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Graphs with convex-QP stability number

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Summary

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- Adverse graphs and (k, τ)-regular sets.
- Analysis of particular families of graphs.
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Introduction

Let us consider the simple graph

$$G = (V, E)$$

of order n, where $V = V(G)$ is the set of nodes and $E = E(G)$ is the set of edges.

A_G will denote the adjacency matrix of the graph G and $\lambda_{min}(A_G)$ the minimum eigenvalue of A_G.
It is well known that if G has at least one edge, then $\lambda_{min}(A_G) \leq -1$. Actually

- $\lambda_{min}(A_G) = 0$ iff G has no edges,
- $\lambda_{min}(A_G) = -1$ iff G has at least one edge and every component complete,
- $\lambda_{min}(A_G) \leq -\sqrt{2}$ otherwise.

Introduction (cont.)

A graph G is $(H_1, \ldots, H_k)$-free if G contains no copy of the graphs $H_1, \ldots, H_k$, as induced subgraphs.

- In particular, G is H-free if G has no copy of H as an induced subgraph.
- A claw-free graph is a $K_{1,3}$-free graph.

Let us define the quadratic programming problem $(P_G(\tau))$:

$$\upsilon_G(\tau) = \max\{2\hat{e}^T x - x^T(\frac{1}{\tau}A_G + I_n)x : x \geq 0\},$$

with $\tau > 0$.

If $x^*(\tau)$ is an optimal solution for $(P_G(\tau))$ then

$$0 \leq x^*(\tau) \leq 1.$$

Introduction (cont.)

$$\forall \tau > 0 \qquad 1 \leq \upsilon_G(\tau) \leq n.$$

The fucntion $\upsilon_G :]0, +\infty[\mapsto [1, n]$ has the following properties:

- $\forall \tau > 0 \;\; \alpha(G) \leq \upsilon_G(\tau)$.
- $0 < \tau_1 < \tau_2 \Rightarrow \upsilon_G(\tau_1) \leq \upsilon_G(\tau_2)$.
- $\upsilon_G(1) = \alpha(G)$.
- If $\tau^* > 0$, then the following are equivalent.
  - $\exists \bar{\tau} \in]0, \tau^*[$ such that $\upsilon_G(\bar{\tau}) = \upsilon_G(\tau^*)$;
  - $\upsilon_G(\tau^*) = \alpha(G)$;
  - $\forall \tau \in]0, \tau^*[\;\; x^*(\tau)$ is not spurious;
  - $\forall \tau \in]0, \tau^*] \; \upsilon_G(\tau) = \alpha(G)$.
- $\forall U \subset V(G) \;\; \forall \tau > 0 \;\; \upsilon_{G-U}(\tau) \leq \upsilon_G(\tau)$.

Introduction (cont.)

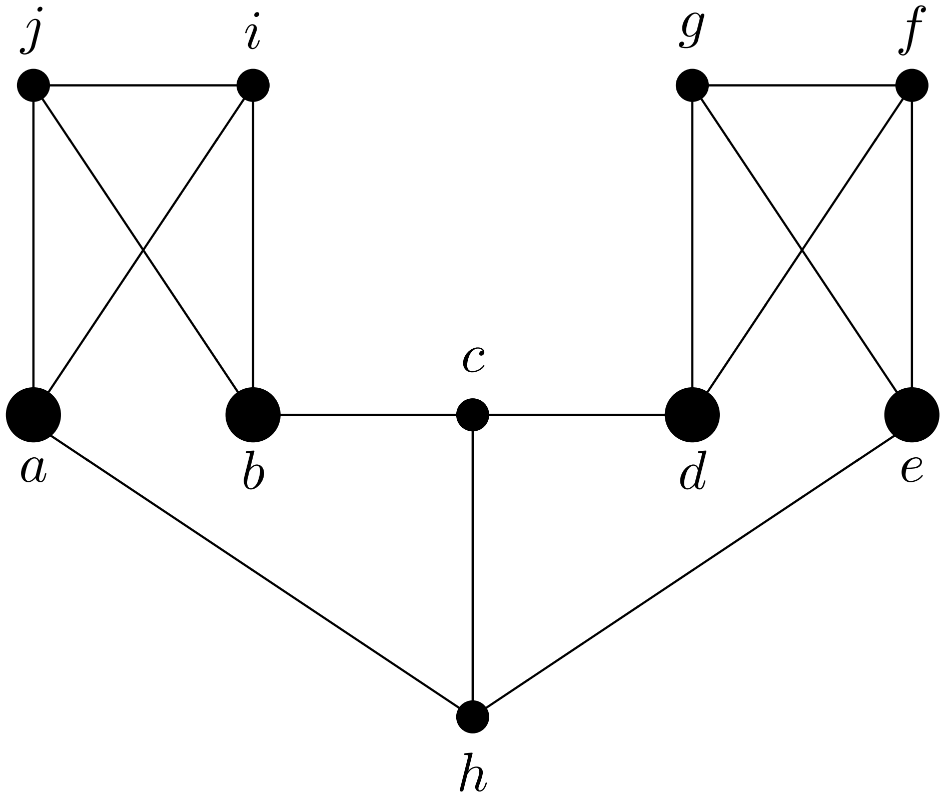

Figure 1: A graph G with $\lambda_{min}(A_G) = -2$ and $\upsilon_G(2) = \alpha(G) = 4$.

Introduction (cont.)

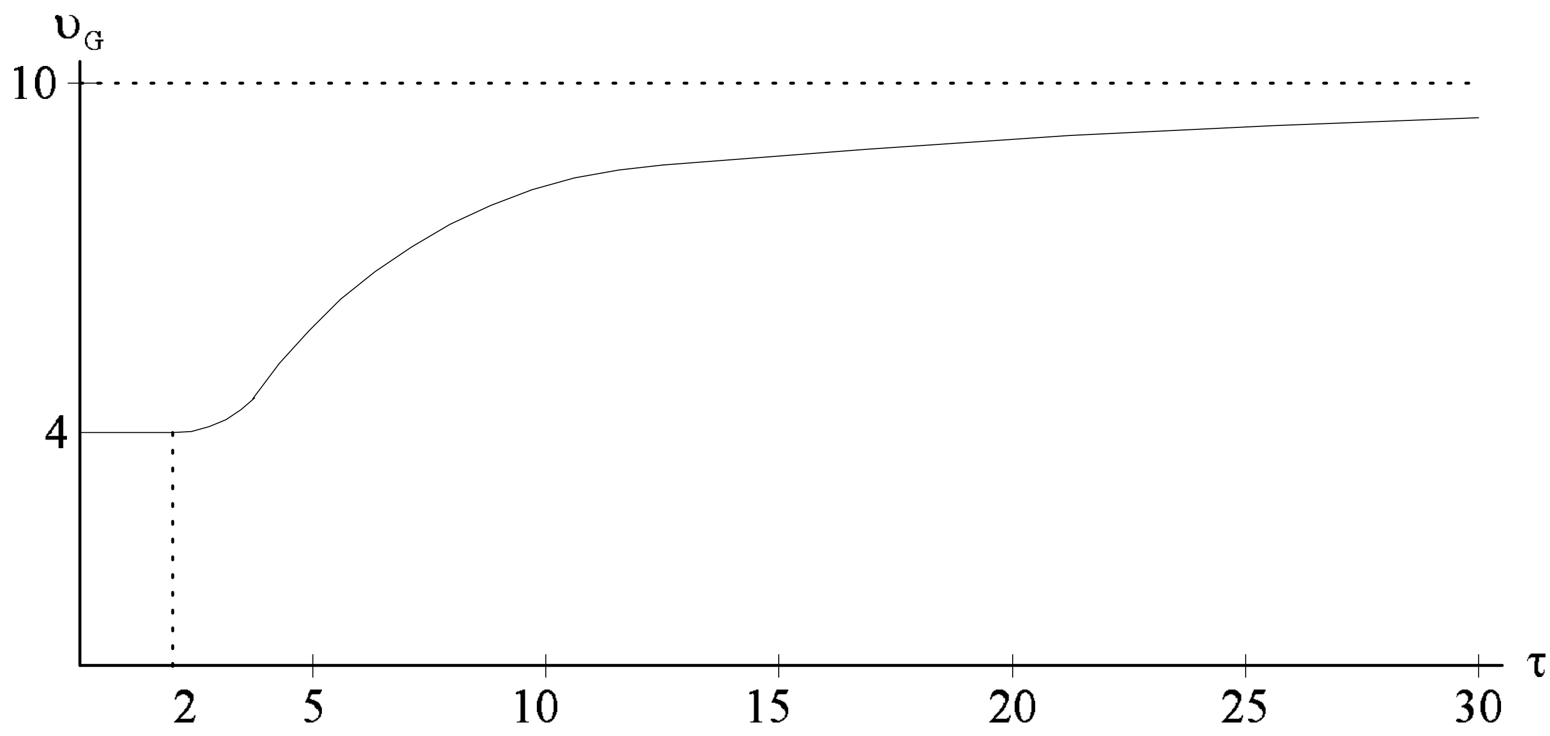

Figure 2: Function $\upsilon_G(\tau)$, where G is the above graph.

The class of $\mathcal{Q}$-graphs

- The graphs G such that $\upsilon_G(-\lambda_{min}(A_G)) = \alpha(G)$ are called graphs with *convex-QP stability number* where QP means quadratic program.

- The class of these graphs will be denoted by $\mathcal{Q}$ and its elements called $\mathcal{Q}$-graphs.

- Since the components of the optimal solutions of $(P_G(\tau))$ are between 0 and 1, then $\upsilon_G(\tau) = \alpha(G)$ if and only if $(P_G(\tau))$ has an integer optimal solution.

  Theorem[Luz, 1995]
  If G has at least one edge then $G \in \mathcal{Q}$ if and only if, for a maximum stable set S (and then for all),

$$-\lambda_{\min}(A_G) \leq \min\{|N_G(i) \cap S| : i \notin S\}. \tag{1}$$

The class of $\mathcal{Q}$-graphs (cont.)

There exists an infinite number of graphs with convex-QP stability number.

Theorem[Cardoso, 2001]
A connected graph with at least one edge, which is nor a star neither a triangle, has a perfect matching if and only if its line graph has convex-QP stability number.

As immediate consequence, we have the following corollary.

Corollary[Cardoso, 2001]
If G is a connected graph with an even number of edges then $L(L(G))$ has convex-QP stability number.

The class of $\mathcal{Q}$-graphs (cont.)

There are several famous $\mathcal{Q}$-graphs.

- The Petersen graph P, where $\lambda_{min}(A_P) = -2$ and $\alpha(P) = \upsilon_P(2) = 4$.
- The Hoffman-Singleton graph HS, where $\lambda_{min}(A_{HS}) = -3$ and $\alpha(HS) = \upsilon_{HS}(3) = 15$.
- If the fourth graph of Moore M_4 there exists with $\alpha(M_4) = 400$, as it is expected, then it is a $\mathcal{Q}$-graph.
- Additionally, taking into account (**??**), graphs defined by the disjoint union of complete subgraphs and complete bipartite graphs are trivial examples of $\mathcal{Q}$-graphs.

The class of $\mathcal{Q}$-graphs (cont.)

Additional examples of $\mathcal{Q}$-graphs

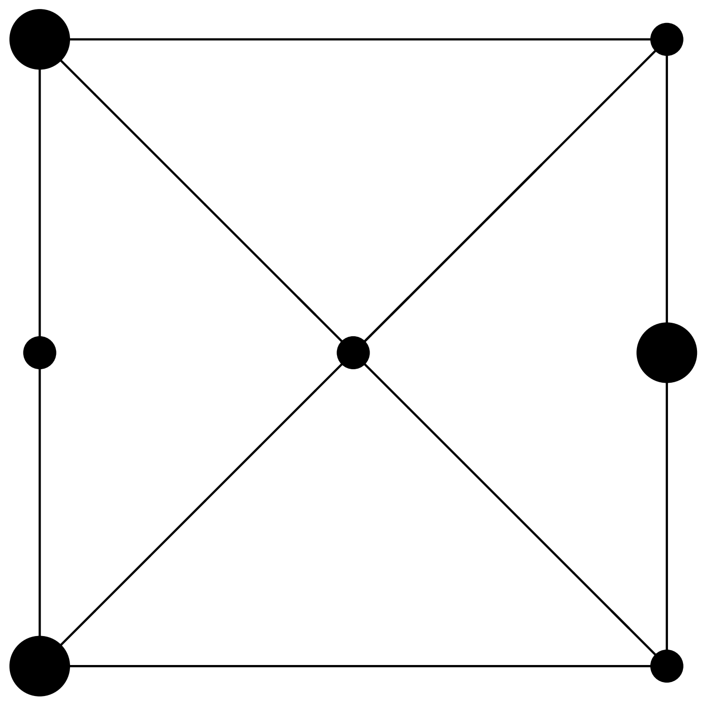

Figure 3: Graph G such that $\lambda_{min}(A_G) = -2$ and $\upsilon_G(2) = 3 = \alpha(G)$.

The class of $\mathcal{Q}$-graphs (cont.)

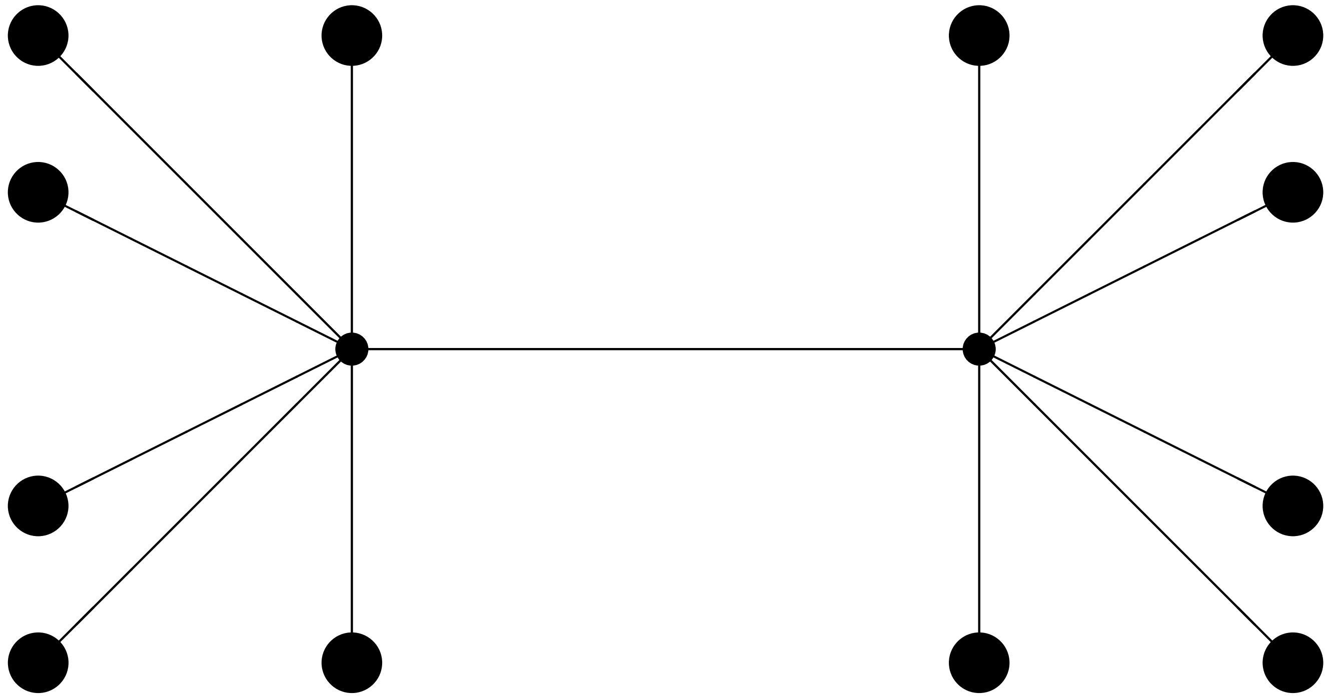

Figure 4: Graph G such that $\lambda_{min}(A_G) = -3$ and $\upsilon_G(3) = 12 = \alpha(G)$.

The class of $\mathcal{Q}$-graphs (cont.)

- A graph belongs to $\mathcal{Q}$ if and only if each of its components belongs to $\mathcal{Q}$.
- Every graph G has a subgraph $H \in \mathcal{Q}$ such that $\alpha(G) = \alpha(H)$.
- If $G \in \mathcal{Q}$ and $\exists U \subseteq V(G)$ such that

$$\alpha(G) = \alpha(G - U)$$

  then $G - U \in \mathcal{Q}$.
- If $\exists v \in V(G)$ such that

$$\upsilon_G(\tau) \neq \max\{\upsilon_{G-\{v\}}(\tau), \upsilon_{G-N_G(v)}(\tau)\},$$

  with $\tau = -\lambda_{min}(A_G)$, then $G \notin \mathcal{Q}$.

The class of $\mathcal{Q}$-graphs (cont.)

- Consider that $\exists v \in V(G)$ such that

$$\upsilon_{G-\{v\}}(\tau) \neq \upsilon_{G-N_G(v)}(\tau)$$

and $\tau = -\lambda_{min}(A_G)$.

1. If $\upsilon_G(\tau) = \upsilon_{G-\{v\}}(\tau)$ then

$$G \in \mathcal{Q} \text{ iff } G - \{v\} \in \mathcal{Q}.$$

2. If $\upsilon_G(\tau) = \upsilon_{G-N_G(v)}(\tau)$ then

$$G \in \mathcal{Q} \text{ iff } G - N_G(v) \in \mathcal{Q}.$$

The class of $\mathcal{Q}$-graphs (cont.)

- Assuming that

$$\tau_1 = -\lambda_{min}(A_G) > -\lambda_{min}(A_{G-U}) = \tau_2,$$

with $U \subset V(G)$. Then

$$\begin{aligned} \upsilon_G(\tau_1) = \upsilon_{G-U}(\tau_2) \quad &\Rightarrow \quad G \in \mathcal{Q}, \\ \upsilon_G(\tau_1) > \upsilon_{G-U}(\tau_2) \quad &\Rightarrow \quad G \notin \mathcal{Q} \text{ or } U \cap S \neq \emptyset, \end{aligned}$$

for each maximum stable set S of G.

Adverse graphs and (k, τ)-regular sets

- Using the above results, we may recognize if a graph G is (or not) a $\mathcal{Q}$-graph, unless an induced subgraph $H = G - U$ (where $U \subset V(G)$ can be empty) is obtained, such that

$$
\begin{aligned}
\tau &= \lambda_{min}(A_G) = \lambda_{min}(A_H), && (2)\\
& \upsilon_G(\tau) = \upsilon_H(\tau), && (3)\\
\forall v \in V(H) \quad & \lambda_{min}(A_H) = \lambda_{min}(A_{H-N_G(v)}), && (4)\\
\forall v \in V(H) \quad & \upsilon_H(\tau) = \upsilon_{H-N_G(v)}(\tau). && (5)
\end{aligned}
$$

- A subgraph H of G without isolated vertices, for which the conditions (**??**)-(**??**) are fulfilled is called *adverse*.

Adverse graphs and (k, τ)-regular sets (cont.)

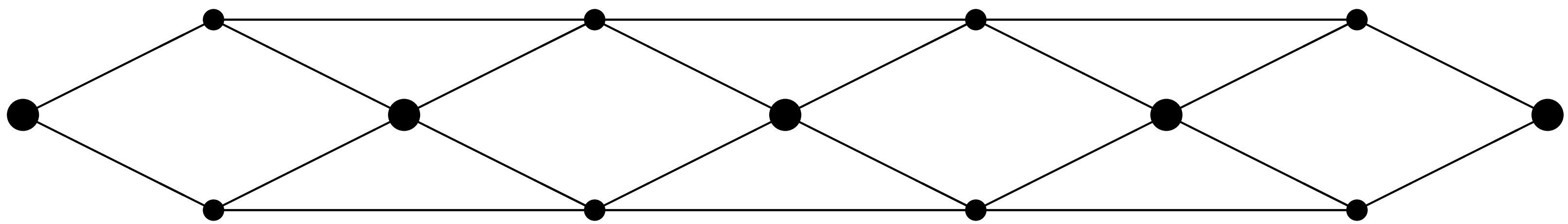

Figure 5: Adverse graph G, with $\lambda_{min}(A_G) = -2$ and $\upsilon_G(2) = \alpha(G) = 5$.

Adverse graphs and (k, τ)-regular sets (cont.)

- Based in the above results, a procedure which recognizes if a graph G is (or not) in $\mathcal{Q}$ or determines an adverse subgraph can be implemented.
- A subset of vertices $S \subset V(G)$ is (k, τ)-*regular* if induces in G a k-regular subgraph and $\forall v \notin S$

$$|N_G(v) \cap S| = \tau.$$

- The maximum stable sets of the graphs of figures 1, **??** and **??** are $(0, 2)$-regular and the maximum stable set of the graph of figure **??** is $(0, 6)$-regular.

Adverse graphs and (k, τ)-regular sets (cont.)

- The Petersen graph P includes the $(0, 2)$-regular set $S = \{1, 2, 3, 4\}$ and the $(2, 1)$-regular sets $T_1 = \{1, 2, 5, 7, 8\}$ and $T_2 = \{3, 4, 6, 9, 10\}$.

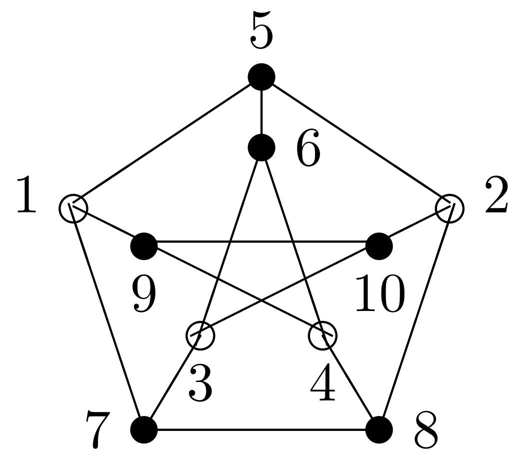

Figure 6: The Petersen graph.

- $L(P)$ includes the $(0, 2)$-regular set $\{\{1, 9\}, \{5, 6\}, \{2, 10\}, \{4, 8\}, \{3, 7\}\}$ (a perfect matching) and the $(0, 1)$-regular set $\{\{5, 6\}, \{9, 10\}, \{7, 8\}\}$ (a perfect induced matching).

Adverse graphs and (k, τ)-regular sets (cont.)

Theorem

Let G be adverse and $\tau = -\lambda_{min}(A_G)$. Then $G \in \mathcal{Q}$ if and only if $\exists S \subset V(G)$ which is $(0, \tau)$-regular.

Theorem

Let G be p-regular, with $p > 0$. Then $G \in \mathcal{Q}$ if and only if $\exists S \subset V(G)$ which is $(0, \tau)$-regular, with $\tau = -\lambda_{min}(A_G)$.

Theorem[Thompson, 1981]

Let G be a p-regular graph and $x(S)$ the characteristic vector of $S \subset V(G)$. Then S is (k, τ)-regular if and only if

$$(\hat{e} - \frac{p - (k - \tau)}{\tau} x(S)) \in Ker(A_G - (k - \tau)I_n),$$

where $\hat{e}$ is the all-ones vector.

Analysis of particular families of graphs

There are several families of graphs in which we can recognise (in polynomial-time) $\mathcal{Q}$-graphs.

1. Bipartite graphs
   - Since the minimum eigenvalue of a connected bipartite graph G is simple, then $\exists v \in V(G)$ such that $\lambda_{min}(A_G) < \lambda_{min}(A_{G-\{v\}})$.
2. Dismantlable graphs
   - The one-vertex graph is dismantlable. A graph G with at least two vertices is dismantlable if $\exists x, y \in V(G)$ such that $N_G[x] \subseteq N_G[y]$ and $G - \{x\}$ is dismantlable

     Theorem

     Given a graph G and $\tau > 1$, if $\exists p, q \in V(G)$ such that $N_G[q] \subseteq N_G[p]$ then $\upsilon_G(\tau) > \upsilon_{G-N_G(p)}(\tau)$.

Analysis of particular families of graphs (cont.)

3. Graphs with low Dilworth number

   – Given two vertices $x, y \in V(G)$, if $N_G(y) \subseteq N_G[x]$ then we say that the vertices x and y are comparable (according to the vicinal preorder). The Dilworth number of a graph G, $\text{dilw}(G)$, is the largest number of pairwise incomparable vertices of G.

     Theorem

     Let G be a not complete graph. If dilw$(G) < \omega(G)$ then G is not adverse.

   A threshold graph has Dilworth number equal to 1.

Analysis of particular families of graphs (cont.)

4. (C_4, P_5)-free graphs

Theorem

Let G be a graph and $\tau > 1$. If $\exists pq \in E(G)$ such that

$$\upsilon_G(\tau) = \upsilon_{G-N_G(p)}(\tau) = \upsilon_{G-N_G(q)}(\tau)$$

then pq belongs to a C_4 or p and q are the midpoints of a P_4.

Combining the above theorem with a result obtained from (Brandstädt and Lozin, 2001), where it is stated that "if a graph is (banner,P_5)-free then any midpoint of a P_4 is α-redundant", the next theorem follows.

Theorem

*Let G be a graph without isolated vertices, for which the equalities (**??**) hold, with $\tau > 1$. If G is (C_4, P_5)-free, then*

$$\forall v \in V(G) \;\; \alpha(G) = \alpha(G - \{v\}).$$

Analysis of particular families of graphs (cont.)

5. Claw-free graphs

Theorem

Let G be a claw-free graph and $\tau > 1$. If $\exists pq \in E(G)$ such that p and q are not the midpoints of a P_4 and

$$\upsilon_G(\tau) = \upsilon_{G-N_G(p)}(\tau) = \upsilon_{G-N_G(q)}(\tau)$$

then neither p nor q are α-critical.

Theorem

Let G be a $(claw, P_5)$-free graph without isolated vertices. If G is adverse then $\forall v \in V(G)\ \ \alpha(G) = \alpha(G - \{v\})$.

Theorem

Let G be a claw-free graph and $p, q \in V(G)$ such that $pq \notin E(G)$. If $N_G(p) \subseteq N_G(q)$ then $\forall v \in N_G(p)$

$$\alpha(G) = \alpha(G - \{v\}).$$

Relations with the Lovász's ϑ-function

- It is well known (Lovász,1986) that the Lovász's ϑ-number of a graph G of order n, can be obtained from the equality

$$\vartheta(G) \quad = \quad \min\{\lambda_{max}(C) : C \in \mathcal{C}(G)\}, \tag{6}$$

where $\mathcal{C}(G)$ is the set of all symmetric $n \times n$ matrices for which $(C)_{ij} = 1$ if $i = j$ or $ij \notin E(G)$ and the entries corresponding to adjacent vertices are free to choose.

- On the other hand, the Lovász's Sandwich Theorem, states the very useful property:

  Theorem[Lovász, 1986]
  For every graph G,

$$\alpha(G) \leq \vartheta(G) \leq \bar{\chi}(G),$$

  where $\bar{\chi}(G)$ denotes the minimum number of cliques covering $V(G)$.

Relations with the Lovász's ϑ-function (cont.)

- Let G be a non null p-regular graph, $\tau = -\lambda_{min}(A_G)$ and $C_G = \hat{e}\hat{e}^T - \frac{\upsilon_G(\tau)}{\tau} A_G$ (then $C_G \in \mathcal{C}(G)$).
  1. If x^* is optimal for $(P_G(\tau))$ then $C_G x^* = \upsilon_G(\tau) x^*$.
  2. $\alpha(G) \leq \vartheta(G) \leq \upsilon_G(\tau) = \lambda_{max}(C_G)$.
  3. $\alpha(G) = \vartheta(G) = \upsilon_G(\tau)$ if and only if there exists a $(0, \tau)$-regular set.

- According to (Luz, 2003), for every graph G

$$\tau \geq -\lambda_{min}(A_G) \;\Rightarrow\; \upsilon_G(\tau) \geq \vartheta(G).$$

  Therefore, when $\tau \geq -\lambda_{min}(A_G)$, $\alpha(G) = \vartheta(G) = \upsilon_G(\tau)$ if and only if $\tau \leq |N_G(v) \cap S| \;\; \forall v \notin S$.

Final remarks and open problems

- When $\tau \in]1, -\lambda_{min}(A_G)[$, if $\alpha(G) = \upsilon_G(\tau)$ (from the **Karush-Khun-Tucker** conditions) we may conclude that for every maximum stable set S of G

$$\tau \quad \leq \quad |N_G(v) \cap S| \;\; \forall v \notin S. \tag{7}$$

However, despite the existence of graphs G with a maximum stable set S for which the condition (**??**) is fulfilled but the equality $\upsilon_G(\tau) = \alpha(G)$ does not holds, remains open to know:

(1) if the condition (**??**), with $\tau \in]1, -\lambda_{min}(A_G)[$, fulfilled for every maximum stable set S of G is sufficient to obtain the equality

$$\upsilon_G(\tau) = \alpha(G).$$

Final remarks and open problems (cont.)

- It is proved that an adverse graph $G \in \mathcal{Q}$ if and only if $\exists S \subset V(G)$ which is $(0, \tau)$-regular, with $\tau = -\lambda_{min}(A_G)$. However,

  (2) it is open to know the complexity of the recognition of $(0, \tau)$-regular sets, with $\tau = -\lambda_{min}(A_G)$, in adverse graphs G.

- Several families of graphs in which the $\mathcal{Q}$-graphs can be recognized in polynomial-time were introduced, as it was the case of bipartite graphs, dismantlable graphs, threshold graphs, (C_4, P_5)-free graphs and $(claw, P_5)$-free graphs.

Final remarks and open problems (cont.)

- According to (Cardoso, 2003) the recognition of $\mathcal{Q}$-graphs which are line graphs of forests can be done also in polynomial-time. However,

  (3) there are many other families of graphs (as it is the case of claw-free graphs) in which it is not known if the $\mathcal{Q}$-graphs are polynomial-time recognizable;

  (4) furthermore, it is an open problem to know if there exists an adverse graph without convex-QP stability number, even when the graph is claw-free;

  (5) another interesting question is about the characterization of hereditary claw-free graphs G with Dilworth number less than $|V(G)|$ (note that if such family there exists then the $\mathcal{Q}$-graphs belonging to it are polynomial-time recognizable).

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