TRIVIAL SOLUTIONS IN GENERALIZED FERMAT EQUATIONS

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Abstract. The modular method is a powerful tool to study generalized Fermat equations. Trivial solutions often become an obstruction to the method. While using hypergeometric motives to construct Galois representations, some of these solutions give rise to newforms with complex multiplication. In the present article, assuming a standard conjecture in Darmon's program, we prove that asymptotically any newform with complex multiplication cannot be related to a non-trivial primitive solution. This allows to discard all such newforms, including those associated to trivial solutions.

1. Introduction

Since the proof of Fermat's Last Theorem by Wiles [Wil95], an increasing amount of interest has been devoted to the study of *generalized Fermat equations*. These are equations of the form

$$x^q + y^p + z^r = 0, \tag{1}$$

where the triple $(q,p,r) \in \mathbb{Z}^3_{>0}$ is known as the *signature*. A solution $(a,b,c) \in \mathbb{Z}^3$ of (1) is called *primitive* if $\gcd(a,b,c) = 1$ and *trivial* if $abc = 0$. From [DG95, Theorem 2], (1) has only finitely many non-trivial primitive solutions for a fixed signature (q,p,r) satisfying $\frac{1}{q}+\frac{1}{p}+\frac{1}{r} < 1$. However, Beal's conjecture states the non-existence of non-trivial solutions when $q,p,r \geq 3$ are free variables.

The approach used by Wiles to prove Fermat's Last Theorem is now commonly known as the *modular method*, and is well-suited for studying solutions of (1) when the exponents move along a line. Restricting to prime exponents, the signatures belong to one of the following four families (up to relabeling of the variables): (p,p,p)[1], (p,p,r), (r,p,r) and (q,r,p), where q,r are fixed and different, and p varies. Of these, the case (q,r,p) is the least understood in the literature.

Let us briefly discuss how the modular method works. For this, let ζ_k be a primitive k-th root of unity and let $K = \mathbb{Q}(\zeta_q)^+ \cdot \mathbb{Q}(\zeta_r)^+$. Finally, let $\mathfrak{p}$ be a prime of K above p. The modular method proceeds by contradiction and its main steps are the following:

Step I: To a hypothetical solution (a,b,c) of (1), attach a geometric Galois representation

$$\rho_{\mathfrak{p}} : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \to \mathrm{GL}_2(K_{\mathfrak{p}}),$$

such that the conductor $\mathfrak{n}$ of the residual representation $\overline{\rho}_{\mathfrak{p}}$ is independent of the solution (a,b,c).

Step II: Prove that $\overline{\rho}_{\mathfrak{p}}$ is modular, that is, that there exists a Hilbert modular form g such that $\overline{\rho}_{\mathfrak{p}} \simeq \overline{\rho}_{g,\mathfrak{p}}$.

Step III: Prove that $\overline{\rho}_{\mathfrak{p}}$ is absolutely irreducible. Then, by level-lowering results, we conclude that there exists a Hilbert newform f of parallel weight 2 and level $\mathfrak{n}$ such that $\overline{\rho}_{\mathfrak{p}} \simeq \overline{\rho}_{f,\mathfrak{p}}$.

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[1]This case corresponds to Fermat's Last Theorem.

Step IV: Compute $S_2(\mathfrak{n})$ (which, by Step I, is independent of the solution (a,b,c)) and show that $\overline{\rho}_{\mathfrak{p}} \not\simeq \overline{\rho}_{f,\mathfrak{p}}$ for all newforms f in $S_2(\mathfrak{n})$.

Historically, Step I was performed by associating to a solution (a,b,c) a rational elliptic curve, so that $\rho_{\mathfrak{p}}$ is its $\mathfrak{p}$-adic Galois representation. In more recent articles, following Darmon's pioneering paper [Dar00], the elliptic curve was replaced by a GL_2-type Jacobian of a hyperelliptic curve. These curves are referred to as Frey hyperelliptic curves and have been used to study the families (p,p,r) [Dar00, BCDF17, CK22], (r,r,p) [BCDF19, BCDF25a, BCDF25b], and $(2,r,p)$ [GKVT26].

A further generalization is due to Golfieri and the second author [GP24], where the general signature case was addressed by using hypergeometric motives (as implicitly used by Darmon in [Dar00]), solving Step I completely. More precisely, the following holds.

Theorem 1.1. *Let $q > r > 2$ be prime numbers and $K = \mathbb{Q}(\zeta_q)^+ \cdot \mathbb{Q}(\zeta_r)^+$. Let $p > 3$ be a prime number and $\mathfrak{p}$ a prime ideal of K above p. For every hypothetical primitive solution $(a,b,c) \in \mathbb{Z}^3$ to*

$$x^q + y^p + z^r = 0,$$

there is a Galois representation

$$\rho_{\mathfrak{p}} : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \to \mathrm{GL}_2(K_{\mathfrak{p}}),$$

attached to (a,b,c) satisfying that its residual representation $\overline{\rho}_{\mathfrak{p}}$ is unramified outside primes dividing pqr and finite at p.

The representation $\rho_{\mathfrak{p}}$ comes from the hypergeometric motive $\mathcal{H}((\frac{1}{r},-\frac{1}{r}),(\frac{1}{q},-\frac{1}{q})|t)$ (see §2.1 for details). Regarding Step II, in [GP24, §5 and §6] some modularity results were obtained. For example, the case $r = 3$ and $q \geq 5$ is completely solved in Theorem 6.2 in loc. citt., while the general case is proved in [MPVT26].

Step III is particularly difficult to approach in full generality. In [PV25, Corollary 6.2], the irreducibility of $\overline{\rho}_{\mathfrak{p}}$ is proved only under certain divisibility conditions on the solution (a,b,c). In order to fulfill Step III, we assume the veracity of the following "large-image" conjecture (Conjecture 4.1 in [Dar00]).

Conjecture 1. *Let K be a totally real field and L any number field. Then there exists a constant $C(L,K)$, depending only on L and K, such that for any abelian variety A/L of GL_2-type with*

$$\mathrm{End}_L(A) \otimes \mathbb{Q} = \mathrm{End}_{\overline{L}}(A) \otimes \mathbb{Q} \simeq K,$$

and all prime ideals $\mathfrak{p}$ of K with norm larger than $C(L,K)$, the image of the mod $\mathfrak{p}$-representation associated to A contains $\mathrm{SL}_2(\mathbb{F}_{\mathfrak{p}})$.

Assuming Conjecture 1, if the hypergeometric motive does not have complex multiplication, it follows that $\overline{\rho}_{\mathfrak{p}}$ is absolutely irreducible when the norm of $\mathfrak{p}$ is sufficiently large. Then [BD14, Theorem 3.2.2] proves the existence of the newform f given in Step III (see also [Jar99, Raj01, Fuj06]).

With the first three steps completed, it remains to perform Step IV. For this, there are two different difficulties. The first one is computational. Indeed, when the norm of the conductor $\mathfrak{n}$ is large, the dimension of the space of newforms in $S_2(\mathfrak{n})$ makes the computation unfeasible. For instance, this already occurs for $q = 7$ and $r = 3$.

The second challenge is more theoretical and is related to the existence of newforms with complex multiplication in the space $S_2(\mathfrak{n})$ (these forms tend to consistently pass the elimination process due to the large number of zeroes in their Fourier coefficients). The *trivial solutions*

$$(\pm 1, \mp 1, 0),\ (\pm 1, 0, \mp 1),\ (0, \pm 1, \mp 1),$$

are solutions of (1) irrespective of the signature (q,p,r). In [PV25, Section 4], it is proved that the second pair of solutions induces a singular motive, so it does not correspond to a newform,

whereas the first and last pairs are related to newforms with complex multiplication. This leads to the following question.

Question. *Is it true that if (a,b,c) is a primitive non-trivial solution to (1) and f is a newform with complex multiplication then $\overline{\rho}_{\mathfrak{p}} \not\simeq \overline{\rho}_{f,\mathfrak{p}}$?*

When working with Frey hyperelliptic curves, it is usually not hard to prove that the curve constructed from a non-trivial primitive solution has at least one prime of potentially multiplicative reduction, and, consequently, it cannot have complex multiplication by [ST68, Theorem 6]. Therefore, a positive answer to the posted Question follows from Conjecture 1, when p is large enough. We prove a similar result even when no Frey hyperelliptic curve is available.

Theorem A. *Let $q > r > 2$ and $p > 3$ be prime numbers, with q and r fixed. Let (a,b,c) be a hypothetical non-trivial primitive solution to*

$$x^q + y^p + z^r = 0,$$

and let $\rho_{\mathfrak{p}}$ be the Galois representation attached to (a,b,c) in Theorem 1.1. Assume that Conjecture 1 holds. Then there exists a constant $C(q,r)$ (depending only on q and r) such that if $p > C(q,r)$, for any newform f with complex multiplication we have that $\overline{\rho}_{\mathfrak{p}} \not\simeq \overline{\rho}_{f,\mathfrak{p}}$.

In [PV25] it was explained how a solution $(a,c,s,\ell,m,n,u,v) \in \mathbb{Z}^2 \times \mathbb{Z}^6_{\geq 0}$ to the equation

$$q^s r^\ell a^q \pm q^m r^n + q^u r^v c^r = 0 \tag{2}$$

gives rise to a Galois representation with properties that mimic the ones coming from solutions to (1), making the modular method to fail, even if solutions of (2) do not necessarily correspond to solutions of (1). If all modular forms bypassing the elimination process of the modular method happen to correspond to solutions of (2) or to have complex multiplication (as usually occurs in the literature), then Theorem A implies that any solution to (1) comes from a solution to the simpler equation (2), which is akin to a Thue-Mahler equation.

A key ingredient for the proof of Theorem A is the following result on hypergeometric motives (see §2.1 for details). Let q and r be prime numbers such that $q > r > 2$ and let $F = \mathbb{Q}(\zeta_{qr})$.

Theorem B. *Let $t \in \mathbb{Q} \setminus \{0,1\}$. If $v_r(t-1) > 2 + \frac{2}{r-1}$, then the hypergeometric motive $\mathcal{H}((\frac{1}{r}, -\frac{1}{r}), (\frac{1}{q}, -\frac{1}{q})|t)$ has multiplicative reduction at prime ideals of F dividing r.*

The result is of particular interest, since it involves studying the action of inertia on cyclic covers of $\mathbb{P}^1$ at primes dividing the degree. Its proof follows the strategy described in [Dok21].

Organization of the paper. In Section 2, we introduce the tools needed to prove Theorem A, including results on hypergeometric motives and models of curves over discrete valuation rings. In Section 3, we prove the key result, Theorem 3.1, from which Theorem B follows. Finally, in Section 4, we prove Theorem A, thereby giving a positive answer to the Question on page 3. We want to stress that there is a subtlety while verifying the hypothesis of Conjecture 1, which has not been previously addressed in the literature. In this section, we explain the issue and show how to solve it.

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2. Preliminaries

We present the two main technical tools that will be needed throughout this paper: hypergeometric motives and models of curves over discrete valuation rings. We will just recall the details that we need and we do not intend to provide a full reference for either object. Instead, we refer the interested reader to [GP24, Section 2] and [Dok21] for references.

2.1. **Hypergeometric motives.** In this section, we recall the main properties of the rank-2 hypergeometric motive $\mathcal{H}((\frac{1}{r},-\frac{1}{r}),(\frac{1}{q},-\frac{1}{q})|t)$ used to construct the representation $\rho_{\mathfrak{p}}$ of Theorem 1.1.

Let $q \neq r$ be two rational odd primes. Let $F = \mathbb{Q}(\zeta_{qr})$ and $K = \mathbb{Q}(\zeta_q)^+ \cdot \mathbb{Q}(\zeta_r)^+$ be the compositum of $\mathbb{Q}(\zeta_q)^+$ with $\mathbb{Q}(\zeta_r)^+$. Let $t \in \mathbb{Q} \setminus \{0,1\}$. Then there exists a rank-2 hypergeometric motive $\mathcal{H}((\frac{1}{r},-\frac{1}{r}),(\frac{1}{q},-\frac{1}{q})|t)$ defined over K. Its set of primes of bad reduction is contained in $S_{\mathrm{pw}} \cup S_{\mathrm{pt}}$, where

$$S_{\mathrm{pw}} := \{\lambda \subset K \ : \ \lambda \mid qr\},$$

is the set of potentially wild primes and

$$S_{\mathrm{pt}} := \{\lambda \subset K \ : \ \lambda \nmid qr \text{ and } v_\lambda(t(t-1)) \neq 0\},$$

is the set of potentially tame primes. At the potentially tame primes, the image of inertia can be easily computed (see [GP26, Corollary 6.14]). Our goal is to describe the behaviour at the potentially wild primes when $v_q(t-1)$ (respectively $v_r(t-1)$) is sufficiently large.

We note that the aforementioned motive can be realized by using the middle cohomology of an Euler curve. Concretely, set

$$A := q-r, \quad B := -q-r, \quad C := q+r, \quad D := -r, \quad N := qr. \tag{3}$$

The *Euler curve* is defined by

$$\mathcal{C} : y^N = x^A(1-x)^B(1-tx)^C t^D. \tag{4}$$

Over F, the multiplicative group μ of N-th roots of unity acts on $\mathcal{C}$ via $\zeta_N \cdot (x,y) = (x,\zeta_N y)$. The de Rham cohomology group $H^1(\mathcal{C},F) := H^1(\mathcal{C},\mathbb{Q}) \otimes_{\mathbb{Q}} F$ admits a decomposition

$$H^1(\mathcal{C},F) = \bigoplus_{\chi} H^1_\chi(\mathcal{C},F), \tag{5}$$

where the sum ranges over characters of μ and $H^1_\chi(\mathcal{C},F)$ denotes the χ-isotypical component. Let $\chi_0 : \mu \to F^\times$ denote the identity character. Up to a Tate twist, the hypergeometric motive $\mathcal{H}((\frac{1}{r},-\frac{1}{r}),(\frac{1}{q},-\frac{1}{q})|t)$ is defined as the χ_0-isotypical component of (5) (see [GP26, Definition 6.10] for extra details). To define the motive over K rather than F, one needs to consider quotients of $\mathcal{C}$ given by some involutions. However, for our purposes (to prove that the reduction type of the motive is potentially multiplicative) it is enough to work with the curve $\mathcal{C}$ itself.

2.2. **Models of curves over DVRs.** As seen in the previous section, we need to study the curve $\mathcal{C}$, which is a singular non-hyperelliptic curve. To do so, we shall apply the techniques developed in [Dok21]. We begin with a brief overview of the results and definitions of Dokchitser's article. We refer the reader to §3 of loc. cit. for further details.

Let K be a discretely valued field with valuation v, ring of integers $\mathcal{O}_K$, uniformizer π, and residue field k. Consider a bivariate polynomial

$$P = \sum_{i,j} a_{i,j} x^i y^j \in K[x,y].$$

Its Newton polygon is defined by the convex hull of the points (i,j), that is

$$\Delta = \mathrm{conv}\{(i,j) \in \mathbb{Z}^2 \mid a_{i,j} \neq 0\} \subset \mathbb{R}^2.$$

Similarly, define the Newton polytope of P with respect to the valuation v by

$$\Delta_v = \mathrm{lch}\{(i, j, v(a_{i,j})) \mid a_{i,j} \neq 0\} \subset \mathbb{R}^2 \times \mathbb{R},$$

where lch stands for the lower convex hull, i.e., the points (a_0, b_0, c_0) in the convex hull such that $(a_0, b_0, c_0) - (0, 0, \varepsilon)$ does not lie in the convex hull for any $\varepsilon > 0$. Finally, we note that the projection $p : \Delta_v \to \Delta$ is a homeomorphism onto its image, and thus the valuation induces a piecewise affine function $v : \Delta \to \mathbb{R}$.

Definition 2.1. *The images in Δ of the open faces of Δ_v under the above projection are called the* v-vertices, v-edges, *and* v-faces, *according to whether the corresponding face of Δ_v has dimension* 0, 1, *or* 2. *Thus every v-vertex is a lattice point, every v-edge (denoted by L) is homeomorphic to an open interval, and every v-face (denoted by F) is homeomorphic to an open disc.*

For any v-edge L and v-face F, define

$$L(\mathbb{Z}) = L \cap \mathbb{Z}^2, \qquad F(\mathbb{Z}) = F \cap \mathbb{Z}^2.$$

Similarly, $\Delta(\mathbb{Z}) = \mathrm{int}(\Delta) \cap \mathbb{Z}^2$. The notation $\bar{L}(\mathbb{Z})$, $\bar{F}(\mathbb{Z})$, and $\bar{\Delta}(\mathbb{Z})$ refers to the previous objects together with the boundary points. The subscript $\mathbb{Z}$ on a set indicates the subset of elements satisfying the additional condition that $v(P) \in \mathbb{Z}$. For instance,

$$F(\mathbb{Z})_{\mathbb{Z}} = \{P \in F(\mathbb{Z}) \mid v(P) \in \mathbb{Z}\}.$$

Finally, for a v-edge or v-face λ, we denote by δ_λ the common denominator of the numbers $v(P)$ with $P \in \bar{\lambda}(\mathbb{Z})$.

Definition 2.2 (Restriction)**.** *Let $g = \sum_{\mathbf{i} \in \mathbb{Z}^n} c_{\mathbf{i}} \mathbf{x}^{\mathbf{i}} \in K[\mathbf{x}]$ where $\mathbf{x} = (x_1, \ldots, x_n)$, and let $\emptyset \neq S \subset \mathbb{Z}^n$. Let Λ be the smallest affine lattice satisfying $S \subset \Lambda \subset \mathbb{Z}^n$, and suppose that Λ has rank r. Choose an affine lattice isomorphism $\phi : \mathbb{Z}^r \longrightarrow \Lambda$, write $\mathbf{y} = (y_1, \ldots, y_r)$, and define*

$$g|_S = \sum_{\mathbf{i} \in \phi^{-1}(S)} c_{\phi(\mathbf{i})} \mathbf{y}^{\mathbf{i}}.$$

Different choices of ϕ modify $g|_S$ only by a monomial factor and a $\mathrm{GL}_r(\mathbb{Z})$-change of variables. For a v-vertex, v-edge, or v-face λ, we set

$$P|_\lambda = P|_{\bar{\lambda}(\mathbb{Z})_{\mathbb{Z}}}.$$

Definition 2.3 (Reduction)**.** *Let $h \in K[x, y]$. Assume that there exist integers c, m, n such that $\tilde{h}(x, y) = \pi^c h(\pi^m x, \pi^n y)$ has coefficients in $\mathcal{O}_K$, and that $\tilde{h} \bmod \pi$ has the same Newton polygon as h. The polynomial $\tilde{h} \bmod \pi \in k[x, y]$ is then called the* reduction *of h, and will be denoted by $\bar{h}$.*

Given a v-edge or v-face λ, define the k-scheme

$$X_\lambda \colon \{\overline{P|_\lambda} = 0\}. \tag{6}$$

Definition 2.4. *A v-face F is* Δ_v-regular *if the variety X_F is smooth over the residue field k. The same definition applies to edges L that are not on the boundary of Δ.*

The definition at edges lying on the boundary is a little more involved (see [Dok21, Definition 3.9]). The zero locus of P is said to be Δ_v-regular if all its faces and edges are Δ_v-regular.

The model of the Euler curve that we shall consider in Section 3 is not Δ_v-regular, but, as explained in §10 of [Dok21], having Δ_v-regular faces and edges is sometimes enough to apply [Dok21, Theorem 3.14]. By gluing the desingularizations, this allows to determine the semistable reduction of $\mathcal{C}$ (see Remark 10.1 of loc. cit.).

3. Reduction at potentially wild primes

Given two distinct odd prime numbers q, r and $t \in \mathbb{Q} \setminus \{0,1\}$ we consider the hypergeometric motive

$$\mathcal{H}\left(\left(\frac{1}{r}, -\frac{1}{r}\right), \left(\frac{1}{q}, -\frac{1}{q}\right) \middle| t\right). \tag{7}$$

Let (a,b,c) be a hypothetical solution to (1). The specialization of (7) at $t = -\frac{a^q}{c^r}$ gives rise to the representation $\rho_{\mathfrak{p}}$ in Theorem 1.1 [GP24].

It follows from [GP26, Proposition 4.2] that for any pair of integers m, n prime to qr, the motive in (7) and the motive

$$\mathcal{H}\left(\left(\frac{n}{r}, -\frac{n}{r}\right), \left(\frac{m}{q}, -\frac{m}{q}\right) \middle| t\right) \tag{8}$$

are Galois conjugate. Then it is enough to study the case where m and n satisfy that $nq - mr = 1$, so the Euler curve associated to the motive in (8) has equation

$$\mathcal{C} : y^{qr} = x(1-x)^B(1-tx)^C t^D, \tag{9}$$

where $B = -mr - nq$, $C = mr + nq$ and $D = -mr$. After a simple change of variables, we may furthermore assume that $0 < B, C < qr$ and that $B + C = qr$.

Let ζ_n be a primitive n-th root of unity and let $F = \mathbb{Q}(\zeta_{qr})$. Fix a prime ideal $\mathfrak{r}$ of F lying above r, and write $F_{\mathfrak{r}}$ for the completion of F at $\mathfrak{r}$. Let π be a local uniformizer. We denote by $v_{\mathfrak{r}}$ the normalized valuation at $\mathfrak{r}$ over F and by v_r the valuation at r over $\mathbb{Q}$.

Theorem 3.1. *Let $t \in \mathbb{Q} \setminus \{0,1\}$ satisfy $v_r(t-1) > 2 + \frac{2}{r-1}$. Then the curve $\mathcal{C}$ has a semistable model over $F_{\mathfrak{r}}$ whose toric part has full rank.*

Proof. Set $k := v_{\mathfrak{r}}(t-1) = (r-1)v_r(t-1) > 2r$. Since t is a unit, we can make a change of variables and remove the factor t^D in (9) (because the extension $F_{\mathfrak{r}}(\sqrt[q]{t})/F_{\mathfrak{r}}$ is unramified at $\mathfrak{r}$). Translate the singular point $(1,0)$ to the origin (using the change of variables $x \mapsto x+1,\ y \mapsto -y$), so the new equation reads $P(x,y) = 0$ for

$$P(x,y) = y^{qr} + (-x)^B \sum_{i=0}^{C+1} (1-t)^{C-i}(-t)^{i-1}\left(\binom{C}{i-1} - t\binom{C+1}{i}\right) x^i. \tag{10}$$

Keeping the terminology of Section 2.2, let Δ denote the Newton polygon of $P(x,y)$. Let L_1 be the line connecting the points $(qr, 0)$ and $(0, qr)$, with equation

$$L_1 : x + y = qr. \tag{11}$$

This line splits the Newton polygon Δ as $\Delta = F_1 \cup F_2$, where F_1 is determined by the points where $x + y \leq qr$ and F_2 by the points where $x + y \geq qr$. Let $v_1 : F_1 \to \mathbb{R}$ be the affine function defined by

$$v_1(x,y) = qrk - k(x+y), \tag{12}$$

and $v_2 : F_2 \to \mathbb{R}$ be the null function.

Claim 1: F_1 and F_2 are the v-faces of Δ, as defined in Definition 2.1 (see Figure 1[2]).

Proof of Claim 1. Write $P(x,y) = \sum_{i,j} a_{i,j} x^i y^j$. The non-zero coefficients of $P(x,y)$ are $a_{0,qr}$ and $a_{i,0}$, for $i = B, \ldots, qr+1$. Then, Claim 1 follows easily from the data in Table 3.1, which contains information on the valuation of the non-zero coefficients $a_{i,j}$, together with the fact that $B + C = qr$.

[2]Large numbers indicate that the corresponding coefficient of P has exactly the specified valuation.

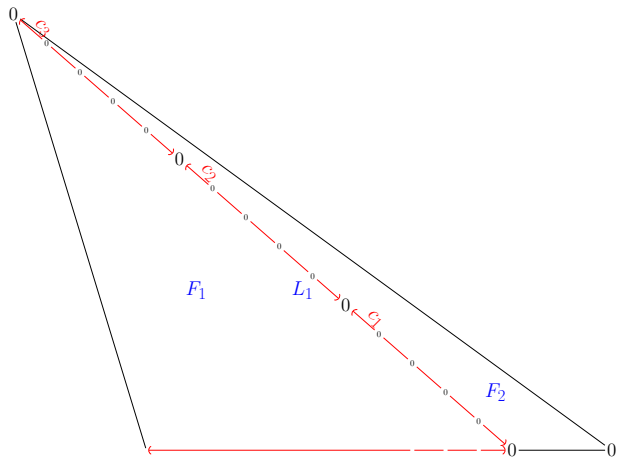


FIGURE 1. Newton polytope of $P(x, y)$ for $r = 5$ and $q = 3$.

i	j	$v_{\mathfrak{r}}(a_{i,j})$	i	j	$v_{\mathfrak{r}}(a_{i,j})$
0	qr	0	$qr-1$	0	k
qr	0	0	$qr+1$	0	0
B	0	Ck	$[B, qr-2]$	0	$\geq (qr-i)k$

TABLE 3.1. Valuations of the coefficients appearing in (10).

Claim 2: The two faces F_1 and F_2 are Δ_v-regular.

Proof of Claim 2. By Definition 2.2, we have

$$P|_{F_1} = y^{qr} + (-x)^B \sum_{i=0}^{C} (1-t)^{C-i}(-t)^{i-1} \left(\binom{C}{i-1} - t\binom{C+1}{i} \right) x^i$$

and

$$P|_{F_2} = t^{C-1}(C - t(C+1))x^{qr} - t^C x^{qr+1} + y^{qr}.$$

Let $u \equiv \frac{1-t}{\pi^k} \pmod{\mathfrak{r}}$. Then the reduction $\overline{P|_{F_1}}$ equals

$$\overline{P|_{F_1}} = y^{qr} + u^C(-1)^B x^B \sum_{i=0}^{C} \binom{C}{i} (-x/u)^i = y^{qr} + (-x)^B (u-x)^C,$$

which defines a smooth curve in the $\mathbb{G}_m^2$, where $\mathbb{G}_m$ stands for the multiplicative group. A similar computation, using the fact that $C \equiv 1 \pmod{r}$, proves the result for F_2.

If P_1, P_2 are points in $\mathbb{Z}^2$, we use the notation $L := P_1 \mapsto P_2$ to define the segment that joins P_1 and P_2. Consider the following edges (appearing in Figure 1).

$$L_2 := (0, qr) \mapsto (qr+1, 0), \qquad L_3 := (0, qr) \mapsto (B, 0),$$
$$L_4 := (B, 0) \mapsto (qr, 0), \qquad L_5 := (qr, 0) \mapsto (qr+1, 0),$$

along with the edge $L_1 = (qr, 0) \mapsto (0, qr)$ defined in (11). Then the reductions of P at the edges are

$$\overline{P|_{L_1}} = T^{qr} - 1 = \prod_{j=1}^{q} (T - \zeta_q^j)^r, \quad \text{and} \quad \overline{P|_{L_4}} = (-T)^B (u-T)^C, \tag{13}$$

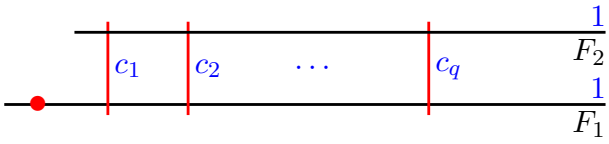


FIGURE 2. Graph of components of the Euler curve $\mathcal{C} : P(x, y) = 0$.

corresponding to edges which are not Δ_v-regular and

$$\overline{P|_{L_2}} = 1 - T, \quad \overline{P|_{L_3}} = 1 - T, \quad \overline{P|_{L_5}} = -1 - T,$$

which are Δ_v-regular. The edge L_1 lies on the boundary of F_1 and F_2 and, by (13), it consists of q singular points. Then, the graph of components is illustrated in Figure 2, where the red edges mean that the resulting model is not Δ_v-regular.

Fix $j \in \{1, \dots, q\}$. To resolve the line c_j in Figure 2 we need to find a model that satisfies the hypothesis of [Dok21, Theorem 3.14]. To do so, we translate the singular point ζ_q^j of (13) to a point not lying in $\mathbb{G}_m$. The change of variable $y \to y + \zeta_q^j x$ transforms equation (10) into the model $Q(x, y) = 0$, where

$$Q(x,y) = (y + \zeta_q^j x)^{qr} + (-x)^B \sum_{i=0}^{C+1} (1-t)^{C-i}(-t)^{i-1} \left(\binom{C}{i-1} - t\binom{C+1}{i} \right) x^i.$$

Define the following segments:

$$\begin{aligned}
L'_1 &:= (0, qr) \mapsto (qr - r, r), & L'_2 &:= (qr - r, r) \mapsto (qr - 1, 1),\\
L'_3 &:= (qr - 1, 1) \mapsto (qr - 1, 0), & L'_4 &:= (qr - 1, 1) \mapsto (qr + 1, 0),\\
L'_5 &:= (0, qr) \mapsto (B, 0), & L'_6 &:= (0, qr) \mapsto (qr + 1, 0),\\
L'_7 &:= (B, 0) \mapsto (qr - 1, 0), & L'_8 &:= (qr - r, r) \mapsto (qr - 1, 0),\\
L'_9 &:= (qr - r, r) \mapsto (qr + 1, 0), & L'_{10} &:= (qr - 1, 0) \mapsto (qr + 1, 0).
\end{aligned}$$

The segments decompose the polygon Δ in the form

$$\Delta = F'_1 \cup F'_2 \cup F'_3 \cup F'_4 \cup F'_5,$$

where

- F'_1 is the polygon with boundary $L'_3 \cup L'_4 \cup L'_{10}$.
- F'_2 is the polygon with boundary $L'_2 \cup L'_4 \cup L'_9$.
- F'_3 is the polygon with boundary $L'_1 \cup L'_8 \cup L'_7 \cup L'_5$.
- F'_4 is the polygon with boundary $L'_2 \cup L'_3 \cup L'_8$.
- F'_5 is the polygon with boundary $L'_1 \cup L'_9 \cup L'_6$.

Define the following affine functions $v'_i : F'_i \to \mathbb{R}$:

$$v'_1(x,y) = -\frac{k}{2}x + (r - 1 - k)y + \frac{k(qr+1)}{2}, \quad v'_2(x,y) = -rx - (r+1)y + r(qr+1),$$

$$v'_3(x,y) = qrk - k(x+y), \quad v'_4(x,y) = (r-k)x + (r-1-k)y + r(qk - qr + 1), \quad v'_5(x,y) = 0.$$

Claim 3: *F_i's are the v-faces of the Newton polytope of the polynomial $Q(x, y)$ (see Figure 3).*

Proof of Claim 3. Write $Q(x,y) = \sum b_{i,j} x^i y^j$. The indices (i, j) for which $b_{i,j} \neq 0$ belong either to the x-axis or to the line $x + y = qr$. In Table 3.2, we summarize information on the valuation of the coefficients $b_{i,j}$. Then, Claim 3 follows from the data. Note that it is here where the hypothesis $k > 2r$ is used. This is in order to ensure that the point $(qr - 1, 1, r - 1)$ belongs to the lower convex hull and thus the face F'_1 is contractible (see [Dok21, Definition 5.1]).

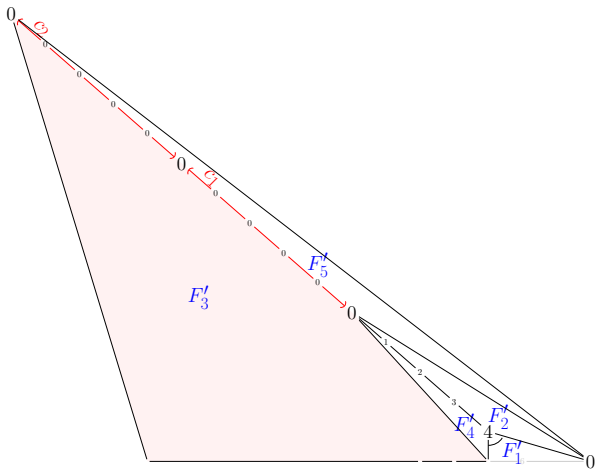


FIGURE 3. Desingularization along one segment for $r = 5$ and $q = 3$.

i	j	$v_{\mathfrak{r}}(b_{i,j})$	i	j	$v_{\mathfrak{r}}(b_{i,j})$
0	qr	0	$qr-1$	1	$r-1$
$qr-1$	0	k	$qr-r$	r	0
B	0	Ck	$qr+1$	0	0
$[B, qr-2]$	0	$\geq (qr-i)k$	$[qr-2, qr-r+1]$	$qr-i$	$\geq r-1$

TABLE 3.2. Valuations of the coefficients appearing in $Q(x, y)$.

Claim 4: The faces F_2' and F_4' are Δ_v-regular and the same holds for the line L_2'.

Proof of Claim 4. A direct computation give that the reduced equations are

$$\overline{Q|_{F_4'}} = x^{qr-1}y\zeta_q^{-j}q\frac{r}{\pi^{r-1}} + x^{qr-r}y^r\zeta_q^{-rj}q + ux^{qr-1},$$

$$\overline{Q|_{F_2'}} = x^{qr-1}y\zeta_q^{-j}q\frac{r}{\pi^{r-1}} + x^{qr-r}y^r\zeta_q^{-rj}q - x^{qr+1}$$

and

$$\overline{Q|_{L_2'}} = q\zeta_q^{-j}\left(\frac{r}{\pi^{r-1}}1 + T^{r-1}\zeta_q^{(1-r)j}\right).$$

All of them are Δ_v-regular.

Then Theorem 3.14 in [Dok21] implies that the desingularization along c_j consists of $r-1$ chains of $\mathbb{P}^1$'s joining the component F_1 with the component F_2 as in Figure 2. The number of projective lines depends on, following Dokchitser's notation, the values $s_2^{L_2'}$ and $s_4^{L_2'}$ of Definition 3.12 in loc. cit. Indeed, take $P_1 = (qr+1, 0)$ and $P_0 = (qr, 0)$. Since $\delta_{L_2'} = 1$, we have that

$$s_2^{L_2'} = v_2(P_1) - v_2(P_0) = -r \quad \text{and} \quad s_4^{L_2'} = v_4(P_1) - v_4(P_0) = r - k.$$

The hypothesis $v_r(t-1) > 2 + \frac{r-1}{2}$ implies that $-r > r - k$, so the number of projective lines is $k - 2r - 1$ by [Dok21, Theorem 3.14]. The desingularization looks like Figure 4, where the dashed blue line connecting F_2' and F_4' comes from the contractible face F_1'.

The graph of components is obtained by gluing together each of the desingularizations. The toric rank equals the number of independent cycles in the dual graph of the graph of components (see [Gro72, Proposition 3.5]). It can be computed using the well-known formula

$$\#E - \#V + 1,$$

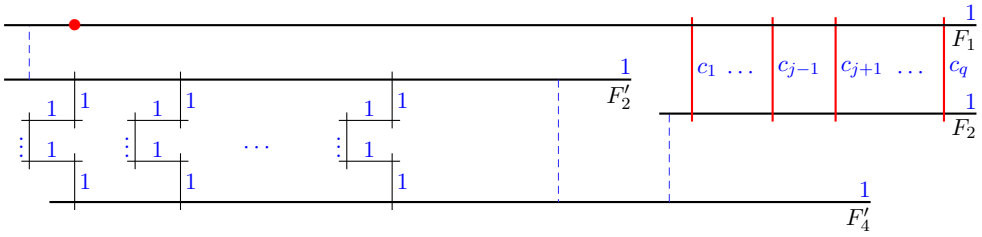


FIGURE 4. Graph of components of the desingularization.

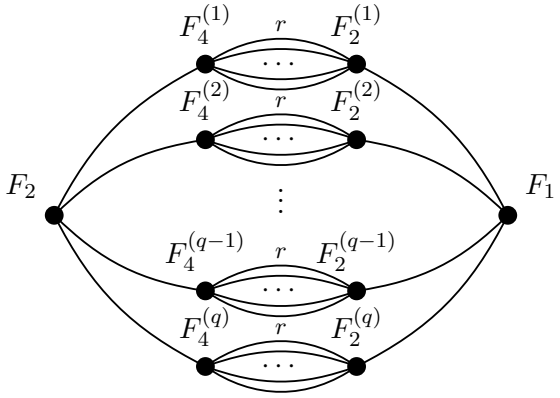


FIGURE 5. Simplified dual graph.

where E denotes the set of edges and V the set of vertices (see, for example, [Die17, Theorem 1.9.5(ii)]). Note that the formula is invariant if in Figure 4 we collapse the chain of $\mathbb{P}^1$'s into a single one, allowing to consider the simplified dual graph of Figure 5 (the vertices $F_4^{(j)}$ and $F_2^{(j)}$ correspond to the components appearing while taking the desingularization at c_j).

It readily follows from Figure 5 that $\#E = qr + 2q$, $\#V = 2q + 2$ and so the total of linearly independent cycles is

$$qr + 2q - (2q + 2) + 1 = qr - 1,$$

which matches the genus of the curve (see [Arc03, Theorem 4.1]). □

Theorem B. *Let $q, r > 2$ be primes. Let $t \in \mathbb{Q} \setminus \{0, 1\}$. If $v_r(t-1) > 2 + \frac{2}{r-1}$, then the hypergeometric motive $\mathcal{H}((\frac{1}{r}, -\frac{1}{r}), (\frac{1}{q}, -\frac{1}{q})|t)$ has multiplicative reduction at prime ideals of F dividing r.*

Proof. From Definition 6.8 in [GP26], the motive $\mathcal{H}((\frac{1}{r}, -\frac{1}{r}), (\frac{1}{q}, -\frac{1}{q})|t)$ is a Tate twist of the motive associated to the Euler curve (9). The result then follows from Theorem 3.1. □

Remark 3.2. Theorem B also holds for the prime q. This is due to the fact that, by [GP26, Theorem 6.2], we have that

$$\mathcal{H}\left(\left(\frac{1}{r}, -\frac{1}{r}\right), \left(\frac{1}{q}, -\frac{1}{q}\right) \middle| t\right) \simeq \mathcal{H}\left(\left(\frac{1}{q}, -\frac{1}{q}\right), \left(\frac{1}{r}, -\frac{1}{r}\right) \middle| t^{-1}\right),$$

and if $v_q(t-1) > 0$ then $v_q(t-1) = v_q(t^{-1} - 1)$, so that the roles of q and r are symmetric in Theorem B.

4. Proof of the main theorem

Let us start by discussing a subtle point that was overlooked in the literature. As explained in the introduction, Step I consists on, given a solution (a, b, c) to (1), constructing a Galois representation attached to a rank-2 motive inside an abelian variety A. For example, in the pioneering work of Darmon [Dar00], the variety A is the Jacobian of the hyperelliptic curves studied in [TTV91].

Let g denote the dimension of this variety A. Then, there is a totally real field K of degree g contained in its endomorphism algebra $\mathrm{End}^0_K(A) := \mathrm{End}_K(A) \otimes \mathbb{Q}$. To apply Conjecture 1 to A one needs to determine its full endomorphism algebra $\mathrm{End}^0_{\overline{\mathbb{Q}}}(A)$. If A is absolutely simple, the condition $K \subset \mathrm{End}^0_K(A)$ implies that either A has complex multiplication, or $\mathrm{End}^0_K(A) = \mathrm{End}^0_{\overline{\mathbb{Q}}}(A)$. Consequently, if A does not have complex multiplication, Conjecture 1 can be used.

But we remark that it is not necessarily the case that A is absolutely simple. In [TTV91, Corollary 6] the authors prove that for generic values, this in indeed the case. However, this does not imply that the variety is absolutely simple for all value of (a, b, c)!

A natural solution to the problem is to replace the variety A by a *building block* B. For this idea to work, one needs to have explicit control on a *complete field of definition* of B (i.e. a field where not only the variety B is defined but also all of its endomorphisms).

In the rest of this section, given two abelian varieties A_1 and A_2, we will write $A_1 \sim A_2$ to denote that A_1 and A_2 are isogenous over $\overline{\mathbb{Q}}$.

Lemma 4.1. *Let A be an abelian variety of dimension g defined over a number field L. Suppose there exists a prime ideal $\mathfrak{p}$ of L where A has toric reduction of full rank. Suppose furthermore that there exists a number field K with $[K : \mathbb{Q}] = g$ and $K \subset \mathrm{End}^0_L(A)$. Then A has at most one building block of type I in Albert's classification of GL_2-type, i.e. there exists a simple abelian variety B defined over $\overline{\mathbb{Q}}$ and a positive integer r such that $A \sim B^r$ and $\mathrm{End}^0_{\overline{\mathbb{Q}}}(B)$ is a totally real number field of degree $\dim(B)$.*

Proof. The proof follows the ideas of [FFG26]. By Poincaré decomposition theorem, there exist simple abelian varieties $B_1, \dots, B_n$ and non-negative integers $r_1, \dots, r_n$ such that

$$A \sim B_1^{r_1} \times \cdots \times B_n^{r_n}.$$

Hence $g = r_1 \dim(B_1) + \cdots + r_n \dim(B_n)$, and

$$K \subset \mathrm{End}^0_{\overline{\mathbb{Q}}}(A) \simeq M_{r_1}(\mathrm{End}^0_{\overline{\mathbb{Q}}}(B_1)) \times \cdots \times M_{r_n}(\mathrm{End}^0_{\overline{\mathbb{Q}}}(B_n)).$$

Since K is a field, it embeds into one of the factors, say the first one (denote it by B and r_1 as r, to ease notation). Let H denote the center of $\mathrm{End}^0_{\overline{\mathbb{Q}}}(B)$. Since B is simple, this is a number field, and $[\mathrm{End}^0_{\overline{\mathbb{Q}}}(B) : H] = d^2$ for some positive integer d. By Albert's classification (see, e.g., [Mum08, Theorem 2 in §21]), we have the following possibilities for H:

- type I: $H = \mathrm{End}^0_{\overline{\mathbb{Q}}}(B)$, and so $\mathrm{End}^0_{\overline{\mathbb{Q}}}(B)$ is a totally real number field and $d = 1$.
- type II: H is a totally real number field and $\mathrm{End}^0_{\overline{\mathbb{Q}}}(B)$ is a totally indefinite quaternion algebra over H (so $d = 2$).
- type III: H is a totally real number field and $\mathrm{End}^0_{\overline{\mathbb{Q}}}(B)$ is a totally definite quaternion algebra over H (so $d = 2$).
- type IV: H is a CM field, $\mathrm{End}^0_{\overline{\mathbb{Q}}}(B)$ is a division algebra and $[H : \mathbb{Q}]d^2 \mid 2\dim(B)$.

It follows from the previous description that if M is any subfield of $\mathrm{End}^0_{\overline{\mathbb{Q}}}(B)$ it satisfies

$$[M : \mathbb{Q}] \leq \begin{cases} [H : \mathbb{Q}] & \text{for type I,} \\ \dim(B) & \text{for types II and III,} \\ 2\dim(B)/d & \text{for type IV.} \end{cases} \tag{14}$$

Since, by assumption, there exists a prime of multiplicative reduction, B does not have complex multiplication (see [ST68]) and thus does not correspond to type IV with $d = 1$. Since we have that $K \subset M_r(\mathrm{End}^0_{\overline{\mathbb{Q}}}(B))$:

$$\dim(A) = g = [K : \mathbb{Q}] \leq r \dim(B) \leq g, \tag{15}$$

Consequently, $A \sim B^r$.

If B is of type IV, then (14) and (15) give that $d = 2$ and (15) forces K to be a maximal commutative subalgebra. Consequently, it contains the center H which is a CM extension, giving a contradiction. The existence of a prime of multiplicative reduction for B discards the types II and III (as explained in [DR04, pg. 126]). Then B is of type I and $[H : \mathbb{Q}] = \dim(B)$ as claimed. □

If A is an abelian variety as in the lemma, we denote by H the algebra $\mathrm{End}^0_{\overline{\mathbb{Q}}}(B)$ and by S the set of prime ideals of L where A has bad but non-multiplicative reduction.

Proposition 4.2. *Keep the hypotheses of Lemma 4.1. Then there exists a polyquadratic extension M/L unramified outside S such that the variety B and all of its endomorphisms are defined over M.*

Proof. The fact that M can be taken such that $\mathrm{Gal}(M/L)$ is an abelian group of exponent 2 follows from the work of Ribet [Rib94, Theorem 1.2] (see also Theorem 3.24 in Guitart's Ph.D. thesis [Gui10]). The ramification statement follows from [Rib75] (see also [DR04, Proposition 1.2]). □

Theorem A. *Let $q > r > 2$ and $p > 3$ be prime numbers, with q and r fixed. Let (a, b, c) be a hypothetical non-trivial primitive solution to*

$$x^q + y^p + z^r = 0,$$

and let $\rho_{\mathfrak{p}}$ be the Galois representation attached to (a, b, c) in Theorem 1.1. Assume that Conjecture 1 holds. Then there exists a constant $C(q, r)$ (depending only on q and r) such that if $p > C(q, r)$, for any newform f with complex multiplication we have that $\overline{\rho}_{\mathfrak{p}} \not\simeq \overline{\rho}_{f,\mathfrak{p}}$.

Proof. Let $\mathcal{M}$ denote the maximal polyquadratic extension of K which is unramified outside the prime ideals of K dividing qr. Note that $\mathcal{M}/K$ is a finite extension. Define

$$C(q, r) := \max\{C(\mathcal{M}, H) \ : \ H \subset K\},$$

where $C(\mathcal{M}, H)$ is the constant of Conjecture 1.

Let (a, b, c) be a non-trivial primitive solution to (1), where $q > r > 2$ and $p > C(q, r)$, and let $K = \mathbb{Q}(\zeta_q)^+ \cdot \mathbb{Q}(\zeta_r)^+$. Let $\mathcal{C}$ denote the Euler curve in (9) specialized at $t = -\frac{a^q}{c^r}$. The curve $\mathcal{C}$ has two involutions (see [GP26, Proposition 7.5]), both defined over K. Let $\mathcal{C}'$ denote the quotient of $\mathcal{C}$ by the subgroup generated by these involutions and let A be the new part of its Jacobian (so A is an abelian variety of dimension $(q-1)(r-1)/4$). The field K is contained in $\mathrm{End}^0_K(A)$ and the motive $\mathcal{H}$ of (7) has a realization in A.

By Catalan's conjecture [Mih04], $b \neq \pm 1$, so there exists a prime number $\ell \mid b$. Let λ be a prime ideal of K above ℓ. If $\lambda \nmid qr$, then the motive has multiplicative reduction at λ by [GP24, Theorem 1.4]. If $\lambda \mid qr$, since $t - 1 = \frac{b^p}{c^r}$, the same holds by Theorem B and Remark 3.2. Then the variety A has a prime ideal of multiplicative reduction.

By [GP24, Theorem 1.4], the variety A is semistable at all primes except possibly those dividing qr. By Lemma 4.1 and Proposition 4.2, there exists a simple GL_2-type abelian variety B of type I totally defined over $\mathcal{M}$ such that $A \sim B^d$ for some d (if A is simple, $\mathcal{M} = K$ and $B = A$). Let $H = \mathrm{End}^0_{\mathcal{M}}(B) \subset K$.

Let $\mathfrak{p}$ be a prime ideal of K above p. As recalled in §3, the Galois representation $\rho_{\mathfrak{p}}$ considered in Theorem 1.1 is the one attached to $\mathcal{H}$. Let $\mathfrak{P} = \mathfrak{p} \cap H$. Then, by definition,

$$\overline{\rho}_{B,\mathfrak{P}} \simeq \overline{\rho}_{\mathfrak{p}}|_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathcal{M})}. \tag{16}$$

By Lemma 4.1, the variety B satisfies the hypotheses of Conjecture 1. Then, by (16), the image of $\overline{\rho}_{\mathfrak{p}}$ restricted to $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathcal{M})$ contains $\mathrm{SL}_2(\mathbb{F}_{\mathfrak{p}})$ and, in particular, the same holds for $\overline{\rho}_{\mathfrak{p}}$. Then, $\overline{\rho}_{\mathfrak{p}}$ cannot be isomorphic to the residual representation of any form f with complex multiplication, since the latter has image contained in the normalizer of a Cartan subgroup. □

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