# On the kernels of Wiener-Hopf-Hankel operators on variable exponent Lebesgue spaces[1]

L.P. Castro[†,⋆], A.S. Silva[†]

[†] CIDMA – Center for Research and Development in Mathematics and Applications,
Department of Mathematics, University of Aveiro, Aveiro, Portugal.

[⋆] *Corresponding Author*. Castro@ua.pt.

*Dedicated to honoring the memory of Professor A.V. Balakrishnan,*
*an eminent researcher and an inspirational communicator.*

**Abstract.** We investigate properties of the kernels (and cokernels) of Wiener-Hopf plus and minus Hankel operators on variable exponent Lebesgue spaces. Constructive operator identities are used in view to describe those kernels upon the consideration of auxiliary operators. Moreover, a Coburn-Simonenko type theorem is obtained for Wiener-Hopf plus and minus Hankel operators in the framework of variable exponent Lebesgue spaces.

## 1 Introduction

Convolution type operators [3, 6, 12] arise in a great diversity of applied problems and are useful in a significant amount of different areas of science. Having on their structure a convolution, this can be reflected on the definition of those operators in different ways. Moreover, the convolution itself can be defined in a large amount of varieties, but having always a factorization property associated with a certain integral transform [5, 21, 22, 31]. For the classical convolution case, it is clear that such transform is the Fourier transform, and that the corresponding factorization property ends with the product of the two Fourier transforms of the elements initially used in the convolution. Anyway, now-a-days different convolutions are known and considered in a great diversity of situations and associated with different integral transforms (e.g. like the fractional Fourier transform, Hankel and Hartley type transforms); cf. [4, 5, 21, 22, 23, 31].

[2010] **Mathematics Subject Classification**: 42B25; 45P05; 46E30; 47A53; 47B35; 47G10.
Keywords: Wiener-Hopf operator, Hankel operator, kernel, cokernel, variable exponent Lebesgue space, equivalence after extension relation.
[1] Accepted author's manuscript (AAM) published in [MESA - Mathematics in Engineering, Science and Aerospace 7(2) (2016), 375–386]. Final source webpage in http://nonlinearstudies.com/index.php/mesa/

Not rarely, the convolution type operators that arise in the applications appear not isolated but, instead, multiplied by other types of operators. Even if sometimes those other operators are very simple ones, their multiplications by the convolution type operators change significantly the final structural properties of the resulting operators (e.g., their Fredholm and invertibility properties, as well as the structure of their inverses and the corresponding solutions of the equations characterized by them); see [15, 16, 17, 20].

That is the case e.g. in the theory of classical convolution operators firstly defined within some spaces on the real line, which afterwards are truncated to the half-line, and known as Wiener-Hopf operators; or even the previous ones composed with the reflection operator (and therefore giving rise to a Hankel type operator). For the last two types of operators, it also occurs that algebraic sums of those again have special invertibility properties and are also very important in some applications. This is the case in the phenomena of wave diffraction by configurations which exhibit certain symmetries, and where Wiener-Hopf plus Hankel and Wiener-Hopf minus Hankel (i.e. Wiener-Hopf-Hankel) operators appear in a very natural manner.

The classes of Wiener-Hopf plus and minus Hankel operators that initially appear in those wave diffraction problems are considered within a framework of Bessel potential (or Sobolev) spaces; see [7, 8, 9]. Anyway, a lifting procedure is possible to be done in the smoothness of the spaces so that the initial operators are related with new Wiener-Hopf plus and minus Hankel operators acting already between classical Lebesgue spaces.

Having in our disposal the recent huge development in variable exponent Lebesgue spaces [19], a corresponding research has been also started on the analysis of the properties of several types of operators within the framework of variable exponent Lebesgue spaces (cf. [14, 24, 26, 27]), in contrast with the previous knowledge of their properties when considered on the classical Lebesgue space.

In some cases, concrete applications are already known for such more global framework of variable exponents. This is the case e.g. in the modeling of so-called electrorheological fluids [30], thermo-rheological fluids [1] and image processing [13]. In some other cases, the research is just being moved by the natural search for a larger mathematical formal knowledge.

Having all that in mind, in the present paper we will analyse certain invertibility properties of Wiener-Hopf plus Hankel and Wiener-Hopf minus Hankel operators in the framework of variable exponent Lebesgue spaces [10, 11]. Namely, after introducing, in the next section, the formal definition of those operators and spaces, as well as known results of some of their associated operators, in Section 3 we will analyse the kernel of those operators with the help of certain auxiliary operators. To this end, we will use several operator identities between that operators in analysis. In Section 4, we deduce a Coburn-Simonenko type theorem [18] for Wiener-Hopf plus Hankel and Wiener-Hopf minus Hankel operators on variable exponent Lebesgue spaces (which basically describes an alternative result in the dimensions of theirs kernels and cokernels).

## 2 Definitions and basic results

Let $p : \mathbb{R} \to [1,\infty]$ be measurable a.e. finite function. We denote by $L^{p(\cdot)}(\mathbb{R})$ the set of all complex-valued functions $f$ on $\mathbb{R}$ such that

$$I_{p(\cdot)}\left(\frac{f}{\lambda}\right) := \int_{\mathbb{R}} \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx \ < \ \infty$$

for some $\lambda > 0$. This set becomes a Banach space when equipped with the norm

$$\|f\|_{p(\cdot)} := \inf\left\{\lambda > 0 \,:\, I_{p(\cdot)}\left(\frac{f}{\lambda}\right) \leq 1\right\}.$$

The space $L^{p(\cdot)}(\mathbb{R})$ is precisely what we are refereing as the *variable exponent Lebesgue space*, since it generalizes the standard Lebesgue space.

Throughout this paper we will be always assuming that

$$p_- := \mathrm{ess}_{x\in\mathbb{R}} \inf p(x) > 1\,, \qquad p_+ := \mathrm{ess}_{x\in\mathbb{R}} \sup p(x) < \infty. \tag{2.1}$$

Under these conditions, the space $L^{p(\cdot)}(\mathbb{R})$ is separable and reflexive, and its dual space is isomorphic to $L^{q(\cdot)}(\mathbb{R})$, where $q(\cdot)$ is the conjugate exponent function defined by

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1 \qquad (x \in \mathbb{R}).$$

Additionally, with condition (2.1) we have that $\|\phi I\|_{\mathcal{L}(L^{p(\cdot)}(\mathbb{R}))} \leq \|\phi\|_{L^\infty(\mathbb{R})}$ for a function $\phi \in L^\infty(\mathbb{R})$, and where $I$ denotes the identity operator.

We shall denote by $L^{p(\cdot)}(\mathbb{R}_+)$ the variable exponent Lebesgue space of complex-valued functions on the positive half-line $\mathbb{R}_+ = (0,+\infty)$. The subspace of $L^{p(\cdot)}(\mathbb{R})$ formed by all functions supported on the closure of $\mathbb{R}_+$ is denoted by $L_+^{p(\cdot)}(\mathbb{R})$ and $L_-^{p(\cdot)}(\mathbb{R})$ represents the subspace of $L^{p(\cdot)}(\mathbb{R})$ formed by all the functions supported on the closure of $\mathbb{R}_- := (-\infty, 0)$.

We will make an extensive use of the Fourier transformation $\mathcal{F}$, defined in the Schwartz space $\mathcal{S}(\mathbb{R})$ of rapidly decreasing functions by

$$\mathcal{F}\varphi(\xi) := \int_{\mathbb{R}} e^{i\eta\xi}\varphi(\eta)\, d\eta \qquad (\xi \in \mathbb{R}),$$

and which has an inverse $\mathcal{F}^{-1}$, also given on $\mathcal{S}(\mathbb{R})$ by

$$\mathcal{F}^{-1}\psi(\eta) := \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\eta\xi}\psi(\xi)\, d\xi \qquad (\eta \in \mathbb{R}).$$

**Definition 1.** *(cf. [25, 29]) If $1 < p_- \leq p_+ < \infty$, a function $\phi \in L^\infty(\mathbb{R})$ is said to be an $L^{p(\cdot)}$-Fourier multiplier if there is a constant $C$ such that for all $f \in \mathcal{S}(\mathbb{R})$ we have*

$$\|\mathcal{F}^{-1}\phi \cdot \mathcal{F} f\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)}.$$

For any $L^{p(\cdot)}$-Fourier multiplier $\phi$ the operator $f \mapsto \mathcal{F}^{-1}\phi \cdot \mathcal{F} f$ extends uniquely to a bounded operator on $L^{p(\cdot)}(\mathbb{R})$ which will be denoted by $W^0_\phi$. The set of all $L^{p(\cdot)}$-Fourier multipliers will be denoted by $\mathcal{M}_{p(\cdot)}$. It is clear that $\mathcal{M}_{p(\cdot)}$ is a unital normed algebra under pointwise operations and the norm

$$\|\phi\|_{\mathcal{M}_{p(\cdot)}} := \|W^0_\phi\|_{\mathcal{L}(L^{p(\cdot)}(\mathbb{R}))} = \|\mathcal{F}^{-1}\phi \cdot \mathcal{F}\|_{\mathcal{L}(L^{p(\cdot)}(\mathbb{R}))}.$$

We are now in condition to identify in a mathematical way the main objects of this work.

We will consider Wiener-Hopf plus and minus Hankel operators, acting between Lebesgue spaces with variable exponent $p(\cdot)$, denoted by

$$W_\phi \pm H_\phi : L_+^{p(\cdot)}(\mathbb{R}) \to L^{p(\cdot)}(\mathbb{R}_+),$$

with $W_\phi$ and $H_\phi$ being Wiener-Hopf and Hankel operators defined by

$$W_\phi = r_+\mathcal{F}^{-1}\phi\cdot\mathcal{F}, \qquad H_\phi = r_+\mathcal{F}^{-1}\phi\cdot\mathcal{F}J, \tag{2.2}$$

respectively. Here, $r_+$ represents the operator of restriction from $L^{p(\cdot)}(\mathbb{R})$ onto $L^{p(\cdot)}(\mathbb{R}_+)$, $\phi$ is the so-called Fourier symbol which is assumed to belong to $\mathcal{M}_{p(\cdot)}$, and

$$J : L_+^{p(\cdot)}(\mathbb{R}) \to L^{p(\cdot)}(\mathbb{R})$$

is the reflection operator given by the rule

$$J\varphi(x) = \widetilde{\varphi}(x) = \varphi(-x)$$

which throughout the paper will be always defined for even functions $p(\cdot)$ (so that $J$ will therefore be a bounded operator in those variable exponent Lebesgue spaces).

Under those conditions, the Wiener-Hopf and the Hankel operators, defined in (2.2), are bounded linear operators.

It is also interesting to remark that the boundedness of a wide variety of operators follows from the boundedness of the Hardy-Littlewood maximal operator on variable exponent Lebesgue spaces. Given $f \in L^1_{\text{loc}}(\mathbb{R})$, we recall that the Hardy-Littlewood maximal operator $M$ is defined by

$$(Mf)(x) := \sup_{x\in\Omega} \frac{1}{|\Omega|}\int_\Omega |f(y)|dy,$$

where the supremum is taken over all intervals $\Omega \subset \mathbb{R}$ containing $x$, and the Cauchy singular integral operator $S$ is defined by

$$(Sf)(x) := \frac{1}{\pi i}\int_{\mathbb{R}} \frac{f(\tau)}{\tau - x}d\tau,$$

where the integral is understood in the principal value sense.

**Theorem 1.** *(cf., e.g., [26, Theorem 2.1.])* Let $p : \mathbb{R} \to [1,\infty]$ be a measurable function satisfying (2.1). If the Hardy-Littlewood maximal operator $M$ is bounded on $L^{p(\cdot)}(\mathbb{R})$, then the Cauchy singular integral operator $S$ is bounded on $L^{p(\cdot)}(\mathbb{R})$.

The following result states a sufficient condition on $p(\cdot)$ for $M$ to be bounded on $L^{p(\cdot)}(\mathbb{R})$.

**Theorem 2.** *(cf. e.g. [29, Theorem 2.5.], [19])* Let $p : \mathbb{R} \to [1,\infty]$ satisfy (2.1). In addition, suppose that there exist constants $A_0$ and $A_\infty$ such that $p(\cdot)$ satisfies

$$|p(x) - p(y)| \le \frac{A_0}{-\log|x-y|}, \qquad |x-y| \le \frac{1}{2}, \tag{2.3}$$

and

$$|p(x) - p(y)| \le \frac{A_\infty}{\log(e+|x|)}, \qquad |x| \le |y|. \tag{2.4}$$

Then, the Hardy-Littlewood maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R})$.

It is important to stress that the log-Hölder continuity conditions (2.3)-(2.4) are not necessary conditions for the last purpose. The exponent $p(\cdot)$ can even be discontinuous. In fact, Lerner have shown in [28] that there exist discontinuous variable exponents $p(\cdot)$ (without limit at infinity) for which the Hardy-Littlewood maximal operator is bounded on the $L^{p(\cdot)}(\mathbb{R})$ space. Under conditions

(2.1), (2.3) and (2.4), from [25] it is also known that $\mathcal{M}_{p(\cdot)}$ is a Banach algebra under pointwise operations and the norm $\|\cdot\|_{\mathcal{M}_{p(\cdot)}}$, and that $\mathcal{M}_{p(\cdot)}$ is continuously embedded into $L^\infty(\mathbb{R})$.

Let $\mathcal{B}(\mathbb{R})$ denote the class of exponents $p:\mathbb{R}\to[1,\infty]$ continuous on $\mathbb{R}$ satisfying (2.1), (2.3) and (2.4) (and therefore with $M$ and $S$ being bounded operators on $L^{p(\cdot)}(\mathbb{R})$). Additionally, $\mathcal{B}_e(\mathbb{R})$ represents the set of all even functions $p(\cdot)\in\mathcal{B}(\mathbb{R})$.

Let $\ell_0$ denote the zero extension operator from the space $L^{p(\cdot)}(\mathbb{R}_+)$ into the space $L_+^{p(\cdot)}(\mathbb{R})$,

$$\ell_0 : L^{p(\cdot)}(\mathbb{R}_+) \to L_+^{p(\cdot)}(\mathbb{R}). \tag{2.5}$$

We will denote by $P$ the canonical projection of $L^{p(\cdot)}(\mathbb{R})$ onto $L_+^{p(\cdot)}(\mathbb{R})$, and its complementary projection $Q := I-P$ of $L^{p(\cdot)}(\mathbb{R})$ onto $L_-^{p(\cdot)}(\mathbb{R})$. We have that

$$P := \ell_0 r_+ .$$

Note that $P^2 = P$ and $Q^2 = Q$.

Additionally, we will use the notation $\mathcal{P}$ to denote the Riesz projection defined by

$$\mathcal{P} = \mathcal{F}\ell_0 r_+ \mathcal{F}^{-1}$$

and denote by $\mathcal{Q}$ its complementary projection, $\mathcal{Q} = I - \mathcal{P}$.

By analogy with [3, Proposition 2.10], we derive the following relations between Wiener-Hopf and Hankel operators acting on variable exponent Lebesgue spaces.

**Proposition 1.** Let $\phi$, $\varphi \in \mathcal{M}_{p(\cdot)}$ and $p(\cdot)\in\mathcal{B}_e(\mathbb{R})$. Then

$$W_{\phi\varphi} = W_\phi \ell_0 W_\varphi + H_\phi \ell_0 H_{\widetilde{\varphi}} \tag{2.6}$$

$$H_{\phi\varphi} = W_\phi \ell_0 H_\varphi + H_\phi \ell_0 W_{\widetilde{\varphi}}. \tag{2.7}$$

In what follows, we will also make use of the relations

$$JQ = PJ,\ \ JP = QJ,\ \ J^2 = I,\ \ JW_\phi^0 J = W_{\widetilde{\phi}}^0 .$$

## 3 Kernels of Wiener-Hopf-Hankel operators

We will denote by $\mathcal{GM}_{p(\cdot)}$ the group of all invertible elements of $\mathcal{M}_{p(\cdot)}$. Let $\phi\in\mathcal{GM}_{p(\cdot)}$, $p(\cdot)\in\mathcal{B}_e(\mathbb{R})$ and consider the Wiener-Hopf plus and minus Hankel operators,

$$W_\phi \pm H_\phi : L_+^{p(\cdot)}(\mathbb{R}) \to L^{p(\cdot)}(\mathbb{R}_+).$$

With some computations, we can obtain an important formula to "associate" these operators with a (pure) Wiener-Hopf operator. Namely, multiplying $W_\phi \pm H_\phi$ on the left by the zero extension operator $\ell_0 : L^{p(\cdot)}(\mathbb{R}_+) \to L_+^{p(\cdot)}(\mathbb{R})$, and on the right by the projection $P : L^{p(\cdot)}(\mathbb{R}) \to L_+^{p(\cdot)}(\mathbb{R})$, we obtain

$$\ell_0(W_\phi \pm H_\phi)P = P(W_\phi^0 \pm W_\phi^0 J)P : L^{p(\cdot)}(\mathbb{R}) \to L_+^{p(\cdot)}(\mathbb{R}),$$

where we recall that $W_\phi^0 = \mathcal{F}^{-1}\phi\cdot\mathcal{F}$.

To extend these operators to the full $L^{p(\cdot)}(\mathbb{R})$ space, we use the projection $Q : L^{p(\cdot)}(\mathbb{R}) \to L_-^{p(\cdot)}(\mathbb{R})$, obtaining the equivalent after extension [2] operators

$$PW^0_\phi P \pm PW^0_\phi JP + Q : L^{p(\cdot)}(\mathbb{R}) \to L^{p(\cdot)}(\mathbb{R}).$$

Again, let $I := I_{L^{p(\cdot)}(\mathbb{R})}$ denote the identity operator on $L^{p(\cdot)}(\mathbb{R})$. Observe that

$$PW^0_\phi P \pm PW^0_\phi JP + Q = (I - PW^0_\phi Q \mp PW^0_\phi JQ)(PW^0_\phi \pm PW^0_\phi J + Q),$$

where the operators $I - PW^0_\phi Q \mp PW^0_\phi JQ$ are invertible (with inverses given by $I + PW^0_\phi Q \pm PW^0_\phi JQ$, respectively). Let

$$\Psi = \begin{bmatrix} 0 & \phi\widetilde{\phi^{-1}} \\ -1 & \widetilde{\phi^{-1}} \end{bmatrix}. \tag{3.1}$$

We can state the following

$$\begin{bmatrix} PW^0_\phi P + PW^0_\phi JP + Q & 0 \\ 0 & PW^0_\phi P - PW^0_\phi JP + Q \end{bmatrix} = A\left(PW^0_\Psi P + Q\right)B \tag{3.2}$$

where $A := A_1 A_2$ and $B$ are invertible operators with

$$\begin{aligned} A_1 &= \frac{1}{2}\begin{bmatrix} I - PW^0_\phi Q - PW^0_\phi JQ & 0 \\ 0 & I - PW^0_\phi Q + PW^0_\phi JQ \end{bmatrix}\begin{bmatrix} I & J \\ I & -J \end{bmatrix} \\ A_2 &= I + PW^0_\Psi Q \\ B &= \begin{bmatrix} I & 0 \\ W^0_{\widetilde{\phi}} & W^0_{\widetilde{\phi}} \end{bmatrix}\begin{bmatrix} I & I \\ J & -J \end{bmatrix}, \end{aligned}$$

with inverses given by

$$\begin{aligned} A_1^{-1} &= \begin{bmatrix} I & I \\ J & -J \end{bmatrix}\begin{bmatrix} I + PW^0_\phi Q + PW^0_\phi JQ & 0 \\ 0 & I + PW^0_\phi Q - PW^0_\phi JQ \end{bmatrix} \\ A_2^{-1} &= I - PW^0_\Psi Q \\ B^{-1} &= \frac{1}{2}\begin{bmatrix} I & J \\ I & -J \end{bmatrix}\begin{bmatrix} I & 0 \\ -I & W^0_{\widetilde{\phi^{-1}}} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} I - J & W^0_{\phi^{-1}} J \\ I + J & -W^0_{\phi^{-1}} J \end{bmatrix}, \end{aligned}$$

respectively.

Moreover, we can rewrite the operator $PW^0_\Psi P + Q$ as a product of two matrix operators. Namely, for the first component, we have

$$PW^0_\Psi P = \begin{bmatrix} PW^0_{\phi\widetilde{\phi^{-1}}} P & 0 \\ 0 & I_{L_+^{p(\cdot)}(\mathbb{R})} \end{bmatrix}\begin{bmatrix} 0 & I_{L_+^{p(\cdot)}(\mathbb{R})} \\ -I_{L_+^{p(\cdot)}(\mathbb{R})} & PW^0_{\widetilde{\phi^{-1}}} P \end{bmatrix}, \tag{3.3}$$

where

$$C := \begin{bmatrix} 0 & I_{L_+^{p(\cdot)}(\mathbb{R})} \\ -I_{L_+^{p(\cdot)}(\mathbb{R})} & PW^0_{\widetilde{\phi^{-1}}} P \end{bmatrix}$$

is invertible with inverse

$$C^{-1} = \begin{bmatrix} PW^0_{\widetilde{\phi}^{-1}}P & -I_{L_+^{p(\cdot)}(\mathbb{R})} \\ I_{L_+^{p(\cdot)}(\mathbb{R})} & 0 \end{bmatrix}.$$

Thus, we can rewrite (3.2) as

$$\begin{bmatrix} PW^0_\phi P + PW^0_\phi JP + Q & 0 \\ 0 & PW^0_\phi P - PW^0_\phi JP + Q \end{bmatrix} = A\left( \begin{bmatrix} PW^0_{\widetilde{\phi\phi^{-1}}}P & 0 \\ 0 & I_{L_+^{p(\cdot)}(\mathbb{R})} \end{bmatrix} C + Q \right) B. \tag{3.4}$$

**Theorem 3.** Let $\phi \in \mathcal{GM}_{p(\cdot)}$, $p(\cdot) \in \mathcal{B}_e(\mathbb{R})$ and consider the operators

$$W_\phi \pm H_\phi : L_+^{p(\cdot)}(\mathbb{R}) \to L^{p(\cdot)}(\mathbb{R}_+).$$

(i) If $f \in \ker W_{\phi\widetilde{\phi}^{-1}}$, then

$$\begin{aligned} (F,G)^T &:= \frac{1}{2}\left( PW^0_{\widetilde{\phi}^{-1}}f + JQW^0_{\widetilde{\phi}^{-1}}f,\ PW^0_{\widetilde{\phi}^{-1}}f - JQW^0_{\widetilde{\phi}^{-1}}f \right)^T \\ &\in \ker\ \mathrm{diag}(W_\phi + H_\phi, W_\phi - H_\phi); \end{aligned} \tag{3.5}$$

(ii) If $(F,G)^T \in \ker\ \mathrm{diag}(W_\phi + H_\phi, W_\phi - H_\phi)$, then

$$f := PW^0_{\widetilde{\phi}}(F+G) + PW^0_{\widetilde{\phi}}J(F-G) \in \ker W_{\phi\widetilde{\phi}^{-1}}. \tag{3.6}$$

Moreover, the operators

$$\begin{aligned} L_1 &: \ker W_{\phi\widetilde{\phi}^{-1}} \to \ker\ \mathrm{diag}(W_\phi + H_\phi,\ W_\phi - H_\phi), \\ L_2 &: \ker\ \mathrm{diag}(W_\phi + H_\phi,\ W_\phi - H_\phi) \to \ker\ W_{\phi\widetilde{\phi}^{-1}}, \end{aligned}$$

defined by the actions in (3.5) and (3.6) are invertible operators satisfying $L_2 = L_1^{-1}$.

**Proof.** The result follows from relation (3.4). In fact, from (3.4), if $f \in \ker W_{\phi\widetilde{\phi}^{-1}}$, and consequently

$$(f,0)^T \in \ker\ \mathrm{diag}(W_{\widetilde{\phi\phi^{-1}}}, I),$$

then $B^{-1}PC^{-1}((f,0)^T)$ belongs to

$$\ker(W_\phi + H_\phi + Q, W_\phi - H_\phi + Q) \cong \ker(W_\phi + H_\phi, W_\phi - H_\phi).$$

Moreover,

$$B^{-1}PC^{-1}((f,0)^T) = \frac{1}{2}\left( PW^0_{\widetilde{\phi}^{-1}}Pf + JQW^0_{\widetilde{\phi}^{-1}}Pf,\ PW^0_{\widetilde{\phi}^{-1}}Pf - JQW^0_{\widetilde{\phi}^{-1}}Pf \right)^T,$$

which is equal to $\frac{1}{2}\left( PW^0_{\widetilde{\phi}^{-1}}f + JQW^0_{\widetilde{\phi}^{-1}}f,\ PW^0_{\widetilde{\phi}^{-1}}f - JQW^0_{\widetilde{\phi}^{-1}}f \right)^T$ since $f \in L_+^{p(\cdot)}(\mathbb{R})$. Thus, we obtain (3.5).

On the other hand, if $(F,G)^T \in \ker(W_\phi + H_\phi, W_\phi - H_\phi)$, then

$$CPB((F,G)^T) \in \ker\ \mathrm{diag}(W_{\phi\widetilde{\phi}^{-1}}, I).$$

Thus,

$$CPB((F,G)^T) = \begin{bmatrix} PW^0_{\tilde{\phi}}(F+G) + PW^0_{\tilde{\phi}}J(F-G) \\ -(F+G) + PW^0_{\tilde{\phi}^{-1}}PW^0_{\tilde{\phi}}(F+G) + PW^0_{\tilde{\phi}^{-1}}PW^0_{\tilde{\phi}}J(F-G) \end{bmatrix} \in \ker \operatorname{diag}(W_{\phi\tilde{\phi}^{-1}}, I),$$

which proves (3.6).

Finally, let $f$ and $(F,G)^T$ as defined in (3.5) and (3.6). We observe that

$$CPB(F,G)^T = (f,0)^T,$$

which completes the proof.

**Proposition 2.** Let $\phi \in \mathcal{GM}_{p(\cdot)}$, $p(\cdot) \in \mathcal{B}_e(\mathbb{R})$. If $f \in \ker W_{\phi\tilde{\phi}^{-1}}$, then

$$JQW^0_{\phi\tilde{\phi}^{-1}}f \in \ker W_{\phi\tilde{\phi}^{-1}}$$

and $(JQW^0_{\phi\tilde{\phi}^{-1}})^2 f = f$.

**Proof.** Let $\phi \in \mathcal{GM}_{p(\cdot)}$ $(p(\cdot) \in \mathcal{B}_e(\mathbb{R}))$ and $f \in \ker W_{\phi\tilde{\phi}^{-1}}$. Then

$$\begin{aligned} \ell_0 W_{\phi\tilde{\phi}^{-1}}(JQW^0_{\phi\tilde{\phi}^{-1}}f) &= PW^0_{\phi\tilde{\phi}^{-1}}PJQW^0_{\phi\tilde{\phi}^{-1}}f = JQW^0_{\tilde{\phi}\phi^{-1}}QW^0_{\phi\tilde{\phi}^{-1}}f \\ &= JQW^0_{\tilde{\phi}\phi^{-1}}W^0_{\phi\tilde{\phi}^{-1}}f - JQW^0_{\tilde{\phi}\phi^{-1}}PW^0_{\phi\tilde{\phi}^{-1}}f \\ &= JQf - JQW^0_{\tilde{\phi}\phi^{-1}}0 \\ &= 0. \end{aligned}$$

Additionally, for any $f \in \ker W_{\phi\tilde{\phi}^{-1}}$, we have

$$\begin{aligned} (JQW^0_{\phi\tilde{\phi}^{-1}})^2 f &= JQW^0_{\phi\tilde{\phi}^{-1}}PJQW^0_{\phi\tilde{\phi}^{-1}}Pf = PW^0_{\tilde{\phi}\phi^{-1}}QW^0_{\phi\tilde{\phi}^{-1}}f \\ &= PW^0_{\tilde{\phi}\phi^{-1}}W^0_{\phi\tilde{\phi}^{-1}}f - PW^0_{\tilde{\phi}\phi^{-1}}PW^0_{\phi\tilde{\phi}^{-1}}f = f, \end{aligned}$$

which completes the proof.

Consider the operator $P_{\phi\tilde{\phi}^{-1}} := JQW^0_{\phi\tilde{\phi}^{-1}}P|_{\ker W_{\phi\tilde{\phi}^{-1}}}$. From Proposition 2,

$$P_{\phi\tilde{\phi}^{-1}} : \ker W_{\phi\tilde{\phi}^{-1}} \to \ker W_{\phi\tilde{\phi}^{-1}}$$

and

$$P^2_{\phi\tilde{\phi}^{-1}} = I.$$

Additionally, $P^-_{\phi\tilde{\phi}^{-1}} := (1/2)(I - P_{\phi\tilde{\phi}^{-1}})$ and $P^+_{\phi\tilde{\phi}^{-1}} := (1/2)(I + P_{\phi\tilde{\phi}^{-1}})$ are complementary projections generating a decomposition of $\ker W_{\phi\tilde{\phi}^{-1}}$.

For $f \in \ker W_{\phi\tilde{\phi}^{-1}}$, let

$$\varphi^{\pm}(f) := \frac{1}{2}(PW^0_{\tilde{\phi}^{-1}}f \pm JQW^0_{\tilde{\phi}^{-1}}f).$$

By Theorem 3, $\varphi^{\pm}(f) \in \ker(W_\phi \pm H_\phi)$.

**Lemma 1.** Let $\phi \in \mathcal{GM}_{p(\cdot)}$, $p(\cdot) \in \mathcal{B}_e(\mathbb{R})$. For every $f \in \ker W_{\phi\tilde{\phi}^{-1}}$, the following relations

$$(W_{\tilde{\phi}} + H_{\tilde{\phi}})\varphi_+(f) = P^+_{\phi\tilde{\phi}^{-1}} f, \qquad (W_{\tilde{\phi}} - H_{\tilde{\phi}})\varphi_-(f) = P^-_{\phi\tilde{\phi}^{-1}} f$$

hold. The corresponding mapping $\varphi_+ : \operatorname{im} P^+_{\phi\tilde{\phi}^{-1}} \to \operatorname{im} P^+_{\phi\tilde{\phi}^{-1}}$ and $\varphi_- : \operatorname{im} P^-_{\phi\tilde{\phi}^{-1}} \to \operatorname{im} P^-_{\phi\tilde{\phi}^{-1}}$ are injective operators.

**Proof.** Let $f \in \ker W_{\phi\tilde{\phi}^{-1}}$. We will show that the operator $W_{\tilde{\phi}} + H_{\tilde{\phi}}$ maps $\varphi^+(f)$ into $P^+_{\phi\tilde{\phi}^{-1}} f$ and the operator $W_{\tilde{\phi}} - H_{\tilde{\phi}}$ maps $\varphi^-(f)$ into $P^-_{\phi\tilde{\phi}^{-1}} f$.

For the first case, using Proposition 1, we have

$$\begin{aligned}
\ell_0(W_{\tilde{\phi}} + H_{\tilde{\phi}})\varphi^+(f) &= \frac{1}{2}\ell_0(W_{\tilde{\phi}}\ell_0 W_{\tilde{\phi}^{-1}} f + W_{\tilde{\phi}}\ell_0 JQW^0_{\tilde{\phi}^{-1}} f + H_{\tilde{\phi}}\ell_0 W_{\tilde{\phi}^{-1}} f + H_{\tilde{\phi}}\ell_0 JQW^0_{\tilde{\phi}^{-1}} f) \\
&= \frac{1}{2}(\ell_0 W_{\tilde{\phi}}\ell_0 W_{\tilde{\phi}^{-1}} f + \ell_0 W_{\tilde{\phi}} PW^0_{\phi^{-1}} Jf + \ell_0 H_{\tilde{\phi}}\ell_0 W_{\tilde{\phi}^{-1}} f + \ell_0 H_{\tilde{\phi}} PW^0_{\phi^{-1}} Jf) \\
&= \frac{1}{2}(\ell_0 W_{\tilde{\phi}}\ell_0 W_{\tilde{\phi}^{-1}} f + \ell_0 W_{\tilde{\phi}}\ell_0 H_{\phi^{-1}} f + \ell_0 H_{\tilde{\phi}}\ell_0 W_{\tilde{\phi}^{-1}} f + \ell_0 H_{\tilde{\phi}}\ell_0 H_{\phi^{-1}} f) \\
&= \frac{1}{2}(\ell_0 W_{\tilde{\phi}\tilde{\phi}^{-1}} f + \ell_0 H_{\tilde{\phi}\phi^{-1}} f) \\
&= \frac{1}{2}(f + PW^0_{\tilde{\phi}\phi^{-1}} Jf) \\
&= \frac{1}{2}(f + PJW^0_{\phi\tilde{\phi}^{-1}} f) \\
&= \frac{1}{2}(f + JQW^0_{\phi\tilde{\phi}^{-1}} f) = P^+_{\phi\tilde{\phi}^{-1}} f.
\end{aligned}$$

The other case is proved analogously.

**Proposition 3.** Let $\phi \in \mathcal{GM}_{p(\cdot)}$, $p(\cdot) \in \mathcal{B}_e(\mathbb{R})$. Then

$$\ker(W_\phi + H_\phi) = \varphi^+(\operatorname{im} P^+_{\phi\tilde{\phi}^{-1}}),$$
$$\ker(W_\phi - H_\phi) = \varphi^-(\operatorname{im} P^-_{\phi\tilde{\phi}^{-1}})$$

**Proof.** Using the invertible operator $L_1$ defined in Theorem 3, we obtain

$$\ker \operatorname{diag}(W_\phi + H_\phi, W_\phi - H_\phi) = L_1(\ker W_{\phi\tilde{\phi}^{-1}}),$$

where

$$\ker W_{\phi\tilde{\phi}^{-1}} = \operatorname{im} P^+_{\phi\tilde{\phi}^{-1}} \dot{+} \operatorname{im} P^-_{\phi\tilde{\phi}^{-1}}.$$

Hence,

$$\ker \operatorname{diag}(W_\phi + H_\phi, W_\phi - H_\phi) = L_1(\ker W_{\phi\tilde{\phi}^{-1}}) = L_1(\operatorname{im} P^+_{\phi\tilde{\phi}^{-1}}) \dot{+} L_1(\operatorname{im} P^-_{\phi\tilde{\phi}^{-1}}).$$

Thus, having in mind the definitions of $L_1$, $L_2$, $\varphi^+$ and $\varphi^-$, the result follows.

Notice that from the equivalence relation (3.4) we have that if $W_{\phi\tilde{\phi}^{-1}}$ is a Fredholm operator (and continuing to assume that $\phi \in \mathcal{GM}_{p(\cdot)}$ and $p(\cdot) \in \mathcal{B}_e(\mathbb{R})$), then both operators $W_\phi + H_\phi$ and $W_\phi - H_\phi$ are Fredholm and it holds the following identity for the Fredholm indices of the related operators:

$$\operatorname{ind}(W_\phi + H_\phi) + \operatorname{ind}(W_\phi - H_\phi) = \operatorname{ind} W_{\phi\tilde{\phi}^{-1}}.$$

In more detail, and now from Proposition 3, we immediately have the following result for the dimensions of the kernels of $W_\phi + H_\phi$ and $W_\phi - H_\phi$.

**Corollary 1.** Let $\phi \in \mathcal{GM}_{p(\cdot)}$, $p(\cdot) \in \mathcal{B}_e(\mathbb{R})$. Then

$$\dim\ker(W_\phi + H_\phi) = \dim \operatorname{im} P^{+}_{\phi\tilde{\phi}^{-1}},$$
$$\dim\ker(W_\phi - H_\phi) = \dim \operatorname{im} P^{-}_{\phi\tilde{\phi}^{-1}}.$$

## 4 Coburn-Simonenko type theorem for Wiener-Hopf-Hankel operators

The following theorem is the analogue of Coburn-Simonenko Theorem for Wiener-Hopf operators on variable exponent Lebesgue spaces, which states that a nonzero bounded Wiener-Hopf operator has a trivial kernel or a dense range.

**Theorem 4.** If $\phi \in \mathcal{M}_{p(\cdot)}$, $p(\cdot) \in \mathcal{B}(\mathbb{R})$, and if $\phi$ does not vanish identically, then the kernel of $W_\phi$ in $L^{p(\cdot)}_{+}(\mathbb{R})$ is trivial or the image of $W_\phi$ is dense in $L^{p(\cdot)}_{+}(\mathbb{R})$.

**Proof.** First, recall that $W_\phi$ is equivalent after extension [2] to $\phi\mathcal{P} + \mathcal{Q}$.

We will show that $\phi\mathcal{P} + \mathcal{Q}$ is injective on $L^{p(\cdot)}(\mathbb{R})$ or $\mathcal{Q}\phi I + \mathcal{P}$ is injective on $L^{q(\cdot)}(\mathbb{R})$.

Assume the contrary, i.e., assume that there are nonzero $f \in L^{p(\cdot)}(\mathbb{R})$ and $g \in L^{q(\cdot)}(\mathbb{R})$ such that

$$(\phi\mathcal{P} + \mathcal{Q})f = 0 \qquad \text{and} \qquad (\mathcal{Q}\phi I + \mathcal{P})g = 0.$$

The first identity implies that

$$f_- := \mathcal{Q}f \in \mathcal{Q}L^{p(\cdot)}(\mathbb{R}), \qquad f_+ := \mathcal{P}f \in \mathcal{P}L^{p(\cdot)}(\mathbb{R})$$

and

$$\phi f_+ + f_- = 0.$$

From the second equality, we have that $\mathcal{Q}\phi g = 0$ and $\mathcal{P}g = 0$, whence $g_+ := \phi g \in \mathcal{P}L^{q(\cdot)}(\mathbb{R})$ and $g_- := g \in \mathcal{Q}L^{q(\cdot)}(\mathbb{R})$. Multiplying the equality $\phi f_+ = -f_-$ by $g$ we get $\phi g f_+ = -f_- g_-$ and hence,

$$g_+ f_+ = -f_- g_-. \tag{4.1}$$

The left-hand side of (4.1) belongs to $\mathcal{P}L^1(\mathbb{R})$ while its right-hand side belongs to $\mathcal{Q}L^1(\mathbb{R})$. Consequently, $g_+ f_+ = 0$ and $f_- g_- = 0$. Since $g_- = g \neq 0$, then $f_- = 0$. It follows that $f_+ \neq 0$, otherwise, $f = f_- + f_+ = 0$.

Now, the equality $g_+ f_+ = 0$ shows that $0 = g_+ = \phi g$. Since $\phi \in \mathcal{M}_{p(\cdot)} \setminus \{0\}$, we arrive at conclusion that $g = g_- = 0$, which contradicts the hypothesis.

If $\phi\mathcal{P} + \mathcal{Q}$ is injective on $L^{p(\cdot)}(\mathbb{R})$, then $W_\phi$ is injective on $L^{p(\cdot)}(\mathbb{R})$. On the other hand, if $\mathcal{Q}\phi I + \mathcal{P}$ is injective on $L^{q(\cdot)}(\mathbb{R})$, then $C(\mathcal{Q}\phi I + \mathcal{P})C$ is also injective on $L^{p(\cdot)}(\mathbb{R})$ (where $C : \varphi \mapsto \overline{\varphi}$ denote the operator of complex conjugation in $L^{p(\cdot)}(\mathbb{R})$). Since $C(\mathcal{Q}\phi I + \mathcal{P})C = (\phi\mathcal{P} + \mathcal{Q})^*$, we infer that $\phi\mathcal{P} + \mathcal{Q}$ has a dense range on $L^{p(\cdot)}$ and thus, $W_\phi$ has a dense range on $L^{p(\cdot)}(\mathbb{R}_+)$.

**Theorem 5.** Let $\phi \in \mathcal{GM}_{p(\cdot)}$, $p(\cdot) \in \mathcal{B}_e(\mathbb{R})$, and let $T$ denote any of the operators $W_\phi - H_\phi$, $W_\phi + H_\phi$. Then, at least, one of the spaces $\ker T$ or $\operatorname{coker} T$ is trivial.

**Proof.** If $\dim\ker W_{\phi\tilde{\phi}^{-1}} > 0$, then the Coburn-Simonenko theorem for Wiener-Hopf operators gives that

$$\operatorname{coker} W_{\phi\tilde{\phi}^{-1}} = \{0\}.$$

From identity (3.4), we obtain that $\operatorname{coker}(W_\phi \pm H_\phi) = \{0\}$.

On the other hand, assume that $\ker W_{\phi\tilde{\phi}^{-1}} = \{0\}$. Then, the equivalence after extension relation (3.4) implies that

$$\ker(W_\phi \pm H_\phi) = \{0\}.$$

Thus, we conclude that the Coburn-Simonenko theorem is valid for Wiener-Hopf plus Hankel, and Wiener-Hopf minus Hankel operators acting on variable exponent Lebesgue spaces.

**Corollary 2.** Let $\phi \in \mathcal{GM}_{p(\cdot)}$, $p(\cdot) \in \mathcal{B}_e(\mathbb{R})$. If $\dim\ker W_{\phi\tilde{\phi}^{-1}} = 0$, then $\ker(W_\phi \pm H_\phi) = \{0\}$ and if $\dim\ker W_{\phi\tilde{\phi}^{-1}} > 0$, then $\operatorname{coker}(W_\phi \pm H_\phi) = \{0\}$.

**Proof.** This statement is directly derived from the proof of the last result.

**Acknowledgement**: This work was supported in part by the Portuguese Foundation for Science and Technology ("FCT–Fundação para a Ciência e a Tecnologia"), through CIDMA – *Center for Research and Development in Mathematics and Applications*, within project UID/MAT/04106/2013. A.S. Silva also acknowledges the support of FCT through the postdoctoral scholarship SFRH/BPD/96763/2013.

## References

[1] S.N. Antontsev, J.F. Rodrigues, On stationary thermo-rheological viscous flows, *Ann. Univ. Ferrara Sez. VII Sci. Mat.* **52**(1) (2006), 19–36.

[2] H. Bart, V.E. Tsekanovskii, Matricial coupling and equivalence after extension, *Oper. Theory Adv. Appl.* **59** (1992), 143–160.

[3] A. Böttcher, Yu.I. Karlovich, I.M. Spitkovsky, *Convolution Operators and Factorization of Almost Periodic Matrix Functions*, Birkhäuser, Basel, 2002.

[4] L.E. Britvina, On polyconvolutions generated by the Hankel transform, *Mathematical Notes* **76**(1) (2004), 18–24.

[5] L.E. Britvina, Generalized convolutions for the Hankel transform and related integral operators, *Math. Nachr.* **280**(9-10) (2007), 962–970.

[6] L.P. Castro, Regularity of convolution type operators with PC symbols in Bessel potential spaces over two finite intervals, *Math. Nachr.* **261-262**(1) (2003), 23–36.

[7] L.P. Castro, D. Kapanadze, The impedance boundary-value problem of diffraction by a strip, *J. Math. Anal. Appl.* **337**(2) (2008), 1031–1040.

[8] L.P. Castro, D. Kapanadze, Wave diffraction by wedges having arbitrary aperture angle, *J. Math. Anal. Appl.* **421**(2) (2015), 1295–1314.

[9] L.P. Castro, D. Kapanadze, Crack impedance-Dirichlet boundary value problems of diffraction in a half-plane, *MESA – Mathematics in Engineering, Science and Aerospace* **6**(3) (2015), 551–566.

[10] L.P. Castro, A.S. Silva, Wiener-Hopf-Hankel operators with piecewise continuous Fourier Symbols on variable exponent Lebesgue spaces, L.H. Son, W. Tutschke, edts, *Interactions between Real and Complex Analysis*, pp. 243–255, Science and Technics Publishing House, Ministry for Science and Technology of Vietnam, Ha Noi, 2012.

[11] L.P. Castro, A.S. Silva, Interplay of Wiener-Hopf and Hankel operators with almost periodic Fourier symbols on standard and variable exponent Lebesgue spaces, *Annals of Functional Analysis* **6**(2) (2015), 49–59.

[12] L.P. Castro, F.-O. Speck, *Inversion of matrix convolution type operators with symmetry*, Port. Math. **62** (2005), 193–216.

[13] Y. Chen, S. Levine, M. Rao, Variable exponent, linear growth functionals in image restoration, *SIAM J. Appl. Math.* **66** (2006), 1383–1406.

[14] D. Cruz-Uribe, A. Fiorenza, J.M. Martell, C. Pérez, The boundedness of classical operators on variable $L^p$ spaces, *Ann. Acad. Sci. Fenn. Math.* **31**(1) (2006), 239–264.

[15] V. D. Didenko, B. Silbermann, Index calculation for Toeplitz plus Hankel operators with piecewise quasi-continuous generating functions, *Bull. London Math. Soc.* **45**(3) (2013), 633–650.

[16] V. D. Didenko, B. Silbermann, Structure of kernels and cokernels of Toeplitz plus Hankel operators, *Integral Equation Oper. Theory* **10**(1) (2014), 1–31.

[17] V. D. Didenko, B. Silbermann, Some results on the invertibility of Toeplitz plus Hankel operators, *Ann. Acad. Sci. Fenn. Math.* **39** (2014), 443–461.

[18] V. D. Didenko, B. Silbermann, The Coburn-Simonenko Theorem for some classes of Wiener-Hopf plus Hankel operators, *Publications de l'Institute Mathe'matique* **96**(110) (2014), 443–461.

[19] L. Diening, P. Harjulehto, P. Hästö, M. Růžička, *Lebesgue and Sobolev Spaces with Variable Exponents*, Lecture Notes in Math. **2017**, Springer, Heidelberg, 2011.

[20] T. Ehrhardt, *Factorization Theory for Toeplitz plus Hankel Operators and Singular Integral Operators with Flip*, Habilitation Thesis, Technischen Universitität Chemnitz, Chemnitz, 2004.

[21] B.T. Giang, N. Van Mau, N.M. Tuan, Convolutions for the Fourier transforms with geometric variables and applications, *Math. Nachr.* **283**(12) (2010), 1758–1770.

[22] B.T. Giang, N.M. Tuan, Generalized convolutions for the integral transforms of Fourier type and applications, *Fract. Calc. Appl. Anal.* **12**(3) (2009), 253–268.

[23] V.A. Kakichev, On the convolution for integral transforms, *Izv. Akad. Nauk BSSR, Ser. Fiz.-Mat. Nauk* **1967**(2) (1967), 48–57.

[24] A.Yu. Karlovich, Semi-Fredholm singular integral operators with piecewise continuous coefficients on weighted variable Lebesgue spaces are Fredholm, *Oper. Matrices* **1** (2007), 427–444.

[25] A. Karlovich, Banach algebra of the Fourier multipliers on weighted Banach function spaces, *Concr. Oper.* **2** (2015), 27–36.

[26] A.Yu. Karlovich, I.M. Spitkovsky, On singular integral operators with semi-almost periodic coefficients on variable Lebesgue spaces, *J. Math. Anal. Appl.* **384**(2) (2011), 706–725.

[27] V.M. Kokilashvili, S.G. Samko, Operators of harmonic analysis in weighted spaces with non-standard growth, *J. Math. Anal. Appl.* **352**(1) (2009), 15–34.

[28] A.K. Lerner, Some remarks on the HardyLittlewood maximal function on variable $L^p$ spaces, *Math. Z.* **251** (2005), 509–521.

[29] J. Motos, M.J. Planells, C.F. Talavera, On variable exponent Lebesgue spaces of entire analytic functions, *J. Math. Anal. Appl.* **388** (2012), 775–787.

[30] M. Růžička, *Electrorheological Fluids: Modeling and Mathematical Theory*, Lecture Notes in Mathematics **1748**, Springer-Verlag, Berlin, 2000.

[31] N.M. Tuan, P.D. Tuan, Generalized convolutions relative to the Hartley transforms with applications, *Sci. Math. Jpn.* **70** (2009), 77–89 (e-2009, p. 351–363).