Reproducing Kernels and Discretization†

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Abstract. We give a short survey of a general discretization method based on the theory of reproducing kernels. We believe our method will become the next generation method for solving analytical problems by computers. For example, for solving linear PDEs with general boundary or initial value conditions, independently of the domains. Furthermore, we give an ultimate sampling formula and a realization of reproducing kernel Hilbert spaces.

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1. The inverse by using a finite number of data

Let $\mathcal{H}$ be a Hilbert (possibly finite-dimensional) space, and consider E to be an abstract set and $\mathbf{h}$ a Hilbert $\mathcal{H}$-valued function on E. Then, we consider the linear transform

$$f(p) = (\mathbf{f}, \mathbf{h}(p))_{\mathcal{H}}, \quad \mathbf{f} \in \mathcal{H}\,, \tag{1.1}$$

from $\mathcal{H}$ into the linear space $\mathcal{F}(E)$ comprising all the complex valued functions on E. We form a positive definite quadratic form function $K(p,q)$ on $E \times E$ defined by

$$K(p,q) = (\mathbf{h}(q), \mathbf{h}(p))_{\mathcal{H}} \quad \text{on} \quad E{\times}E. \tag{1.2}$$

Proposition 1.1.

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(*I*) *The range of the linear mapping* (1.1) *on* $\mathcal{H}$ *is characterized as the reproducing kernel Hilbert space* H_K *admitting the reproducing kernel* $K(p,q)$.

(*II*) *In general, we have the inequality* $\|f\|_{H_K} \le \|\mathbf{f}\|_{\mathcal{H}}$. *Here, for any member* f *of* H_K *there exists a uniquely determined* $\mathbf{f}^* \in \mathcal{H}$ *satisfying* $f(p) = (\mathbf{f}^*, \mathbf{h}(p))_{\mathcal{H}} \quad \text{on } E$ *and*

$$\|f\|_{H_K} = \|\mathbf{f}^*\|_{\mathcal{H}}. \tag{1.3}$$

(*III*) *In general, we have the inversion formula in* (1.1) *in the form*

$$f \mapsto \mathbf{f}^* \tag{1.4}$$

in (*II*) *by using the reproducing kernel Hilbert space* H_K.

The typical ill-posed problem (1.1) in $\mathcal{H}$ will become a well-posed problem in H_K, see the details [7, 8, 9].

Our idea is based on the approximate realization of the abstract Hilbert space H_K by taking a finite number of points of E. This is done because, in general, the reproducing kernel Hilbert space H_K has a complicated structure.

By taking a finite number of points $\{p_j\}_{j=1}^n$, we set

$$K(p_j, p_{j'}) := a_{jj'}. \tag{1.5}$$

Then, if the matrix $A_n :=\| a_{jj'} \|$ is positive definite, then, the corresponding norm in H_{A_n} comprising the vectors $\mathbf{x} = (x_1, x_2, ..., x_n)^T$ is determined by $\|\mathbf{x}\|^2_{H_{A_n}} = \mathbf{x}^* \widetilde{A_n} \mathbf{x}$, where $\widetilde{A_n} = \overline{A_n^{-1}} = \|\widetilde{a_{jj'}}\|$ (see [8], p. 250).

Proposition 1.2. *In the linear mapping*

$$f(p) = (\mathbf{f}, \mathbf{h}(p))_{\mathcal{H}}, \quad \mathbf{f} \in \mathcal{H} \tag{1.6}$$

for A_n, *the minimum norm inverse* $\mathbf{f}^*_{A_n}$ *satisfying*

$$f(p_j) = (\mathbf{f}, \mathbf{h}(p_j))_{\mathcal{H}}, \quad \mathbf{f} \in \mathcal{H} \tag{1.7}$$

is given by

$$\mathbf{f}^*_{A_n} = \sum_j \sum_{j'} f(p_j) \widetilde{a_{jj'}} \mathbf{h}(p_{j'}), \tag{1.8}$$

where $\widetilde{a_{jj'}}$ *are assumed the elements of the complex conjugate inverse of the positive definite Hermitian matrix* A_n *constituted by the elements* $a_{jj'} = (\mathbf{h}(p_{j'}), \mathbf{h}(p_j))_{\mathcal{H}}$. *Here, the positive definiteness of* A_n *is a basic assumption.*

2. Convergence of the approximate inverses

The following proposition deals with the convergence of our approximate inverses in Proposition 1.2. See [1, 2] for the details.

Proposition 2.1. *Let $\{p_j\}_{j=1}^{\infty}$ be a sequence of distinct points on E, that is the positive definiteness of A_n for any n and a uniqueness set for the space H_K. Then, in the space $\mathcal{H}$*

$$\lim_{n\to\infty} \mathbf{f}^*_{A_n} = \mathbf{f}^*. \tag{2.1}$$

Proposition 2.2. *(***Ultimate realization of reproducing kernel Hilbert spaces***). In our general situation and for a uniqueness set $\{p_j\}$ of the set E satisfying the linearly independence in Proposition 1.2, we obtain*

$$\|f\|^2_{H_K} = \|\mathbf{f}^*\|^2_{\mathcal{H}} = \lim_{n\to\infty} \sum_j \sum_{j'} f(p_j)\widetilde{a_{jj'}}\overline{f(p_{j'})}. \tag{2.2}$$

Proposition 2.3. *(***Ultimate sampling theory***). In our general situation and for a uniqueness set $\{p_j\}$ of the set E satisfying the linearly independence, we obtain*

$$\begin{aligned} f(p) &= \lim_{n\to\infty} (\mathbf{f}^*_{A_n}, \mathbf{h}(p))_{\mathcal{H}} = \lim_{n\to\infty} \left(\sum_j \sum_{j'} f(p_j)\widetilde{a_{jj'}}\mathbf{h}(p_{j'}), \mathbf{h}(p) \right)_{\mathcal{H}} \\ &= \lim_{n\to\infty} \sum_j \sum_{j'} f(p_j)\widetilde{a_{jj'}} K(p, p_{j'}). \end{aligned} \tag{2.3}$$

3. Ordinary linear differential equations

In view to have a concrete exemplification of the method, let us consider a prototype differential operator

$$Ly := \alpha y'' + \beta y' + \gamma y. \tag{3.1}$$

Here, we shall consider a very general situation that the coefficients are arbitrary functions (no continuity requirement) and on a general interval I. We wish to construct some natural solution of

$$Ly = g \tag{3.2}$$

for a very general function g on a general interval I.

Proposition 3.1. *([3, 1]). Let us fix a positive number h and take a finite number of points $\{t_j\}_{j=1}^n$ of I such that $|\alpha(t_j)|^2 + |\beta(t_j)|^2 + |\gamma(t_j)|^2 \neq 0$ for each j. Then, the optimal solution $y_h^{A_n}$ of the equation (3.2) is given by*

$$y_h^{A_n}(t) = \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} F_h^{A_n}(\xi) e^{-it\xi} d\xi$$

in terms of the function $F_h^{A_n} \in L_2(-\pi/h, +\pi/h)$ in the sense that $F_h^{A_n}$ has the minimum norm in $L_2(-\pi/h, +\pi/h)$ among the functions $F \in L_2(-\pi/h, +\pi/h)$ satisfying, for the characteristic function $\chi_h(t)$ of the interval $(-\pi/h, +\pi/h)$:

$$\frac{1}{2\pi} \int_{\mathbb{R}} F(\xi)[\alpha(t)(-\xi^2) + \beta(t)(-i\xi) + \gamma(t)]\chi_h(\xi) \exp(-it\xi) d\xi = g(t) \tag{3.3}$$

for all $t = t_j$ and for the function space $L_2(-\pi/h, +\pi/h)$.

The minimal norm function $F_h^{A_n}$ is given by

$$F_h^{A_n}(\xi) = \sum_{j,j'=1}^{n} g(t_j)\widetilde{a_{jj'}}\overline{(\alpha(t_{j'})(-\xi^2) + \beta(t_{j'})(-i\xi) + \gamma(t_{j'}))}\exp(it_{j'}\xi).$$

Here, the matrix $A_n = \{a_{jj'}\}_{j,j'=1}^n$ formed by the elements $a_{jj'} = K_{hh}(t_j, t_{j'})$ with $K_{hh}(t,t') =$

$$\frac{1}{2\pi}\int_{\mathbb{R}}[\alpha(t)(-\xi^2) + \beta(t)(-i\xi) + \gamma(t)]\overline{[\alpha(t')(-\xi^2) + \beta(t')(-i\xi) + \gamma(t')]}$$
$$\cdot\chi_h(\xi)\exp(-i(t-t')\xi)d\xi$$

is positive definite and the $\widetilde{a_{jj'}}$ are the elements of the inverse of $\overline{A_n}$ (the complex conjugate of A_n).

The minimal norm solution $y_h^{A_n}$ of the equation (3.2) is given by

$$\begin{aligned} y_h^{A_n}(t) &= \sum_{j,j'=1}^{n} g(t_j)\widetilde{a_{jj'}}\frac{1}{2\pi}[-\overline{\alpha(t_{j'})}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \xi^2 e^{-i(t-t_{j'})\xi}d\xi \\ &\quad + i\overline{\beta(t_{j'})}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \xi e^{-i(t-t_{j'})\xi}d\xi + \overline{\gamma(t_{j'})}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{-i(t-t_{j'})\xi}d\xi]. \end{aligned}$$

As about general linear operator equations, we consider the equations in some reproducing kernel Hilbert spaces. These spaces can be considered as the images of some Hilbert spaces as in Proposition 1.1 (see [8, 9]). Then, the linear operator equation may be reduced to Proposition 1.2 by *the backward transformation* as in Proposition 3.1. So, we will be able to consider our method as *a fundamental theory for linear operator equations in the framework of Hilbert spaces.*

4. Numerical examples

We set $h = 1$; we seek our solution in the Paley-Wiener space $W(\pi)$ with equi-spaced collocation points.

Example. We consider an initial value problem

$$t^3y''(t) + ty'(t) = -25t^3\sin(5t) + 5t\cos(5t), \ \ (-1 < t \leq 1),$$
$$y(-1) = \sin(5), \quad y'(-1) = 5\cos(5),$$

and we set collocation points to $t_j = -1 + 2j/(n-2), j = 1, 2, \ldots, n-2$.

Numerical results shown in Fig. 1 have a good coincide with the exact solution $y(t) = \sin(5t)$.

Example. We consider an initial value problem

$$y''(t) = g(t), \ \ (-1 < t \leq 1), \quad y(-1) = y'(-1) = 0,$$

where

$$g(t) = \begin{cases} 0, & t < 0; \\ t, & t \geq 0. \end{cases}$$

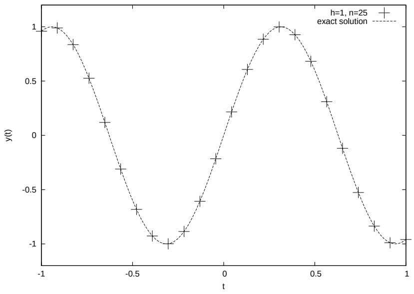
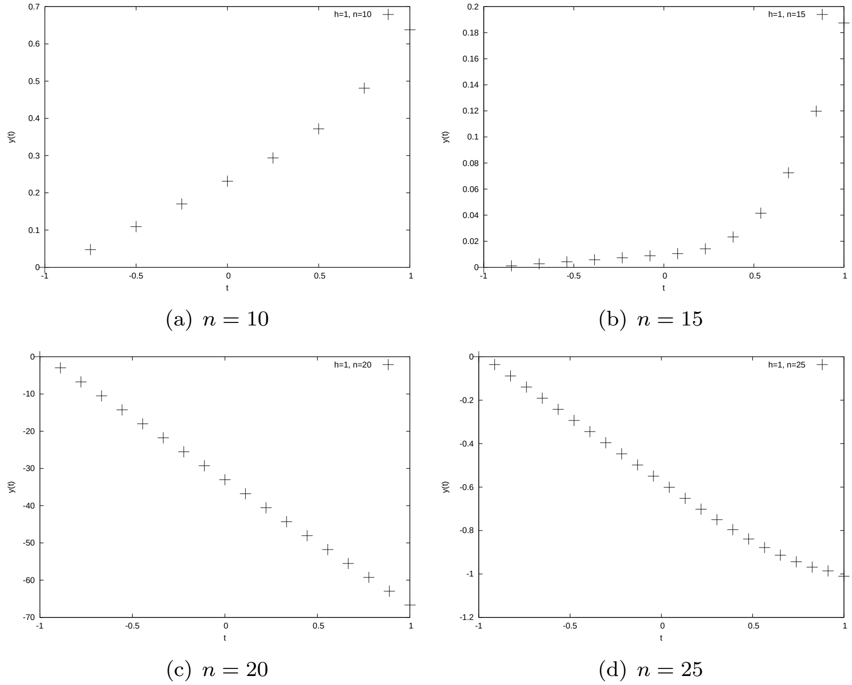


FIGURE 1. Numerical Results for the Initial Value Problem for $n = 25$ with 100 Decimal Digits. Maximum Error Is Approximately 10^{-13}.



(a) $n = 10$

(b) $n = 15$

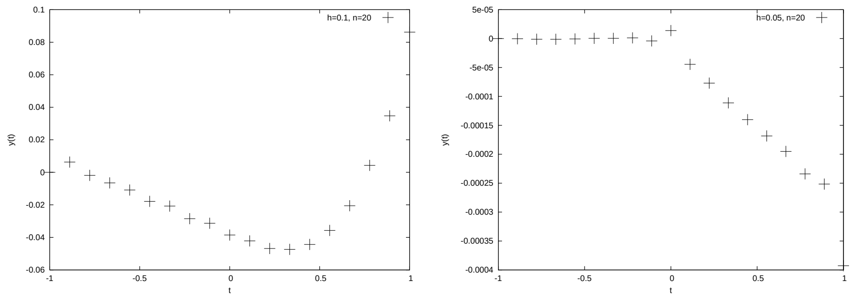
(c) $n = 20$

(d) $n = 25$

FIGURE 2. Numerical Results by 500 Decimal Digits Precision, $h = 1$

We know that there exists a unique solution

$$y(t) = \begin{cases} 0, & t < 0; \\ \dfrac{t^3}{6}, & t \geq 0. \end{cases}$$



(a) $h = 0.1$ (b) $h = 0.05$

FIGURE 3. Numerical Results by 500 Decimal Digits Precision, $n = 20$

The proposed method assumes that the solution belongs to the Paley-Wiener space $W(\frac{\pi}{h})$, where h is an approximation parameter. Our numerical results imply that the Paley-Wiener space seems to show the need of application of suitable Sobolev spaces as basic approximate function spaces.

In our new discretization method we will need the precision in some deep way and huge computer resources. However, these both requirements were prepared by Fujiwara already (e.g., recall the case of the inverse Laplace transform). See [4, 5, 6] for the details.

We are looking for some optimal solutions satisfying the differential equations at the given discrete points and so, we are free from important restrictions on the domains which occur on ordinary methods. For instance, this is not the case of the *Finite Element Method* and the *Difference Method* which are depending seriously on the domains. In our case, we can consider the problems on any domains. See [1, 2] for the details.

Anyhow, error estimates for our approximate solutions are entirely new open problems.

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