A survey on (k, τ)-regular sets and their applications

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Outline

Graphs

- In this presentation we deal with simple graphs G (just called graphs) of order n with a set of vertices $V(G)$ and a set of edges $E(G)$.
- An element of $E(G)$, which has the vertices i and j as end-vertices, is denoted by ij.
- If $x \in V(G)$, then the neighborhood of x is

$$N_G(x) = \{y : xy \in E(G)\}$$

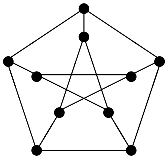
and the closed neighborhood is

$$N_G[x] = N_G(x) \cup \{x\}.$$

- A graph G is p-regular if $\forall v \in V(G)\ |N_G(v)| = p$.

Strongly regular graphs

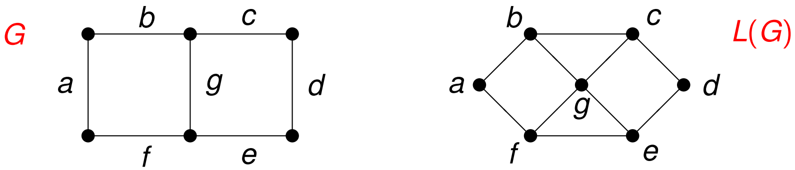
A **strongly regular graph** with parameters (n, p, a, c) is a p-regular graph, neither complete nor null ($0 < p < n - 1$) such that each pair of adjacent vertices has a common neighbors and each pair of nonadjacent vertices has c common neighbors.



The strongly regular graph with parameters $(10, 3, 0, 1)$

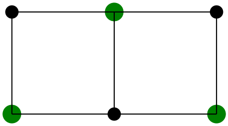
Line graphs

- The **line graph** $L(G)$ of a graph G has the edges of G as its vertices, with two vertices of $L(G)$ being adjacent if and only if the corresponding edges of G have a vertex in common.
- An example:



Stable sets and cliques

A vertex subset $S \subseteq V(G)$ is a **stable set (clique)** if no (every) pair of vertices in S is connected by an edge.

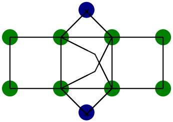


A stable set (clique) with maximum cardinality is a **maximum stable set (*maximum clique*)** and its cardinality is the **stability (clique) number**. The stability (clique) number is denoted by $\alpha(G)$ ($\omega(G)$).

Simplicial vertex, simplex and simplicial graph

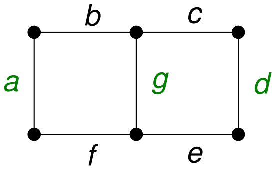
Let G be a graph.

- A vertex $v \in V(G)$ is **simplicial** if it appears in exactly one maximal clique of G (that is, if $N_G[v]$ induces a complete graph).
- A maximal clique containing at least one simplicial vertex is called a **simplex** of G.



Matching

A edge subset $M \subseteq E(G)$ is a **matching** if there are no two edges with a common vertex. A matching of maximum size is a **maximum matching**.



A **perfect matching** is a matching M such that each vertex $v \in V(G)$ is incident in one edge of M.
An **induced matching** of a graph is a matching having no two edges joined by an edge.

Graph eigenvalues

- By graph eigenvalues we mean adjacency eigenvalues. Where, as usually, the adjacency matrix of a graph G is the $n \times n$ matrix $A_G = (a_{ij})$ such that $a_{ij} = \begin{cases} 1, & \text{if } ij \in E(G) \\ 0, & \text{otherwise.} \end{cases}$
- Thus A_G is symmetric and has n real eigenvalues

$$\lambda_{max}(A_G) = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = \lambda_{min}(A_G).$$

- We use the terminology for G and A_G interchangeably and then $Sp(G)$ denotes the set of eigenvalues of G.
- For each eigenvalue $\lambda \in Sp(G)$, the associated eigenspace is denoted by $\mathcal{E}_G(\lambda)$.

Efficient edge dominating sets

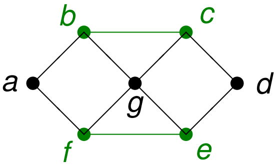
An **efficient edge dominating set** $D \subseteq E(G)$ of a graph G is an induced maching such that every edge of $E(G) \setminus D$ has just one end vertex in common with exactly one edge of D.
A **cut** set defined by $S \subset V(G)$ is the edge subset $\partial(S) \subseteq E(G)$ were each edge $e \in \partial(S)$ has exactly one end vertex in S.

Edge partition property

Let G be a graph and $M = \{e_1, \ldots, e_m\} \subseteq E(G)$. Then M is an efficient edge dominating set if and only if $E(G)$ can be partitioned into the subsets E_{e_j}, for $j = 1, \ldots, m$, such that $e_j = u_j v_j$ and $E_{e_j} = \partial(\{u_j, v_j\}) \cup \{e_j\}$.

An example

A graph with an efficient edge dominating set



The edge subset $M = \{bc, ef\}$ is an efficient edge dominating set. Notice that

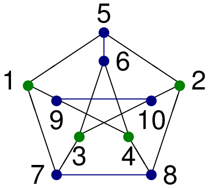
$$E_{bc} = \partial(\{b, c\}) \cup \{bc\} = \{ba, bg, cg, cd, bc\}$$
$$E_{ef} = \partial(\{e, f\}) \cup \{ef\} = \{eg, ed, fg, fa, ef\},$$

$E_{bc} \cap E_{ef} = \emptyset$ and $E(G) = E_{bc} \cup E_{ef}$.

Definition of (k, τ)-regular set

A vertex subset $S \subseteq V(G)$ is (k, τ)**-regular** if induces a k-regular subgraph and

$$\forall v \notin S \ \ |N_G(v) \cap S| = \tau.$$



$S_1 = \{1, 2, 3, 4\}$ is $(0, 2)$-regular.
$S_2 = \{5, 6, 7, 8, 9, 10\}$ is $(1, 3)$-regular.

The origens of (k, τ)-regular sets

The (k, τ)-regular sets emerged

- related to strongly regular graphs and designs in
  - D. M. Thompson, 1981.
  - A. Neumaier, 1982.
- in the study of graphs with domination constraints
  - J. A. Telle, 1993.
  - M. M. Halldórsson, J. Kratochvíl and J. A. Telle, 2000.

Particular cases

- If G has a (k, τ)-regular set S then in its complement, $\bar{G}$, S is $(|S| - k - 1, |S| - \tau)$-regular.
- According to the definition, if a graph G is k-regular, then $V(G)$ is a (k, τ)-regular for every nonnegative integer τ.
- By convention, if G is k-regular, then we say that $V(G)$ is $(k, 0)$-regular.
- If a p-regular graph G has a (k, τ)-regular set S, then $V(G) \setminus S$ is $(p - \tau, p - k)$-regular.

Independent sets

Theorem[Barbosa and C, 2004]

A graph G has a maximum stable set which is $(0, 1)$-regular if and only if each vertex belongs to exactly one simplex.

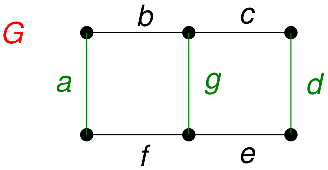
Theorem

Let G be a graph and $S \subset V(G)$. If S is $(0, \tau)$-regular with $\tau = -\lambda_{min}(A_G)$, then S is a maximum stable set and every maximum stable set is $(0, \tau)$-regular.

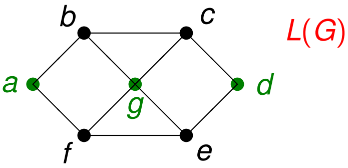
- The Petersen graph P has a $(0, 2)$-regular set with cardinality 4 and $\lambda_{min}(A_P) = -2$. Then $\alpha(P) = 4$.
- The Hoffman-Singleton graph HS has a $(0, 3)$-regular set with cardinality 15 and $\lambda_{min}(A_{HS}) = -3$. Then $\alpha(HS) = 15$.

Characterization of perfect matching

A connected graph G, with more than one edge, has a perfect matching M if and only if $L(M)$ is a $(0, 2)$-regular set of the line graph $L(G)$.



$M = \{a, d, g\}$

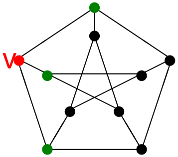


$L(M) = \{a, d, g\}$

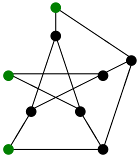
Strongly regular graphs

Theorem

A k-regular graph G of order n is strongly regular with parameters (n, k, a, c) if and only if $\forall v \in V(G)$, $S = N_G(v)$ is (a, c)-regular in $H = G - \{v\}$.



The Petersen graph P



The graph $P - \{v\}$

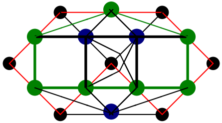
Efficient edge dominating sets and Hamilton Cycles

Theorem

A connected graph G with more than one edge has an efficient edge dominating set $D \subset E(G)$ if and only if $L(D)$ is $(0, 1)$-regular in the line graph $L(G)$.

Theorem

A graph G has an Hamilton cycle C if and only if $L(C)$ is $(2, 4)$-regular in the line graph $L(G)$.



Eigenvalues and (k, τ)-regular sets

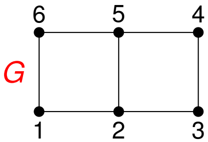
Theorem[Thompson, 1981]

Let G be a p-regular graph and $\mathbf{x}$ the characteristic vector of $S \subseteq V(G)$. Then S is a (k, τ)-regular set, with $\tau > 0$, if and only if $k - \tau \in Sp(G)$ with corresponding eigenvector $(p - k + \tau)\mathbf{x} - \tau j$, where j is the all-one vector.

Theorem[C and Rama, 2007]

Let $\lambda \in \mathbb{Z}$ and G be a graph with a (k_1, τ_1)-regular set S_1 ($\tau_1 > 0$) and a (k_2, τ_2)-regular set S_2, such that $S_1 \neq S_2$ and $k_1 - \tau_1 = k_2 - \tau_2 = \lambda$. Then $\lambda \in Sp(G)$ with corresponding eigenvector $\frac{\tau_2}{\tau_1}\mathbf{x}_1 - \mathbf{x}_2$, where $\mathbf{x}_1$ and $\mathbf{x}_2$ are the characteristic vectors of S_1 and S_2, respectively.

Eigenvalues and eigenvectors



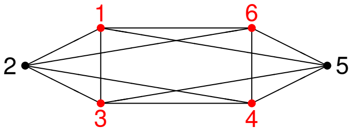
$S_1 = \{1, 2, 5, 6\}$ and $S_2 = \{2, 3, 4, 5\}$ are $(2, 1)$-regular.
$T_1 = \{1, 4\}$ and $T_2 = \{3, 6\}$ are $(0, 1)$-regular.
Then $\{-1, 1\} \subset Sp(A_G) = \{-2.41, -1, -0.41, 0.41, 1, 2.41\}$
and

$$\begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \in \mathcal{E}_G(1) \qquad \text{and} \qquad \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \in \mathcal{E}_G(-1).$$

Maximum induced k-regular subgraphs

Theorem[C, Kamiński and Lozin, 2007]

Let G be a graph and $\tau = -\lambda_{min}(A_G)$. If $S \subseteq V(G)$ is $(k, k+\tau)$-regular, then S is a maximum cardinality set inducing a k-regular subgraph.



Graph G with $Sp(G) = \{-2, 0, 4\}$ and the $(2,4)$-regular set $\{1, 3, 4, 6\}$.

The main characteristic polynomial

Definition

Each distinct eigenvalue $\mu_1, \ldots, \mu_p,\ 1 \leq p \leq n$, of a graph G such that $\mathcal{E}_G(\mu_i)$ is *not* orthogonal to the all-one vector $\mathbf{j}$ is said to be **main.** The remaining distinct eigenvalues are **non-main.**

Lemma [D. Cvetković and M. Petrić, 1984]

If G is a graph with p distinct main eigenvalues $\mu_1, \ldots, \mu_p$, then the **main characteristic polynomial** of G

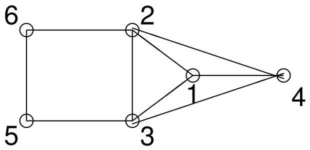
$$m_G(\lambda) \quad = \quad \lambda^p - c_0\lambda^{p-1} - c_1\lambda^{p-2} - \cdots - c_{p-2}\lambda - c_{p-1}$$

has integer coefficients.

The walk matrix

Definition

Given a graph G of order n, the $n \times k$ walk matrix of G is the matrix $\mathbf{W'} = \left(\mathbf{j}, \mathbf{A_G j}, \mathbf{A}_G^2\mathbf{j}, \ldots, \mathbf{A}_G^{k-1}\mathbf{j}\right)$.



$$\mathbf{W'} = (j, A_G j, A_G^2 j) = \begin{pmatrix} 1 & 3 & 11 \\ 1 & 4 & 12 \\ 1 & 4 & 12 \\ 1 & 3 & 11 \\ 1 & 2 & 6 \\ 1 & 2 & 6 \end{pmatrix}$$

The column space of the walking matrix

Theorem (Hagos, 2002)

Let G be a graph of order n with p distinct main eigenvalues. The rank of its $n \times k$ walk matrix $\mathbf{W}'$, is p for $k \geq p$.

Then the number of distinct main eigenvalues p is such that

$$p = \min\{k : \{j, Aj, A^2j, \ldots, A^kj\} \text{ is linear dependent}\}.$$

The pth column of $\mathbf{AW}$ is $\mathbf{A}^p\mathbf{j} = \mathbf{W}\begin{pmatrix} c_{p-1} \\ \vdots \\ c_1 \\ c_0 \end{pmatrix}$, where c_j, for $j = 0, \ldots, p-1$, are the coefficients of $m_G(\lambda)$.

Main (non-main) eigenvalues of graphs with (κ, τ)-regular sets

Definition

The main eigenspace of a graph G, $Main(G)$, is the subspace spanned by the eigenvectors not orthogonal to the all one vector j.

Theorem[C, Sciriha and Zerafa, 2008]

Let G be a graph with a (k, τ)-regular set S, where $\tau > 0$, and $\lambda \in Sp(G)$.

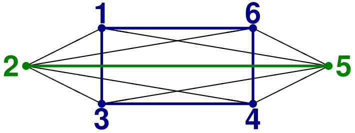
1. The eigenvalue λ is non-main if and only if
$$\lambda = k - \tau \qquad \text{or} \qquad \mathbf{x}_S \in (\mathcal{E}_G(\lambda))^{\perp}.$$
2. If λ is main with associated eigenvector $\mathbf{u} \in Main(G)$, then
$$\mathbf{u}^t\mathbf{x}_S \neq 0 \text{ and } \lambda = \tau\frac{\mathbf{u}^t\mathbf{x}_{\bar{S}}}{\mathbf{u}^t\mathbf{x}_S} + k.$$

Graph operations

Definition

The graph $H = G_1 \bigoplus_s^\tau G_2$ is obtained from a k_1-regular graph G_1 and a k_2-regular graph G_2, connecting each vertex of $V(G_1) = \{x_1, \ldots, x_{n_1}\}$ to $\tau > 0$ vertices in $V(G_2) = \{y_1, \ldots, y_{n_2}\}$, such that $B_i = N_H(x_i) \cap V(G_2), i = 1, \ldots, n_1$, is a $1-(n_2, \tau, s)$ combinatorial design.

Graph $H = G_1 \bigoplus_2^4 G_2$, with $G_1 = H[\{2, 5\}] = K_2$ and $G_2 = H[\{1, 3, 4, 6\}] = C_4$.

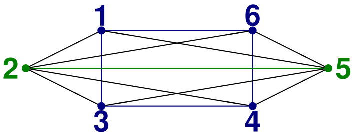


Graphs with two main eigenvalues

As direct consequence:

Considering a k_1-regular graph G_1 and a k_2-regular graph G_2, let $H = G_1 \bigoplus_s^\tau G_2$ be the graph obtained as above described. If $\lambda \in Sp(H)$ is main, then $\lambda = \frac{k_1+k_2 \pm \sqrt{(k_2-k_1)^2+4s\tau}}{2}$.

Graph $H = K_2 \bigoplus_s^\tau C_2$, where $\tau = 4$ and $s = 2$.



$$\lambda = \frac{k_1 + k_2 \pm \sqrt{(k_2 - k_1)^2 + 4s\tau}}{2} = \frac{1 + 2 \pm \sqrt{1 + 32}}{2}.$$

Another example

The n-wheel

Considering the particular case of a n-wheel, W_n, it follows that $W_n = G_1 \bigoplus_1^{n-1} G_2$, with $G_1 = K_1$ and $G_2 = C_{n-1}$. Then

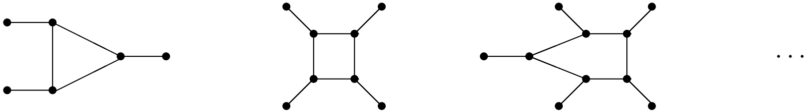
$$\lambda_{max}(W_n) = \frac{2 + \sqrt{4 + 4(n-1)}}{2} = 1 + \sqrt{n}.$$

Furthermore, if $n > 4$, then there exists another main eigenvalue

$$\lambda(W_n) = 1 - \sqrt{n}.$$

Families of non-regular graphs with constant index

Attaching s pendent vertices to each of the vertices of a k-regular graph G of order n, then the main eigenvalues of $H = nK_1 \bigoplus_s^1 G$ are $\lambda(H) = \frac{k \pm \sqrt{k^2+4s}}{2}$.



$H_6 = 3K_1 \bigoplus_1^1 C_3$ $\quad H_8 = 4K_1 \bigoplus_1^1 C_4$ $\quad H_{10} = 5K_1 \bigoplus_1^1 C_5$ $\quad \ldots$

Theorem (Hou and Tian, 2005)

The graphs attained from a cycle C_n by attaching $s > 0$ pendent vertices to every vertex of C_n are all unicyclic graphs with exactly two main eigenvalues.

Determination of (κ, τ)-regular sets

Another example of a graph with exactly two main eigenvalues is the graph the SGT in Rio announcement (see abstract book), $H = G_1 \bigoplus_1^1 G_2$, where G_1 is 2-regular and G_2 is 4-regular. Then the main eigenvalues of H are $\lambda = \frac{(2+4)\pm\sqrt{(4-2)^2+4}}{2} = 3 \pm \sqrt{2}$.

Lemma[C and Rama, 2004]

A graph G has a (k, τ)-regular set S if and only if its characteristic vector, $\mathbf{x}$, is a solution of the linear system $(A_G - (k - \tau)I)\mathbf{x} = \tau\hat{e}$.

Determination of (κ, τ)-regular sets (cont.)

Theorem[C, Sciriha and Zerafa, 2008]

Let G be a graph with p distinct main eigenvalues and a (κ, τ)-regular set S, with $\tau > 0$, and its characteristic vector

$$\mathbf{x}_S = \sum_{i=0}^{p-1} \alpha_i \mathbf{A}^i \mathbf{j} + \mathbf{q}, \qquad (1)$$

where $\mathbf{q} \in Main(G)^{\perp}$. If $k - \tau = 0$ or $k - \tau \neq \frac{c_0 \pm \sqrt{c_0^2 + 4c_1}}{2}$, then $\alpha_0, \ldots, \alpha_{p-1}$ can be determined using $(\kappa - \tau)$ and the coefficients of $m_G(\lambda)$, $c_0, c_1, \ldots, c_{p-1}$. Furthermore,

1. if $(\kappa - \tau) \notin Sp(G)$, then $\mathbf{q} = 0$.
2. If $(\kappa - \tau) \in Sp(G)$ then $\mathbf{q} \in \mathcal{E}_G(\kappa - \tau)$.

Determination of the characteristic vector of a (k, τ)-regular set

The determination of the characteristic vector x_S in (1) can be done using the linear independence of the vectors $\mathbf{A}^i\mathbf{j}$, for $0 \leq i \leq p-1$, and $\mathbf{q}$, and then the following system of equations is obtained.

$$
\begin{array}{llrcll}
\text{along } \mathbf{j} & : & (\kappa-\tau)\alpha_0 & = & -\tau+\alpha_{p-1}c_{p-1}; & (0) \\
\text{along } \mathbf{Aj} & : & (\kappa-\tau)\alpha_1 & = & \alpha_0+\alpha_{p-1}c_{p-2}; & (1) \\
\text{along } \mathbf{A}^2\mathbf{j} & : & (\kappa-\tau)\alpha_2 & = & \alpha_1+\alpha_{p-1}c_{p-3}; & (2) \\
\vdots & & & & & \\
\text{along } \mathbf{A}^{p-1}\mathbf{j} & : & (\kappa-\tau)\alpha_{p-1} & = & \alpha_{p-2}+\alpha_{p-1}c_0; & (p-1) \\
\text{along } \mathbf{q} & : & (\kappa-\tau)\mathbf{q} & = & \mathbf{Aq} & (p)
\end{array}
$$

Taking into account that $k-\tau=0$ or $k-\tau \neq \frac{c_0 \pm \sqrt{c_0^2+4c_1}}{2}$, we may conclude that the system of equations $(0), \ldots, (p-1)$ has a unique solution.

The $(k - \tau)$-parametric vector g

Definition

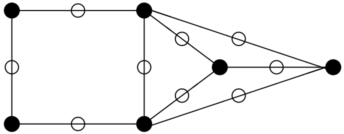
Let G be a graph with p distinct main eigenvalues, $\kappa, \tau \in \mathbb{Z}^+ \cup \{0\}$, and $\alpha_0, \ldots, \alpha_{p-1}$, a solution of the above system of equations $(0), (1), \ldots, (p-1)$. Then the vector

$$\mathbf{g} = W\widehat{\alpha} = \sum_{i=0}^{p-1} \alpha_i \mathbf{A}^i \mathbf{j}$$

is referred to as a $(\kappa - \tau)$–**parametric vector** of G.

Graph subdivisions

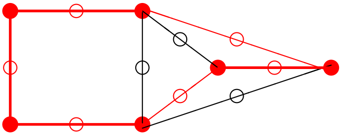
- **Definition**
  The subdivision of a graph G, denoted by G^*, is obtained by inserting a vertex in each edge of G.
  - This subdivision produces a bipartite graph where one vertex subset of the bipartition is formed by inserted vertices and the other vertex subset is formed by original vertices.
  - Example:



Hamiltonian graphs and $(2,2)$-regular sets

Theorem[C, Sciriha and Zerafa, 2008]

A graph G is Hamiltonian if and only if its subdivision G^* has a $(2,2)$-regular set S inducing a connected subgraph.



Existence of Hamiltonian graphs

Theorem[C, Sciriha and Zerafa, 2008]

Let G be a graph and G^* its subdivision.

1. If zero is a main eigenvalue of G^*, then G is not Hamiltonian.
2. If $0 \in Sp(G^*)$ and $\mathbf{g}$ is the 0–parametric vector of G^*, then G is Hamiltonian if and only if $\exists \mathbf{q} \in \mathcal{E}_{G^*}(0)$ such that $\mathbf{g} + \mathbf{q}$ is the characteristic vector of a $(2, 2)$-regular set of G^* inducing a connected subgraph.
3. If $0 \notin Sp(G^*)$, then G is Hamiltonian if and only if the 0–parametric vector of G^* is the characteristic vector of a $(2, 2)$-regular set of G^* inducing a connected subgraph.

An algorithm

- **Algorithm for the recognition of Hamiltonian graphs**(G)

  1. Determine the subdivision G^*;
  2. **If** 0 is a main eigenvalue of G^*
     - **then** STOP - G is not Hamiltonian
     - **else** Determine the 0–parametric vector g of G^*;
  3. **If** g is the characteristic vector of a $(2, 2)$-regular set of G^* inducing a connected subgraph
     - **then** STOP G is Hamiltonian;
     - **else** Determine a 0-eigenvector q such that $g + q$ is the characteristic vector x_S of a $(2, 2)$-regular set of G^* inducing a connected subgraph, if there exists.

- **End**

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