

# Multiplicative Number Theory

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# Generating functions and asymptotics

The general rationale of analytic number theory is to derive statistical information about a sequence  $\{a_n\}$  from the analytic behaviour of an appropriate generating function, such as a power series  $\sum a_n z^n$  or a Dirichlet series  $\sum a_n n^{-s}$ .

The type of generating function employed depends on the problem being investigated. There are no rigid rules governing the kind of generating function that is appropriate – the success of a method justifies its use – but we usually deal with **additive questions by means of power series or trigonometric sums**, and with **multiplicative questions by Dirichlet series**.

To appreciate why power series are useful in dealing with additive problems, note that if  $A(z) = \sum a_k z^k$  and  $B(z) = \sum b_m z^m$  then the power series coefficients of  $C(z) = A(z)B(z)$  are given by

$$c_n = \sum_{k+m=n} a_k b_m.$$

# Additive number theory - Example

If  $f(z) = \sum_{n=1}^{\infty} z^{n^k}$ , for  $|z| < 1$ , then the  $n^{\text{th}}$  power series coefficient of  $f(z)^s$  is the number  $r_{k,s}$  of representations of  $n$  as the sum of  $s$  positive  $k^{\text{th}}$  powers,

$$n = m_1^k + \cdots + m_s^k.$$

We can recover  $r_{k,s}$  from  $(f(z))^s$ , by means of Cauchy coefficient formula

$$r_{s,k} = \frac{1}{2\pi i} \oint \frac{(f(z))^s}{z^{n+1}} dz.$$

By choosing an appropriate contour, and estimating the integrand, we can determine the asymptotic size of  $r_{s,k}$  as  $n \rightarrow \infty$ , provided that  $s$  is sufficiently large. This is the germ of the **Hardy–Littlewood circle method**.

# Dirichlet series

A **Dirichlet series** is a series of the form  $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ , where  $s$  is a complex variable.

If  $\beta(s) = \sum_{m=1}^{\infty} b_m m^{-s}$  is another Dirichlet series and  $\gamma(s) = \alpha(s)\beta(s)$  then (ignoring questions relating to the arrangement of terms of infinite series)  $\gamma(s) = \sum_{k=1}^{\infty} c_k k^{-s}$ , where

$$c_k = \sum_{nm=k} a_n b_m.$$

This explains why Dirichlet series are useful in dealing with multiplicative problems.

We will use the standard notation  $s = \sigma + it$ .

# Riemann zeta function

Among the Dirichlet series we consider the **Riemann zeta function** which, for  $\sigma > 1$  is defined by the absolute convergent series

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

## Lemma

*The series  $\zeta(s)$  is absolutely convergent for all  $s \in \mathbb{C}$  with  $\sigma > 1$ , and uniformly convergent in any compact subset of  $\{s \mid \sigma > 1\}$ . In particular, by Weierstrass Theorem<sup>a</sup>,  $\zeta(s)$  is an analytic function in  $\{s \mid \sigma > 1\}$ .*

**Remember**  $|n^{-s}| = |e^{-s \log n}| = |e^{-\sigma \log n} e^{-it \log n}| = |e^{-\sigma \log n}|.$

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<sup>a</sup>Weierstrass Theorem states that if  $\{f_k\}$  is a sequence of analytic functions in an open set  $\Omega \subset \mathbb{C}$  and if  $f$  is a function on  $\Omega$  such that  $f_k \rightarrow f$  uniformly in any compact subset of  $\Omega$ , then  $f$  is analytic in  $\Omega$ , and also  $f_k^{(n)} \rightarrow f^{(n)}$  uniformly on compact subsets of  $\Omega$ .

# Euler Identity

The reason why  $\zeta(s)$  is important in the study of primes is the following identity, which is a consequence of the **unique prime factorization**:

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1},$$

for all  $s \in \mathbb{C}$  with  $\sigma > 1$ .

$$\begin{aligned} & (1 + 2^{-s} + 2^{-2s} + \dots) (1 + 3^{-s} + 3^{-2s} + \dots) (1 + 5^{-s} + 5^{-2s} + \dots) (1 + 7^{-s} + 7^{-2s} + \dots) \dots \\ & = 1 + 2^{-s} + 3^{-s} + (2^2)^{-s} + 5^{-s} + (2 \cdot 3)^{-s} + 7^{-s} + \dots \end{aligned}$$

None of the factors on the right hand side vanishes, since  $|p^{-s}| = |p^{-\sigma}| < 1$ , when  $\sigma > 1$ .

Hence, it seems reasonable that we have  $\zeta(s) \neq 0$ , for  $\sigma > 1$  (this can be rigorously proved).

# $\log \zeta(s)$

It follows that  $\log \zeta(s)$  can be defined for each  $s \in \mathbb{C}$  with  $\sigma > 1$ . Although  $\log$  is a multivalued function, we may define

$$\log \zeta(s) = - \sum_p \log (1 - p^{-s}),$$

where each logarithm on the right is taken in the principal branch. Using the Euler identity, this is the most natural choice of the logarithm.

Using the Taylor expansion  $-\log(1 - z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \dots$ , valid for all  $z \in \mathbb{C}$  with  $|z| < 1$ , we may write

$$\log \zeta(s) = \sum_p \sum_{m=1}^{\infty} m^{-1} p^{-ms}.$$

# Infinitude of primes

Now, we restrict to considering real  $s > 1$ . Directly from the definition of  $\zeta(s)$  we have

$$\lim_{s \rightarrow 1^+} \zeta(s) = \infty \quad \text{and so} \quad \lim_{s \rightarrow 1^+} \log \zeta(s) = \infty.$$

Also note that, for  $s > 1$ ,

$$\begin{aligned} \sum_p \sum_{m=2}^{\infty} m^{-1} p^{-ms} &< \sum_p \sum_{m=2}^{\infty} p^{-m} = \sum_p \frac{1}{p(p-1)} \\ &< \sum_{n=2}^{\infty} \frac{1}{n-1} - \frac{1}{n} = 1. \end{aligned}$$

Hence,  $\lim_{s \rightarrow 1^+} \sum_p p^{-s} = \infty$ , and so, there are an infinitude of primes.

## Euler summation formula

The simplest types of sums  $\sum_{n \leq x} f(n)$  are those in which  $f$  is a smooth function that is defined for real arguments  $x$ .

The basic idea for handling such sums is to approximate the sum by a corresponding integral and investigate the error made in the process. The following important result, known as Euler's summation formula, gives an exact formula for the difference between such a sum and the corresponding integral.

### Theorem (Euler summation formula)

*Let  $0 < y \leq x$  and suppose  $f(x)$  is a function defined on the interval  $[y, x]$  and having a continuous derivative there. Then*

$$\sum_{y < n \leq x} f(n) = \int_y^x f(t) dt + \int_y^x \{t\} f'(t) dt - \{x\} f(x) + \{y\} f(y),$$

*where  $\{x\} = x - \lfloor x \rfloor$  denotes the fractional part of  $x$ .*

## Euler summation formula (contd)

Taking  $y = 1$ , we obtain the special case

$$\sum_{n \leq x} f(n) = \int_1^x f(t) dt + \int_1^x \{t\} f'(t) dt - \{x\} f(x) + f(1).$$

Proof.

Consider  $F(x) = \lfloor x \rfloor$ . Then, we can write the given sum as a Stieltjes integral

$$\sum_{y < n \leq x} f(n) = \int_y^x f(t) dF(t).$$

Now, we have  $dF(t) = dt - d\{t\}$  and  $d\{t\} = \frac{d\{t\}}{dt} dt$ , hence, using integration by parts, we obtain

$$\sum_{y < n \leq x} f(n) = \int_y^x f(t) dt + \int_y^x \{t\} f'(t) dt - \{x\} f(x) + \{y\} f(y).$$

# Example 1: Partial sums of the harmonic series

Theorem

We have

$$\sum_{n \leq x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right),$$

where

$$\gamma = \lim_{x \rightarrow \infty} \left( \sum_{n \leq x} \frac{1}{n} - \log x \right) \simeq 0.5772\dots,$$

is the Euler constant.

Proof.

By Euler summation formula,

$$\sum_{n \leq x} \frac{1}{n} = \int_1^x \frac{1}{t} dt - \int_1^x \frac{\{t\}}{t^2} dt + \frac{\{x\}}{x} + 1.$$

# Example 1: Partial sums of the harmonic series (contd)

Proof.

Now,  $0 \leq \frac{\{t\}}{t^2} \leq \frac{1}{t^2}$  and the improper integral  $\int_1^\infty \frac{1}{t^2} dt$  is convergent and, in fact equal to 1. So, the improper integral

$$I = \int_1^\infty \frac{\{t\}}{t^2} dt$$

is also convergent and its value is between 0 and 1. Therefore,

$$\int_1^x \frac{\{t\}}{t^2} dt = \int_1^\infty \frac{\{t\}}{t^2} dt - \int_x^\infty \frac{\{t\}}{t^2} dt = I + O\left(\frac{1}{x}\right).$$

Hence

$$\sum_{n \leq x} \frac{1}{n} = \log x + 1 - I + O\left(\frac{1}{x}\right),$$

for  $x \geq 1$ . In particular, this implies  $\gamma = \lim_{x \rightarrow \infty} \left( \sum_{n \leq x} \frac{1}{n} - \log x \right) = 1 - I$ .



## Example 2: Stirling formula

Theorem (Partial sums of the logarithmic function)

We have

$$\sum_{n \leq N} \log n = N(\log N - 1) + \frac{1}{2} \log N + \frac{1}{2} \log(2\pi) + O\left(\frac{1}{N}\right).$$

Corollary (Stirling formula)

$$n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + O\left(\frac{1}{n}\right)\right).$$

Corollary

For  $x \geq 2$ , we have

$$\sum_{n \leq x} \log n = x(\log x - 1) + O(\log x).$$

## Example 3: Integral representation of the Riemann zeta function

Theorem (Integral representation of the Riemann zeta function)

For  $\sigma > 1$ ,

$$\zeta(s) = \frac{s}{s-1} - s \int_1^{\infty} \frac{\{x\}}{x^{s+1}} dx.$$

Proof.

By Euler summation formula,

$$\sum_{n \leq x} \frac{1}{n^s} = \int_1^x \frac{1}{y^s} dy - \int_1^x \{y\} \frac{s}{y^{s+1}} dy - \frac{\{x\}}{x^s} + 1.$$

Now,

$$\int_1^x \frac{1}{y^s} dy = \left[ \frac{y^{1-s}}{1-s} \right]_1^x = \frac{x^{1-s} - 1}{1-s} = \frac{1}{s-1} + O_s(x^{1-\sigma}).$$



## Example 3: Integral representation of the Riemann zeta function (contd)

Proof.

Also,

$$\begin{aligned} - \int_1^x \{y\} \frac{s}{y^{s+1}} dy &= -s \int_1^\infty \frac{\{y\}}{y^{s+1}} dy + \int_x^\infty \{y\} \frac{s}{y^{s+1}} dy = \\ &= -s \int_1^\infty \frac{\{y\}}{y^{s+1}} dy + O_s(x^{-\sigma}) \end{aligned}$$

Letting  $x \rightarrow \infty$  we obtain

$$\zeta(s) = \frac{s}{s-1} - s \int_1^\infty \frac{\{x\}}{x^{s+1}} dx.$$



# Abel summation

Abel summation is the analogue for sums of integration by parts. Given a sum of the form  $\sum_{n \leq x} a_n f(n)$ , where  $a_n$  is an arithmetic function with summatory function

$$A(x) = \sum_{n \leq x} a_n$$

and  $f(n)$  a smooth weight, the Abel summation allows one to remove the weight  $f(n)$  and reduce the evaluation or estimation of the above sum to that of an integral over  $A(t)$ .

## Theorem (Abel summation)

*Let  $a : \mathbb{N} \rightarrow \mathbb{C}$  be an arithmetic function,  $0 < y < x$  be real numbers and  $f : [y, x] \rightarrow \mathbb{C}$  be a function with continuous derivative in  $[y, x]$ . Then*

$$\sum_{y < n \leq x} a_n f(n) = A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)dt.$$

# Abel summation

In particular, considering  $y = 1$ , we have

$$\sum_{n \leq x} a_n f(n) = A(x)f(x) - \int_1^x A(t)f'(t)dt.$$

**Exercise** Deduce Euler summation from Abel summation.

## Abel summation - Proof

Proof.

Set  $\chi(n, t) = 1$  if  $n \leq t$  and  $\chi(n, t) = 0$  otherwise. Then

$$\begin{aligned}
 \int_y^x A(t)f'(t)dt &= \int_y^x \sum_{n \leq x} a_n \chi(n, t) f'(t) dt \\
 &= \sum_{n \leq x} a_n \int_y^x \chi(n, t) f'(t) dt \\
 &= \sum_{n \leq x} a_n \int_{\max(n, y)}^x f'(t) dt \\
 &= \sum_{n \leq x} a_n f(x) - \sum_{n \leq x} a_n f(\max(n, y)) \\
 &= \sum_{n \leq x} a_n f(x) - \sum_{n \leq y} a_n f(y) - \sum_{y < n \leq x} a_n f(n) \\
 &= A(x)f(x) - A(y)f(y) - \sum_{y < n \leq x} a_n f(n)
 \end{aligned}$$



# Example: Mellin transform representation of Dirichlet series

## Theorem (Mellin transformation representation of Dirichlet series)

Let  $f$  be an arithmetic function,  $S_f(x) = \sum_{n \leq x} f(n)$  and let  $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$  be the Dirichlet series associated with  $f$ , whenever the series converge. Then, for any complex number  $s$  with  $\sigma > 0$  such that the  $F(s)$  converges, we have

$$F(s) = s \int_1^{\infty} \frac{S_f(x)}{x^{s+1}} dx.$$

# Dirichlet product

## Definition (Dirichlet product)

Given two arithmetic functions  $f$  and  $g$ , the *Dirichlet product* (or *Dirichlet convolution*) of  $f$  and  $g$ , denoted by  $f \star g$  is the arithmetic function defined by

$$(f \star g)(n) = \sum_{d|n} f(d)g\left(\frac{n}{d}\right).$$

- ①  $\tau(n) = \mathbf{1} \star \mathbf{1}$ , where  $\mathbf{1}(n) = 1$ , for any  $n$ ;
- ②  $\sigma(n) = \text{Id} \star \mathbf{1}$ , where  $\text{Id}(n) = n$ , for any  $n$ ;
- ③  $\mathbf{e} = \mu \star \mathbf{1}$ , where  $\mathbf{e}(1) = 1$  and  $\mathbf{e}(n) = 0$  for  $n > 1$ ;
- ④  $\varphi = \mu \star \text{Id}$ ;
- ⑤  $\text{Id} = \varphi \star \mathbf{1}$ ;
- ⑥  $\log = \Lambda \star \mathbf{1}$ , where  $\Lambda(p^m) = \log p$  and  $\Lambda(n) = 0$  if  $n$  is not a prime power.

# The convolution method

Given an arithmetic function  $f$  whose partial sums  $F(x) = \sum_{n \leq x} f(n)$  we want to estimate, we try to express  $f$  as a convolution  $f = f_0 \star g$ , where  $f_0$  is a function that approximates  $f$ , good estimates for the partial sums  $F_0(x) = \sum_{n \leq x} f_0(n)$  are available and  $g$  is a small perturbation.

We have

$$\begin{aligned}
 F(x) &= \sum_{n \leq x} f(n) = \sum_{n \leq x} \sum_{d|n} g(d) f_0\left(\frac{n}{d}\right) \\
 &= \sum_{d \leq x} g(d) \sum_{\substack{n \leq x \\ d|n}} f_0\left(\frac{n}{d}\right) \\
 &= \sum_{d \leq x} g(d) \sum_{n' \leq \frac{x}{d}} f_0(n') \\
 &= \sum_{d \leq x} g(d) F_0\left(\frac{x}{d}\right).
 \end{aligned}$$

# Partial sums of the Euler $\varphi$ function

## Theorem (Partial sums of the Euler $\varphi$ function)

We have

$$\sum_{n \leq x} \varphi(n) = \frac{3}{\pi^2} x^2 + O(x \log x),$$

for  $x \geq 2$ .

**Proof.**

We have  $\varphi = \mu \star \text{Id}$ . If we consider  $f = \varphi$ ,  $g = \mu$  and  $f_0 = \text{Id}$  then

$$F_0(x) = \frac{1}{2} [x] ([x] + 1) = \frac{1}{2} x^2 + O(x).$$



# Partial sums of the Euler $\varphi$ function (contd)

Proof.

Therefore,

$$\begin{aligned}
 \sum_{n \leq x} \varphi(n) &= \sum_{d \leq x} \mu(d) \left( \frac{1}{2} \frac{x^2}{d^2} + O\left(\frac{x}{d}\right) \right) \\
 &= \frac{1}{2} x^2 \sum_{d \leq x} \frac{\mu(d)}{d^2} + O\left(\sum_{d \leq x} \frac{x}{d}\right) \\
 &= \frac{1}{2} x^2 \sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} + O\left(x^2 \sum_{d > x} \frac{1}{d^2}\right) + O(x \log x) \\
 &= \frac{1}{2} x^2 \sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} + O(x \log x) \\
 &= \frac{3}{\pi^2} x^2 + O(x \log x)
 \end{aligned}$$

# The Dirichlet series $\frac{1}{\zeta(s)}$

For  $\sigma > 1$ , we have

$$\begin{aligned}
 1 &= \sum_{n=1}^{\infty} \frac{\mathbf{e}(n)}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{d|n} \mu(d) \\
 &= \sum_{d=1}^{\infty} \sum_{m=1}^{\infty} \frac{\mu(d)}{(dm)^s} \\
 &= \sum_{d=1}^{\infty} \frac{\mu(d)}{d^s} \sum_{m=1}^{\infty} \frac{1}{(m)^s}.
 \end{aligned}$$

Therefore,

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}.$$

Hence,

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} = \frac{6}{\pi^2}.$$

# Algebraic properties of Dirichlet series

## Definition

Let  $f$  be arithmetic function. The Dirichlet series associated to  $f$  is the series

$$D_f(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

## Theorem (Dirichlet series of convolution products)

*Let  $f$  and  $g$  be arithmetic functions with associated Dirichlet series  $D_f(s)$  and  $D_g(s)$ . Let  $h = f \star g$  with associated Dirichlet series  $D_h(s)$ . If  $D_f(s)$  and  $D_g(s)$  converges absolutely at some point  $s$  then so does  $D_h(s)$  and  $D_h(s) = D_f(s)D_g(s)$ .*

## Dirichlet series of arithmetic functions

- 1 **Unit function:**  $D_e(s) = 1$ ;
- 2 **Möbius function:**  $D_\mu(s) = \frac{1}{\zeta(s)}$ , for  $\sigma > 1$ ;
- 3 **Characteristic function of squares**  $s(n)$ :  $D_{s(n)}(s) = \zeta(2s)$ , for  $\sigma > \frac{1}{2}$ ;
- 4 **Logarithm:**  $D_{\log} = -\zeta'(s)$ , for  $\sigma > 1$ ;
- 5 **Identity function:**  $D_{\text{Id}}(s) = \zeta(s-1)$ , for  $\sigma > 2$ ;
- 6 **Euler  $\varphi$  function:**  $D_\varphi(s) = \frac{\zeta(s-1)}{\zeta(s)}$ , for  $\sigma > 2$ , since  $\varphi = \mu \star \text{Id}$ ;
- 7 **Divisor function:**  $D_\tau(s) = \zeta^2(s)$ , for  $\sigma > 1$ , since  $\tau(n) = \mathbf{1} \star \mathbf{1}$ ;
- 8 **Sum of divisors function:**  $D_\sigma(s) = \zeta(s)\zeta(s-1)$ , for  $\sigma > 2$ , since  $\sigma(n) = \text{Id} \star \mathbf{1}$ ;
- 9 **Characteristic function of squarefree numbers:**  $D_{\mu^2}(s) = \frac{\zeta(s)}{\zeta(2s)}$ , for  $\sigma > 1$ , since  $\mathbf{1} = \mu^2 \star s$ ;
- 10 **Von Mangoldt function:**  $D_\Lambda(s) = \frac{\zeta'(s)}{\zeta(s)}$ , for  $\sigma > 1$ , since  $\log = \Lambda \star \mathbf{1}$ .

# Partial sums of the divisor function

## Theorem (Partial sums of the divisor function)

We have

$$\sum_{n \leq x} \tau(n) = x \log x + (2\gamma - 1)x + O(\sqrt{x}).$$

## Proof.

We have  $\tau = \mathbf{1} \star \mathbf{1}$ . If we consider  $f = \tau$ ,  $g = \mathbf{1}$  and  $f_0 = \mathbf{1}$  then  $F_0(x) = \lfloor x \rfloor$ . Therefore,

$$\sum_{n \leq x} \tau(n) = \sum_{d \leq x} 1 \left\lfloor \frac{x}{d} \right\rfloor.$$



# Dirichlet hyperbola method

Proof.

Notice that

$$\sum_{d \leq x} 1 \left\lfloor \frac{x}{d} \right\rfloor = \sum_{\substack{m, d \leq x \\ md \leq x}} 1.$$

This sum can be separated in three sums,

$$\sum_{\substack{m, d \leq x \\ md \leq x}} 1 = \sum_{\substack{m \leq \sqrt{x} \\ d \leq \frac{x}{m}}} 1 + \sum_{\substack{d \leq \sqrt{x} \\ m \leq \frac{x}{d}}} 1 - \sum_{\substack{d \leq \sqrt{x} \\ m \leq \sqrt{x}}} 1.$$

The last sum is equal to  $[\sqrt{x}]^2$ , which is  $x + O(\sqrt{x})$ . □

## Partial sums of the divisor function (contd)

Proof.

Now,

$$\begin{aligned}\sum_{\substack{m \leq \sqrt{x} \\ d \leq \frac{x}{m}}} 1 &= \sum_{m \leq \sqrt{x}} \left\lfloor \frac{x}{m} \right\rfloor \\ &= x \sum_{m \leq \sqrt{x}} \frac{1}{m} + O\left(\sum_{m \leq \sqrt{x}} 1\right) \\ &= x \log(\sqrt{x}) + \gamma x + O(\sqrt{x}).\end{aligned}$$

Therefore,

$$\sum_{n \leq x} \tau(n) = x \log x + (2\gamma - 1)x + O(\sqrt{x}).$$



# Dirichlet divisor problem

Dirichlet divisor problem consists in finding an estimation of the function  $\Delta(x)$ , given by

$$\sum_{n \leq x} d(n) = x \log x + (2\gamma - 1)x + \Delta(x).$$

Let  $\psi(x) = \{x\} - \frac{1}{2}$ . Using better estimations of the partial sums of the harmonic series, of  $\log \lfloor \sqrt{x} \rfloor$  and of  $\lfloor \sqrt{x} \rfloor^{-1}$ , it can be shown that

$$\Delta(x) = -2 \sum_{n \leq \sqrt{x}} \psi\left(\frac{x}{n}\right) - \frac{1}{6} + 2(\{\sqrt{x}\} - \{\sqrt{x}\}^2) + O\left(\frac{1}{\sqrt{x}}\right).$$

So, the main (oscillating) term of  $\Delta(x)$  is related to  $\psi(x)$ . Hardy proved that

$$\Delta(x) = \Omega_{\pm}\left(x^{\frac{1}{4}}\right)$$

and conjectured that

$$\Delta(x) = O\left(x^{\frac{1}{4} + \delta}\right)$$

# Dirichlet divisor problem

Dirichlet(1849):  $1/2=0.5$

Voronoi(1903):  $1/3=0.(3)$

van der Corput(1922):  $33/100=0.33$

van der Corput(1928):  $27/82 = 0.3(29268)$

Chih(1950) and Richert(1953):  $15/46 \sim 0.326086$

Kolesnik(1982):  $35/108 = 0.32(407)$

Iwaniec and Mozzochi (1988):  $7/22 = 0.3(18)$

Huxley(2003):  $131/416 \sim 0.314903$

# Lower bounds for $\pi(n)$

Getting upper and lower bounds for the prime counting function  $\pi(x)$  is surprisingly difficult. Euclid's result that there are infinitely many primes shows that  $\pi(x)$  tends to infinity, but standard proofs of this result give at most very weak lower bounds for  $\pi(x)$ . Euclid's argument, shows that  $p_n \leq p_1 p_2 \cdots p_{n-1} + 1$ . By induction, this implies  $p_n \leq 2^{2^{n-1}}$ . Hence,  $\pi(x) \geq \log \log x$ .

Getting upper bounds for  $\pi(x)$  is not easy either, even to get  $\pi(x) = o(x)$  is by no means easy.

# Chebyshev estimates

Consider

$$\theta(x) = \sum_{p \leq x} \log p \quad \text{and} \quad \psi(x) = \sum_{n \leq x} \Lambda(n).$$

## Theorem (Chebyshev estimates)

For  $x \geq 2$ , we have

- 1  $\psi(x) \asymp x$ ;
- 2  $\theta(x) \asymp x$ ;
- 3  $\pi(x) \asymp \frac{x}{\log x}$ .

## Chebyshev estimates - Proof

Proof.

Define  $S(x) = \sum_{n \leq x} \log n$  and  $D(x) = S(x) - 2S\left(\frac{x}{2}\right)$ . We will evaluate  $D(x)$  in two different ways. As we saw before,

$$\begin{aligned} D(x) &= x(\log x - 1) + O(\log x) - 2\frac{x}{2}\left(\log \frac{x}{2} - 1\right) + O(\log x) \\ &= x \log 2 + O(\log x). \end{aligned}$$

This implies  $\frac{x}{2} \leq D(x) \leq x$ , for sufficiently large  $x$ .

On the other hand, using  $\log = \Lambda \star \mathbf{1}$  and the Convolution Method, we obtain

$$S(x) = \sum_{d \leq x} \Lambda(d) \sum_{\substack{n \leq x \\ d|n}} 1 = \sum_{d \leq x} \Lambda(d) \left\lfloor \frac{x}{d} \right\rfloor.$$



## Chebyshev estimates - Proof

Proof.

Now,

$$\begin{aligned}
 D(x) &= S(x) - 2S\left(\frac{x}{2}\right) \\
 &= \sum_{d \leq x} \Lambda(d) \left\lfloor \frac{x}{d} \right\rfloor - 2 \sum_{d \leq \frac{x}{2}} \Lambda(d) \left\lfloor \frac{\frac{x}{2}}{d} \right\rfloor \\
 &= \sum_{d \leq x} \Lambda(d) \left( \left\lfloor \frac{x}{d} \right\rfloor - 2 \left\lfloor \frac{x}{2d} \right\rfloor \right)
 \end{aligned}$$

Notice that the function  $f(t) = \lfloor t \rfloor - 2 \lfloor \frac{t}{2} \rfloor$  is always an integer which is 1 when  $1 \leq t < 2$  and  $f(t) \in \{0, 1\}$  if  $t \geq 2$ . Therefore, for  $x \geq 2$ ,  
 $D(x) \leq \sum_{d \leq x} \Lambda(d) = \psi(x)$  or  $D(x) \geq \sum_{\frac{x}{2} < d \leq x} \Lambda(d) = \psi(x) - \psi\left(\frac{x}{2}\right)$ .

□

# Chebyshev estimates - Proof

Proof.

Hence  $\psi(x) \geq \frac{x}{2}$  and  $\psi(x) \leq x + \psi\left(\frac{x}{2}\right)$ , for  $x$  sufficiently large. The last inequality implies  $\psi(x) \leq 2x$ . Whence,

$$\psi(x) \asymp x.$$

Now, to get  $\theta(x) \asymp x$ , notice that  $\theta(x) \leq \psi(x)$  and

$$\begin{aligned} \psi(x) - \theta(x) &= \sum_{p^m \leq x} \log p - \sum_{p \leq x} \log p = \sum_{p \leq \sqrt{x}} \log p \sum_{m=2}^{\frac{\log x}{\log p}} 1 \\ &\leq \sum_{p \leq \sqrt{x}} \log p \left\lfloor \frac{\log x}{\log p} \right\rfloor \\ &\leq \sqrt{x} \log x. \end{aligned}$$

## Chebyshev estimates - Proof

Proof.

Finally, we prove that  $\pi(x) \asymp \frac{x}{\log x}$ . Firstly,

$$\pi(x) \geq \frac{1}{\log x} \sum_{p \leq x} \log p = \frac{\theta(x)}{\log x} \geq c_1 \frac{x}{\log x}$$

and

$$\pi(x) \leq \pi(\sqrt{x}) + \frac{1}{\log \sqrt{x}} \sum_{\sqrt{x} < p \leq x} \log p \leq \sqrt{x} + \frac{2\theta(x)}{\log x}.$$



# Chebyshev estimates - Comments

Chebyshev used a more complicated version of this argument, in which the linear combination  $S(x) - 2S\left(\frac{x}{2}\right)$  is replaced by

$$S(x) - S\left(\frac{x}{2}\right) - S\left(\frac{x}{3}\right) - S\left(\frac{x}{5}\right) + S\left(\frac{x}{30}\right)$$

to obtain the constants  $c_1 = 0.92\dots$  and  $c_2 = 1.10\dots$ . All other proofs of these estimates use some kind of trickery.

For example, a commonly seen argument, which may be a bit shorter than the one given here, but has more the character of pulling something out of the air, is based on an analysis of the middle binomial coefficient  $\binom{2n}{n}$ .

# Relation between $\psi(x)$ , $\theta(x)$ and $\pi(x)$ .

## Theorem

We have  $\theta(x) = \psi(x) + O(\sqrt{x})$  and  $\pi(x) = \frac{\psi(x)}{\log x} + O\left(\frac{x}{\log^2 x}\right)$ .

## Proof.

The first relation results from the Chebyshev estimate  $\pi(x) \asymp \frac{x}{\log x}$  applied to the expression  $\sum_{p \leq \sqrt{x}} \log p \left\lfloor \frac{\log x}{\log p} \right\rfloor$  above. The second relation results from Abel summation, considering  $a_n = \log n$  if  $n$  is prime and  $a_n = 0$  otherwise, and  $f(x) = \frac{1}{\log x}$ . We have

$$\begin{aligned} \pi(x) &= \sum_{n \leq x} a_n f(n) = \frac{\theta(x)}{\log x} - \int_1^x \frac{\theta(t)}{t \log^2 t} dt \\ &= \frac{\theta(x)}{\log x} - O\left(\frac{x}{\log^2 x}\right). \end{aligned}$$

## Mertens estimates

## Theorem

1

$$\sum_{n \leq x} \frac{\Lambda(n)}{n} = \log x + O(1);$$

2

$$\sum_{p \leq x} \frac{\log p}{p} = \log x + O(1);$$

3

$$\sum_{p \leq x} \frac{1}{p} = \log \log x + \gamma - \sum_p \sum_{k=2}^{\infty} \frac{1}{kp^k} + O\left(\frac{1}{\log x}\right);$$

4

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) = \frac{e^{-\gamma}}{\log x} + O\left(\frac{1}{\log^2 x}\right).$$

# Mertens estimates

Proof.

By Chebyshev estimate  $\psi(x) \asymp x$ , we have

$$x \log x + O(x) = S(x) = \sum_{n \leq x} \log(n) = \sum_{d \leq x} \Lambda(d) \left\lfloor \frac{x}{d} \right\rfloor = x \sum_{d \leq x} \frac{\Lambda(d)}{d} + O(x).$$

Now,

$$\sum_{d \leq x} \frac{\Lambda(d)}{d} - \sum_{p \leq x} \frac{\log(p)}{p} = O(1).$$

For the third estimate, we use Abel summation.

$$\sum_{p \leq x} \frac{1}{p} = \frac{\sum_{p \leq x} \frac{\log p}{p}}{\log x} - \int_2^x \left( \sum_{p \leq t} \frac{\log p}{p} \right) \frac{-1}{t \log^2 t} dt.$$

For the fourth estimate we take the logarithm and use the third. □

# Mertens type estimates

## Theorem

For any  $x \geq 1$ ,

$$\sum_{n \leq x} \frac{\mu(n)}{n} \leq 1$$

## Proof.

We have  $\mathbf{e} = \mu \star \mathbf{1}$  and, without loss of generality, we may assume  $x = N$  an integer. So, by the convolution method, we have

$$\sum_{n \leq N} \mathbf{e}(n) = \sum_{d \leq N} \mu(d) \left[ \frac{N}{d} \right] = N \sum_{d \leq N} \frac{\mu(d)}{d} - \sum_{d \leq N} \mu(d) \left\{ \frac{N}{d} \right\}.$$



## Mertens type estimates - contd

Proof.

Now,  $\left\{\frac{N}{d}\right\} = 0$  if  $d = N$ , hence

$$\sum_{d \leq N} \mu(d) \left\{\frac{N}{d}\right\} \leq N - 1.$$

Therefore,

$$N \sum_{d \leq N} \frac{\mu(d)}{d} \leq 1 + N - 1 = N.$$

and we obtain the stated result. □

# Relation between PNT and the $\mu(n)$ function

## Theorem

*The Prime Number Theorem is equivalent to*

$$\lim_{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} \mu(n) = 0.$$

What is so surprising about this result is that there does not seem to be any obvious connection between the distribution of primes (which is described by the PNT), and the distribution of values of the Moebius function.

# Prime Number Theorem

In its original form, the Prime Number Theorem states that  $\pi(x) \sim \frac{x}{\log x}$ .  
Which is equivalent to the following relations:

- 1  $\pi(x) \sim \text{Li}(x)$ ;
- 2  $\psi(x) \sim x$ ;
- 3  $\theta(x) \sim x$ .

Where

$$\text{Li}(x) = \int_2^x \frac{dt}{\log t}.$$

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Where

$$\text{Li}(x) = \int_2^x \frac{dt}{\log t}.$$

Theorem (Prime Number Theorem with error term)

We have

$$\psi(x) = x + O(x \exp(-c(\log x)^a)),$$

where  $c$  and  $a$  are constants (various values for  $a$  have been considered, for example Hildebrand has the proof for  $a = 0.1$ ).

## The error term

Hadamard and Poussin (1896):	$o(x)$	Complex
De la Vallée Poussin (1899):	$O(x \exp(-c(\log x)^{0.5}))$	Complex
Vinogradov–Korobov (1958):	$O(x \exp(-c(\log x)^{0.6-\epsilon}))$	Complex
Erdős–Selberg (1949):	$o(x)$	Elementary
Lavrik–Sobirov (1973):	$O(x \exp(-c(\log x)^{1/6-\epsilon}))$	Elementary
?	$O(x^{0.5+\epsilon})?$	?

# Perron's formulas

## Theorem

Let  $f(n)$  be an arithmetic function and suppose the Dirichlet series  $D_f(s)$  has finite abscissa of absolute convergence  $\sigma_a$ . Then, for any  $c \geq \max(0, \sigma_a)$  and any non-integral value  $x > 1$ ,

$$\sum_{n \leq x} f(n) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} D_f(s) \frac{x^s}{s} ds,$$

where the improper integral  $\int_{c-i\infty}^{c+i\infty}$  is to be interpreted as  $\lim_{T \rightarrow \infty} \int_{c-iT}^{c+iT}$ .

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where the improper integral  $\int_{c-i\infty}^{c+i\infty}$  is to be interpreted as  $\lim_{T \rightarrow \infty} \int_{c-iT}^{c+iT}$ . Moreover, given  $T > 0$ , we have

$$\sum_{n \leq x} f(n) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} D_f(s) \frac{x^s}{s} ds + R(T),$$

where  $R(T) = O\left(\frac{x^c}{T} \sum_{n=1}^{\infty} \frac{|f(n)|}{n^c |\log \frac{x}{n}|\right)$ .

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$$\sum_{n \leq x} f(n)(x - n) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} D_f(s) \frac{x^{s+1}}{s(s+1)} ds.$$

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**Remark:** Notice that

$$\sum_{n \leq x} f(n)(x - n) = \sum_{n \leq x} f(n) \int_n^x 1 dy = \int_1^x \sum_{n \leq y} f(n) dy.$$

# Sketch of the proof of PNT

The starting point is Perron's formula for the function  $\Lambda(n)$ . Since  $D_{\Lambda} = -\frac{\zeta'(s)}{\zeta(s)}$ , we have

$$\sum_{n \leq x} \Lambda(n)(x - n) = -\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{\zeta'(s)}{\zeta(s)} \frac{x^{s+1}}{s(s+1)} ds,$$

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for any  $a > 1$ . We apply this formula initially with a value of  $a$  depending on  $x$  and slightly larger than 1, (namely  $a = 1 + \frac{1}{\log x}$ ) and then move the contour of integration to the left  $\sigma = 1$ .

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Since  $\zeta(s)$  has a pole at  $s = 1$ ,  $\frac{\zeta'(s)}{\zeta(s)}$  will also have a pole at  $s = 1$  and passing over this point, by the Residue Theorem we pick a contribution of  $\frac{x^2}{2}$  from the residue of the integrand at  $s = 1$ .

## Sketch of the proof of PNT (contd)

Notice that if the contour for the Residue Theorem encloses any zero of  $\zeta(s)$ , we would have other poles in the integrand, and we would have to consider their contribution.

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If one proves that  $\frac{x^2}{2}$  is the main term of  $\sum_{n \leq x} \Lambda(n)(x - n)$ , since

$$\sum_{n \leq x} \Lambda(n)(x - n) = \int_1^x \sum_{n \leq y} \Lambda(n) dy,$$

then it is possible to prove that the main term of  $\sum_{n \leq x} \Lambda(n)$  is  $x$ .

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then it is possible to prove that the main term of  $\sum_{n \leq x} \Lambda(n)$  is  $x$ . While from an estimate for a function one can easily derive a corresponding estimate for the integral of this function, a similar derivation in the other direction is in general not possible. However, in this case we are able to do so by exploiting the fact that the function  $\psi(x)$  is non decreasing.

# Sketch of the proof of PNT (contd)

The error term will come from bounding the integral over the shifted path of integration.

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In order to obtain good estimates for the integrand, we need to, on the one hand, move as far to the left of  $\sigma = 1$  as possible (so that  $|x^s| = x^\sigma$  is small compared with  $x$ ).

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The region in which we can establish such bounds consists of points  $s$  bounded on the left by a curve of the form  $\sigma = 1 - c \log^9 t$ .

# Zero-free regions for $\zeta(s)$

## Theorem

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- 3 There exists positive constants  $c_2$  and  $A_2$  such that  $\zeta(s)$  has no zeros in the region  $\sigma > 1 - c_2 \log^9 |t|$  and  $|t| \geq 2$ , and in this region

$$\left| \frac{1}{\zeta(s)} \right| \leq A_2 \log^7 |t|.$$

# 3 – 4 – 1 Lemma

## Lemma

For any real  $\alpha$  we have  $3 + 4 \cos \alpha + \cos 2\alpha \geq 0$ .

## Proof.

$$0 \leq (1 + \cos \alpha)^2 = 1 + 2 \cos \alpha + \frac{1}{2}(1 + \cos 2\alpha).$$



## 3 – 4 – 1 Lemma

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### Lemma (Lemma 3-4-1 for the $\zeta$ function)

If  $\sigma > 1$  and  $t \in \mathbb{R}$ , we have

$$|\zeta^3(\sigma)\zeta^4(\sigma + it)\zeta(\sigma + 2it)| \geq 1.$$

## 3 – 4 – 1 Lemma - Proof

Proof.

For  $\sigma > 1$ , use

$$\begin{aligned}\log |\zeta(s)| &= \left| \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \right| \\ &= -\operatorname{Re} \sum_p \log \left(1 - \frac{1}{p^s}\right) \\ &= \operatorname{Re} \sum_p \sum_{m=1}^{\infty} \frac{1}{mp^{ms}} \\ &= \sum_p \sum_{m=1}^{\infty} \frac{\cos(t \log p^m)}{mp^{m\sigma}}\end{aligned}$$



# Zero-free regions for $\zeta(s)$ - Idea of the Proof

The key ingredient to prove that  $\zeta(s)$  has no zeros on the vertical line  $\sigma = 1$  is to use the  $3 - 4 - 1$  Lemma. Assuming we had a zero in this line, say at  $1 + it$ , by the Lemma this would imply that  $\zeta(s)$  should have another pole at  $1 + 2it$ .

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The 3 – 4 – 1 Lemma is also used to obtain the third zero-free region, now using upper bounds of  $\zeta(s)$  in the region indicated.

# Shifting the path of integration

We have

$$\sum_{n \leq x} \Lambda(n)(x - n) = -\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{\zeta'(s)}{\zeta(s)} \frac{x^{s+1}}{s(s+1)} ds,$$

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where the path of integration is a vertical line located within the half-plane  $\sigma > 1$ . We move the portion  $|t| \leq T$  of this path to the left of the line  $\sigma = 1$ , replacing it by a rectangular path. With this new path we will obtain the main term  $x^2/2$  which is the contribution of the residue at the singularity inside the enclosed region and the integral on all the other lines can be bounded and becomes the error term.

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We end the study of the PNT with a result that shows how connected are the zero-free regions of  $\zeta(s)$  with the error term of  $\pi(x)$ .

## Theorem

*Let  $1/2 < \alpha < 1$ . Then the Riemann zeta function has no zeros in the half-plane  $\sigma > \alpha$  if and only if  $\psi(x) = x + O_\epsilon(x^{\alpha+\epsilon})$ .*

# Dirichlet's Theorem

## Theorem

*Given any positive integers  $q$  and  $a$  with  $(a, q) = 1$ , there exist infinitely many primes congruent to  $a$  modulo  $q$ .*

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A natural attempt to prove Dirichlet's theorem would be to try to mimick Euclid's proof of the infinitude of primes. In certain special cases, this does indeed succeed. For example, to show that there are infinitely many primes congruent to 3 modulo 4, assume there are only finitely many, say  $p_1, \dots, p_k$  and consider  $N = p_1^2 \cdots p_k^2 + 2$ .

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Similar, though more complicated, elementary arguments can be given for some other special arithmetic progressions, but these methods do not seem to be able to prove the general case of Dirichlet's theorem.

# Dirichlet's Theorem

We could also try to prove Dirichlet's Theorem, by replacing the function  $\zeta(s)$  by the series

$$\sum_{\substack{n=1 \\ n \equiv a \pmod{q}}} \frac{1}{n^s}.$$

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But unfortunately, the Euler product for  $\zeta(s)$  does not generalize to this series.

# Dirichlet's characters

## Definition

Let  $q$  be a positive integer. A *Dirichlet character modulo  $q$*  is an arithmetic function  $\chi$  with the following properties:

- 1  $\chi$  is periodic modulo  $q$  ( $\chi(q + a) = \chi(a)$ );
- 2  $\chi(mn) = \chi(m)\chi(n)$  for any integers  $m$  and  $n$ , and  $\chi(1) = 1$ ;
- 3  $\chi(n) \neq 0$  if and only if  $(n, q) = 1$ .

The arithmetic function  $\chi_0$  defined by  $\chi_0(n) = 1$  if  $(n, q) = 1$  and equal to zero otherwise, is the *principal character modulo  $q$* .

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The arithmetic function  $\chi_0$  defined by  $\chi_0(n) = 1$  if  $(n, q) = 1$  and equal to zero otherwise, is the *principal character modulo  $q$* .

The Legendre symbol  $\left(\frac{a}{q}\right)$  is a Dirichlet character (it is in fact a quadratic character, because  $\left(\frac{a}{q}\right)^2 = \chi_0$ ).

# Elementary properties of Dirichlet's characters

## Theorem

We have

- 1 The values of a Dirichlet character modulo  $q$  are either 0 or a  $\varphi(q)$ -root of unity;
- 2 The characters modulo  $q$  form a group with respect to pointwise multiplication;
- 3 There exist exactly  $\varphi(q)$  Dirichlet characters modulo  $q$ .

# Dirichlet's characters

There are three different cases for the Dirichlet's characters modulo  $q$ :

- 1 If  $q = p^m$  odd  $(\mathbb{Z}/q\mathbb{Z})^*$  is cyclic and has generator  $g$  (called *primitive root modulo  $q$* ). For  $(a, q) = 1$ , define  $\nu$  by  $a \equiv g^{\nu(a)} \pmod{q}$ . Let  $\omega$  be a primitive  $\varphi(q)$ -root of unity and for  $(r, q) = 1$  define  $\chi_{q,r}(a) = \omega^{\nu(r)\nu(a)}$  if  $(a, q) = 1$ , and equal to zero, otherwise.

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- 2 If  $q = 2^m$ , with  $m \geq 2$ ,  $(\mathbb{Z}/q\mathbb{Z})^* = (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/\frac{\varphi(q)}{2}\mathbb{Z})$  where each of these cyclic groups have the generators  $(-1)$  and  $5$ , respectively. For  $a$  odd define  $\nu_0$  and  $\nu$  by  $a \equiv (-1)^{\nu_0(a)} 5^{\nu(a)} \pmod{q}$ . Let  $\omega$  be a primitive  $\frac{\varphi(q)}{2}$ -root of unity. Define  $\chi_{q,r}(a) = (-1)^{\nu_0(r)\nu_0(a)} \omega^{\nu(r)\nu(a)}$  if  $a$  is odd, and equal to zero, if  $a$  is even, for odd  $r$ .

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- 3 If  $q = p_1^{b_1} \cdots p_k^{b_k}$ , with  $k \geq 2$  then  $(\mathbb{Z}/q\mathbb{Z})^* = (\mathbb{Z}/\varphi(p_1^{b_1})\mathbb{Z}) \cdots (\mathbb{Z}/\varphi(p_k^{b_k})\mathbb{Z})$ . Using the Chinese remainder theorem define, for  $(a, q) = 1$ ,  $\chi_{q,a} = \prod_{i=1}^k \chi_{p_i^{b_i}, a_i}$ , where  $a \equiv a_i \pmod{p_i^{b_i}}$ .

# Orthogonality relations

1 for any  $\chi \in X_q$

$$\frac{1}{\varphi(q)} \sum_{a=1}^q \chi(a) = \begin{cases} 1 & \text{if } \chi = \chi_0 \\ 0 & \text{otherwise} \end{cases}$$

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- 3 for any integers  $a$  and  $n$  with  $(a, q) = 1$ , we have

$$\frac{1}{\varphi(q)} \sum_{\chi \in X_q} \overline{\chi(a)} \chi(n) = \begin{cases} 1 & \text{if } n \equiv a \pmod{q} \\ 0 & \text{otherwise} \end{cases}$$

# L functions

## Definition

Given a Dirichlet character  $\chi$  its Dirichlet series is called a  $L$ - function and denoted by  $L(s, \chi)$ , i. e.,

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for all  $s$  with  $\sigma > 1$ . Similarly to the zeta function we can define the  $\log L(s, \chi)$  and obtain

$$\log L(s, \chi) = \sum_p \sum_{m=1}^{\infty} \frac{\chi(p^m)}{mp^{ms}}$$

## Dirichlet's theorem - proof

Let  $a$  be a fixed integer with  $(a, q) = 1$ , then

$$\begin{aligned}
 \frac{1}{\varphi(q)} \sum_{\chi \in X_q} \overline{\chi(a)} \log L(s, \chi) &= \frac{1}{\varphi(q)} \sum_{\chi \in X_q} \overline{\chi(a)} \sum_p \sum_{m=1}^{\infty} \frac{\chi(p^m)}{mp^s} \\
 &= \sum_p \sum_{m=1}^{\infty} \sum_{\chi \in X_q} \overline{\chi(a)} \chi(p^m) \frac{1}{mp^{ms}} \\
 &= \sum_p \sum_{\substack{m=1 \\ p^m \equiv a \pmod{q}}}^{\infty} \frac{1}{mp^{ms}} \\
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for all complex  $s$  with  $\sigma > 1$ . So, if we prove that the left hand side tends to infinity, when  $\sigma \rightarrow 1^+$ , we prove Dirichlet Theorem.

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Hence,

$$\prod_{\chi \in X_q} L(s, \chi) \geq 1.$$

# Dirichlet's theorem - proof

Suppose  $\chi$  is complex and  $L(1, \chi) = 0$ . Then  $L(1, \overline{\chi}) = 0$ . But then we get a contradiction, as

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$$\lim_{\sigma \rightarrow 1^+} \sum_{p \equiv a \pmod{q}}^{\infty} \frac{1}{p^\sigma} = \infty,$$

and the arithmetic progression  $a + qb$  has infinitely many primes.

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