min}(A_G) = \lambda_{min}(A_{G-N_G(v)})$.

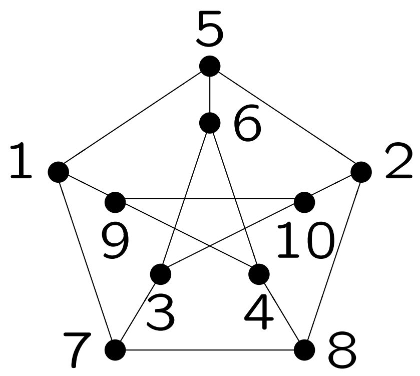
The graph G depicted in the next figure is an adverse graph (which is a $\mathcal{Q}$-graph, since $\upsilon(G) = 5 = \alpha(G)$).



A vertex subset $S \subseteq V(G)$ is (k, τ)-regular if induces a k-regular subgraph and

$$\forall v \notin S \ \ |N_G(v) \cap S| = \tau.$$

For instance, consider the Pertersen graph.

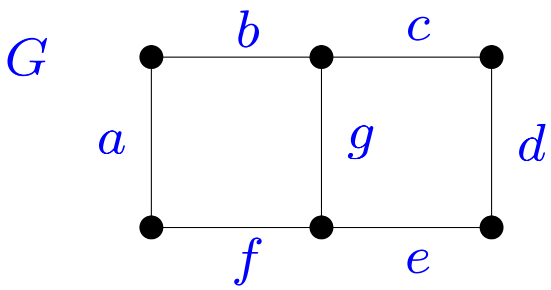


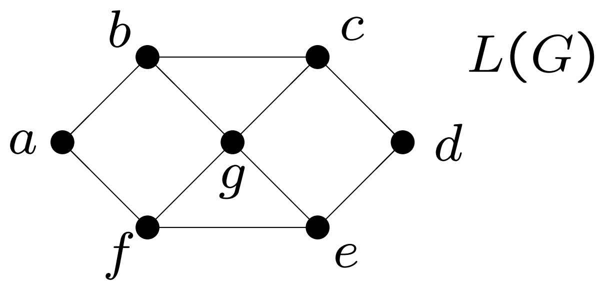
- $S_1 = \{1, 2, 3, 4\}$ is $(0, 2)$-regular,

- $S_2 = \{5, 6, 7, 8, 9, 10\}$ is $(1, 3)$-regular,

- $S_3 = \{1, 2, 5, 7, 8\}$ is $(2, 1)$-regular.

Each Hamilton cycle in a graph defines a $(2,4)$-regular set in its line graph. For instance, in the next figure, the edge set $\{a,b,c,d,e,f\} \subset E(G)$ defines a $(2,4)$-regular set in $L(G)$.





Theorem 4 (C and Cvetković, 2006) *A regular graph G with at least one edge is a $\mathcal{Q}$-graph if and only if there exists a $(0,\tau)$-regular set $S \subset V(G)$, with $\tau = -\lambda_{min}(A_G)$. Furthermore, S is a maximum stable set and then every maximum stable set is $(0,\tau)$-regular.*

- An adverse graph G is a $\mathcal{Q}$-graph if and only if $\exists S \subseteq V(G)$ which is $(0,\tau)$-regular, with $\tau = -\lambda_{min}(A_G)$.

4. Related properties and extensions

Despite the recognition of (k, τ)-regular sets is to be NP-hard, we have the following useful results.

Theorem 5 (Thompson, 1981) *A p-regular graph has a (k, τ)-regular set S, with $\tau > 0$, if and only if $k - \tau$ is an adjacency eigenvalue and $(p - k + \tau)x(S) - \tau\hat{e}$ is a $(k - \tau)$-eigenvector.*

Theorem 6 (C and Rama, 2004) *A graph G has a (k, τ)-regular set $S \subseteq V(G)$ if and only if the characteristic vector x of S is a solution for the linear system*

$$(A_G - (k - \tau)I)x = \tau\hat{e}.$$

Given a graph G with at least one edge, consider the modified convex quadratic programming problem depending on a parameter k, where by τ we denote $-\lambda_{min}(A_G)$,

$$v_k(G) = \max_{x \geq 0} 2\hat{e}^T x - \frac{\tau}{k+\tau} x^T (\frac{A_G}{\tau} + I_n) x.$$

Then, as proved in (C., Kamiński and Lozin, 2007), the following properties hold:

- If $\exists S \subseteq V(G)$ inducing a subgraph of G such that $\overline{d}_{G[S]} = k$ (where $\overline{d}_H$ denotes the average degree of H), then $|S| \leq v_k(G)$.

- If $\exists S \subseteq V(G)$ inducing a k-regular subgraph, then $|S| = v_k(G)$ if and only if

$$\tau + k \ \leq \ |N_G(v) \cap S| \ \forall v \notin S.$$

Theorem 7 (C, Kamiński and Lozin, 2007) *If G is a p-regular graph of order n, with $p > 0$, then*

$$\upsilon_k(G) = n\frac{k - \lambda_{min}(A_G)}{\lambda_{max}(A_G) - \lambda_{min}(A_G)}.$$

Furthermore, there exists a vertex subset S inducing a k-regular subgraph such that

$$|S| = \upsilon_k(G)$$

if and only if S is $(k, k+\tau)$-regular, with $\tau = -\lambda_{min}(A_G)$.

Then we have the following extension of the Hoffman bound.

Corollary 8 (C, Kamiński and Lozin, 2007) *Let G be a p-regular graph with n vertices ($p > 0$) and $S \subseteq V(G)$ inducing a k-regular subgraph, then*

$$|S| \leq n\frac{k - \lambda_{min}(A_G)}{\lambda_{max}(A_G) - \lambda_{min}(A_G)}.$$

5. REFERENCES

- V. E. Alekseev, On influence of local restrictions to the complexity of finding the independence number of the graph, Combinatorial and Algebraic Methods in Applied Mathematics, Gorkie University Press, Gorky, (1982): 3-13 (in Russian).

- C. Berge, *Two theorems in graph theory*. Proc. Nat. Acad. Sci. USA 43 (1957): 842-844.

- D. M. Cardoso, Convex quadratic programming approach to the maximum matching problem, *Journal of Global Optimization*, 21 (2001): 91-106.

- D. M. Cardoso, On graphs with stability number equal to the optimal value of a convex quadratic program, *Matemática Contemporânea*, 25 (2003): 9-24.

- D. M. Cardoso, P. Rama, Equitable bipartitions of graphs and related results, *Journal of Mathematical Sciences*, 120 (2004): 869-880.

- D. M. Cardoso and D. Cvetković, Graphs with eigenvalue -2 attaining a convex quadratic upper bound for the stability number, *Bull. T.CXXXIII de l'Acad. Serbe Sci. Arts, Cl. Sci. Math. Natur., Sci. Math.*, 31 (2006): 41-55.

- D. M. Cardoso, M. Kaminski, V. Lozin, Maximum k-regular induced subgraphs, to appear in Journal of Combinatorial Optimization (2007) doi: 10.1007/s10878-007-9045-9.

- R. M. Karp, Reducibility among combinatorial problems, In: *Complexity of Computer Computations*, eds. R. E. Miller and J. W. Thatcher, Plenum Press, New York, (1972): 85-104.

- V. V. Lozin and D. M. Cardoso, On hereditary properties of the class of graphs with convex quadratic stability number, *Cadernos de Matemática*, CM/I-50, Departamento de Matemática da Universidade de Aveiro (1999).

- C. J. Luz, An upper bound on the independence number of a graph computable in polynomial time, *Operations Research Letters*, 18 (1995): 139-145.

- G. J. Minty, On maximal independent sets of vertices in claw-free graphs, *J. Comb. Theory* Ser. B 28 (1980): 284-304.

- R. Mosca, Polynomial algorithms for the maximum independent set problem on particular classes of P_5-free graphs, *Inform. Process. Lett.* 61 (1997): 137-144.

- R. Mosca, Independent sets in certain P_6-free graphs, Discrete Appl. Math. 92 (1999): 177-191.
- T. S. Motzkin and E. G. Straus, Maxima for graphs and a new proof of a theorem of Túran, Canadian Journal of Mathematics, 17 (1965): 533-540.
- N. Sbihi, Algorithm de recherche d'un independent de cardinalité maximum dans une graphe sans étoiles, *Discrete Mathematics* 29 (1980): 53-76.
- D. M. Thompson, Eigengraphs: constructing strongly regular graphs with block designs, *Utilitas Math.*, 20 (1981): 83-115.

http://www.mat.ua.pt/dcardoso