THE CLASS OF GRAPHS WITH CONVEX-QP STABILITY NUMBER

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Outline

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3. The class of $\mathcal{Q}$-graphs.

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1. Basic definitions and notation.

- We consider simple graphs G with at least one edge;
- $E(G)$ and $V(G)$ will denote the edge set and the vertex set of G, respectively;
- Given a vertex subset $S \subseteq V(G)$ a subgraph induced by S, $G' = G[S]$ is such that $V(G') = S$ and $E(G')$ are the edges of G connecting vertices of S;
- The neighborhood of $v \in V(G)$, denoted by $N_G(v)$, is the subset of vertices adjacent to v, and the degree of v is

$$d_G(v) = |N_G(v)|.$$

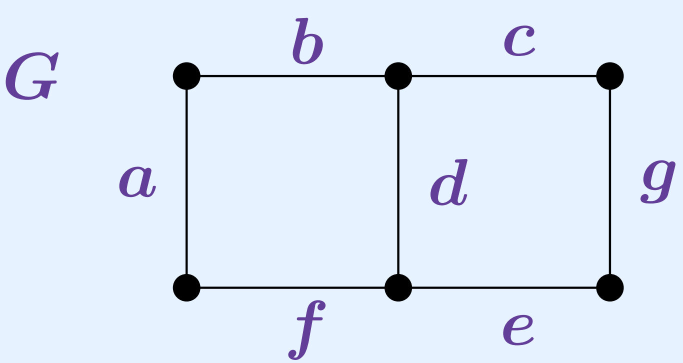
1. Basic definitions and notation (cont.)

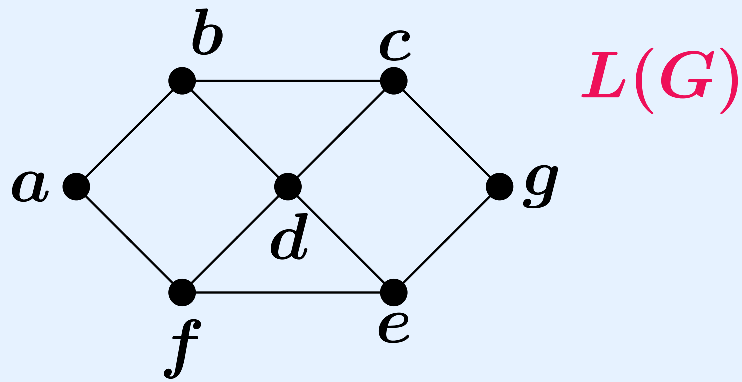
- If $d_G(v) = p \;\; \forall v \in V(G)$ then we say that G is p-regular.
- A stable set (clique) is a vertex subset inducing a null (complete) subgraph. The cardinality of a maximum size stable set (clique) of a graph G is called stability (clique) number of G and it is denoted by $\alpha(G)$ ($\omega(G)$);
- The complement of a graph G, denoted by $\bar{G}$, is such that $V(\bar{G}) = V(G)$ and $E(\bar{G}) = \{ij : i,j \in V(G) \wedge ij \notin E(G)\}$.
- Then $\alpha(G) = \omega(\bar{G})$ and the problem of determining the stability number is equivalent to the problem of determining the clique number.

1. Basic definitions and notation (cont.)

- A matching in a graph G is a subset of edges $M \subseteq E(G)$, no two of which have a common vertex. A matching with maximum cardinality is designated maximum matching.
- If M is a matching and for each vertex $v \in V(G)$ there is one edge $e \in M$ such that v is incident with e, then M is called a perfect matching.
- The *line graph* $L(G)$ of a graph G has the edges of G as its vertices, with two vertices of $L(G)$ being adjacent if and only if the corresponding edges of G have a vertex in common.
- Then a matching in G corresponds to a stable set in $L(G)$.

1. Basic definitions and notation (cont.)





- **The edge subset $\{a, d, g\}$ is a perfect matching for G,**
- **and then $\{a, d, g\}$ is a maximum stable set for $L(G)$.**

1. Basic definitions and notation (cont.)

- **The *adjacency matrix* of a graph G, denoted by A_G, is such that $A_G = (a_{ij})_{n \times n}$, with $n > 1$, and**

$$a_{ij} = \begin{cases} 1, & \text{if } ij \in E(G) \\ 0, & \text{otherwise.} \end{cases}$$

- **Thus A_G is symmetric and then it has n real eigenvalues**

$$\lambda_{max}(A_G) = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = \lambda_{min}(A_G).$$

- **Furthermore, since G has at least one edge, the minimum eigenvalue of A_G, $\lambda_{min}(A_G)$, is not greater than -1.**

2.Quadratic programming upper bounds for $\alpha(G)$

- Given a nonnegative integer k, to determine if a graph G has a stable set of size k is NP-complete [Karp, 1972].
- Therefore, taking into account the increasing number of applications, it is crucial to obtain efective computable approximations for the stability number of a graph.
- Consider the convex quadratic programming problem

$$v(G) = \max_{x \geq 0} \; 2\hat{e}^T x - x^T \left(\frac{A_G}{-\lambda_{min}(A_G)} + I_n \right) x, \qquad (1)$$

where $\hat{e}$ is the all ones vector and I_n the identity matrix of order n [Luz, 1995].

2. Quadratic programming upper bounds for $\alpha(G)$ (cont.)

- The optimal value of (1) has the following properties [Luz, 1995]:

  1. $\alpha(G) \leq \upsilon(G)$;
  2. $\alpha(G) = \upsilon(G)$ if and only if for a maximum stable set S (and then for all)

$$-\lambda_{min}(A_G) \leq \min\{|N_G(v) \cap S| : v \notin S\}; \qquad (2)$$

- Actually, $\alpha(G) = \upsilon(G)$ if and only if there exists a stable set S for which (2) holds [C and D. Cvetković, 2006].

2. Quadratic programming upper bounds for $\alpha(G)$ (cont.)

- **[Luz, 1995]** There exists an optimal solution x^* for **(1)** such that $x_i^* = 0$ for some vertex $i \in V(G)$.

- **[Luz and C, 1998]** If $\tilde{x}$ and $\bar{x}$ are distinct optimal solutions for **(1)**, then the vector $\tilde{x} - \bar{x}$ belongs to the $\lambda_{min}(A_G)$-eigensubspace.

- Assuming that G is regular, we may conclude that

  1. **[Luz, 1995]** $\upsilon(G) = n\frac{-\lambda_{min}(A_G)}{\lambda_{max}(A_G)-\lambda_{min}(A_G)}$.

  2. **[C and D. Cvetković, 2006]** $\upsilon(G) = \alpha(G)$ if and only if there exists a stable set S for which **(2)** holds as equality.

2. Quadratic programming upper bounds for $\alpha(G)$ (cont.)

- **Considering a family of quadratic programming problems depending on a parameter $\tau > 0$ [C, 2003]:**

$$v_G(\tau) = \max\{2\hat{e}^T x - x^T(\frac{A_G}{\tau} + I_n)x : x \geq 0\}, \qquad (3)$$

for each $\tau > 0$, we may conclude that

1. $\alpha(G) \leq v_G(\tau)$;
2. $1 \leq v_G(\tau) \leq n$, $v_G(\tau) = 1$ **if and only if** G **is complete, and** $v_G(\tau) = n$ **if and only if** G **has no edges.**
3. **Furthermore,** $v_G(-\lambda_{min}(A_G)) = v(G)$ **and (3) is a convex program for** $\tau \geq -\lambda_{min}(A_G)$.

2. Quadratic programming upper bounds for $\alpha(G)$ (cont.)

[C, 2003] The function $\upsilon_G :]0, +\infty[\mapsto [1, n]$ verifies:

1. $0 < \tau_1 < \tau_2 \Rightarrow \upsilon_G(\tau_1) \leq \upsilon_G(\tau_2)$.
2. The following statements are equivalent:
   - $\exists \bar{\tau} \in]0, \tau^*[$ such that $\upsilon_G(\bar{\tau}) = \upsilon_G(\tau^*)$;
   - $\upsilon_G(\tau^*) = \alpha(G)$;
   - $\forall \tau \in]0, \tau^*]\ \upsilon_G(\tau) = \alpha(G)$.
3. $\forall U \subset V(G)\ \ \forall \tau > 0\ \ \upsilon_{G-U}(\tau) \leq \upsilon_G(\tau)$.

- For $\tau = 1$, (3) is equivalent to the Motzkin-Straus program [Motzkin and Straus, 1965], and then $\upsilon_G(1) = \alpha(G)$.

2.Quadratic programming upper bounds for $\alpha(G)$(cont.)

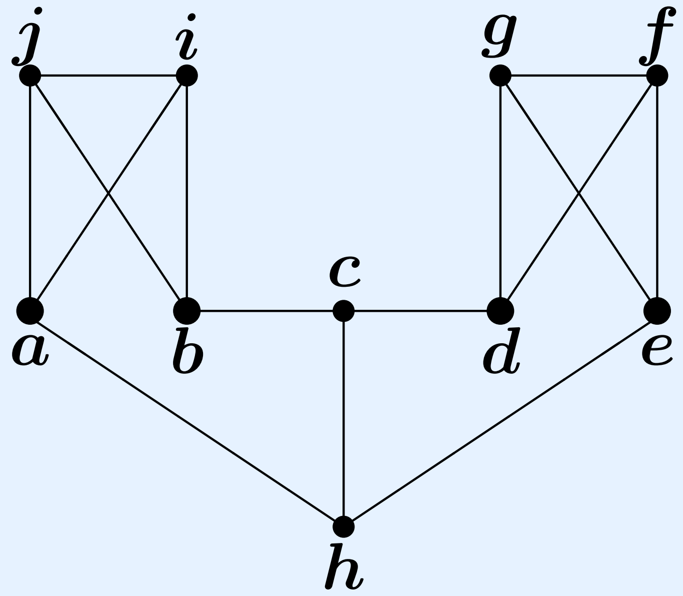


Figure 2: A cubic graph G such that $\lambda_{min}(A_G) = -2$ and $\upsilon_G(2) = 4 = \alpha(G)$.

2. Quadratic programming upper bounds for $\alpha(G)$ (cont.)

- **Now consider the quadratic programming problem**

$$v(G,C) = \max\{2\hat{e}^T x - x^T(\frac{C}{\lambda_{min}(C)} + I_n)x : x \geq 0\}, \quad (4)$$

where C is a real symmetric matrix, $C = (c_{ij})$, such that $c_{ij} = 0$ if $i = j$ or $ij \notin E(G)$.

- **The above defined matrix C is called weighted adjacency matrix of G.**

2. Quadratic programming upper bounds for $\alpha(G)$ (cont.)

- **[Luz and Schrijver, 2005] The Lovász theta number, introduced in [Lovász, 1979] as a bound on the Shannon capacity of a graph, may be redefined as follows:**

$$\vartheta(G) = \min_{C} \upsilon(G, C),$$

where C ranges over all weighted adjacency matrices of G.

- **As it is well known, for every graph G,**

$$\alpha(G) \leq \vartheta(G).$$

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2. Quadratic programming upper bounds for $\alpha(G)$ (cont.)

- **Then, for every graph G, $\vartheta(G) \leq \upsilon(G)$.**
- **Additionally, according to the sandwich theorem,**

1. $\alpha(G) \leq \vartheta(G) \leq \bar{\chi}(G)$, where $\bar{\chi}(G)$ denotes the cardinality of a minimum clique cover of G (that is, a minimum vertex set partition such that each subset is a clique);
2. and therefore, if G is a perfect graph then $\alpha(G) = \vartheta(G)$ (note that a perfect graph is a graph G such that $\alpha(H) = \bar{\chi}(H)$ for every induced subgraph H of G).

3. The class of $\mathcal{Q}$-graphs

- **The graphs G such that $\alpha(G) = \upsilon(G)$ are called graphs with convex-QP stability number, where QP means quadratic program. The class of these graphs is denoted by $\mathcal{Q}$ and its elements called $\mathcal{Q}$-graphs.**

- **The graph depicted in Figure 2 is an example of a $\mathcal{Q}$-graph.**

- **[Lozin and C, 2001] The class $\mathcal{Q}$ is not hereditary, in the sense that is not closed under vertex deletion. However, if $G \in \mathcal{Q}$ and $\exists U \subseteq V(G)$ such that $\alpha(G) = \alpha(G - U)$, then $G - U \in \mathcal{Q}$.**

3. The class of $\mathcal{Q}$-graphs(cont.)

[C, 2001] There exists an infinite number of $\mathcal{Q}$-graphs.

1. A connected graph with at least one edge, which is nor a star neither a triangle, has a perfect matching if and only if its line graph is a $\mathcal{Q}$-graph.
2. If each component of G has a no null even number of edges then $L(L(G))$ is a $\mathcal{Q}$-graph.

- There are several famous $\mathcal{Q}$-graphs. For instance, the Petersen graph P (where $\lambda_{min}(A_P) = -2$ and $\alpha(P) = 4$) and the Hoffman-Singleton graph H (where $\lambda_{min}(A_H) = -3$ and $\alpha(H) = 15$).

3. The class of $\mathcal{Q}$-graphs(cont.)

Regarding generalized line graphs (GLGs), which are line graphes of simple graphs or line graphs of graphs with blossoms, that is, graphs with petals (pendent double edges) attached at its vertices, it is convenient to redefine the concept of line graph.

- **The** *line graph* $L(H)$ **of a graph** H **has the edges of** H **as its vertices, with two vertices of** $L(H)$ **being adjacent if and only if the corresponding edges of** H **have precisely one vertex in common.**

3. The class of $\mathcal{Q}$-graphs(cont.)

- A GLG may be denoted by $L(H; a_1, \ldots, a_n)$, where H is a graph with a_i petals attached at vertex i, for $i = 1, \ldots, n$.

Then we have the following recent result:

[C and D. Cvetković, 2006] Let $G = L(H, a_1, \ldots, a_n)$ be a GLG different from K_n. Let $V(H) = V_1 \cup V_2$, where $V_1 = \{i \in V(H) : a_i > 0\}$ and $V_2 = V(H) \setminus V_1$. If $V_2 = \emptyset$ or $H[V_2]$ has no edges then $G \in \mathcal{Q}$ otherwise $G \in \mathcal{Q}$ if and only if the subgraph $H[V_2]$, after deleting its isolated vertices (if they exists), has a perfect matching.

3. The class of $\mathcal{Q}$-graphs(cont.)

Related with the recognition of $\mathcal{Q}$-graphs, we may refer the following results [C, 2001]:

- A graph G belongs to $\mathcal{Q}$ if and only if each of its components belongs to $\mathcal{Q}$.

- Every graph G has an induced subgraph $H \in \mathcal{Q}$.

- If $\exists U \subseteq V(G)$ such that $\upsilon(G) = \upsilon(G - U)$ and $\lambda_{min}(A_G) < \lambda_{min}(A_{G-U})$, then $G \in \mathcal{Q}$.

3. The class of $\mathcal{Q}$-graphs(cont.)

- If $\exists v \in V(G)$ such that

$$\upsilon(G) \neq \max\{\upsilon(G-v), \upsilon(G-N_G(v))\},$$

then $G \notin \mathcal{Q}$.

- Consider that $\exists v \in V(G)$ $\upsilon(G-v) \neq \upsilon(G-N_G(v))$.
  - If $\upsilon(G) = \upsilon(G-v)$ then

$$G \in \mathcal{Q} \Leftrightarrow G-v \in \mathcal{Q};$$

  - If $\upsilon(G) = \upsilon(G-N_G(v))$ then

$$G \in \mathcal{Q} \Leftrightarrow G-N_G(v) \in \mathcal{Q}.$$

3. The class of $\mathcal{Q}$-graphs(cont.)

The above results allows the recognition of $\mathcal{Q}$-graphs, except for adverse graphs, which are graphs having an induced subgraph G without isolated vertices and such that $\upsilon(G)$ is not integer, for which for every vertex $v \in V(G)$ the following conditions hold:

- $\upsilon(G) = \upsilon(G - N_G(v))$.
- $\lambda_{min}(A_G) = \lambda_{min}(A_{G-N_G(v)})$.

3. The class of $\mathcal{Q}$-graphs(cont.)

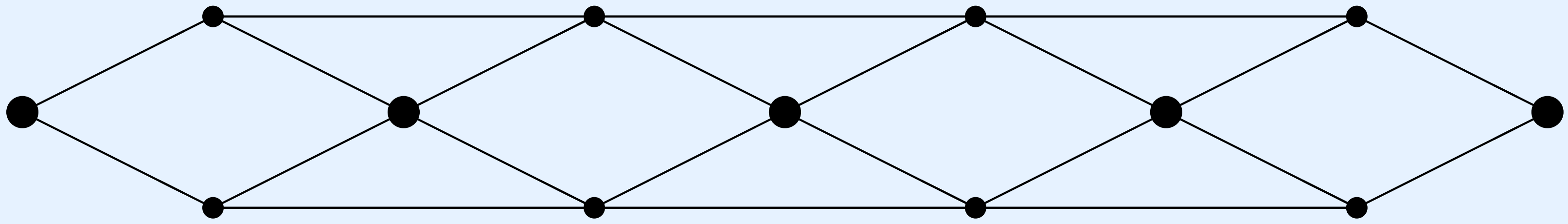
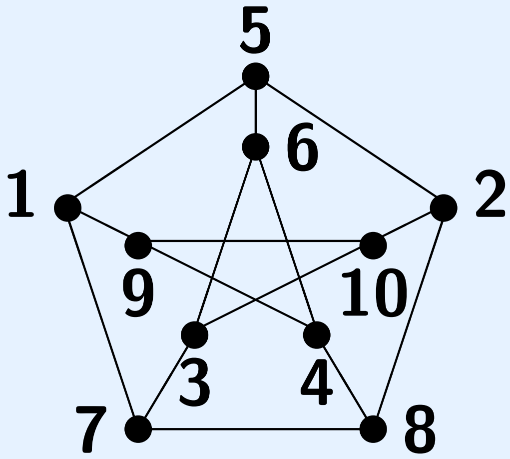


Figure 4: Adverse graph G, where $\lambda_{min}(A_G) = -2$ and $\upsilon(G) = \alpha(G) = 5$.

4. Recognition of $\mathcal{Q}$-graphs and (k, τ)-regular sets

- **A vertex subset $S \subseteq V(G)$ is (k, τ)-regular if induces a k-regular subgraph and $\forall v \notin S \ \ |N_G(v) \cap S| = \tau$.**

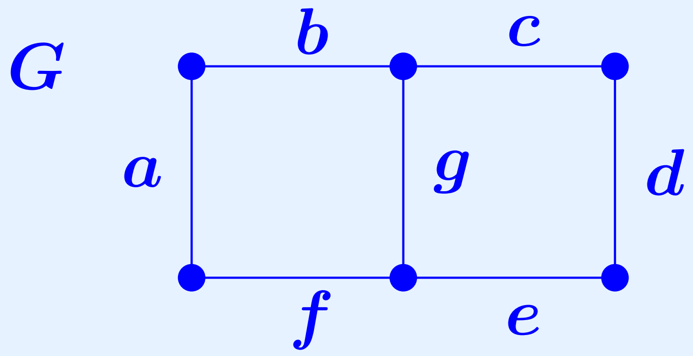


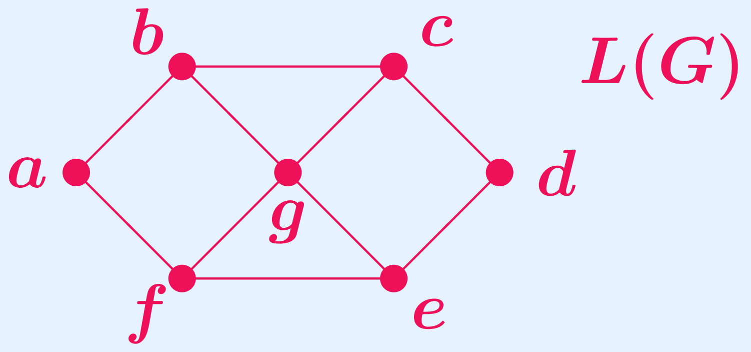
Considering the Petersen graph, $S_1 = \{1, 2, 3, 4\}$ is $(0, 2)$-regular, $S_2 = \{5, 6, 7, 8, 9, 10\}$ is $(1, 3)$-regular and $S_3 = \{1, 2, 5, 7, 8\}$ is $(2, 1)$-regular.

4. Recognition of $\mathcal{Q}$-graphs and (k, τ)-regular sets (cont.)

- Each Hamilton cycle of an Hamiltonian graph G defines a $(2, 4)$-regular set in $L(G)$.

- For instance, in the graph G of the next figure, the edge set $\{a, b, c, d, e, f\} \subset E(G)$ defines a $(2, 4)$-regular set in $L(G)$.





4. Recognition of $\mathcal{Q}$-graphs and (k, τ)-regular sets (cont.)

- **In the opposite direction, if a line graph $L(G)$ has a $(2, 4)$-regular set $S \subset V(L(G))$ such that the subgraph induced by S is connected, then G is Hamiltonian.**

- **A graph G has a perfect matching if and only if $L(G)$ has a $(0, 2)$-regular set.**

4. Recognition of $\mathcal{Q}$-graphs and (k, τ)-regular sets (cont.)

[C and Rama, 2004] A graph G has a (k, τ)-regular set S if and only if the characteristic vector x of S is a solution for the linear system $(A_G - (k - \tau)I)x = \tau \hat{e}$.

- In the case of regular graphs we have the following result:

[Thompson, 1981] A p-regular graph has a (k, τ)-regular set S, with $\tau > 0$, if and only if $k - \tau$ is an adjacency eigenvalue and $(p - k + \tau)x(S) - \tau \hat{e}$ belongs to the corresponding eigenspace.

4. Recognition of $\mathcal{Q}$-graphs and (k, τ)-regular sets (cont.)

[C, 2003] If G is an adverse graph then $G \in \mathcal{Q}$ if and only if $\exists S \subseteq V(G)$ which is $(0, \tau)$-regular, with $\tau = -\lambda_{min}(A_G)$.

- **In general, the determination of (k, τ)-regular sets is NP-hard.**
- **However, it is open to know if there is a polynomial-time algorithm for the recognition $(0, \tau)$-regular sets in adverse graphs and then the polynomial-time recognition of $\mathcal{Q}$-graphs is open.**

4. Recognition of $\mathcal{Q}$-graphs and (k, τ)-regular sets (cont.)

There are several families of graphs for which we may recognize (in polynomial-time) $\mathcal{Q}$-graphs. For instance,

- Bipartite graphs.

It should be noted that the minimum eigenvalue of a connected bipartite graph G is simple, then $\exists v \in V(G)$ such that $\lambda_{min}(A_G) < \lambda_{min}(A_{G-\{v\}})$.

4. Recognition of $\mathcal{Q}$-graphs and (k, τ)-regular sets (cont.)

- Dismantable graphs, that is, graphes with the following recursive definition:

One-vertex graph is dismantable and a graph G with at least two vertices is dismantable if $\exists x, y \in V(G)$ such that $N_G[x] \subseteq N_G[y]$ (where $N_G[v] = N_G(v) \cup \{v\}$) and $G - \{x\}$ is dismantlable.

[C, 2003] Given a graph G and $\tau > 1$, if $\exists p, q \in V(G)$ such that $N_G[q] \subseteq N_G[p]$ then $\upsilon_G(\tau) > \upsilon_{G-N_G(p)}(\tau)$.

4. Recognition of $\mathcal{Q}$-graphs and (k, τ)-regular sets (cont.)

• **Graphs without induced connected subgraphs H such that dilw$(H) > \omega(H)$.**

Given two vertices $x, y \in V(G)$, if $N_G(y) \subseteq N_G[x]$ then we say that the vertices x and y are comparable. The Dilworth number of a graph G, dilw(G), is the largest number of pairwise incomparable vertices of G.

[C, 2003] Let G be a not complete graph. If dilw$(G) \leq \omega(G)$ then G is not adverse.

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