Crack impedance-Dirichlet boundary value problems of diffraction in a half-plane[0]

L.P. Castro[1,⋆], D. Kapanadze[2]

[1] CIDMA – Center for Research and Development in Mathematics and Applications, Department of Mathematics, University of Aveiro, Aveiro, Portugal.
[2] A. Razmadze Mathematical Institute, Tbilisi State University, Tbilisi, Georgia.

⋆ *Corresponding Author*. Castro@ua.pt.

Abstract. We study two wave diffraction problems modeled by the Helmholtz equation in a half-plane with a crack characterized by Dirichlet and impedance boundary conditions. The existence and uniqueness of solutions is proved by an appropriate combination of general operator theory, Fredholm theory, potential theory and boundary integral equation methods. This combination of methods leads also to integral representations of solutions. Moreover, in Sobolev spaces, a range of smoothness parameters is obtained in which the solutions of the problems are valid.

1 Introduction

Problems of wave diffraction by geometrical configurations involving cracks have been object of great interest in the scientific literature in the last years. In part, this is due to their fundamental relevance in a great variety of concrete applications. Indeed, they serve as models for a significant number of complex situations in different sciences. In the present paper, we will be considering problems of wave diffraction by a half-plane configuration to which we are adding a perpendicular crack to the main boundary involving Dirichlet and impedance boundary conditions.

For different types of boundary value problems in domains with cuts or cracks, the specialized papers [22]–[30] presented integral representation of solutions in the form of potentials. Some other general works (cf. [5]–[12], [18]–[19], [31]–[40]) provided convenient settings, explained and detailed justified why different classes of problems of diffraction by plane sectors admit exact analytic solutions. Moreover, in some of these works, consequent exact analytic solutions were obtained.

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In our present case, the geometry of a half-plane with a perpendicular crack makes it possible to separate the Helmholtz equation in two quadrants and to "reduce" (in a certain sense) each of the original wave diffraction problems into corresponding boundary value problems involving also some transmission conditions in a contact half-line which is comum to both quadrants. This allows extra possibilities to derive the consequent solutions of the problems in appropriate Sobolev space settings, as well as some other qualitative properties. Thus, in here, we continue our development of operator theory methods to deal with wave diffraction problems involving cracks, and apply it to the important problem of diffraction by a screen occupying a half-plane containing a crack characterized by having impedance and Dirichlet boundary conditions.

The formulation of the problems in Bessel potential spaces and the derivation of conditions which ensure corresponding uniqueness of solutions is presented in the next section. In section 3 we will rewrite our original problems in convenient Wiener-Hopf-Hankel equations. Section 4 is devoted to a Fredholm and invertibility analysis of Wiener-Hopf operators which we associate to the previous Wiener-Hopf-Hankel equations. The main result is presented in the last section and arises as a natural consequence of the previous constructions and results. It exhibits formulas for the solutions of the original problems in Bessel potential spaces and a range of increased smoothness of the spaces where that solutions are still valid.

2 Formulation of the problems and uniqueness of solutions

We start by introducing some general notation so that we will be able to present the formulation of our problems from the mathematical point of view.

We use the notation $\mathcal{S}(\mathbb{R}^n)$ for the Schwartz space of all rapidly decreasing functions and $\mathcal{S}'(\mathbb{R}^n)$ for the dual space of tempered distributions on $\mathbb{R}^n$. The Bessel potential space $H^s(\mathbb{R}^n)$, with $s \in \mathbb{R}$, is formed by the elements $\varphi \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|\varphi\|_{H^s(\mathbb{R}^n)} = \|\mathcal{F}^{-1}(1+|\xi|^2)^{s/2} \cdot \mathcal{F}\varphi\|_{L_2(\mathbb{R}^n)}$$

is finite. As the notation indicates, $\|\cdot\|_{H^s(\mathbb{R}^n)}$ is a norm for the space $H^s(\mathbb{R}^n)$ which makes it a Banach space. Here, $\mathcal{F} = \mathcal{F}_{x\mapsto\xi}$ denotes the Fourier transformation in $\mathbb{R}^n$. For a given Lipschitz domain $\mathcal{D}$, on $\mathbb{R}^n$, we denote by $\widetilde{H}^s(\mathcal{D})$ the closed subspace of $H^s(\mathbb{R}^n)$ whose elements have supports in $\overline{\mathcal{D}}$, and $H^s(\mathcal{D})$ denotes the space of generalized functions on $\mathcal{D}$ which have extensions into $\mathbb{R}^n$ that belong to $H^s(\mathbb{R}^n)$. The space $\widetilde{H}^s(\mathcal{D})$ is endowed with the subspace topology, and on $H^s(\mathcal{D})$ we introduce the norm of the quotient space $H^s(\mathbb{R}^n)/\widetilde{H}^s(\mathbb{R}^n\backslash\overline{\mathcal{D}})$. Throughout the paper we will use the notation $\mathbb{R}^n_\pm := \{x = (x_1,\ldots,x_{n-1},x_n) \in \mathbb{R}^n : \pm x_n > 0\}$. Note that the spaces $H^0(\mathbb{R}^n_+)$ and $\widetilde{H}^0(\mathbb{R}^n_+)$ can be identified, and we will denote them by $L_2(\mathbb{R}^n_+)$. For a comprehensive treatment of Sobolev spaces we refer to [1], for unbounded Lipschitz domains see also [33], and for domains with conical points, edges, polyhedra, cuts (or cracks), slits or holes we cite [20].

Let

$$\begin{aligned}
\Omega &:= \{(x_1,x_2) \in \mathbb{R}^2 \,:\, x_1 > 0,\, x_2 \in \mathbb{R}\},\\
\Gamma_1 &:= \{(x_1,0) \,:\, x_1 \in \mathbb{R}\},\\
\Gamma_2 &:= \{(0,x_2) \,:\, x_2 \in \mathbb{R}\},
\end{aligned}$$

and

$$\mathcal{C} := \{(x_1,0) \,:\, 0 < x_1 < a\} \,\subset\, \Gamma_1$$

for a certain positive number a and $\Omega_{\mathcal{C}} := \Omega\backslash\overline{\mathcal{C}}$. Clearly, $\partial\Omega = \Gamma_2$ and $\partial\Omega_{\mathcal{C}} = \Gamma_2 \cup \overline{\mathcal{C}}$.

For our purposes below we introduce further notations:

$$\Omega_1 := \{(x_1,x_2) \in \mathbb{R}^2 \,:\, x_1 > 0,\, x_2 > 0\},$$
$$\Omega_2 := \{(x_1,x_2) \in \mathbb{R}^2 \,:\, x_1 > 0,\, x_2 < 0\};$$

then, $\partial\Omega_j = \mathcal{S}_j \cup \mathcal{S}$, for $j = 1,2$, where

$$\mathcal{S} := \{(x_1,0) \,:\, x_1 \geq 0\} \,\subset\, \Gamma_1,$$
$$\mathcal{S}_1 := \{(0,x_2) \,:\, x_2 \geq 0\} \,\subset\, \Gamma_2,$$
$$\mathcal{S}_2 := \{(0,x_2) \,:\, x_2 \leq 0\} \,\subset\, \Gamma_2.$$

Finally, we introduce the following unit normal vectors $n_1 = \overrightarrow{(0,-1)}$ on Γ_1 and $n_2 = \overrightarrow{(-1,0)}$ on Γ_2.

Let $\varepsilon \in [0,\frac{1}{2})$. We are interested in studying the problem of existence and uniqueness of an element $u \in H^{1+\varepsilon}(\Omega_{\mathcal{C}})$, such that

$$\left(\Delta + k^2\right) u = 0 \quad \text{in} \quad \Omega_{\mathcal{C}}, \tag{2.1}$$

and u satisfies one of the following two representative boundary conditions:

$$\begin{cases} [\partial_{n_1} u]^+_{\mathcal{C}} - p\,[u]^+_{\mathcal{C}} = g_1^+ & \text{on} \quad \mathcal{C}, \\ [u]^-_{\mathcal{C}} = g_0^- \quad \text{on} \quad \mathcal{C}, \end{cases} \quad \text{and} \quad \begin{cases} [u]^+_{\mathcal{S}_1} = h_1 & \text{on} \quad \mathcal{S}_1, \\ [u]^+_{\mathcal{S}_2} = h_2 & \text{on} \quad \mathcal{S}_2, \end{cases} \tag{2.2}$$

$$\begin{cases} [\partial_{n_1} u]^+_{\mathcal{C}} - p\,[u]^+_{\mathcal{C}} = g_1^+ & \text{on} \quad \mathcal{C}, \\ [u]^-_{\mathcal{C}} = g_0^- \quad \text{on} \quad \mathcal{C}, \end{cases} \quad \text{and} \quad \begin{cases} [\partial_{n_2} u]^+_{\mathcal{S}_1} = f_1 & \text{on} \quad \mathcal{S}_1, \\ [\partial_{n_2} u]^+_{\mathcal{S}_2} = f_2 & \text{on} \quad \mathcal{S}_2, \end{cases} \tag{2.3}$$

for $j = 1,2$. Here the wave number $k \in \mathbb{C}\setminus\mathbb{R}$ and the number $p \in \mathbb{C}$ are given. The elements $[u]^+_{\mathcal{S}_j}$ and $[\partial_{n_2} u]^+_{\mathcal{S}_j}$ denote the Dirichlet and the Neumann traces on $\mathcal{S}_j$, respectively, while by $[u]^{\pm}_{\mathcal{C}}$ we denote the Dirichlet traces on $\mathcal{C}$ from both sides of the crack and by $[\partial_{n_1} u]^+_{\mathcal{C}}$ we denote the Neumann trace on $\mathcal{C}$ from the upper side of the crack.

Throughout the paper, on the given data, we assume that $h_j \in H^{1/2+\varepsilon}(\mathcal{S}_j)$, $f_j \in H^{-1/2+\varepsilon}(\mathcal{S}_j)$, for $j = 1,2$, and $g_j^{\pm} \in H^{1/2-j+\varepsilon}(\mathcal{C})$, for $j = 0,1$. Furthermore, we suppose that they satisfy the following compatibility conditions:

$$\chi_0 \left(g_0^- - r_{\mathcal{C}} h_2 \circ e^{-i\frac{\pi}{2}}\right) \in r_{\mathcal{C}} \widetilde{H}^{1/2+\varepsilon}(\mathcal{C}), \tag{2.4}$$

$$\chi_0 \left(g_1^+ + r_{\mathcal{C}} f_1 \circ e^{i\frac{\pi}{2}}\right) \in r_{\mathcal{C}} \widetilde{H}^{-1/2+\varepsilon}(\mathcal{C}). \tag{2.5}$$

Here, $r_{\mathcal{C}}$ denotes the restriction operator to $\mathcal{C}$ and $\chi_0 \in C^\infty([0,a])$ is such that $\chi_0(x) \equiv 1$ for $x \in [0,a/3]$ and $\chi_0(x) \equiv 0$ for $x \in [2a/3,a]$.

From now on we will refer to:

- Problem $\mathcal{P}_{I-D-D}$ as the problem characterized by (2.1), (2.2), and (2.4);
- Problem $\mathcal{P}_{I-D-N}$ as the one characterized by (2.1), (2.3), and (2.5).

As about the just stated compatibility conditions, note that they are necessary conditions for the well-posedness of the corresponding problems. Note also that, the compatibility condition (2.5) included in Problem $\mathcal{P}_{I-D-N}$ is an additional restriction only for $\varepsilon = 0$.

Now, having formulated the problems in a rigorous mathematical way and having considered the necessary compatibility conditions, we are in a position to look for conditions which will guarantee the uniqueness result for the solutions of the problems in consideration.

Theorem 1. If one of the following situations holds:

$$\begin{aligned}
&\text{(a)} \quad (\Re\mathrm{e}\, k)(\Im\mathrm{m}\, k) > 0, \quad \Im\mathrm{m}\, p \geq 0,\\
&\text{(b)} \quad (\Re\mathrm{e}\, k)(\Im\mathrm{m}\, k) < 0, \quad \Im\mathrm{m}\, p \leq 0,\\
&\text{(c)} \quad |\Im\mathrm{m}\, k| \geq |\Re\mathrm{e}\, k|, \quad \Re\mathrm{e}\, p \leq 0,\\
&\text{(d)} \quad \Re\mathrm{e}\, k = 0, \quad \Im\mathrm{m}\, p > 0,\\
&\text{(e)} \quad \Im\mathrm{m}\, p \neq 0, \quad (\Im\mathrm{m}\, k)^2 - (\Re\mathrm{e}\, k)^2 + 2(\Re\mathrm{e}\, k)(\Im\mathrm{m}\, k)\tfrac{\Re\mathrm{e}\, p}{\Im\mathrm{m}\, p} > 0,
\end{aligned}$$

then problems $\mathcal{P}_{I-D-D}$ and $\mathcal{P}_{I-D-N}$ have at most one solution.

Proof. The proof is somehow standard and uses the Green's formula (being sufficient to consider the case $\varepsilon = 0$). Let R be a sufficiently large positive number and $B(R)$ be the disk centered at the origin with radius R. Set $\Omega_R := \Omega_C \cap B(R)$. Note that the domain Ω_R has a piecewise smooth boundary S_R including both sides of $\mathcal{C}$ and denote by $n(x)$ the outward unit normal vector at the non-singular points $x \in S_R$.

Let u be a solution of the homogeneous problem. Then the first Green's identity for u and its complex conjugate $\bar{u}$ in the domain Ω_R, together with zero boundary conditions on S_R yields

$$\int_{\Omega_R} \left[|\nabla u|^2 - k^2|u|^2\right] dx = p \int_{\mathcal{C}} |[u]^+|^2 dx + \int_{\partial B(R)\cap\Omega_C} (\partial_n u)\, \bar{u}\, dS_R\,. \tag{2.6}$$

From the real and imaginary parts of the last identity, we obtain

$$\int_{\Omega_C} \left[|\nabla u|^2 + \left((\Im\mathrm{m}\, k)^2 - (\Re\mathrm{e}\, k)^2\right)|u|^2\right] dx - (\Re\mathrm{e}\, p)\int_{\mathcal{C}} |[u]^+|^2 dx = \Re\mathrm{e} \int_{\partial B(R)\cap\Omega_C} (\partial_n u)\, \bar{u}\, dS_R\,,$$

$$-2(\Re\mathrm{e}\, k)(\Im\mathrm{m}\, k)\int_{\Omega_C} |u|^2\, dx - (\Im\mathrm{m}\, p)\int_{\mathcal{C}} |[u]^+|^2 dx = \Im\mathrm{m} \int_{\partial B(R)\cap\Omega_C} (\partial_n u)\, \bar{u}\, dS_R\,.$$

Further, for each of the conditions (a)–(e), arguing similarly as in the proof of [13, Theorem 3.1], we get that $u = 0$ in Ω_C.

3 Potentials and Wiener-Hopf-Hankel formulation of the problems

In the present section, we will start by recalling some results from potential theory. Then, using such results, we will be able to rewrite the original problems as Wiener-Hopf-Hankel equations.

From now on, throughout the remaining part of the paper, and without loss of generality, we assume that $\Im\mathrm{m}\, k > 0$; the complementary case $\Im\mathrm{m}\, k < 0$ runs with obvious changes. Let us denote the standard fundamental solution of the Helmholtz equation (in two dimensions) by

$$\mathcal{K}(x) := -\frac{i}{4} H_0^{(1)}(k|x|)\,,$$

where $H_0^{(1)}(k|x|)$ is the Hankel function of the first kind of order zero (cf. [19, §3.4]). Furthermore, we introduce the single and double layer potentials on Γ_j:

$$V_j(\psi)(x) = \int_{\Gamma_j} \mathcal{K}(x-y)\psi(y)d_y\Gamma_j\,, \quad x \notin \Gamma_j\,,$$
$$W_j(\varphi)(x) = \int_{\Gamma_j} [\partial_{n_j(y)}\mathcal{K}(x-y)]\varphi(y)d_y\Gamma_j\,, \quad x \notin \Gamma_j\,,$$

where $j = 1,2$ and ψ, φ are density functions. Note that for $j = 1$ sometimes we will write $\mathbb{R}$ instead of Γ_1. In this case, for example, the single layer potential defined above has the form

$$V_1(\psi)(x_1,x_2) = \int_{\mathbb{R}} \mathcal{K}(x_1-y,x_2)\psi(y)dy, \quad x_2 \neq 0.$$

Set $\mathbb{R}^2_\pm := \{(x_1,x_2) \in \mathbb{R}^2 : x_2 \gtrless 0\}$ and let us first consider the operators $V := V_1$ and $W := W_1$.

Theorem 2 (cf. [8]). The single and double layer potentials V and W are continuous operators

$$V : H^s(\mathbb{R}) \to H^{s+1+\frac{1}{2}}(\mathbb{R}^2_\pm), \qquad W : H^{s+1}(\mathbb{R}) \to H^{s+1+\frac{1}{2}}(\mathbb{R}^2_\pm)$$

for all $s \in \mathbb{R}$.

Clearly, a similar result holds true for the operators V_2 and W_2.

Let us now recall some properties of the above introduced potentials. The following limit relations are well-known (cf. [8]):

$$[V(\psi)]^+_{\mathbb{R}} = [V(\psi)]^-_{\mathbb{R}} =: \mathcal{H}(\psi), \qquad [\partial_n V(\psi)]^\pm_{\mathbb{R}} =: [\mp\frac{1}{2}I](\psi)\,,$$

$$[W(\varphi)]^\pm_{\mathbb{R}} =: [\pm\frac{1}{2}I](\varphi), \qquad [\partial_n W(\varphi)]^+_{\mathbb{R}} = [\partial_n W(\varphi)]^-_{\mathbb{R}} =: \mathcal{L}(\varphi)\,,$$

where

$$\mathcal{H}(\psi)(z) := \int_{\mathbb{R}} \mathcal{K}(z-y)\psi(y)dy\,, \quad z \in \mathbb{R}\,,$$
$$\mathcal{L}(\varphi)(z) := \lim_{\mathbb{R}^2_\mp \ni x \to z \in \mathbb{R}} \partial_{n(x)} \int_{\mathbb{R}} [\partial_{n(y)}\mathcal{K}(y-x)]\varphi(y)dy\,, \quad z \in \mathbb{R}\,,$$

and I denotes the identity operator.

In our further reasoning we will make a convenient use of the even and odd extension operators defined by

$$\ell^e\varphi(y) = \begin{cases} \varphi(y), & y \in \mathbb{R}_\pm \\ \varphi(-y), & y \in \mathbb{R}_\mp \end{cases} \qquad \text{and} \qquad \ell^o\varphi(y) = \begin{cases} \varphi(y), & y \in \mathbb{R}_\pm \\ -\varphi(-y), & y \in \mathbb{R}_\mp \end{cases},$$

respectively.

Remark 1 (cf. [18]). The following operators

$$\ell^e : H^{\varepsilon+\frac{1}{2}}(\mathbb{R}_\pm) \longrightarrow H^{\varepsilon+\frac{1}{2}}(\mathbb{R}), \qquad \ell^o : r_{\mathbb{R}_\pm}\widetilde{H}^{\varepsilon+\frac{1}{2}}(\mathbb{R}_\pm) \longrightarrow H^{\varepsilon+\frac{1}{2}}(\mathbb{R}),$$
$$\ell^o : H^{\varepsilon-\frac{1}{2}}(\mathbb{R}_\pm) \longrightarrow H^{\varepsilon-\frac{1}{2}}(\mathbb{R}), \qquad \ell^e : r_{\mathbb{R}_\pm}\widetilde{H}^{\varepsilon-\frac{1}{2}}(\mathbb{R}_\pm) \longrightarrow H^{\varepsilon-\frac{1}{2}}(\mathbb{R}),$$

are continous for all $\varepsilon \in [0,1/2)$.

Lemma 1 (cf. [8]). If $0 \le \varepsilon < 1/2$, then

$$r_{\Gamma_2} \circ V \circ \ell^o \psi = 0, \qquad r_{\Gamma_2} \circ W \circ \ell^o \tilde{\varphi} = 0,$$

$$r_{\Gamma_2} \circ \partial_{n_2} V \circ \ell^e \tilde{\psi} = 0, \qquad r_{\Gamma_2} \circ \partial_{n_2} W \circ \ell^e \varphi = 0$$

for all $\psi \in H^{\varepsilon-\frac{1}{2}}(\mathcal{S})$, $\tilde{\psi} \in r_{\mathcal{S}} \widetilde{H}^{\varepsilon-\frac{1}{2}}(\mathcal{S})$, $\varphi \in H^{\varepsilon+\frac{1}{2}}(\mathcal{S})$, and $\tilde{\varphi} \in r_{\mathcal{S}} \widetilde{H}^{\varepsilon+\frac{1}{2}}(\mathcal{S})$.

Note that analogous results are valid for the operators V_2 and W_2.

Finally, let us assume that one of the conditions (a)–(e) of Theorem 1 is satisfied and note that the operator

$$\mathcal{A}_p := \mathcal{L} - \frac{p}{2} : H^{\frac{1}{2}+\varepsilon}(\mathbb{R}) \longrightarrow H^{-\frac{1}{2}+\varepsilon}(\mathbb{R}), \tag{3.1}$$

is invertible, cf. [8].

Now, we will equivalently write our problems in the form of single equations characterized by Wiener-Hopf plus Hankel operators. In view of this, the use of the pseudodifferential operators introduced in the last section together with an appropriate use of odd and even extension operators will be quite important. In addition, the reflection operator J given by the rule

$$J\psi(y) = \psi(-y) \quad \text{for all} \quad y \in \mathbb{R}.$$

will also play an important role here.

We start with the $\mathcal{P}_{I-D-D}$ problem. This boundary value problem can equivalently be rewritten in the following form: Find $u_j \in H^{1+\varepsilon}(\Omega_j)$, $j = 1,2$, such that

$$\left(\Delta + k^2\right) u_j = 0 \quad \text{in} \quad \Omega_j, \tag{3.2}$$

$$[u_j]^+_{\mathcal{S}_j} = h_j \quad \text{on} \quad \mathcal{S}_j, \tag{3.3}$$

$$[\partial_{n_1} u_1]^+_{\mathcal{C}} - p\,[u_1]^+_{\mathcal{C}} = g_1^+, \quad [u_2]^-_{\mathcal{C}} = g_0^- \quad \text{on} \quad \mathcal{C}, \tag{3.4}$$

and

$$[u_1]^+_{\mathcal{C}^c} - [u_2]^-_{\mathcal{C}^c} = 0, \quad [\partial_{n_1} u_1]^+_{\mathcal{C}^c} - [\partial_{n_1} u_2]^-_{\mathcal{C}^c} = 0 \quad \text{on} \quad \mathcal{C}^c, \tag{3.5}$$

where $\mathcal{C}^c = \mathcal{S} \backslash \overline{\mathcal{C}}$.

Let us consider the following functions

$$u_1 := W_1 \mathcal{A}_p^{-1} \ell^o r_{\mathcal{S}} \psi + H_1 \quad \text{in } \Omega_1, \tag{3.6}$$

and

$$u_2 = -2W_1 \ell^o r_{\mathcal{S}} \varphi + H_2 \quad \text{in } \Omega_2, \tag{3.7}$$

where

$$H_1 := W_1 \mathcal{A}_p^{-1} \left(\ell^o(\ell_+ g_1^+ + 2p[W_2(\ell^e h_1)]^+_{\mathcal{S}})\right) + 2W_2(\ell^e h_1) \quad \text{in } \Omega_1,$$

and

$$H_2 := 2W_2(\ell^e h_2) - 2W_1 \left(\ell^o(\ell_+ g_0^- - 2[W_2(\ell^e h_2)]^-_{\mathcal{S}})\right) \quad \text{in } \Omega_2;$$

here ψ and φ are arbitrary elements of the spaces $\widetilde{H}^{-\frac{1}{2}+\varepsilon}(\mathcal{C}^c)$ and $\widetilde{H}^{\frac{1}{2}+\varepsilon}(\mathcal{C}^c)$, respectively; $\ell_+ g_1 \in H^{-\frac{1}{2}+\varepsilon}(\mathcal{S})$ is any fixed extension of $g_1^+ \in H^{-\frac{1}{2}+\varepsilon}(\mathcal{C})$, $\ell_+ g_0^- \in H^{\frac{1}{2}+\varepsilon}(\mathcal{S})$ is any fixed extension of $g_0^- \in H^{\frac{1}{2}+\varepsilon}(\mathcal{C})$, while $\mathcal{A}_p^{-1}$ denotes the inverse of the operator $\mathcal{A}_p$, cf. (3.1). Note that, the functions H_1 and H_2 are well defined (cf. Remark 1 and compatibility condition (2.4)) and are known.

Using the properties of the operators introduced above (see also [8]) it is easy to verify that u_j, $j=1,2$, belong to the spaces $H^{1+\varepsilon}(\Omega_j)$ and satisfy equations (3.2)-(3.4). Thus it remains to fulfil the conditions (3.5), which lead us to the following equation

$$r_{\mathcal{C}^c}\mathcal{K}\ell^o r_{\mathcal{S}}\Upsilon = H_{ID}, \tag{3.8}$$

where

$$\mathcal{K} := \begin{pmatrix} I & -\frac{1}{2}\mathcal{A}_p^{-1} \\ 2\mathcal{L} & \mathcal{L}\mathcal{A}_p^{-1} \end{pmatrix}, \quad \Upsilon := \begin{pmatrix} -2\varphi \\ -2\psi \end{pmatrix},$$

I is the identity operator and $H_{ID} = (H_{ID}^1, H_{ID}^2)^\top$ is a known vector function with

$$H_{ID}^1 := 2[H_2]_{\mathcal{C}^c}^- - 2[H_1]_{\mathcal{C}^c}^+ \in H^{\frac{1}{2}+\varepsilon}(\mathcal{C}^c),$$

$$H_{ID}^2 := 2[\partial_{n_1}H_1]_{\mathcal{C}^c}^+ - 2[\partial_{n_1}H_2]_{\mathcal{C}^c}^- \in H^{-\frac{1}{2}+\varepsilon}(\mathcal{C}^c).$$

Note that to simplify further arguments in Section 4 we prefer to have the equation (3.8) in the just derived form.

As a consequence of the equation (3.8), in view to obtain more information on the elements ψ and φ, we need to investigate the invertibility of the operator

$$r_{\mathcal{C}^c}\mathcal{K}\ell^o r_{\mathcal{S}} : \begin{array}{c} \widetilde{H}^{\frac{1}{2}+\varepsilon}(\mathcal{C}^c) \\ \oplus \\ \widetilde{H}^{-\frac{1}{2}+\varepsilon}(\mathcal{C}^c) \end{array} \longrightarrow \begin{array}{c} H^{\frac{1}{2}+\varepsilon}(\mathcal{C}^c) \\ \oplus \\ H^{-\frac{1}{2}+\varepsilon}(\mathcal{C}^c) \end{array}$$

With the help of the operator J and the shift convolution operators

$$\mathrm{Op}(\tau_{\pm a}) := \mathcal{F}^{-1}\tau_{\pm a}\cdot\mathcal{F}$$

where $\tau_b(\xi) := e^{ib\xi}$, $\xi \in \mathbb{R}$, we equivalently reduce the problem to the invertibility of the operator

$$r_{\mathbb{R}_+}\mathcal{K}_{--} : \begin{array}{c} \widetilde{H}^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \\ \oplus \\ \widetilde{H}^{-\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \end{array} \longrightarrow \begin{array}{c} H^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \\ \oplus \\ H^{-\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \end{array} \tag{3.9}$$

where

$$\mathcal{K}_{--} := \mathcal{K}\mathrm{diag}\{I - \mathrm{Op}(\tau_{-2a})J, I - \mathrm{Op}(\tau_{-2a})J\}.$$

Let us note here that because of Theorem 1 and having in mind the exhibited limit relations of the potentials, we already know that $\mathrm{Ker}\, r_{\mathbb{R}_+}\mathcal{K}_{--} = \{0\}$.

Let us now turn to the boundary value problem $\mathcal{P}_{I-D-N}$. This can equivalently be rewritten in the following form: Find $u_j \in H^{1+\varepsilon}(\Omega_j)$, $j=1,2$, such that

$$\left(\Delta + k^2\right)u_j = 0 \quad \text{in} \quad \Omega_j, \tag{3.10}$$

$$[\partial_{n_2}u_j]_{\mathcal{S}_j}^+ = f_j \quad \text{on} \quad \mathcal{S}_j, \tag{3.11}$$

$$[\partial_{n_1}u_1]_{\mathcal{C}}^+ - p\,[u_1]_{\mathcal{C}}^+ = g_1^+, \quad [u_2]_{\mathcal{C}}^- = g_0^- \quad \text{on} \quad \mathcal{C}, \tag{3.12}$$

and

$$[u_1]^+_{\mathcal{C}^c} - [u_2]^-_{\mathcal{C}^c} = 0, \quad [\partial_{n_1} u_1]^+_{\mathcal{C}^c} - [\partial_{n_1} u_2]^-_{\mathcal{C}^c} = 0 \quad \text{on} \quad \mathcal{C}^c, \tag{3.13}$$

where $\mathcal{C}^c = \mathcal{S} \setminus \overline{\mathcal{C}}$.

Let us consider the following functions

$$u_1 := W_1 \mathcal{A}_p^{-1} \ell^e r_{\mathcal{S}} \psi + F_1 \quad \text{in } \Omega_1, \tag{3.14}$$

and

$$u_2 = -2W_1 \ell^e r_{\mathcal{S}} \varphi + F_2 \quad \text{in } \Omega_2, \tag{3.15}$$

where

$$F_1 := W_1 \mathcal{A}_p^{-1} \left(\ell^e (\ell_+ g_1^+ + 2[\partial_{n_1} V_2(\ell^o f_1)]^+_{\mathcal{S}}) \right) - 2V_2(\ell^o f_2) \quad \text{in } \Omega_1,$$

and

$$F_2 := -2V_2(\ell^o f_2) - 2W_1 \left(\ell^e (\ell_+ g_0^-) \right) \quad \text{in } \Omega_2.$$

Here $\psi \in \widetilde{H}^{-\frac{1}{2}+\varepsilon}(\mathcal{C}^c)$ and $\varphi \in \widetilde{H}^{-\frac{1}{2}+\varepsilon}(\mathcal{C}^c)$ are arbitrary elements as above. Due to (2.5) the functions F_1 and F_2 are well defined and known. Note that u_j, $j = 1, 2$, belong to the spaces $H^{1+\varepsilon}(\Omega_j)$ and satisfy equations (3.10)-(3.12). The conditions (3.13) lead us to the following equation

$$r_{\mathcal{C}^c} \mathcal{K} \ell^e r_{\mathcal{S}} \Upsilon = F_{ID}, \tag{3.16}$$

where

$$\mathcal{K} := \begin{pmatrix} I & -\dfrac{1}{2}\mathcal{A}_p^{-1} \\ 2\mathcal{L} & \mathcal{L}\mathcal{A}_p^{-1} \end{pmatrix}, \quad \Upsilon := \begin{pmatrix} -2\varphi \\ -2\psi \end{pmatrix},$$

I is the identity operator and $F_{ID} = (F^1_{ID}, F^2_{ID})^\top$ is a known vector function with

$$F^1_{ID} := 2\,[F_2]^-_{\mathcal{C}^c} - 2\,[F_1]^+_{\mathcal{C}^c},$$
$$F^2_{ID} := 2\,[\partial_{n_1} F_1]^+_{\mathcal{C}^c} - 2\,[\partial_{n_1} F_2]^-_{\mathcal{C}^c}.$$

Thus, we need to investigate the invertibility of the operator

$$r_{\mathcal{C}^c} \mathcal{K} \ell^e r_{\mathcal{S}} : \begin{array}{c} \widetilde{H}^{\frac{1}{2}+\varepsilon}(\mathcal{C}^c) \\ \oplus \\ \widetilde{H}^{-\frac{1}{2}+\varepsilon}(\mathcal{C}^c) \end{array} \longrightarrow \begin{array}{c} H^{\frac{1}{2}+\varepsilon}(\mathcal{C}^c) \\ \oplus \\ H^{-\frac{1}{2}+\varepsilon}(\mathcal{C}^c) \end{array}$$

which we equivalently reduce the problem to the invertibility of the operator

$$r_{\mathbb{R}_+} \mathcal{K}_{++} : \begin{array}{c} \widetilde{H}^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \\ \oplus \\ \widetilde{H}^{-\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \end{array} \longrightarrow \begin{array}{c} H^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \\ \oplus \\ H^{-\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \end{array} \tag{3.17}$$

where

$$\mathcal{K}_{++} := \mathcal{K} \operatorname{diag}\{I + \operatorname{Op}(\tau_{-2a})J, I + \operatorname{Op}(\tau_{-2a})J\}.$$

Similarly as before, let us note here that because of Theorem 1 (and the limit relations of the potentials), we already know that $\operatorname{Ker} r_{\mathbb{R}_+} \mathcal{K}_{++} = \{0\}$.

4 Fredholm and invertibility analysis of associated Wiener-Hopf operators

Now, we will exhibit operator relations that will be applied to the operators which appeared in the last section. Such relations will help us in obtaining their Fredholm and invertibility properties.

Having in mind [2, 17], we recall that two bounded linear operators $T : X_1 \to X_2$ and $S : Y_1 \to Y_2$, acting between Banach spaces, are said to be *(toplinear) equivalent after extension* if there are Banach spaces Z_1 and Z_2 and invertible bounded linear operators E and F such that

$$\begin{bmatrix} T & 0 \\ 0 & I_{Z_1} \end{bmatrix} = E \begin{bmatrix} S & 0 \\ 0 & I_{Z_2} \end{bmatrix} F,$$

where I_{Z_1} and I_{Z_2} represent the identity operators in Z_1 and Z_2, respectively. In particular, in case we will simply have $T = ESF$ for some boundedly invertible operators E and F, we will say that T and S are *equivalent* operators. In such a case, we will write $T \sim S$. These operator relations between two operators T and S, if obtained, allow several consequences on the properties of these two operators. Namely, T and S will have the same Fredholm regularity properties (i.e., the properties that directly depend on the kernel and on the image of the operator).

Let us consider

$$\Lambda^s_\pm(\xi) := (\xi \pm i)^s = (1+\xi^2)^{\frac{s}{2}} \exp\left\{ s\, i \arg(\xi \pm i)\right\},$$

with a branch chosen in such a way that $\arg(\xi \pm i) \to 0$ as $\xi \to +\infty$, i.e., with a cut along the negative real axis (see Example 1.7 in [21] for additional information about the properties of these functions). In addition, we will also use the notation

$$\zeta(\xi) := \frac{\Lambda_-(\xi)}{\Lambda_+(\xi)} = \frac{\xi - i}{\xi + i}, \quad \xi \in \mathbb{R}.$$

Lemma 2 (cf. [21, §4])**.** Let $s, r \in \mathbb{R}$, and consider the operators

$$\begin{aligned} \Lambda^s_+(D) &= (D+i)^s \\ \Lambda^s_-(D) &= r_{\mathbb{R}_+}(D-i)^s \ell^{(r)}, \end{aligned}$$

where $(D \pm i)^{\pm s} = \mathcal{F}^{-1}(\xi \pm i)^{\pm s} \cdot \mathcal{F}$, and $\ell^{(r)} : H^r(\mathbb{R}_+) \to H^r(\mathbb{R})$ is any bounded extension operator in these spaces (which particular choice does not change the definition of $\Lambda^s_-(D)$).

These operators arrange isomorphisms in the following space settings

$$\begin{aligned} \Lambda^s_+(D) &: \widetilde{H}^r(\mathbb{R}_+) \to \widetilde{H}^{r-s}(\mathbb{R}_+), \\ \Lambda^s_-(D) &: H^r(\mathbb{R}_+) \to H^{r-s}(\mathbb{R}_+) \end{aligned}$$

(for any $s, r \in \mathbb{R}$).

Bearing in mind the purpose of this section, let $A_{ij} = \mathrm{Op}(a_{ij}) = \mathcal{F}^{-1} a_{ij} \cdot \mathcal{F}$ and $B_{ij} = \mathrm{Op}(b_{ij})$ be pseudodifferential operators of order $\mu_{ij} \in \mathbb{R}$; thus, $\langle \cdot \rangle^{-\mu_{ij}} a_{ij}, \langle \cdot \rangle^{-\mu_{ij}} b_{ij} \in L^\infty(\mathbb{R})$, where $\langle \xi \rangle := (1+\xi^2)^{\frac{1}{2}}$ and $i, j = 1, 2$. Since the operators $r_{\mathbb{R}_+}(A_{ij} + B_{ij} J)$ arrange continuous maps

$$r_{\mathbb{R}_+}(A_{ij} + B_{ij} J) : \widetilde{H}^s(\mathbb{R}_+) \to H^{s-\mu_{ij}}(\mathbb{R}_+)$$

for all $s \in \mathbb{R}$, then 2×2 matrix operator

$$A+BJ=\begin{pmatrix} A_{11}+B_{11}J & A_{12}+B_{12}J \\ A_{21}+B_{21}J & A_{22}+B_{22}J \end{pmatrix}, \quad A=(A_{ij})_{i,j=1,2},\ B=(B_{ij})_{i,j=1,2}$$

arrange continuous maps

$$r_{\mathbb{R}_+}(A+BJ): \begin{matrix} \widetilde{H}^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \\ \oplus \\ \widetilde{H}^{-\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \end{matrix} \to \begin{matrix} H^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \\ \oplus \\ H^{-\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \end{matrix}$$

where $A_{11}=I$, $A_{12}=-\frac{1}{2}\mathcal{A}_p^{-1}$, $A_{21}=2\mathcal{L}$, $A_{22}=\mathcal{L}\mathcal{A}_p^{-1}$, and $B_{ij}=A_{ij}\mathrm{Op}(\tau_{-2a})$, for $i,j=1,2$.

Recall that the complete symbols of the pseudodifferential operators $\mathcal{L}$ and $\mathcal{A}_p$ are (cf. [8, 9]):

$$\sigma(\mathcal{L})(\xi)=-\frac{iw(\xi)}{2} \quad \text{and} \quad \sigma(\mathcal{A}_p)(\xi)=-\frac{iw(\xi)+p}{2}, \tag{4.1}$$

where $w=w(\xi):=(\rho^2+\rho^2)^{\frac{1}{4}}(\cos\frac{\alpha}{2}+i\sin\frac{\alpha}{2})$, with

$$\begin{aligned} \rho=\rho(\xi) &:= (\Re\mathrm{e}\,k)^2-(\Im\mathrm{m}\,k)^2-\xi^2, \\ \rho &:= 2(\Re\mathrm{e}\,k)(\Im\mathrm{m}\,k) \end{aligned}$$

and

$$\alpha:=\begin{cases} \arctan\frac{\rho}{|\rho|} & \text{if} \quad \rho>0,\ \rho>0 \\ \frac{\pi}{2} & \text{if} \quad \rho=0,\ \rho>0 \\ \pi-\arctan\frac{\rho}{|\rho|} & \text{if} \quad \rho<0,\ \rho>0 \\ \pi & \text{if} \quad \rho=0 \\ 2\pi-\arctan\frac{|\rho|}{|\rho|} & \text{if} \quad \rho>0,\ \rho<0 \\ \frac{3\pi}{2} & \text{if} \quad \rho=0,\ \rho<0 \\ \pi+\arctan\frac{|\rho|}{|\rho|} & \text{if} \quad \rho<0,\ \rho<0 \end{cases}. \tag{4.2}$$

Lemma 2 allows us to construct an equivalence relation between $r_{\mathbb{R}_+}(A+BJ)$ and

$$r_{\mathbb{R}_+}(\mathcal{A}+\mathcal{B}J):[L_2(\mathbb{R}_+)]^2\to[L_2(\mathbb{R}_+)]^2, \tag{4.3}$$

which is explicitly given by the following identity

$$r_{\mathbb{R}_+}(\mathcal{A}+\mathcal{B}J):=\mathrm{diag}\{\Lambda_-^{\frac{1}{2}+\varepsilon},\Lambda_-^{-\frac{1}{2}+\varepsilon}\}\,r_{\mathbb{R}_+}(A+BJ)\,\mathrm{diag}\{\Lambda_+^{-\frac{1}{2}-\varepsilon},\Lambda_+^{\frac{1}{2}-\varepsilon}\}, \tag{4.4}$$

where $\mathcal{A}:=(\mathcal{A}_{ij})_{i,j=1,2}$, $\mathcal{B}:=(\mathcal{B}_{ij})_{i,j=1,2}$, with

$$\mathcal{A}_{ij}:=(D-i)^{r_i}A_{ij}(D+i)^{-r_j}, \quad \mathcal{B}_{ij}:=(D-i)^{r_i}B_{ij}J(D+i)^{-r_j}J, \tag{4.5}$$

for $r_1:=\frac{1}{2}+\varepsilon$, $r_2:=-\frac{1}{2}+\varepsilon$. Due to the fact that $\Lambda_-^{s-\mu}:H^{s-\mu}(\mathbb{R}_+)\to L_2(\mathbb{R}_+)$ and $\Lambda_+^{-s}:L_2(\mathbb{R}_+)\to\widetilde{H}^s(\mathbb{R}_+)$ are invertible operators (cf. Lemma 2), the identity (4.4) shows that

$$r_{\mathbb{R}_+}(A+BJ)\sim r_{\mathbb{R}_+}(\mathcal{A}+\mathcal{B}J).$$

Note that

$$\Lambda_+^s(-\xi)=\Lambda_-^s(\xi)\,e^{s\pi i}, \qquad \Lambda_-^s(-\xi)=\Lambda_+^s(\xi)\,e^{-s\pi i}$$

which in particular allow us to describe the operators $\mathcal{A}_{ij}$ and $\mathcal{B}_{ij}$ and their symbols in the following way

$$\mathcal{A}_{ij} = \mathrm{Op}(\tilde{a}_{ij}), \quad \tilde{a}_{ij}(\xi) = \Lambda_-^{r_i}(\xi)\, a_{ij}(\xi)\, \Lambda_+^{-r_j}(\xi),$$
$$\mathcal{B}_{ij} = \mathrm{Op}(\tilde{b}_{ij}), \quad \tilde{b}_{ij}(\xi) = \Lambda_-^{r_i}(\xi)\, b_{ij}(\xi)\, \Lambda_+^{-r_j}(-\xi) = \Lambda_-^{r_i-r_j}(\xi)\, b_{ij}(\xi)\, e^{-r_j\pi i}.$$

In particular, we have $\sigma(\mathcal{A})(\xi) = (\tilde{a}_{ij}(\xi))_{i,j=1,2}$ with

$$\tilde{a}_{11}(\xi) = \zeta^{\frac{1}{2}+\varepsilon}(\xi), \qquad \tilde{a}_{12}(\xi) = -\tfrac{1}{2}\zeta^{\varepsilon}(\xi)\,\langle\xi\rangle\,[\sigma(\mathcal{A}_p)(\xi)]^{-1},$$
$$\tilde{a}_{21}(\xi) = 2\zeta^{\varepsilon}(\xi)\,\sigma(\mathcal{L})(\xi)\,\langle\xi\rangle^{-1}, \quad \tilde{a}_{22}(\xi) = \zeta^{-\frac{1}{2}+\varepsilon}(\xi)\,\sigma(\mathcal{L})(\xi)\,[\sigma(\mathcal{A}_p)(\xi)]^{-1},$$

and $\sigma(\mathcal{B})(\xi) = (\tilde{b}_{ij}(\xi))_{i,j=1,2}$, where

$$\tilde{b}_{11}(\xi) = -i\,\tau_{-2a}(\xi)\,e^{-\varepsilon\pi i},$$
$$\tilde{b}_{12}(\xi) = -\frac{1}{2}i\,(\xi - i)\,\tau_{-2a}(\xi)\,e^{-\varepsilon\pi i}\,[\sigma(\mathcal{A}_p)(\xi)]^{-1},$$
$$\tilde{b}_{21}(\xi) = -2i\,(\xi - i)^{-1}\tau_{-2a}(\xi)\,e^{-\varepsilon\pi i}\,\sigma(\mathcal{L})(\xi),$$
$$\tilde{b}_{22}(\xi) = i\tau_{-2a}(\xi)\,e^{-\varepsilon\pi i}\,\sigma(\mathcal{L})(\xi)\,[\sigma(\mathcal{A}_p)(\xi)]^{-1}.$$

Thus

$$r_{\mathbb{R}_+}\mathcal{K}_{++} \sim r_{\mathbb{R}_+}(\mathcal{A} + \mathcal{B}J) \quad \text{and} \quad r_{\mathbb{R}_+}\mathcal{K}_{--} \sim r_{\mathbb{R}_+}(\mathcal{A} - \mathcal{B}J). \tag{4.6}$$

Further, let us consider a pseudodifferential operator $\mathrm{Op}(\Xi)$ with 4×4 matrix symbol $\Xi(\xi)$ partitioned into four 2×2 blocks α_{ij}, $i,j = 1,2$:

$$\Xi(\xi) = \begin{pmatrix} \alpha_{11}(\xi) & \alpha_{12}(\xi) \\ \alpha_{21}(\xi) & \alpha_{22}(\xi) \end{pmatrix}$$

with

$$\alpha_{11}(\xi) = \sigma(\mathcal{A})(\xi) - \sigma(\mathcal{B})(\xi)[\sigma(\mathcal{A})(-\xi)]^{-1}\sigma(\mathcal{B})(-\xi),$$
$$\alpha_{12}(\xi) = -\sigma(\mathcal{B})(\xi)\,[\sigma(\mathcal{A})(-\xi)]^{-1},$$
$$\alpha_{21}(\xi) = [\sigma(\mathcal{A})(-\xi)]^{-1}\,\sigma(\mathcal{B})(-\xi),$$
$$\alpha_{22}(\xi) = (\sigma(\mathcal{A})(-\xi))^{-1}.$$

The direct calculation shows that α_{11} is the null matrix, i.e., $\alpha_{11}(\xi) \equiv 0$, while

$$\alpha_{12}(\xi) = \begin{pmatrix} -i\tau_{-2a}(\xi)\,e^{\varepsilon\pi i}\zeta^{\frac{1}{2}+\varepsilon}(\xi) & 0 \\ 0 & i\tau_{-2a}(\xi)\,e^{\varepsilon\pi i}\zeta^{-\frac{1}{2}+\varepsilon}(\xi) \end{pmatrix},$$

$$\alpha_{21}(\xi) = \begin{pmatrix} i\tau_{2a}(\xi)\,e^{\varepsilon\pi i}\zeta^{\frac{1}{2}+\varepsilon}(\xi) & 0 \\ 0 & -i\tau_{2a}(\xi)\,e^{\varepsilon\pi i}\zeta^{-\frac{1}{2}+\varepsilon}(\xi) \end{pmatrix},$$

$$\alpha_{22}(\xi) = \begin{pmatrix} -\frac{1}{2}\,e^{2\varepsilon\pi i}\zeta^{\frac{1}{2}+\varepsilon}(\xi) & -e^{2\varepsilon\pi i}\,\langle\xi\rangle\,\sigma(\mathcal{H})(\xi)\,\zeta^{\varepsilon}(\xi) \\ -e^{2\varepsilon\pi i}\,\langle\xi\rangle^{-1}\sigma(\mathcal{A}_p)(\xi)\,\zeta^{\varepsilon}(\xi) & -\frac{1}{2}\,e^{2\varepsilon\pi i}\sigma(\mathcal{A}_p)(\xi)[\sigma(\mathcal{L})(\xi)]^{-1}\zeta^{-\frac{1}{2}+\varepsilon}(\xi) \end{pmatrix}.$$

Under the above conditions it is straightforward to conclude that

$$r_{\mathbb{R}_+}\mathrm{Op}(\Xi) : [L_2(\mathbb{R}_+)]^4 \to [L_2(\mathbb{R}_+)]^4 \tag{4.7}$$

is a continuous operator. Moreover, it is easy to see that the determinant of the symbol of this operator is always nonzero, for all $\xi \in \mathbb{R}$.

The importance of the operator $r_{\mathbb{R}_+}\mathrm{Op}(\Xi)$ is clarified by the next result.

Theorem 3. (i) The operators

$$r_{\mathbb{R}_+}\mathcal{A} \pm r_{\mathbb{R}_+}\mathcal{B}J : [L_2(\mathbb{R}_+)]^2 \to [L_2(\mathbb{R}_+)]^2$$

(defined in (4.3)–(4.5)) are both invertible if and only if the operator $r_{\mathbb{R}_+}\mathrm{Op}(\Xi)$ (given in (4.7)) is invertible.

(ii) The operators $r_{\mathbb{R}_+}\mathcal{A} + r_{\mathbb{R}_+}\mathcal{B}J$ and $r_{\mathbb{R}_+}\mathcal{A} - r_{\mathbb{R}_+}\mathcal{B}J$ have both the Fredholm property if and only if $r_{\mathbb{R}_+}\mathrm{Op}(\Xi)$ has the Fredholm property. In addition, when in the presence of the Fredholm property for these three operators, their Fredholm indices satisfy the identity

$$\mathit{Ind}\,(r_{\mathbb{R}_+}\mathcal{A} + r_{\mathbb{R}_+}\mathcal{B}J) + \mathit{Ind}\,(r_{\mathbb{R}_+}\mathcal{A} - r_{\mathbb{R}_+}\mathcal{B}J) = \mathit{Ind}\, r_{\mathbb{R}_+}\mathrm{Op}(\Xi)\,. \tag{4.8}$$

In fact, this theorem is a consequence of a stronger fact which basically states that $r_{\mathbb{R}_+}\mathrm{Op}(\Xi)$ is (toplinear) equivalent after extension to a diagonal block matrix operator whose diagonal entries are the operators $r_{\mathbb{R}_+}\mathcal{A} + r_{\mathbb{R}_+}\mathcal{B}J$ and $r_{\mathbb{R}_+}\mathcal{A} - r_{\mathbb{R}_+}\mathcal{B}J$. Moreover, it is interesting to clarify that all the necessary operators to identify such (toplinear) equivalence after extension relation can be built in an explicit way (see [14, 15, 16, 17]).

Having in mind the Theorem 3, now we would like to investigate the Wiener-Hopf operator

$$r_{\mathbb{R}_+}\mathrm{Op}(\Xi) : [L_2(\mathbb{R}_+)]^4 \to [L_2(\mathbb{R}_+)]^4.$$

We have that Ξ belongs to the very general C^*-algebra of the semi-almost periodic four by four matrix functions on the real line ($[SAP(\mathbb{R})]^{4\times 4}$); see [41]. We recall that $[SAP(\mathbb{R})]^{4\times 4}$ is the smallest closed subalgebra of $[L^\infty(\mathbb{R})]^{4\times 4}$ that contains the (classical) algebra of (two by two) *almost periodic elements* ($[AP]^{4\times 4}$) and the (four by four) continuous matrices with possible jumps at infinity.

Due to a known characterization of the structure of $[SAP(\mathbb{R})]^{4\times 4}$ (see [3, 4, 41]), we can choose a continuous function on the real line, say γ, such that $\gamma(-\infty) = 0$, $\gamma(+\infty) = 1$ and

$$\Xi = (1-\gamma)\Xi_l + \gamma\Xi_r + \Xi_0,$$

where Ξ_0 is a continuous four by four matrix function with zero limit at infinity, and Ξ_l and Ξ_r are matrices with almost periodic elements, uniquely determined by Ξ, and that in our case have the following form (due to the behavior of Ξ at $\pm\infty$):

$$\Xi_l = \begin{pmatrix} 0 & 0 & i\tau_{-2a}e^{-\varepsilon\pi i} & 0 \\ 0 & 0 & 0 & -i\tau_{-2a}e^{-\varepsilon\pi i} \\ -i\tau_{2a}e^{-\varepsilon\pi i} & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & i\tau_{2a}e^{-\varepsilon\pi i} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

$$\Xi_r = \begin{pmatrix} 0 & 0 & -i\tau_{-2a}e^{\varepsilon\pi i} & 0 \\ 0 & 0 & 0 & i\tau_{-2a}e^{\varepsilon\pi i} \\ i\tau_{2a}e^{\varepsilon\pi i} & 0 & -\frac{1}{2}e^{2\varepsilon\pi i} & \frac{1}{2}e^{2\varepsilon\pi i} \\ 0 & -i\tau_{2a}e^{\varepsilon\pi i} & -\frac{1}{2}e^{2\varepsilon\pi i} & -\frac{1}{2}e^{2\varepsilon\pi i} \end{pmatrix}.$$

Here, it is worth noting that we had in consideration that $\omega(\xi) \to i|\xi|$ as $\xi \to \pm\infty$ (cf. (4.1)–(4.2)), and

$$\zeta^{\nu}(\xi) \to 1 \quad \text{as} \quad \xi \to \infty,$$

and

$$\zeta^{\nu}(\xi) \to e^{-2\pi\nu i} \quad \text{as} \quad \xi \to -\infty.$$

Theorem 4. For $0 \le \varepsilon < 1/4$, the operator $r_{\mathbb{R}_+}\mathrm{Op}(\Xi) : [L_2(\mathbb{R}_+)]^4 \to [L_2(\mathbb{R}_+)]^4$ is a Fredholm operator with zero Fredholm index.

The proof repeats word by word the arguments given in the proof of [12, Theorem 7.4] since the matrices Ξ_l and Ξ_r are exactly the same as corresponding matrices considered in [12, Section 7] and therefore it is omitted here. Note also that if we would allow the case $\varepsilon = 1/4$ then our operators would not have the Fredholm property (and therefore would not be invertible operators).

Corollary 1. Let $0 \le \varepsilon < \frac{1}{4}$ and one of the conditions (a)–(e) in Theorem 1 be satisfied. The Wiener-Hopf plus and minus Hankel operators (3.9) and (3.17) (which characterize our problems) are invertible operators.

Proof. As a consequence of the equivalence relations (4.6), we have:

$$\dim \mathrm{CoKer}\, r_{\mathbb{R}_+} \mathcal{K}_{++} = \dim \mathrm{CoKer}\, r_{\mathbb{R}_+} (\mathcal{A} + \mathcal{B}J), \tag{4.9}$$

$$\dim \mathrm{Ker}\, r_{\mathbb{R}_+} \mathcal{K}_{++} = \dim \mathrm{Ker}\, r_{\mathbb{R}_+} (\mathcal{A} + \mathcal{B}J). \tag{4.10}$$

and

$$\dim \mathrm{CoKer}\, r_{\mathbb{R}_+} \mathcal{K}_{--} = \dim \mathrm{CoKer}\, r_{\mathbb{R}_+} (\mathcal{A} - \mathcal{B}J), \tag{4.11}$$

$$\dim \mathrm{Ker}\, r_{\mathbb{R}_+} \mathcal{K}_{--} = \dim \mathrm{Ker}\, r_{\mathbb{R}_+} (\mathcal{A} - \mathcal{B}J). \tag{4.12}$$

From Theorem 3 and Theorem 4, we obtain that $r_{\mathbb{R}_+}(\mathcal{A} + \mathcal{B}J)$ and $r_{\mathbb{R}_+}(\mathcal{A} - \mathcal{B}J)$ are Fredholm operators. Moreover, recalling that under one of the conditions (a)–(e) in Theorem 1 it holds $\mathrm{Ker}\, r_{\mathbb{R}_+} \mathcal{K}_{++} = \{0\}$ and $\mathrm{Ker}\, r_{\mathbb{R}_+} \mathcal{K}_{--} = \{0\}$, from identities (4.8), (4.9)–(4.12) and Theorem 4, it follows

$$\begin{aligned} 0 &= \mathit{Ind}\, r_{\mathbb{R}_+}(\mathcal{A} + \mathcal{B}J) + \mathit{Ind}\, r_{\mathbb{R}_+}(\mathcal{A} - \mathcal{B}J) = \mathit{Ind}\, r_{\mathbb{R}_+} \mathcal{K}_{++} + \mathit{Ind}\, r_{\mathbb{R}_+} \mathcal{K}_{--} \\ &= (0 - \dim \mathrm{CoKer}\, r_{\mathbb{R}_+} \mathcal{K}_{++}) + (0 - \dim \mathrm{CoKer}\, r_{\mathbb{R}_+} \mathcal{K}_{--}). \end{aligned}$$

Thus, we have

$$\dim \mathrm{CoKer}\, r_{\mathbb{R}_+} \mathcal{K}_{++} = \dim \mathrm{CoKer}\, r_{\mathbb{R}_+} \mathcal{K}_{--} = 0$$

and so we reach to the conclusion that both operators in (3.9) and (3.17) are invertible (under the announced conditions).

5 Main result

We are now in a position to derive the main result of this work. This is obtained as a direct combination of the results and constructions of the last two sections (with special emphasis to Corollary 1).

Theorem 5. If $0 \le \varepsilon < \frac{1}{4}$ and one of the following situations holds

$$\begin{aligned} &(a) \quad (\Re\mathrm{e}\, k)(\Im\mathrm{m}\, k) > 0, \quad \Im\mathrm{m}\, p \geq 0, \\ &(b) \quad (\Re\mathrm{e}\, k)(\Im\mathrm{m}\, k) < 0, \quad \Im\mathrm{m}\, p \leq 0, \\ &(c) \quad |\Im\mathrm{m}\, k| \geq |\Re\mathrm{e}\, k|, \quad \Re\mathrm{e}\, p \leq 0, \\ &(d) \quad \Re\mathrm{e}\, k = 0, \quad \Im\mathrm{m}\, p > 0, \\ &(e) \quad \Im\mathrm{m}\, p \neq 0, \quad (\Im\mathrm{m}\, k)^2 - (\Re\mathrm{e}\, k)^2 + 2(\Re\mathrm{e}\, k)(\Im\mathrm{m}\, k)\frac{\Re\mathrm{e}\, p}{\Im\mathrm{m}\, p} > 0, \end{aligned}$$

then:

(i) the Problem $\mathcal{P}_{I-D-D}$ has a unique solution which is representable as a pair (u_1, u_2) defined by the formulas (3.6) and (3.7), where the components φ and ψ of the unique solution Υ of the equation (3.8) are used.

(ii) the Problem $\mathcal{P}_{I-D-N}$ has a unique solution which is representable as a pair (u_1, u_2) defined by the formulas (3.14) and (3.15), where the components φ and ψ of the unique solution Υ of the equation (3.16) are used.

Moreover, in the present conditions, the two problems $\mathcal{P}_{I-D-D}$ and $\mathcal{P}_{I-D-N}$ are well-posed (since the resolvent operators are continuous).

We conclude by pointing out that although from the natural assumptions in the formulation of the problems (cf. Section 2) we were looking for the eventual possibilities for $\varepsilon \in [0, \frac{1}{2})$, we now realize that the last result is optimal from the point of view of the possible variability of the Bessel potential spaces smoothness orders (in view to have corresponding well-posed problems).

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