ON RANK 2 HYPERGEOMETRIC MOTIVES

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with an Appendix by Elisa Lorenzo García and Ariel Pacetti

Abstract. Hypergeometric motives are a family of motives associated to hypergeometric local systems. Their special features, in particular their rigidity, make them more tractable than general motives. In the present article we prove most of the expected properties satisfied by rank 2 hypergeometric motives.

1. Introduction

Hypergeometric motives provide a very rich family of geometric objects that are well suited for computations (see [25], [20], [7], [15] and the references therein). A lot of progress has been made for rational motives, but when the field of definition is not the rationals, the theory is not so well understood. For example, little is known regarding their field of definition. In many applications (as in [16]) determining whether a motive is defined over a CM field or over a totally real one makes a big difference.

In the present article we restrict to hypergeometric motives of rank 2 given by a pair of rational vectors $\boldsymbol{\alpha} = (a, b)$, $\boldsymbol{\beta} = (c, d)$ and define the motive $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta} \,|\, z)$ as a tensor product of a motive coming from the Euler curve times a Jacobi motive. A first technical subtlety appears with this naive definition. For historical reasons, rank 2 motives were mostly studied when $d = 1$ (as in [15]). In this case, the Euler curve is irreducible, but for general values of d this need not be the case!

Let N be the common denominator of the parameters and let $F = \mathbb{Q}(\zeta_N)$. Let H denote the set of elements $i \in (\mathbb{Z}/N)^\times$ fixing the sets $\{a, b\}$ and $\{c, d\}$ modulo $\mathbb{Z}$; i.e.,

$$\{i \in (\mathbb{Z}/N)^\times : i\{a, b\} \equiv \{a, b\} \bmod \mathbb{Z} \text{ and } i\{c, d\} \equiv \{c, d\} \bmod \mathbb{Z}\} \tag{1}$$

Let $K := F^H$, after identifying H with a subgroup of $\mathrm{Gal}(F/\mathbb{Q})$ in the natural way. The motive $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta} \,|\, z)$ is conjectured to be defined over $K(z)$. We prove this is indeed the case under some hypothesis (see Theorem 7.2).

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Of particular interest is the case of motives defined over totally real number fields (see Corollary 7.8).

Theorem 1. *If K is totally real then for all specializations $\xi \in K$, $\xi \neq 0, 1$, $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta} \,|\, \xi)$ may be defined over the totally real field K.*

Motives defined over totally real fields are conjectured to be related to Hilbert modular forms. There are two types of hypergeometric motives defined over totally real fields (see §8): the ones that correspond to dihedral Artin representations, and motives satisfying that both $a+b$ and $c+d$ are integers. The first ones are modular by a result of Langlands; we prove that the second ones are *potentially modular* (see Theorem 8.2).

Our contribution also includes three important results on properties of the specialized motive $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta} \,|\, \xi)$.

1. A detailed description of the action of inertia at the so called *potentially tame* primes, namely prime ideals $\mathfrak{p}$ not dividing N for which $v_{\mathfrak{p}}(\xi(\xi-1)) \neq 0$. Here is an explicit instance (see Corollary A.3): let M_0 be the companion matrix of the polynomial $(t - \exp(-2\pi i c))(t - \exp(-2\pi i d))$.

Theorem 2. *Let L be a number field containing F and let $\xi \in L$, $\xi \neq 0, 1$. Let $\mathfrak{p}$ be a prime ideal of L not dividing N satisfying $v_{\mathfrak{p}}(\xi) > 0$. Then the image of the inertia group $I_{\mathfrak{p}}$ on $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta} \,|\, \xi)$ is spanned (up to conjugation) by $M_0^{v_{\mathfrak{p}}(\xi)}$.*

The result plays a crucial role while constructing motives attached to solutions of the generalized Fermat equation $x^p + y^q + z^r = 0$ (see [16]).

2. An explicit description of the trace of Frobenii elements at primes of good reduction (see Theorem 6.11).

Theorem 3. *Let $\mathfrak{p}$ be a prime ideal of $\mathbb{Q}(\zeta_N)$ not dividing N satisfying that $v_{\mathfrak{p}}(\xi(\xi-1)) = 0$. Then $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta} \,|\, \xi)$ has good reduction at $\mathfrak{p}$ and the trace of the Frobenius automorphism $\mathrm{Frob}_{\mathfrak{p}}$ acting on $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta} \,|\, \xi)$ equals the finite hypergeometric sum $H_{\mathfrak{p}}(\boldsymbol{\alpha}, \boldsymbol{\beta} \,|\, \xi)$.*

3. Congruence relations (see Theorem 9.3). Hypergeometric motives satisfy many congruences. Let ℓ be a prime number. For a, b rational numbers, denote by $a \sim_\ell b$ the relation defined by the condition that the denominator of $a - b$ is a power of ℓ. Extend this relation to vectors component-wise.

Theorem 4. *Let $\boldsymbol{\alpha}, \boldsymbol{\beta}$ and $\boldsymbol{\alpha}', \boldsymbol{\beta}'$ be generic such that $\boldsymbol{\alpha} \sim_\ell \boldsymbol{\alpha}'$ and $\boldsymbol{\beta} \sim_\ell \boldsymbol{\beta}'$ for some prime ℓ. Then for any $\xi \neq 0, 1$, $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta} \,|\, \xi)$ is congruent to $\mathcal{H}(\boldsymbol{\alpha}', \boldsymbol{\beta}' \,|\, \xi)$ modulo any prime ideal $\mathfrak{l}$ dividing ℓ.*

Congruences are useful in two different ways: they allow in many instances to prove modularity of the motive $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta} \,|\, \xi)$ (see [16]) and also give explicit cases of "raising the level" results.

Example 1. Consider the hypergeometric motive with parameters $(1/2, 1/2), (1, 1)$, corresponding (see Example 3) to a quadratic twist by $\left(\frac{-4}{\cdot}\right)$ of Legendre's family of elliptic curves

$$E_z : y^2 = x(1-x)(1-zx).$$

Subtracting $1/3$ from the first parameter and adding $1/3$ to the second one, we obtain the rational motive with parameters $(1/6, -1/6), (1, 1)$ corresponding (as proved in [18]) to the family of elliptic curves with equation

$$\widetilde{E_z} : y^2 = x^3 - \frac{x^2}{4} + \frac{z}{432}.$$

Then for any specialization ξ of the parameter z the elliptic curves have congruent Galois representations modulo 3.

The article is organized as follows: Section §2 recalls the construction and main properties of the monodromy representation attached to the parameters $(\boldsymbol{\alpha}, \boldsymbol{\beta})$. It also contains a precise statement (Conjecture 2.7) of the properties expected to be satisfied by $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta} \,|\, z)$. We prove a softer version of the conjecture in Corollary 6.14. Section §3 consists of a short survey on Jacobi motives (following [31] and [32]). Section §4 is devoted to the study of finite hypergeometric sums. Proposition 4.2 summarizes their main properties. In Section §5 we compute the zeta function of a general superelliptic curve, relating the number of points of its isotypical components to finite character sums (see Theorem 5.4). Section §6 is devoted to define rank 2 hypergeometric motives. We start studying the case when the Euler curve is irreducible, and use this case for the general definition (see Definition 6.10). The main result is a proof of Theorem 3.

Section §7 is devoted to prove results on the field of definition of the motive (Theorem 1). Section §8 is devoted to the study of hypergeometric motives defined over totally real fields (i.e. parameters satisfying that $-1 \in H$). The last section (Section §9) concerns congruences between hypergeometric motives. The present article ends with two appendices. The first one studies reduction types of the Euler curve, implying Theorem 2. It is also needed to relate the representation of the motive $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta} \,|\, z)$ to its monodromy representation (see Theorem 6.18).

The second appendix addresses the following problem: keeping the previous notation, let $\mathfrak{p}$ be a prime ideal of F not dividing N but (say) dividing the numerator of ξ. Then it is a priori a tame prime for the motive, but it may actually be of good reduction. If this is the case, Theorem 3 does not apply. One can still compute the trace of the Frobenius element geometrically, by computing the stable model of the Euler equation (as explained in Appendix B). This formula is needed while applying Darmon's program to the study of points on the generalized Fermat equation, as explained in [16].

We finish this introduction with a few challenges that we expect to address in the near future:

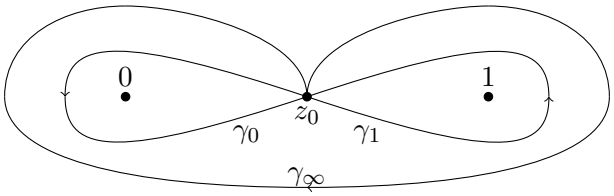

FIGURE 1. Fundamental group of $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\}, z_0)$

- Can the inertial information be used to prove large image results for hypergeometric motives?
- Extend Theorem 3 to primes of the field of definition of $\mathcal{H}(\boldsymbol{\alpha},\boldsymbol{\beta} \mid \xi)$.
- Study larger rank hypergeometric motives.

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2. Generalities

Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be n-tuples of rational numbers. Consider the differential equation

$$D(\boldsymbol{\alpha};\boldsymbol{\beta}) := (\theta+\beta_1-1)\cdots(\theta+\beta_n-1) - z(\theta+\alpha_1)\cdots(\theta+\alpha_n), \tag{2}$$

where $\theta = z\frac{d}{dz}$. This equation has regular singularities at the points $z = 0, 1, \infty$. Let z_0 be a complex number different from $0, 1$. The differential equation $D(\boldsymbol{\alpha};\boldsymbol{\beta})$ has attached, up to isomorphism, a natural *monodromy representation*

$$\rho : \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\}, z_0) \to \mathrm{GL}_n(\mathbb{C}). \tag{3}$$

Definition 2.1. *We say that* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$ *are* generic *if the sets* $\{\exp(2\pi i\alpha_j)\}$, $\{\exp(2\pi i\beta_k)\}$ *are disjoint; equivalently,* $\alpha_i \neq \beta_j \bmod \mathbb{Z}$ *for* $i, j = 1, \ldots, n$.

Proposition 2.2. *The monodromy representation attached to the parameters* $\boldsymbol{\alpha},\boldsymbol{\beta}$ *is irreducible if and only if the parameters are generic.*

Proof. See Propositions 2.7 and 3.3 of [6]. □

In the generic situation, the monodromy representation only depends on the parameters up to translation by integers (by [6, Corollary 2.6]) so we can think of $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ having coordinates in $(\mathbb{Q}/\mathbb{Z})$.

Let $\gamma_0, \gamma_1, \gamma_\infty$ denote paths going through $0, 1, \infty$ respectively in counterclockwise direction as in Figure 1. It follows from Van Kampen's theorem that

$$\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\}, z_0) = \langle \gamma_0, \gamma_1, \gamma_\infty \ : \ \gamma_0\gamma_1\gamma_\infty = 1 \rangle.$$

In particular, the monodromy representation is determined by the matrices

$$M_0 := \rho(\gamma_0), \qquad M_1 := \rho(\gamma_1), \qquad M_\infty := \rho(\gamma_\infty), \tag{4}$$

corresponding to the local monodromy matrices around $0, 1$ and ∞.

Proposition 2.3. *Let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$ and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_n)$ be generic. Let $a_j = \exp(2\pi i \alpha_j)$ and $b_j = \exp(-2\pi i \beta_j)$ for $j = 1, \dots, n$. Define*

$$q_\infty = \prod_{j=1}^{n}(t - a_j), \qquad q_0 = \prod_{j=1}^{n}(t - b_j). \tag{5}$$

Then M_∞ and M_0 are conjugate to the companion matrix of q_∞ and q_0 and M_1 is a pseudo-reflection with special eigenvalue

$$\mu := \exp\left(2\pi i \sum_{j=1}^{n}(\beta_j - \alpha_j)\right).$$

Proof. See Proposition 2.10 and Proposition 3.2 of [6]. □

Let $\boldsymbol{\alpha}, \boldsymbol{\beta}$ be generic rational parameters and let N be their least common denominator. Let $F = \mathbb{Q}(\zeta_N)$ and $\mathcal{O}$ its ring of integers.

Theorem 2.4. *For each prime ideal $\mathfrak{p}$ of F the monodromy representation extends to a continuous representation (unique up to isomorphism)*

$$\rho_{\mathfrak{p}} : \mathrm{Gal}(\overline{\mathbb{Q}(z)}/\overline{\mathbb{Q}}(z)) \to \mathrm{GL}_n(\mathcal{O}_{\mathfrak{p}}), \tag{6}$$

unramified outside $\{0, 1, \infty\}$.

Proof. A construction due to Levelt (see Theorem 3.5 of [6]) proves that we can assume that our monodromy representation takes values in $\mathrm{GL}_n(\mathcal{O})$. The fundamental group $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}, z_0)$ is dense in its étale fundamental group $\widehat{\pi_1}(\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}, z_0)$ and the latter is isomorphic to $\mathrm{Gal}(\Omega/\overline{\mathbb{Q}}(z))$, where Ω denotes the maximal extension of $\overline{\mathbb{Q}}(z)$ unramified outside $\{0, 1, \infty\}$ (see [28, Theorem 6.3.1]). Since $\mathrm{GL}_n(\mathcal{O}_{\mathfrak{p}})$ is profinite, the extension exists by the universal property of the profinite completion (see [24, Lemma 3.2.1]). □

2.1. **Expected properties of $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta} \,|\, z)$.** Let $\boldsymbol{\alpha}, \boldsymbol{\beta}$ be vectors of generic rational numbers, and let N be their least common denominator. Keeping the previous notation, let $F = \mathbb{Q}(\zeta_N)$ and let H be as in (1).

Definition 2.5. *We define the* base field $K := \mathbb{Q}(\zeta_N)^H \subseteq \mathbb{Q}(\zeta_N)$ *of the hypergeometric data $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ as the field fixed by H.*

Definition 2.6. *The* coefficient field *of a motive defined over a number field is the field generated by the traces of all Frobenius elements.*

Let r denote the number of integral coordinates of $\boldsymbol{\alpha}, \boldsymbol{\beta}$.

Conjecture 2.7. *There exists a* hypergeometric motive $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta} | z)$ *attached to the parameters $\boldsymbol{\alpha}, \boldsymbol{\beta}$ satisfying the following properties*

(i) The motive $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta} | z)$ has a realization $\mathcal{H}(z)$ over $K(z)$.

(ii) (Compatibility) The representation of the motive restricted to $\mathrm{Gal}(\overline{K(z)}/\overline{K}(z))$ *is isomorphic to the geometric Galois representation (6).*

Let L *be a finite extension of* K *and let* $\xi \in L$, $\xi \neq 0, 1$. *Let* $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}|\xi)$ *denote the specialization of the motive at* $z = \xi$.

(iii) It is a pure motive defined over L *of degree* n *and weight* $r - 1$.

(iv) The primes of L *of bad reduction belong to* $S_{\mathrm{pt}} \cup S_{\mathrm{pw}}$ *(potentially tame and potentially wild primes respectively), where*

$$S_{\mathrm{pw}} := \{\mathfrak{p} \mid v_{\mathfrak{p}}(N) > 0\}, \quad S_{\mathrm{pt}} := (S_0 \cup S_1 \cup S_\infty) \setminus S_{\mathrm{pw}},$$

and

$$S_0 := \{\mathfrak{p} \mid v_{\mathfrak{p}}(\xi) > 0\}, \quad S_1 := \{\mathfrak{p} \mid v_{\mathfrak{p}}(\xi - 1) > 0\}, \quad S_\infty := \{\mathfrak{p} \mid v_{\mathfrak{p}}(\xi) < 0\}.$$

We let S_g *be the set of primes not in* $S_{\mathrm{pt}} \cup S_{\mathrm{pw}}$.

(v) For a prime $\mathfrak{p} \in S_g$ *the trace of powers of the Frobenius automorphism at* $\mathfrak{p}$ *acting on* $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}|\xi)$ *are given by an explicit finite hypergeometric sum.*

(vi) For generic $\xi \in L$ *it has coefficient field* K.

We use the term motive to refer to a pure motive (as in §4 of [2]), i.e. a pair (X, p) where X is a projective variety and p is a correspondence on X which is an idempotent. We say that the motive has a *realization* over K if both the variety X and the correspondence p are defined over K.

Remark 2.8. The existence of a motive over a number field for general rigid local systems was proved by Katz in [20]. See §6 of [19] for a short exposition of the motivic interpretation.

Remark 2.9. For particular values of the parameter z, we might be able to descend to a motive over a proper subfield $L \subseteq K$. For example the motive of Example 6 has base field $\mathbb{Q}(\sqrt{2})$, but evaluated at squares is defined over $\mathbb{Q}$. It is important to remark that even in this case, the coefficient field matches the base field.

Definition 2.10. *A pair* $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ *of vectors is* rational *if its base field is* $\mathbb{Q}$.

3. Jacobi motives

Fix an integer $N > 1$. For a prime ideal $\mathfrak{p}$ in $F := \mathbb{Q}(\zeta_N) \subseteq \mathbb{C}$ of norm q, where ζ_N is a primitive N-st root of unity, let $\chi_{\mathfrak{p}}$ be the character

$$\chi_{\mathfrak{p}} : (\mathcal{O}/\mathfrak{p})^\times \to \mathcal{O}^\times \tag{7}$$

of order N satisfying

$$\chi_{\mathfrak{p}}(x) \equiv x^{(q-1)/N} \bmod \mathfrak{p}.$$

Extend the definition by setting $\chi_{\mathfrak{p}}(x) = 0$ if $\mathfrak{p} \mid x$.

Remark 3.1. If L is a finite extension of F and $\widehat{\mathfrak{p}}$ is a prime ideal of L dividing $\mathfrak{p}$ we can similarly define a character $\widehat{\chi_{\mathfrak{p}}} : (\mathcal{O}_L/\widehat{\mathfrak{p}})^\times \to \mathbb{C}^\times$ by composing $\chi_{\mathfrak{p}}$ with the norm map. To avoid heavy notation, we will also denote by $\chi_{\mathfrak{p}}$ the character $\widehat{\chi_{\mathfrak{p}}}$.

We define Gauss sums as functions on $\frac{1}{N}\mathbb{Z}/\mathbb{Z}$ as follows. Fix a non-trivial additive character ψ on $\mathbb{F}_q$ and let

$$g(\psi, a, \mathfrak{p}) := \sum_{x\in\mathbb{F}_q^\times} \psi(x)\chi_{\mathfrak{p}}^{Na}(x), \qquad a \in \frac{1}{N}\mathbb{Z}/\mathbb{Z}. \tag{8}$$

Definition 3.2. *Let $\boldsymbol{a} = (a_1, \dots, a_r)$ and $\boldsymbol{b} = (b_1, \dots, b_s)$ be a pair of tuples of rational numbers with common denominator N such that*

$$\sum_{i=1}^{r} a_i \equiv \sum_{i=1}^{s} b_i \bmod \mathbb{Z},$$

The Jacobi sum *attached to $(\boldsymbol{a}, \boldsymbol{b})$ for the prime ideal $\mathfrak{p}$ is defined as*

$$\mathbf{J}(\boldsymbol{a}, \boldsymbol{b}) := (-1)^{r-s+1} \frac{\prod_{i=1}^{r} g(\psi, a_i, \mathfrak{p})}{\prod_{i=1}^{s} g(\psi, b_i, \mathfrak{p})}. \tag{9}$$

It is easy to verify that the definition is independent of the choice of the additive character ψ. By the main result of [32] the map

$$\mathfrak{p} \mapsto \mathbf{J}(\boldsymbol{a}, \boldsymbol{b})(\mathfrak{p})$$

determines a Grössencharacter $\mathbf{J}(\boldsymbol{a}, \boldsymbol{b})$ of F. This is an incarnation of the *Jacobi motive* (see [1, §5]). We will denote this pure motive also by $\mathbf{J}(\boldsymbol{a}, \boldsymbol{b})$ if there is no risk of confusion.

By [1, Theorem 1] (see also Remark 2.3.2 of loc. cit) the infinity type of the Grössencharacter associated to $\mathbf{J}(\boldsymbol{a}, \boldsymbol{b})$ is given by

$$\sigma_j \mapsto \sum_{i=1}^{r} \{ja_i\} - \sum_{i=1}^{s} \{jb_i\}, \qquad \gcd(j, N) = 1, \tag{10}$$

where $\{\cdot\}$ denotes the fractional part of real numbers and $\sigma_j \in \mathrm{Gal}(F/\mathbb{Q})$ is the automorphism satisfying $\sigma_j(\zeta_N) = \zeta_N^j$. If we define

$$p := \sum_i n_i\{\theta_i\}, \qquad q := \sum_i n_i\{-\theta_i\}, \tag{11}$$

then (up to a Tate twist) $\mathbf{J}(\boldsymbol{a}, \boldsymbol{b}) \subseteq H^{p,q}(X, F)$, for X a smooth Fermat hypersurface (see for example §10 of [1]).

Let $T = \{\sigma \in \mathrm{Gal}(F/\mathbb{Q}) \ : \ \sigma(\mathbf{J}(\boldsymbol{a}, \boldsymbol{b})) = \mathbf{J}(\boldsymbol{a}, \boldsymbol{b})\}$. Then the base field of the motive $\mathbf{J}(\boldsymbol{a}, \boldsymbol{b})$ is F^T (see §2.3 of [31]).

4. Finite Hypergeometric Sums

Let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$ and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_n) \in \mathbb{Q}^n$ be arbitrary vectors of rational numbers and let N be the common denominator of the α's and β's. For the remaining of the section we fix a prime power $q \equiv 1 \bmod N$ and a generator ϖ of $\widehat{\mathbb{F}_q^\times}$. To simplify the notation set $g(m) := g(\psi, \varpi^m)$ for $m \in \mathbb{Z}$.

Definition 4.1. *For $z \in \mathbb{F}_q$, define the finite hypergeometric sum $H_q(\boldsymbol{\alpha}, \boldsymbol{\beta}|z)$ by*

$$\frac{1}{1-q}\sum_{m=0}^{q-2}\prod_{i=1}^{n}\left(\frac{g(m+\alpha_i(q-1))g(-m-\beta_i(q-1))}{g(\alpha_i(q-1))g(-\beta_i(q-1))}\right)\varpi((-1)^n z)^m. \tag{12}$$

The finite hypergeometric sum $H_q(\boldsymbol{\alpha}, \boldsymbol{\beta}|z)$ does not depend on the choice of the additive character ψ, but it does (in general) depend on the choice of the character ϖ.

Proposition 4.2. *The hypergeometric sum $H_q(\boldsymbol{\alpha}, \boldsymbol{\beta}|z)$ satisfies the following basic properties*

1. *(Ordering) $H_q(\boldsymbol{\alpha}, \boldsymbol{\beta}|z)$ is independent of the ordering of $\{\alpha_1, \dots, \alpha_n\}$ and $\{\beta_1, \dots, \beta_n\}$.*
2. *(Inversion) For $z \in \mathbb{F}_q^\times$ we have*
$$H_q(\boldsymbol{\alpha}, \boldsymbol{\beta}|z) = H_q(-\boldsymbol{\beta}, -\boldsymbol{\alpha}|z^{-1}).$$
3. *(Galois action) Let $z \in \mathbb{F}_q^\times$ and $\sigma \in \mathrm{Gal}_{\mathbb{Q}}$ with $\sigma(\zeta_N) = \zeta_N^j$ for some j coprime with N then*
$$H_q(\boldsymbol{\alpha}, \boldsymbol{\beta}|z)^\sigma = H_q(j\boldsymbol{\alpha}, j\boldsymbol{\beta}|z).$$
4. *(Coefficient Field) Let $z \in \mathbb{F}_q^\times$ and let H be the subgroup of $(\mathbb{Z}/N)^\times$ as before. We have*
$$H_q(\boldsymbol{\alpha}, \boldsymbol{\beta}|z) \in \mathbb{Q}(\zeta_N)^H.$$

Proof. The first three properties follow basically from its definition. To prove the last one, note that the standard properties of Gauss sums imply that the value $H_q(\boldsymbol{\alpha}, \boldsymbol{\beta}|z)$ is an element of $\mathbb{Q}(\zeta_{q-1})$ (see also [7, Proposition 3.2]). All elements of $(\mathbb{Z}/(q-1))^\times$ that are congruent to 1 modulo N as well as the elements of H leave the sets $\{\alpha_1, \dots, \alpha_n\}$ and $\{\beta_1, \dots, \beta_n\}$ stable under multiplication (up to translation by integers), hence the second statement (and Galois theory) imply the result. □

Recall that for L a finite extension of $F = \mathbb{Q}(\zeta_N)$ and $\hat{\mathfrak{p}}$ a prime ideal of L, we defined in Remark 3.1 an order N character $\widehat{\chi_{\mathfrak{p}}}$ of $(\mathcal{O}_L/\hat{\mathfrak{p}})^\times$.

Definition 4.3. *Let $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{Q}^n$ be generic parameters and let N be their least common denominator. Let L be a finite extension of F and let $\hat{\mathfrak{p}}$ be a prime ideal of L of norm q not dividing N. Let $\xi \in L$ satisfy $v_{\hat{\mathfrak{p}}}(\xi) = v_{\hat{\mathfrak{p}}}(\xi - 1) = 0$. The* $\hat{\mathfrak{p}}$-hypergeometric sum*, denoted $H_{\hat{\mathfrak{p}}}(\boldsymbol{\alpha}, \boldsymbol{\beta}|\xi)$ is the value $H_q(\boldsymbol{\alpha}, \boldsymbol{\beta}|\xi)$ for a generator ϖ of $\widehat{\mathbb{F}_q^\times}$ satisfying $\varpi^{\frac{q-1}{N}} = \widehat{\chi_{\mathfrak{p}}}^{-1}$.*

4.1. **Hypergeometric character sums.** Recall the following well known definition.

Definition 4.4. *Let φ, η be two multiplicative characters on $\mathbb{F}_q^\times$. The Jacobi sum attached to them is defined by*

$$J(\varphi,\eta) = \sum_{x\in\mathbb{F}_q} \varphi(x)\eta(1-x),$$

where $\varphi(0) = \eta(0) = 0$.

If φ, η are characters of $\mathbb{F}_q$ and η is non-trivial, then

$$J(\varphi,\eta) = \varphi(-1)\frac{g(\varphi^{-1}\eta^{-1})g(\varphi)}{g(\eta^{-1})}. \tag{13}$$

Let $\varepsilon_1, \eta_1, \dots, \varepsilon_{n-1}, \eta_{n-1}, \chi$ be characters of $\mathbb{F}_q^\times$ (extended to $\mathbb{F}_q$ by setting their value at 0 to be 0). For $z \in \mathbb{F}_q$ let

$$H(z) = \sum_{x_1,\dots,x_{n-1}} \prod_{i=1}^{n-1} \varepsilon_i(x_i)\eta_i(1-x_i)\chi^{-1}(1-zx_1\cdots x_{n-1}). \tag{14}$$

Theorem 4.5. *Keep the previous notation, and set $\varepsilon_n = \chi$ and $\varepsilon_n\eta_n = 1$. Let $\boldsymbol{\alpha}, \boldsymbol{\beta}$ be rational numbers such that $\varpi^{(q-1)\alpha_i} = \varepsilon_i$ and $\varpi^{(q-1)\beta_i} = \varepsilon_i\eta_i$ for $i = 1, \dots, n$. Then $H(z)$ equals*

$$H(z) = (-1)^{n-1}\left(\prod_{i=1}^{n-1}\varepsilon_i(-1)\right)\mathbf{J}((\boldsymbol{\alpha},-\boldsymbol{\beta}),(\boldsymbol{\alpha}-\boldsymbol{\beta}))H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|z).$$

Proof. Add one more variable x_n to $H(z)$ setting

$$x_n := zx_1\cdots x_{n-1}.$$

Then

$$H(z) = \frac{1}{q-1}\sum_{\varphi}\sum_{x_1,\dots,x_n}\prod_{i=1}^{n-1}\varepsilon_i(x_i)\eta_i(1-x_i)\chi^{-1}(1-x_n)\varphi(zx_n^{-1}x_1\cdots x_{n-1}),$$

where the first sum ranges over all characters φ of $\mathbb{F}_q^\times$ (we are abusing notation, declaring x_n^{-1} to be 0 if x_n equals 0). Interchanging sum and product, we get

$$\frac{1}{q-1}\sum_{\varphi}\left(\prod_{i=1}^{n-1}\left(\sum_{x_i}\varepsilon_i\varphi(x_i)\eta_i(1-x_i)\right)\cdot\sum_{x_n}\varphi^{-1}(x_n)\chi^{-1}(1-x_n)\right)\varphi(z).$$

Applying (13) we get that $H(z)$ equals

$$\frac{1}{q-1}\sum_{\varphi}\left(\prod_{i=1}^{n-1}\varepsilon_i(-1)\varphi(-1)\frac{g(\varepsilon_i\varphi)g(\varepsilon_i^{-1}\eta_i^{-1}\varphi^{-1})}{g(\eta_i^{-1})}\right)\cdot\frac{g(\varphi\chi)g(\varphi^{-1})}{g(\chi)}\varphi(-1)\varphi(z)$$

$$= \prod_{i=1}^{n-1} \frac{\varepsilon_i(-1)g(\varepsilon_i)g(\varepsilon_i^{-1}\eta_i^{-1})}{g(\eta_i^{-1})} \cdot \left(\frac{(-1)}{(q-1)} \sum_{\varphi} \left(\prod_{i=1}^{n} \frac{g(\varepsilon_i\varphi)g(\varepsilon_i^{-1}\eta_i^{-1}\varphi^{-1})}{g(\varepsilon_i)g(\varepsilon_i^{-1}\eta_i^{-1})} \right) \varphi(-1)^n \varphi(z) \right),$$

as claimed (with the convention $\varepsilon_n = \chi$ and $\varepsilon_n\eta_n = 1$), the extra -1 coming from the fact that $g(1) = -1$. □

5. Zeta function of Superelliptic curves

For this section let N be a positive integer and L be a number field or a local field containing the N-th roots of unity. Let $f(x) \in L[x]$. Let $\mathcal{C}$ be the superelliptic curve with equation

$$\mathcal{C} : y^N = f(x). \tag{15}$$

Assume that the curve $\mathcal{C}$ is irreducible and the model is *minimal* in the sense that all irreducible factors of $f(x)$ have degree less than N. Let $\widehat{\mathcal{C}}$ be a smooth projective model. For the rest of the section we make the following assumptions.

Assumption 1. *Keeping the previous notation, the polynomial f, the integer N and the residual characteristic p satisfy the following properties:*

(1) $p \nmid N$,
(2) $p \nmid \operatorname{disc}\left(\frac{f}{\gcd(f,f')}\right)$,
(3) The leading coefficient c of f is a unit in $\mathcal{O}$,
(4) The curve $\mathcal{C}$ is irreducible over $\overline{L}$.

These assumptions assure that the reduction of $\widehat{\mathcal{C}}$ modulo $\mathfrak{p}$ is smooth.

Remark 5.1. For $\mathcal{C}$ the Euler curve (23) specialized at $z = \xi \in \mathbb{P}^1(L) \setminus \{0, 1, \infty\}$, a prime ideal $\mathfrak{p}$ satisfies Assumption 1 precisely when $\mathfrak{p} \in S_g$ (see (iv)).

The group $\boldsymbol{\mu}_N$ of N-th roots of unity in L acts on the L-rational points $\mathcal{C}(L)$ of $\mathcal{C}$ by

$$\zeta \cdot (x, y) = (x, \zeta y), \qquad \zeta \in \boldsymbol{\mu}_N.$$

Let λ be a prime ideal of F. Define

$$H^1_{\text{ét}}(\widehat{\mathcal{C}}, F_\lambda)^{\text{new}} := H^1_{\text{ét}}(\widehat{\mathcal{C}}, \mathbb{Q}_\ell)^{\text{new}} \otimes_{\mathbb{Q}_\ell} F_\lambda.$$

The action of the group $\boldsymbol{\mu}_N$ on the curve $\mathcal{C}$ induces a decomposition

$$H^1_{\text{ét}}(\widehat{\mathcal{C}}, F_\lambda)^{\text{new}} := \bigoplus_{\chi} H^1_{\text{ét},\chi}(\widehat{\mathcal{C}}, F_\lambda), \tag{16}$$

where $H^1_{\text{ét},\chi}(\widehat{\mathcal{C}}, F_\lambda)$ is the χ-eigenspace for $\chi : \boldsymbol{\mu}_N \to F_\lambda^\times$ a character of order N. Let $\mathcal{O}$ be the discrete valuation ring consisting of the ring of integers of L (when L is a local field) or a completion of its ring of integers at a prime

ideal (when L is a global field). Let $\mathfrak{p}$ be its maximal ideal and k its residue field, a finite field of characteristic p with q elements (so $N \mid q-1$). Note that $\mathrm{Frob}_{\mathfrak{p}}$ preserves the decomposition (16) as L contains $\boldsymbol{\mu}_N$.

Definition 5.2. *Let ω be a character of $(\mathcal{O}/\mathfrak{p})^\times$. The* counting function *$\mathcal{N}(\omega)$ is defined by*

$$\mathcal{N}(\omega) := \sum_{x\in\mathbb{F}_q} \omega(f(x)) + \begin{cases} \omega(c) & \text{if } N \mid \deg(f), \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

Fix a generator ζ of $\boldsymbol{\mu}_N \subset F \subset L$. Let $k \in \mathbb{F}_q$ be the reduction of ζ modulo $\mathfrak{p}$ and let $\alpha \in \overline{\mathbb{F}_q}$ be such that $\alpha^{q-1} = k$. Then $\varepsilon := \alpha^N \in \mathbb{F}_q$ (since it is a root of $x^{q-1}-1$) and generates $\mathbb{F}_q^\times$.

Lemma 5.3. *The character $\chi_{\mathfrak{p}}$ (defined in (7)) satisfies*

$$\chi_{\mathfrak{p}}(\varepsilon) = \zeta. \tag{18}$$

For a prime ideal λ of F define the group morphism $\chi_{0,\lambda} : \boldsymbol{\mu}_N \to F_\lambda^\times$ to be the restriction of the map $\iota : F \hookrightarrow F_\lambda$ to $\boldsymbol{\mu}_N$.

Theorem 5.4. *Let λ be a prime ideal of F. Let $\mathcal{C}$ be the superelliptic curve given by (15) for $N > 1$ and suppose Assumption 1 holds. Then the trace of $\mathrm{Frob}_{\mathfrak{q}}$ on $H^1_{et,\chi_{0,\lambda}}(\widehat{\mathcal{C}}, F_\lambda)$ equals $-\iota_\lambda(\mathcal{N}(\chi_{\mathfrak{p}}))$.*

Proof. It is not hard to verify that

$$\mathrm{Tr}(\mathrm{Frob})|_{H^1_{\text{ét}}(\widehat{\mathcal{C}},F_\lambda)^{\text{new}}} = -\sum_{\omega} \mathcal{N}(\omega) \tag{19}$$

where ω runs over characters of $k^\times$ of order N. For $\chi : \boldsymbol{\mu}_N \to F_\lambda^\times$ a character of order N, let A_χ be the matrix of $\mathrm{Frob}_{\mathfrak{p}}$ acting on $\mathrm{H}^1_{\text{et},\chi}(\widehat{\mathcal{C}}, \mathrm{F}_\lambda)$. By (19)

$$\sum_{\chi} \mathrm{Tr}(A_\chi) = -\sum_{\omega} \iota_\lambda(\mathcal{N}(\omega)).$$

Let $X/\mathbb{F}_q$ be a non-singular variety and let $\sigma \in \mathrm{Aut}_{\mathbb{F}_q}(X)$. By Lefschetz's trace formula

$$\#\{x \in X(\overline{\mathbb{F}_q}) : \mathrm{Frob}_q(\sigma(x)) = x\} = \sum_{i=0}^{2\dim(X)} (-1)^i \mathrm{Tr}(\mathrm{Frob}_q \circ \sigma)|_{H^i_{\text{ét}}(X,F_\lambda)}.$$

Let σ_i be the automorphism of $\widehat{\mathcal{C}}$ defined by $(x,y) \mapsto (x, \zeta^i y)$. Then

$$\#\{(x,y) \in \widehat{\mathcal{C}}(\mathbb{F}_q) : x \in \mathbb{F}_q, y^{q-1} = k^{-iq}\} = \\ 1 + q - \sum_{\chi} \chi(\zeta)^i \,\mathrm{Tr}(A_\chi) + \text{“old contribution”}, \tag{20}$$

where *“old contribution”* comes from the curves $\mathcal{C}_d$ for a proper divisors $d \mid N$. Since $k \in \mathbb{F}_q$, $k^q = k$ and

$$\{(x,y) \in \widehat{\mathcal{C}}(\mathbb{F}_q) : x \in \mathbb{F}_q, y^{q-1} = k^{-iq}\} = \{(x,y) \in \widehat{\mathcal{C}}(\mathbb{F}_q) : x \in \mathbb{F}_q, y^{q-1} = k^{-i}\}.$$

Recall that $\alpha \in \overline{\mathbb{F}_q}$ satisfies $\alpha^{q-1} = k$ and $\varepsilon := \alpha^N$. Then the map $y \leftrightarrow \tilde{y} := y\alpha^i$ gives a bijection between the sets

$$\{(x,y) \in \widehat{\mathcal{C}}(\mathbb{F}_q) \text{ fixed by } \mathrm{Frob}_q \circ \sigma_i\} \leftrightarrow \{(x,\tilde{y}) \in \mathbb{F}_q^2 : \varepsilon^{-i}\tilde{y}^N = f(x)\}. \tag{21}$$

Indeed,

$$y^{q-1} = k^{-i} \Leftrightarrow \frac{\tilde{y}^{q-1}}{(\alpha^{q-1})^i} = k^{-i} \Leftrightarrow \tilde{y}^q = \tilde{y}.$$

The elements of the right hand side of (21) are the $\mathbb{F}_q$-points of the twisted curve

$$\mathcal{C}_i : y^N = \varepsilon^i f(x).$$

Then the left hand side of (20) equals

$$\#\widehat{\mathcal{C}}_i(\mathbb{F}_q) = 1 + q + \sum_{\omega} \omega(\varepsilon^i)\mathcal{N}(\omega) + \text{“old contribution”}$$

so for all i

$$\sum_{\chi} \chi(\zeta)^i \operatorname{Tr}(A_\chi) = -\sum_{\omega} \iota_\lambda(\omega(\varepsilon))^i \iota_\lambda(\mathcal{N}(\omega)).$$

It follows that $\operatorname{Tr}(A_{\chi_{0,\lambda}}) = -\iota_\lambda(\mathcal{N}(\omega))$ for ω the character of $\mathbb{F}_q$ satisfying $\iota_\lambda(\omega(\varepsilon)) = \chi_{0,\lambda}(\zeta) = \iota_\lambda(\zeta)$, which by Lemma 5.3 corresponds to $\omega = \chi_{\mathfrak{p}}$. □

6. Rank two Hypergeometric Motives

Definition 6.1. *Let (a,b),(c,d) be rational numbers, and let N be their least common denominator. Define the quantities*

$$A = (d-b)N, \quad B = (b-c)N, \quad C = (a-d)N, \quad D = dN. \tag{22}$$

Define the Euler curve attached to $(a,b),(c,d)$ as the affine curve over $\mathbb{Q}(z)$ with equation

$$\mathcal{C} : y^N = x^A(1-x)^B(1-zx)^C z^D. \tag{23}$$

Definition 6.2. *Let $(a,b),(c,d)$ be a pair of generic rationals parameters. The parameters satisfy condition* (**Irr**) *if the Euler curve (23) is irreducible over $\overline{\mathbb{Q}(z)}$.*

Lemma 6.3. *Let (a,b),(c,d) be generic rational numbers and let N be their least common denominator. Then condition* (**Irr**) *holds if and only if*

$$N = \operatorname{lcm}\{\operatorname{den}(d-b), \operatorname{den}(b-c), \operatorname{den}(a-d)\}. \tag{24}$$

Remark 6.4. If r denotes the quotient of N by $\operatorname{lcm}\{\operatorname{den}(d-b), \operatorname{den}(b-c), \operatorname{den}(a-d)\}$, then the curve $\mathcal{C}$ decomposes as the union of r irreducible components defined over $\mathbb{Q}(\zeta_r)[\sqrt[r]{z}]$. This phenomenon does not appear when $d = 1$ since then condition (**Irr**) always holds.

Example 2. Let $a = 1/8, b = 7/8, c = 3/8, d = 5/8$ so $N = 8$. The Euler curve is defined by

$$\mathcal{C}: \quad y^8 = x^6(1-x)^4(1-zx)^4z^5.$$

Over $\mathbb{Q}(w)$, where $w^2 = z$, it becomes the union of the curves

$$\mathcal{C}_\pm: \quad y^4 = \pm wx^3(1-x)^2(1-w^2x)^2.$$

The map $\iota : (x, y) \mapsto (x^{-1}, yx^{-3})$ yields an involution of these curves and it is not too hard to see that the quotient of $\mathcal{C}_\pm$ by ι equals

$$E_\pm : y^2 = x^3 - 4wx^2 \mp w(w-1)^2x.$$

For $d \in \mathbb{Z}$, let θ_d be the quadratic character of the extension $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$. It can be verified (using the involution $(x, y) \to (x^{-1}, -yx^{-3})$) that actually

$$\mathrm{Jac}(\mathcal{C}_\pm) \simeq E_\pm(w) \oplus E_\pm(w) \otimes \theta_{-1}.$$

Definition 6.5. *Let N be a positive integer, and let $\varkappa_N$ be the character of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_N))$ *(of order dividing 2) corresponding to the abelian extension* $\mathbb{Q}(\zeta_{2N})/\mathbb{Q}(\zeta_N)$.

Lemma 6.6. *Let $\mathfrak{p}$ be a prime ideal of $F = \mathbb{Q}(\zeta_N)$ not dividing $2N$. Then*

$$\varkappa_N(\mathfrak{p}) = \chi_{\mathfrak{p}}(-1) = (-1)^{(\mathcal{N}\mathfrak{p}-1)/N}, \tag{25}$$

where $\chi_{\mathfrak{p}}$ is defined in (7).

Proof. The second equality follows from the fact that $\chi_{\mathfrak{p}}(-1) \equiv (-1)^{(\mathcal{N}\mathfrak{p}-1)/N} \bmod \mathfrak{p}$ by definition. Let $t = v_2(N) \geq 0$. Let $\mathcal{N}\mathfrak{p} = p^r$ for p a rational prime number, so r is the order of p modulo N. Then

$$(-1)^{(\mathcal{N}\mathfrak{p}-1)/N} = (-1)^{(p^r-1)/2^t}.$$

Then the right hand side equals 1 if $p^r \equiv 1 \pmod{2^{t+1}}$ and -1 otherwise. By the Chinese remainder theorem it equals 1 if $p^r \equiv 1 \pmod{2N}$ and -1 otherwise; note that $p^r \equiv 1 \pmod{2N}$ precisely when $\mathfrak{p}$ splits in $\mathbb{Q}(\zeta_{2N})/F$. □

Corollary 6.7. *Suppose that $t = v_2(N) \geq 1$. Let κ be a primitive Dirichlet character of conductor 2^{t+1} thought as a character of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. *Then the quadratic character $\varkappa$ equals the restriction of κ to* $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$.

Proof. The fixed field of the character κ corresponds to the Galois extension $\mathbb{Q}(\zeta_{2^{t+1}})$. Then the restriction of κ to $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ has as fixed field $\mathbb{Q}(\zeta_{2N}) = \mathbb{Q}(\zeta_N) \cap \mathbb{Q}(\zeta_{2^{t+1}})$. □

Let $H^1_{\chi_0}(\widehat{\mathcal{C}}, F)^{\mathrm{new}}$ denote the χ_0 isotypical component of the de Rham cohomology group $H^1(\widehat{\mathcal{C}}, F) = H^1(\widehat{\mathcal{C}}, \mathbb{Q}) \otimes_{\mathbb{Q}} F$, where $\chi_0 : \boldsymbol{\mu} \to F^\times$ is the identity character. It is known to be of rank two (see [3] formula (16)).

Definition 6.8. *Let $(a,b),(c,d)$ be a pair of generic rational numbers satisfying condition* (**Irr**) *and let N be their least common denominator. The hypergeometric motive $\mathcal{H}((a,b),(c,d)|z)$ is defined as*

$$H^1_{\chi_0}(\widehat{\mathcal{C}},F)^{new}\otimes \mathbf{J}((-a,-b,c,d),(c-b,d-a))^{-1}\varkappa_N^{(d-b)N}. \tag{26}$$

Example 3. Let $\boldsymbol{\alpha}=(1/2,1/2)$ and $\boldsymbol{\beta}=(1,1)$ as in Example 1. Then the Euler curve corresponds to Legendre's elliptic curve

$$E_z: y^2=x(1-x)(1-zx).$$

The Jacobi motive $\mathbf{J}((-1/2,-1/2,1,1),(1/2,1/2))$ is trivial. Then the hypergeometric motive $\mathcal{H}((1/2,1/2),(1,1)|z)$ corresponds to the quadratic twist by the character $\left(\frac{-4}{\cdot}\right)$ of Legendre's elliptic curve.

Formula (26) involves three different motives, with a priori different fields of definition.

Example 4. Let $a=\frac{1}{15},b=\frac{4}{15},c=\frac{1}{8}$ and $d=\frac{3}{8}$. The Jacobi motive for these parameters is $\mathbf{J}((\frac{14}{15},\frac{11}{15},\frac{1}{8},\frac{3}{8}),(\frac{103}{120},\frac{37}{120}))$. It is defined over $\mathbb{Q}(\zeta_{120})^{\sigma_{19}}$ while the base field of $\mathcal{H}((\frac{1}{15},\frac{4}{15}),(\frac{1}{8},\frac{3}{8})|z)$ corresponds to the field fixed by $H=\langle 49,91\rangle$. There is no relation between both fields.

To define $\mathcal{H}((a,b),(c,d)|z)$ when (**Irr**) does not hold we need an extra twist.

Definition 6.9. *Let $\alpha=\frac{a}{N}$ be a rational number, with a,N coprime integers. Define*

$$\eta_\alpha:\mathrm{Gal}(\overline{\mathbb{Q}(z)}/\mathbb{Q}(z,\zeta_N))\to\overline{\mathbb{Q}}^\times$$

to be the character that factors through $\mathrm{Gal}(\mathbb{Q}(\sqrt[N]{z},\zeta_N)/\mathbb{Q}(z,\zeta_N))$ *and whose value at σ equals*

$$\eta_\alpha(\sigma)=\left(\frac{\sigma(\sqrt[N]{z})}{\sqrt[N]{z}}\right)^a.$$

Let $\xi\in F=\mathbb{Q}(\zeta_N)$, $\xi\neq 0$ and consider the specialization of η_α at $z=\xi$. Let $\mathfrak{p}$ be a prime ideal of F of norm q, not dividing N and such that $v_{\mathfrak{p}}(\xi)=0$. Let $\widetilde{\mathfrak{p}}$ be a prime ideal of $\mathbb{Q}(\sqrt[N]{\xi},\zeta_N)$ dividing $\mathfrak{p}$. The specialization of η_α at ξ satisfies

$$\eta_\alpha(\mathrm{Frob}_{\mathfrak{p}})=\chi_{\mathfrak{p}}(\xi)^a, \tag{27}$$

where $\chi_{\mathfrak{p}}$ is the character defined in (7).

Definition 6.10. *Let $(a,b),(c,d)$ be generic rational numbers, and let N be their least common denominator. Set $F=\mathbb{Q}(\zeta_N)$. The hypergeometric motive $\mathcal{H}((a,b),(c,d)|z)$ is defined by*

$$\mathcal{H}((a-d,b-d),(c-d,1)|z)\otimes\mathbf{J}((-a,-b,c,d),(d-a,d-b,c-d))^{-1}\eta_d(z), \tag{28}$$

where the hypergeometric motive $\mathcal{H}((a-d,b-d),(c-d,1)|z)$ (base changed to $F(z)$ if necessary) is defined in 6.8.

This second definition of $\mathcal{H}((a,b),(c,d)|z)$ has the disadvantage of making it more difficult to understand its descent to subfields of F.

Example 5. Continuing with Example 2, take $a = 1/8, b = 7/8, c = 3/8, d = 5/8$, so $N = 8$. The motive $\mathcal{H}((1/8,7/8),(3/8,5/8)|z)$ has base field $\mathbb{Q}(\sqrt{2})$, but the curve $\mathcal{C}$ is reducible.

The values $\tilde{a} = a - d = 1/2$, $\tilde{b} = b - d = 1/4$ and $\tilde{c} = c - d = 3/4$ have denominator 4. The Euler curve of parameters $(\tilde{a},\tilde{b}),(\tilde{c},1)$ has the affine equation

$$\widetilde{\mathcal{C}} : y^4 = x^3(1-x)^2(1-zx)^2.$$

It is an irreducible curve isomorphic to a twist of an irreducible component of $\mathcal{C}$. The motive $\mathcal{H}((1/2,1/4),(3/4,1)|z)$ has base field $\mathbb{Q}(i)$ and it is not clear how to descend its twist to $\mathbb{Q}(\sqrt{2})(z)$.

Theorem 6.11. *Let $(a,b),(c,d)$ be generic rational parameters and let N be their least common denominator. Let L be a finite extension of F. Let $\xi \in L$ with $\xi \neq 0,1$. Let $\mathfrak{p}$ be a prime ideal of L satisfying that $\mathfrak{p} \nmid N$ and $v_{\mathfrak{p}}(\xi(\xi-1)) = 0$. Then the trace of the Frobenius automorphism $\mathrm{Frob}_{\mathfrak{p}}$ acting on $\mathcal{H}((a,b),(c,d)|\xi)$ equals $H_{\mathfrak{p}}((a,b),(c,d)|\xi)$.*

Before proving the statement, recall that we gave two different definitions of $\mathcal{H}((a,b),(c,d)|\xi)$ depending on whether (**Irr**) holds or not. The result holds for both definitions.

Lemma 6.12. *Let $(a,b),(c,d)$ be rational generic parameters and let N be their least common denominator. Let $\mathfrak{p}$ be a prime ideal of L not dividing N and such that $v_{\mathfrak{p}}(\xi(\xi-1)) = 0$. Then*

$$H_{\mathfrak{p}}((a-d,b-d),(c-d,1)|\xi) = \chi_{\mathfrak{p}}(\xi)^{-dN}\,\mathbf{J}((-a,-b,c,d),(d-a,d-b,c-d))H_{\mathfrak{p}}((a,b),(c,d)|\xi). \tag{29}$$

Proof. Let $q = \mathcal{N}\mathfrak{p}$ and let ϖ be a generator of the character group $\widehat{\mathbb{F}_q^\times}$ so that $\varpi^{(q-1)/N} = \chi_{\mathfrak{p}}^{-1}$. Then [7, Theorem 3.4] implies that

$$H_{\mathfrak{p}}((a-d,b-d),(c-d,1)|\xi) = \varpi(\xi)^{d(q-1)}\frac{g(\chi_{\mathfrak{p}}^{-aN})g(\chi_{\mathfrak{p}}^{cN})}{g(\chi_{\mathfrak{p}}^{(d-a)N})g(\chi_{\mathfrak{p}}^{(c-d)N})}\frac{g(\chi_{\mathfrak{p}}^{-bN})g(\chi_{\mathfrak{p}}^{dN})}{g(\chi_{\mathfrak{p}}^{(d-b)N})g(1)}H_{\mathfrak{p}}((a,b),(c,d)|\xi). \tag{30}$$

□

Proof of Theorem 6.11. Suppose that (**Irr**) holds and let $\mathcal{C}$ denote the Euler curve as defined in (23). The hypothesis on $\mathfrak{p}$ implies (by Remark 5.1) that Assumption 1 holds. The genericity condition on the parameters implies that $a-c, b-c, a-d, b-d \notin \mathbb{Z}$, so $N \nmid A$, $N \nmid B$, $N \nmid C$ and $N \nmid \deg(f(x)) = A+B+C = (a-c)N$. By Theorem 5.4 the trace of Frobenius on $\mathrm{H}^1_{\mathrm{et},\chi}(\widehat{\mathcal{C}},\mathbb{Q}_\ell)$ equals

$$-\mathcal{N}(\chi_{\mathfrak{p}};\xi) = -\chi_{\mathfrak{p}}(\xi)^D\sum_{x\in\mathbb{F}_q}\chi_{\mathfrak{p}}(x^A(1-x)^B(1-\xi x)^C),$$

where $A = (d-b)N$, $B = (b-c)N$, $C = (a-d)N$ and $D = dN$. Set $\alpha_1 = (d-b)N$, $\alpha_2 = (d-a)N$, $\beta_1 = (d-c)N$ and $\beta_2 = 0$. Then Theorem 4.5 implies that

$$-\mathcal{N}(\chi_{\mathfrak{p}};\xi) = \chi_{\mathfrak{p}}(\xi)^D \chi_{\mathfrak{p}}(-1)^A \, \mathbf{J}((\alpha_1,-\beta_1),(\alpha_1-\beta_1)) H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|\xi). \tag{31}$$

By definition (4.3) the value $H_{\mathfrak{p}}(\boldsymbol{\alpha},\boldsymbol{\beta}|\xi)$ equals the value $H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|\xi)$ choosing a generator ϖ satisfying $\varpi^{(q-1)/N} = \chi_{\mathfrak{p}}^{-1}$, hence

$$H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|\xi) = H_{\mathfrak{p}}(((a-d)N,(b-d)N),((c-d)N,1)|\xi).$$

Then (31) equals

$$\chi_{\mathfrak{p}}(\xi)^{dN} \chi_{\mathfrak{p}}^{(d-b)N}(-1) \, \mathbf{J}((d-b,c-d),(c-b)) H_{\mathfrak{p}}\left((a-d,b-d),(c-d,1)|\xi\right).$$

Using Lemma 6.12 this value equals

$$\chi_{\mathfrak{p}}^{(d-b)N}(-1) \, \mathbf{J}((-a,-b,c,d),(c-b,d-a)) H_{\mathfrak{p}}\left((a,b),(c,d)|\xi\right),$$

which matches Definition 6.8 (recall that $\varkappa_N(\mathfrak{p}) = \chi_{\mathfrak{p}}(-1)$ by Lemma 6.6). To prove that $\mathcal{H}((a,b),(c,d)|\xi)$ also matches (28), we apply the proven result to the parameters $(a-d,b-d),(c-d,0)$ and are led to prove the equality

$$H_q((a,b),(c,d)|\xi) = H_q((a-d,b-d),(c-d,1)|\xi) \cdot \mathbf{J}((-a,-b,c,d),(d-a,d-b,c-d))^{-1} \eta_d(\xi).$$

which follows from Lemma 6.12 since $\eta_d(\mathrm{Frob}_{\mathfrak{p}})(\xi) = \chi_{\mathfrak{p}}(\xi)^{dN}$. □

Lemma 6.13. *Let L be a finite extension of F and let $\xi \in L$, with $\xi \neq 0,1$. The coefficient field of the motive $\mathcal{H}((a,b),(c,d)|\xi)$ (i.e. the field extension of $\mathbb{Q}$ generated by the trace of Frobenius elements) is contained in K.*

Proof. Follows from Theorem 6.11 and Proposition 4.2, part (4). □

As a corollary we get the following slightly softer version of Conjecture 2.7 (basically where the field K is replaced by F).

Corollary 6.14. *Let $\boldsymbol{\alpha} = (a,b), \boldsymbol{\beta} = (c,d)$ be rational generic parameters with denominator N. Let $F = \mathbb{Q}(\zeta_N)$ and $K = F^H$, with H as in (1). Then*

1. *The motive $\mathcal{H}(\boldsymbol{\alpha},\boldsymbol{\beta}|z)$ has a realization $\mathcal{H}(z)$ over $F(z)$.*
2. *The representation of the motive restricted to $\mathrm{Gal}(\overline{K(z)}/\overline{F}(z))$ is isomorphic to the geometric Galois representation (6).*
3. *Let L be a finite extension of F and $\xi \in L$. For primes $\mathfrak{p}$ of L of good reduction the trace of Frobenius at $\mathfrak{p}$ on $\mathcal{H}(\xi)$ is given by the finite hypergeometric sum $H_{\mathfrak{p}}(\boldsymbol{\alpha},\boldsymbol{\beta}\,|\,\xi)$.*
4. *The specialization $\mathcal{H}(\xi)$ for generic $\xi \in L$ has coefficient field K.*

Proof. The first two statements follow from Definitions 6.8 and 6.10 and Theorem 6.18; the third statement follows from Theorem 6.11, while the last one follows from the last lemma. □

6.1. **Hodge numbers and normalization.** The Hodge numbers of a rational hypergeometric motive can be computed using a formula conjectured by Corti and Golyshev (proved in [10, 14, 30]) by means of the *zig-zag diagram* (see §5 of [25]). Let us recall the procedure, noting that our motives tend to be non-effective. Assume that $\boldsymbol{\alpha}, \boldsymbol{\beta}$ are generic.

Algorithm 6.15. The zig-zag procedure

Set S the sequence of ordered parameters of $\boldsymbol{\alpha}$ as elements in $(0,1]$ and $\boldsymbol{\beta}$ as elements in $[0,1)$.
To each $s \in S$ associate the color red if $s \in \boldsymbol{\alpha}$ and blue if $s \in \boldsymbol{\beta}$.
$P \leftarrow 0$
for $i = 1 \dots 2n$ ***do***
  Draw a point with the color $S[i]$ at (i, P).
  if *$S[i]$ is blue* ***then***
    $P \leftarrow P - 1$
  else
    $P \leftarrow P + 1$
  end if
end for

Theorem 6.16. *Let $(a,b),(c,d)$ be generic rational parameters. Let r be the number of parameters among a,b,c,d that are in $\mathbb{Z}$. Then:*

(1) The motive $\mathcal{H}((a,b),(c,d)|z)$ is pure of weight $r-1$.

(2) For j coprime to N let $\sigma_j \in \mathrm{Gal}(F/\mathbb{Q})$ be the automorphism sending $\zeta_N \to \zeta_N^j$. Then the Hodge polynomial of $\mathcal{H}((a,b),(c,d)|z)$ for the embedding σ_j equals

$$\sum_P x^{-p_2} y^{p_2+r-1},$$

where the sum runs over the blue points $P = (p_1, p_2)$ of the zig-zag procedure applied to the parameters $(aj,bj),(cj,dj)$.

Proof. Suppose that (**Irr**) holds. By definition, the hypergeometric motive $\mathcal{H}((a,b),(c,d)|z)$ satisfies

$$\mathcal{H}((a,b),(c,d)|z) \otimes \mathbf{J}((-a,-b,c,d),(c-b,d-a)) = H^1_{\chi_0}(\widehat{\mathcal{C}}, F)^{\mathrm{new}} \varkappa_N^{(d-b)N}.$$

The space of differentials of the first kind on $\widehat{\mathcal{C}}$ has an action of $\boldsymbol{\mu}_N$ (the N-th roots of unity); the dimension of the ζ_N^j eigenspace (for $\gcd(j,N)=1$) is given by the formula

$$\left\{\frac{jA}{N}\right\} + \left\{\frac{jB}{N}\right\} + \left\{\frac{jC}{N}\right\} - \left\{\frac{j(A+B+C)}{N}\right\}, \tag{32}$$

where $\{a\}$ denotes the fractional part of a (see §4, formula (21) of [34]).

The motivic weight of the Jacobi motive equals $n = 2 - r$. The Hodge type (corresponding to the embedding σ_j) of the Jacobi motive is given (see (11)) by

$$p := \{-aj\} + \{-bj\} + \{cj\} + \{dj\} - \{(c-b)j\} - \{(d-a)j\}, \quad p + q = n.$$

Table 6.1 contains, for $a \leq b \in (0,1]$ and $c \leq d \in [0,1)$, all possible outcomes of the zig-zag procedure together with the Hodge polynomial of the Jacobi motive and of the Euler curve for the embedding $\sigma(\zeta_N) = \zeta_N$ in the case $r = 0$. The statement follows by comparing the last two columns. It is easy to

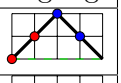
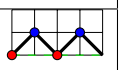
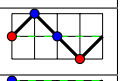
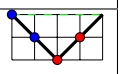
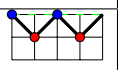
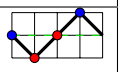
Case	Condition	$\mathbf{J}$	Euler	$\mathcal{H}((a,b),(c,d)\|z)$	ZigZag
I	$a \leq b < c \leq d$	x^2	$x+y$	$x^{-1}+x^{-2}y$	
II	$a < c < b < d$	xy	$2y$	$2x^{-1}$	
III	$a < c \leq d < b$	xy	$x+y$	$y^{-1}+x^{-1}$	
IV	$c \leq d < a \leq b$	y^2	$x+y$	$xy^{-2}+y^{-1}$	
V	$c < a < d < b$	xy	$2x$	$2y^{-1}$	
VI	$c < a \leq b < d$	xy	$x+y$	$x^{-1}+y^{-1}$	

TABLE 6.1. Possible Hodge diagrams in the rank 2 case

verify that the values of the columns (namely $\mathbf{J}$, Euler and $\mathcal{H}((a,b),(c,d)|z)$) remain the same while permuting (a,b) and (c,d) in cases I, III, IV and VI. In case II, when $b < c < a < d$ or when $a < d < b < c$, the Euler curve has Hodge polynomial $2x$ while $\mathbf{J}$ has Hodge polynomial x^2, so the value for $\mathcal{H}$ stays the same. A similar phenomenon occurs in case V.

When r is non-zero, the p-value of the Jacobi motive remains the same, but the motivic weight decreases by r hence the same occurs to q. If (**Irr**) does not hold the proof follows from a similar argument using (28). □

Remark 6.17. In the present article we do not make any additional Tate twist when considering hypergeometric motives unless explicitly mentioned, so our motives might not be *effective*. The reader should bear this in mind when matching a hypergeometric motive to an automorphic form.

6.2. **The Galois representation.** The motive $H^1_{\chi_0}(\widehat{\mathcal{C}}, F)^{\text{new}}$ has attached a family of 2-dimensional Galois representations indexed by prime ideals $\mathfrak{p}$ of F (as defined by Serre in [27]). Denote by

$$\rho_{\mathcal{H}((a,b),(c,d)|z),\mathfrak{p}} : \mathrm{Gal}(\overline{\mathbb{Q}(z)}/F(z)) \to \mathrm{GL}_2(F_{\mathfrak{p}}) \tag{33}$$

the representation of the hypergeometric motive $\mathcal{H}((a,b),(c,d)|z)$.

Theorem 6.18. *The Galois representation $\rho_{\mathcal{H}((a,b),(c,d)|z),\mathfrak{p}}$ restricted to $\mathrm{Gal}(\overline{\mathbb{Q}(z)}/\overline{\mathbb{Q}}(z))$ matches the geometric representation (6).*

Proof. The Galois representation attached to the Jacobi motive does not depend on z and its restriction to $\mathrm{Gal}(\overline{\mathbb{Q}(z)}/\overline{\mathbb{Q}}(z))$ is trivial. Then to prove the result it is enough to prove that the image under $\rho_{\mathcal{H}((a,b),(c,d)|z),\mathfrak{p}}$ of the

inertia group at $z=0$ (respectively $z=1$ and $z=\infty$) of $\overline{\mathbb{Q}}(z)$ is conjugate to the matrix M_0 (respectively M_1 and M_∞), which is proved in Theorems A.2 and A.4. □

Definition 6.19. *The hypergeometric motives $\mathcal{H}((a,b),(c,d)|z)$ and $\mathcal{H}((a',b'),(c',d')|z)$ are called* isogenous *if their Galois representations are isomorphic. We denote it by $\mathcal{H}((a,b),(c,d)|z) \simeq \mathcal{H}((a',b'),(c',d')|z)$.*

Lemma 6.20. *Let $\rho_i : \mathrm{Gal}(\overline{\mathbb{Q}(z)}/F(z)) \to \mathrm{GL}_2(\overline{\mathbb{Q}_p})$ for $i=1,2$ be two irreducible continuous representations satisfying that their restrictions to the subgroup $\mathrm{Gal}(\overline{\mathbb{Q}(z)}/\overline{\mathbb{Q}}(z))$ are irreducible and isomorphic. Then there exists $\chi : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \overline{\mathbb{Q}_p}^\times$ such that $\rho_1 \simeq \rho_2 \otimes \chi$.*

Proof. See [21, Corollary 2.5]. □

Theorem 6.21. *Let $(a,b),(c,d)$ be a pair of generic rational numbers. Then the following properties hold:*

(1) (Twist) With the above notation

$$\mathcal{H}((a,b),(c,d)|z) \simeq \mathcal{H}((a-d,b-d),(c-d,1)|z) \otimes \\ \otimes \mathbf{J}((-a,-b,c,d),(d-a,d-b,c-d))^{-1}\eta_d(z)$$

(2) (Permutation) Let $(a,b),(c,d)$ be generic parameters. Then

$$\mathcal{H}((a,b),(c,d)|z) \simeq \mathcal{H}((b,a),(c,d)|z) \simeq \\ \mathcal{H}((a,b),(d,c)|z) \simeq \mathcal{H}((b,a),(d,c)|z).$$

(3) (Inverse) $\mathcal{H}((a,b),(c,d)|z) \simeq \mathcal{H}((-c,-d),(-a,-b)|z^{-1})$.

Proof. Consider the Galois representation of each of the motives. By Theorem 6.18 its restriction to $\mathrm{Gal}(\overline{\mathbb{Q}(z)}/\overline{\mathbb{Q}}(z))$ is isomorphic to its respective monodromy representation. Using the description of the monodromy matrices given in Proposition 2.3 it is easy to verify that in each case the monodromy representations are isomorphic. The genericity hypothesis on the parameters implies that the monodromy representations are irreducible (by Proposition 2.2) so Lemma 6.20 implies that the two representations are isomorphic up to a twist by a character χ independent of the variable z. Let $\xi \in F$, $\xi \neq 0,1$. By Theorem 6.11 for any prime ideal $\mathfrak{p}$ of F not dividing N nor $\xi(\xi-1)$, the trace of the Frobenius element $\mathrm{Frob}_\mathfrak{p}$ matches the finite hypergeometric sum. The statements for finite hypergeometric sums is either trivial or follows from Proposition 4.2. □

6.3. On the coefficient field of the hypergeometric motive.

Theorem 6.22. *Let a,b,c,d be generic rational parameters such that $c \not\equiv d \bmod \mathbb{Z}$. Let N be their least common denominator. Let L be a finite extension of F. Let $\xi \in L$, $\xi \neq 0,1$, be such that there exists a prime ideal $\mathfrak{p}$ of L satisfying*

- $\mathfrak{p} \nmid N$,

- $r := v_{\mathfrak{p}}(\xi) > 0$ *and prime to* N.

Then both ζ_{c+d} *and* $\zeta_{-c}+\zeta_{-d}$ *belong to the coefficient field of* $\mathcal{H}((a,b),(c,d)|\xi)$. *Similarly, if* $a \not\equiv b$ *and there exists a prime ideal* $\mathfrak{p}$ *of* L *not dividing* N *such that* $v_{\mathfrak{p}}(\xi) < 0$ *and is prime to* N *then both* ζ_{a+b} *and* $\zeta_a + \zeta_b$ *belong to the coefficient field of* $\mathcal{H}((a,b),(c,d)|\xi)$.

Proof. Recall that the geometric representation attached to $\mathcal{H}((a,b),(c,d)|z)$ has monodromy matrices M_∞ conjugate to $\left(\begin{smallmatrix}\zeta_a & 0\\ 0 & \zeta_b\end{smallmatrix}\right)$ at ∞ if $a \not\equiv b \bmod \mathbb{Z}$ and M_0 conjugate to $\left(\begin{smallmatrix}\zeta_{-c} & 0\\ 0 & \zeta_{-d}\end{smallmatrix}\right)$ at 0 if $c \not\equiv d$.

Let $\mathfrak{p}$ be a prime ideal of L satisfying the hypothesis. Then by Corollary A.3 (see also [5, Theorem 1.2]) the image of inertia $I_{\mathfrak{p}}$ is generated by M_0^r (up to conjugation), so [12, Lemma 15] implies that the coefficient field contains both $\zeta_{r(c+d)}$ and $\zeta_{-rc}+\zeta_{-rd}$, which is equivalent to say that it contains ζ_{c+d} and $\zeta_{-c}+\zeta_{-d}$ (because the extension $\mathbb{Q}(\zeta_c,\zeta_d)/\mathbb{Q}$ is abelian). The second statement follows from a similar argument. □

Corollary 6.23. *With the previous hypothesis, let* $\xi \in L$ *be such that there exist prime ideals* $\mathfrak{p}$ *and* $\mathfrak{q}$ *of* L *prime to* N *satisfying:*

- $v_{\mathfrak{p}}(\xi) > 0$ *and prime to* N,
- $v_{\mathfrak{q}}(\xi) < 0$ *and prime to* N.

Then if (**Irr**) *holds and* $|H| = 4$, *the coefficient field of* $\mathcal{H}((a,b),(c,d)|\xi)$ *equals* K.

Proof. Let M denote the coefficient field. By Lemma 6.13, M is contained in K hence $[F : M] \geq |H|$. The hypothesis on ξ (and the last theorem) imply that $\mathbb{Q}(\zeta_{a+b}, \zeta_a+\zeta_b, \zeta_{c+d}, \zeta_c+\zeta_d) \subset M \subset K \subset F$, so $[F : M] \leq 4$ (because $F = \mathbb{Q}(\zeta_a,\zeta_b,\zeta_c,\zeta_d)$). Since $|H| = 4$, $M = K$. □

Example 6. The last corollary suggests that some genericity condition on the specialization parameter ξ is needed for the coefficient field to match K. Consider the case of parameters $(1/8, 7/8), (3/8, 5/8)$ (of Example 2). The group H equals $\langle -1\rangle \subseteq (\mathbb{Z}/8)^\times$, so for any specialization ξ of the parameter, the motive is defined over $\mathbb{Q}(\sqrt{2})$ (by Theorem 7.2). But in Example 2 we showed that when ξ is a square, the field of definition is $\mathbb{Q}$ (the motive actually corresponds to a rational elliptic curve).

7. Extension to K

Let H be the group (1) for generic parameters $(a,b),(c,d)$.

Lemma 7.1. *The group* H *is a subgroup of* $\mathbb{Z}/2 \times \mathbb{Z}/2$.

Proof. For S a finite set, let $\mathrm{Perm}(S)$ denote the group of bijective maps on S. Set $S = \{a,b\} \times \{c,d\}$ (we are not assuming $a \neq b$ nor $c \neq d$). Then there is a group morphism

$$\psi : H \to \mathrm{Perm}(\{a,b\}) \times \mathrm{Perm}(\{c,d\}),$$

sending an element h to the bijective map given by multiplication by h on the sets $\{a,b\}$ and $\{c,d\}$. It is easy to verify that ψ is injective. □

As stated in Conjecture 2.7, it is expected that $K = F^H$ is both the field of definition and the coefficient field of the motive $\mathcal{H}((a,b),(c,d)|z)$ for generic specializations.

Theorem 7.2. *Let $(a,b),(c,d)$ be generic parameters such that $a+b$ and $c+d$ are integers and* (**Irr**) *holds. Then $\mathcal{H}((a,b),(c,d)|z)$ descends to K and for any specialization its coefficient field is contained in K.*

Under the hypothesis we need to understand the three motives appearing in (26). We start proving some preliminary results.

Lemma 7.3. *Let $(a,b),(c,d)$ be generic rational parameters satisfying that $a+b$ and $c+d$ are integers. Let N be their least common denominator. Set*

$$\delta = \begin{cases} 1 & \text{if } a, c \notin \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathfrak{p}$ be a prime ideal of $\mathbb{Q}(\zeta_N)$ of norm q, not dividing N. Then

$$\mathbf{J}((-a,-b,c,d),(c-b,d-a))(\mathfrak{p}) = q^{\delta}.$$

Proof. Follows from the well known formula

$$g(\chi)g(\overline{\chi}) = \chi(-1)\begin{cases} q & \text{if } \chi \neq 1, \\ 1 & \text{if } \chi = 1. \end{cases}$$

□

Corollary 6.7 implies that $\varkappa_N$ matches the restriction to $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ of a character κ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of conductor 2^{t+1}.

Lemma 7.4. *If $(a,b),(c,d)$ are generic parameters such that $a+b$ and $c+d$ are integers, then the character κ is at most quadratic when restricted to $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$.*

Proof. We can assume that $t > 0$ (as otherwise $\varkappa_N$ and κ are trivial). Since $-1 \in H$ (complex conjugation), a simple Galois theory exercise shows that $\mathbb{Q}(\zeta_{2^{t+1}}) \cap F^H = \mathbb{Q}(\zeta_{2^t})^+$. Then κ restricted to $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ factors through $\mathrm{Gal}(\mathbb{Q}(\zeta_{2^{t+1}})/\mathbb{Q}(\zeta_{2^t})^+)$, a group isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$ if $t > 1$ and $\mathbb{Z}/2$ if $t = 1$, hence κ is at most quadratic. □

We are led to construct a motive $\mathfrak{M}$ defined over K whose base change to F is isomorphic to the Euler curve. The motive $\mathfrak{M}$ appears in the quotient of the Euler curve $\mathcal{C}$ by involutions that we now describe.

Proposition 7.5. *Let a,b,c,d be generic parameters such that $a+b = c+d = 0$. For $j \in H$ define a morphism $\iota_j : \mathcal{C} \to \mathcal{C}$ by*

- *If $j = -1$, $\iota_{-1}(x,y) = \left(\frac{1}{zx}, \frac{1}{y}\right)$.*

- *If* $ja = a + r$ *and* $jc = -c + s$, *for some* $r, s \in \mathbb{Z}$,
$$\iota_j(x,y) = \left(\frac{x-1}{zx-1}, \frac{x^{r-s}(1-x)^{-r-s}(1-xz)^{r+s}z^{-s}}{y^j} \right).$$
- *If* $ja = -a + r$ *and* $jc = c + s$, *for* $r, s \in \mathbb{Z}$,
$$\iota_j(x,y) = \left(\frac{zx-1}{z(x-1)}, \frac{x^{r-s}(1-x)^{-r-s}(1-xz)^{r+s}z^{-s}}{y^j} \right).$$

The map ι_j *is an involution of* $\mathcal{C}$ *and the new part of* $\mathcal{C}/\iota_j$ *has genus* $\phi(N)/2$.

Proof. Follows from a standard computation. □

Proof of Theorem 7.2. Let N be the least common denominator of the parameters a, b, c, d. Let $j \in H$ and let ι_j be the involution defined in Proposition 7.5. The new part of $\mathcal{C}/\iota_j$ has genus $\phi(N)/2$. Abusing notation, denote by ζ_N the map on $\mathcal{C}$ given by $\zeta_N(x,y) = (x, \zeta_N y)$. It follows easily from its definition that

$$\iota_j \circ \zeta_N = \begin{cases} \zeta_N^{-j} \circ \iota_j & \text{if } j \neq -1, \\ \zeta_N^{-1} \circ \iota_j & \text{if } j = -1. \end{cases} \tag{34}$$

Then over the field $F^{\langle j \rangle}$ there is an action of the ring $\mathbb{Z}[\zeta_N + \zeta_N^{-j}]$ (respectively $\mathbb{Z}[\zeta_N + \zeta_N^{-1}]$) on $H^1(C/\iota, F)$. Looking at the χ_0 isotypical component, we get a 2-dimensional representation with coefficient field $F^{\langle j \rangle}$. If H is cyclic, this concludes the proof; when H is not cyclic, take the quotient by two different involutions attached to elements on H (getting a representation over F^H). □

Similarly one can prove the following result.

Theorem 7.6. *Let* $(a,b),(c,d)$ *be generic parameters such that* (**Irr**) *holds. Suppose furthermore that* $H = \langle j \rangle$ *where the action of multiplication by* j *is not trivial on both sets* $\{a,b\}$ *and* $\{c,d\}$. *Then* $\mathcal{H}((a,b),(c,d)|z)$ *descends to* F^H *and its coefficient field is contained in* F^H.

7.1. **Extension of specializations.**

Theorem 7.7. *Let* $(a,b),(c,d)$ *be generic parameters with* $|H| \leq 2$. *Then for all* $\xi \in K$, $\xi \neq 0, 1$, *the Galois representation of* $\mathcal{H}((a,b),(c,d)|\xi)$ *extends to* Gal_K *and its coefficient field is contained in a quadratic extension of* K.

Proof. Start assuming that $\rho_{\mathcal{H}((a,b),(c,d)|\xi),\mathfrak{p}}$ is absolutely irreducible. Since H is cyclic, [11, Lemma 3.4] implies that it is enough to prove that for all $\sigma \in H$, the representation $\rho^{\sigma}_{\mathcal{H}((a,b),(c,d)|\xi),\mathfrak{p}}$ is isomorphic to $\rho_{\mathcal{H}((a,b),(c,d)|\xi),\mathfrak{p}}$, or equivalently (since $\rho_{\mathcal{H}((a,b),(c,d)|\xi),\mathfrak{p}}$ is irreducible) that for all prime ideals $\mathfrak{q}$ of F on a density one set both representations have the same trace at $\mathrm{Frob}_{\mathfrak{q}}$. By Theorem 6.11 and Proposition 4.2, if $\sigma(\zeta_N) = \zeta_N^j$, it amounts to verify that

$$H_{\mathfrak{q}}((a,b),(c,d)|\xi) = H_{\mathfrak{q}}((ja,jb),(jc,jd)|\xi),$$

which follows from the definition of H.

If the representation is reducible, then there exists Hecke characters χ_1, χ_2 of Gal_F such that (up to semisimplification)

$$\rho^{\mathrm{ss}}_{\mathcal{H}((a,b),(c,d)|\xi),\mathfrak{p}} \simeq \chi_{1,\mathfrak{p}} \oplus \chi_{2,\mathfrak{p}}.$$

Let σ be the non-trivial element of $\mathrm{Gal}(F/K)$. Either ${}^{\sigma}\chi_1 = \chi_1$ or ${}^{\sigma}\chi_1 = \chi_2$. In the first case, both characters χ_i, $i = 1, 2$, extend to Gal_K, so the extension is defined as the sum of the two extended characters. In the second case, the extension is defined as the induced representation $\mathrm{Ind}_{\mathrm{Gal}_F}^{\mathrm{Gal}_K} \chi_1$ (which is isomorphic to the induction of χ_2). □

Corollary 7.8. *If $a+b$ and $c+d$ are integers, then for $\xi \in K$, $\xi \neq 0, 1$, the Galois representation attached to $\mathcal{H}((a,b),(c,d)|\xi)$ extends to Gal_K.*

Proof. If (**Irr**) holds, the result follow from Theorem 7.2. Otherwise we claim that $H = \{\pm 1\}$ so the result follows from Theorem 7.7.

To prove the claim, by Lemma 6.3 the values $N := \mathrm{lcm}\{\mathrm{den}(a), \mathrm{den}(c)\}$ and $N' := \mathrm{lcm}\{\mathrm{den}(a+c), \mathrm{den}(a-c)\}$ are different so $N = 2N'$. This implies that $v_2(\mathrm{den}(a)) = v_2(\mathrm{den}(c)) \geq 1$. If both valuations are 1, then we can take a quadratic twist of the motive (corresponding to adding $1/2$ to all coefficients) and make all denominators odd, so (**Irr**) holds. Then the sought representation is obtained by taking the quadratic twist of the new motive. Suppose then that $v_2(\mathrm{den}(a)) > 1$. If $j \in H$ and $j \neq \pm 1$ then $ja = a \pmod{\mathbb{Z}}$ and $jc = -c \pmod{\mathbb{Z}}$ or vice-versa. But then $v_2(j-1) > 1$ and $v_2(j+1) > 1$, a contradiction. □

8. Hypergeometric Motives over totally real fields

According to Conjecture 2.7 the motive $\mathcal{H}((a,b),(c,d)|z)$ is defined over a totally real field precisely when $-1 \in H$; i.e. when

$$\{a, b\} = \{-a, -b\}, \qquad \{c, d\} = \{-c, -d\}.$$

For a rational number a, the relation $a \equiv -a \pmod{\mathbb{Z}}$ implies that either $a \in \mathbb{Z}$ or $a \in \frac{1}{2} + \mathbb{Z}$, so we can separate (up to a quadratic twist obtained by adding $1/2$ to all parameters) totally real motives into two disjoint cases:

(1) Motives with parameters $(a,b),(c,d)$ with $a+b \in \mathbb{Z}$ and $c+d \in \mathbb{Z}$,
(2) Motives with parameters $(a,-a),(1/2,1)$ or $(1/2,1),(c,-c)$.

8.1. **Case (1).** We start by computing the Hodge polynomials as explained in §6.1 (using Theorem 6.16). If two of the parameters are integers, then for all embeddings $\sigma : K \hookrightarrow \mathbb{C}$, the zigzag procedure gives case IV in Table 6.1 so the Hodge polynomial equals $x + y$. Then for any $\xi \in K \setminus \{0, 1\}$, the Galois representation of Gal_K attached to $\mathcal{H}((a,b),(c,d)|\xi)$ has Hodge-Tate weights $\{0, 1\}$.

When no parameter is an integer, the zig-zag procedure gives case III when $a < c$ and case VI when $a > c$ of Table 6.1. Both cases have Hodge polynomial $x^{-1} + y^{-1}$ so for any $\xi \in K \setminus \{0, 1\}$ the Galois representation of Gal_K attached to $\mathcal{H}((a,b),(c,d)|\xi)$ has Hodge-Tate weights $\{-1, 0\}$.

8.2. **Case (2).** The zig-zag procedure (for all embeddings $\sigma : K \hookrightarrow \mathbb{C}$) corresponds to case V of the Table 6.1. Its Hodge polynomial is 2, so it should correspond to a parallel weight 1 modular form. This is indeed the case, since such motives have finite monodromy as studied by Schwarz (in [26]), corresponding to dihedral projective image. A detailed description will be given in a subsequent article.

Theorem 8.1. *Let $(a,b),(c,d)$ be generic rational parameters, and let $\xi \in K$, $\xi \neq 0,1$. Suppose that $a+b$ and $c+d$ are integers. Then the Galois representation of the motive $\mathcal{H}((a,b),(c,d)|\xi)$ extended to Gal_K is irreducible.*

Proof. Let $\widetilde{\rho}$ denote the extension. Suppose it is reducible. Let J^{new} denote the abelian variety corresponding to the new part of the Jacobian of the Euler curve used to define the motive. Then (up to semisimplification) the Galois representation of J^{new} decomposes as

$$\rho^{\mathrm{ss}}_{J^{\mathrm{new}},\mathfrak{p}} \simeq \theta_1 \oplus \cdots \oplus \theta_{2s},$$

for θ_i, $1 \leq i \leq 2s$, 1-dimensional representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ (or equivalently Hecke characters, by class field theory) and $s = \varphi(N)$. Let $\sigma : F \to \mathbb{C}$ be an embedding. Then the σ-infinity type of each θ_i is either $(1,0)$ or $(0,1)$.

Recall that any Hecke character of the idèle group of a totally real field is (up to a finite order character) a multiple of the norm character. Then

$$\widetilde{\rho}^{\mathrm{ss}}_{\mathfrak{p}} \simeq \varepsilon_1 \oplus \chi_p^{\pm 1}\varepsilon_2,$$

for $\varepsilon_1, \varepsilon_2$ finite order characters of the idèle group of K, where χ_p denotes the p-cyclotomic character. Then there exists a Hecke character η (corresponding to the Jacobi motive) such that

$$\eta\theta_i = \varepsilon_1, \qquad \eta\theta_j = \chi_p^{\pm 1}\varepsilon_2,$$

for some $1 \leq i,j \leq 2d$. But the σ-infinity type of ε_1 is $(0,0)$ and that of $\varepsilon_2\chi_p$ is $(1,1)$, so the character η cannot exist. □

Theorem 8.2. *Let (a,b) and (c,d) be generic rational parameters satisfying that $a+b$ and $c+d$ are both integers. Then for $\xi \in K$, $\xi \neq 0,1$, the motive $\mathcal{H}((a,b),(c,d)|\xi)$ is potentially modular. In particular, its L-series extends meromorphically to the whole complex plane.*

Proof. Let K be the field of coefficients of the motive $\mathcal{H}((a,b),(c,d)|\xi)$ and let $\{\rho_{\mathfrak{p}}\}_{\mathfrak{p}}$ be the strictly compatible system of Galois representations attached to the motive. Up to a Tate twist, the Hodge-Tate weights of the family are all $\{0,1\}$. For a density 1 set of primes of K, the residual representation $\overline{\rho_{\mathfrak{p}}}|_{\mathrm{Gal}(\overline{\mathbb{Q}}/K(\zeta_p))}$ is absolutely irreducible (by [4, Proposition 5.3.2]). Then by [29, Theorem 1.1.1] the system is potentially modular. □

Remark 8.3. Compare the result with [23, Corollary 2.7], where the authors prove potentially modularity over a CM extension of F.

9. Congruences

Definition 9.1. *Let $\mathcal{H}(\boldsymbol{\alpha},\boldsymbol{\beta}\,|\,\xi)$ and $\mathcal{H}(\boldsymbol{\alpha}',\boldsymbol{\beta}'\,|\,\xi)$ be two hypergeometric motives. Let L be a common coefficient field and let $\mathfrak{l}$ be a prime ideal of L. $\mathcal{H}(\boldsymbol{\alpha},\boldsymbol{\beta}\,|\,\xi)$ is* congruent *to $\mathcal{H}(\boldsymbol{\alpha}',\boldsymbol{\beta}'\,|\,\xi)$ over a number field K modulo $\mathfrak{l}$ if their Galois representations restricted to $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ are.*

Definition 9.2. *Let ℓ be a rational prime. Define on $\mathbb{Q}/\mathbb{Z}$ the relation defined by the condition that the denominator of $a-b$ is an ℓ-th power. Denote it by $a\sim_\ell b$.*

It is easy to verify that the relation $\sim_\ell$ is an equivalence relation. Extend the definition to $(\mathbb{Q}/\mathbb{Z})^n$ component-wise.

Theorem 9.3. *Let $\boldsymbol{\alpha},\boldsymbol{\beta}$ and $\boldsymbol{\alpha}',\boldsymbol{\beta}'$ be generic such that $\boldsymbol{\alpha}\sim_\ell\boldsymbol{\alpha}'$ and $\boldsymbol{\beta}\sim_\ell\boldsymbol{\beta}'$ for some prime ℓ. Let M be the lcm of the denominator of the parameters. Then for any $\xi\neq 0,1$, $\mathcal{H}(\boldsymbol{\alpha},\boldsymbol{\beta}\,|\,\xi)$ is congruent to $\mathcal{H}(\boldsymbol{\alpha}',\boldsymbol{\beta}'\,|\,\xi)$ over $\mathbb{Q}(\zeta_M)$ modulo any prime ideal $\mathfrak{l}$ dividing ℓ.*

Proof. By Theorem 6.11 the trace of both hypergeometric motives over $\mathbb{Q}(\zeta_M)$ at a prime $\mathfrak{p}$ of good reduction match a finite hypergeometric sum. From its very definition (equation (12)) the finite hypergeometric sum is a sum and product/quotients of Gauss sums of the form $g(m+\alpha(q-1),\mathfrak{p})$, where α is one of the parameters (and $q=p^r$ is the order of the residue field $\mathcal{O}_F/\mathfrak{p}$). To prove the statement it is enough to verify that if $\alpha\sim_\ell\beta$ then $g(m+\alpha(q-1),\mathfrak{p})\equiv g(m+\beta(q-1),\mathfrak{p}) \pmod{\mathfrak{l}}$. Recall that if ω denotes a generator of the multiplicative group $\mathbb{F}_q^\times$, then

$$g(m+\alpha(q-1)) = \sum_{x\in\mathcal{O}_F/\mathfrak{p}} \omega^m(x)\omega^{\alpha(q-1)}(x)\psi(x), \tag{35}$$

where ψ is an additive character of $\mathbb{F}_q^\times$. The last sum is an element of $\mathbb{Z}[\zeta_p,\zeta_{q-1}]$. Let $\mathfrak{l}$ be a prime ideal in this ring dividing ℓ. Then

$$\frac{\omega^{\alpha(q-1)}(x)}{\omega^{\beta(q-1)}(x)} = \omega^{(\alpha-\beta)(q-1)}(x) = \omega^{(q-1)/\ell^r}(x),$$

for some non-negative integer r. In particular, it is a root of unity whose order divides ℓ^r, so congruent to 1 modulo $\mathfrak{l}$. This implies that $g(m+\alpha(q-1))\equiv g(m+\beta(q-1)) \pmod{\mathfrak{l}}$ as claimed. □

Appendix A. Tame inertia of a Hypergeometric Motive

by Elisa Lorenzo García and Ariel Pacetti

Let $\mathcal{O}$ be a Henselian discrete valuation ring, let π be a local uniformizer, let K be its field of fractions and let k its residue field. For the present applications, have in mind the case in which $\mathcal{O}=k(z)$ is the coordinate ring of $\mathbb{P}^1_k$. Keeping the article's notation, let (a,b), (c,d) be a pair of generic parameters and let N be their least common denominator. Set

$$A=(d-b)N,\qquad B=(b-c)N,\qquad C=(a-d)N,\qquad D=dN. \tag{36}$$

Define the Euler curve $\mathcal{C}$ over $\mathcal{O}$ by

$$\mathcal{C} : y^N = x^A(1-x)^B(1-\pi x)^C\pi^D. \tag{37}$$

Assume for the rest of the appendix that the following two conditions hold:

- The curve $\mathcal{C}$ is irreducible (i.e. (**Irr**) holds).
- The residual characteristic of K ($\mathrm{char}(k)$) does not divide N.
- K contains the N-th roots of unity.

Fix μ a root of unity in K (a generator of $\boldsymbol{\mu}_N$) and a primitive N-th root of unity ζ in $\mathbb{C}$. Let $\mathcal{Y}^{\mathrm{st}}$ denote the semistable model of $\mathcal{C}$ and let $\mathcal{A}$ be the abelian variety $\mathrm{Pic}^0(\mathcal{Y}^{\mathrm{st}})$. Let $\mathcal{A}_0$ denote its special fiber. It contains a toric part T_0 and an abelian part $\mathcal{B}_0$ so that

$$\mathcal{A}_0^0/T_0 \simeq \mathcal{B}_0$$

(see [8, Example 8, page 246] and also [17, Expose IX, equation (2.1.2)]). Let $n = \dim \mathcal{A}$, $t = \dim T_0$ and $a = \dim \mathcal{B}_0$ so that $n = t + a$.

Let $\ell \equiv 1 \pmod N$ be a prime number different from the characteristic of K (so $\mathbb{Q}_\ell$ contains the N-th roots of unity). Let $T_\ell(\mathcal{A})$ denote the Tate module of the generic fiber of $\mathcal{A}$. Since $\mathcal{Y}^{\mathrm{st}}$ is semistable, the action of the inertia group I_K on $T_\ell(\mathcal{A})$ is unipotent with echelon rank 2, i.e. there exists a subspace U such that I_K acts trivially on U and on $T_\ell(\mathcal{A})/U$ (see [17, Proposition 3.5], page 350).

Let Sp_2 denote the $\mathbb{Q}_\ell[I_K]$-module of dimension 2 where the action of the inertia group I_K factors through the maximal pro-ℓ-group $I_K(\ell)$, and a generator g acts like the matrix $\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$. Let $\Upsilon = (V, E)$ denote the dual graph of $\overline{\mathcal{Y}^{\mathrm{st}}}$ (the special fiber of $\mathcal{Y}^{\mathrm{st}}$). Then μ (the rank of the toric part T_0) equals the rank of $\mathrm{H}^1(\Upsilon, \mathbb{Z})$ and there is an isomorphism of $\mathbb{Q}_\ell[I_K]$-modules

$$T_\ell(\mathcal{A}) \simeq \left(\left(\mathrm{H}^1(\Upsilon, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell\right) \otimes \mathrm{Sp}_2\right) \oplus \bigoplus_{\tilde{Y} \in V} T_\ell(\mathrm{Pic}^0(\tilde{Y})), \tag{38}$$

where the sum runs over the irreducible components of $\overline{\mathcal{Y}^{\mathrm{st}}}$. To describe the action of I_K on $T_\ell(\mathcal{A})$ it suffices to compute the semistable model of $\mathcal{C}$ and its dual graph.

The (**Irr**) assumption and the generic condition on the parameters imply that

(i) $\gcd(N, A, B, C) = 1$,
(ii) $N \nmid A$, $N \nmid B$, $N \nmid C$ and $N \nmid A + B + C$.

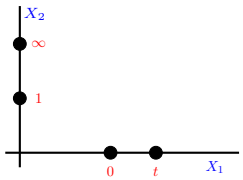
A simple change of variables transforms equation (37) into

$$\pi^{A+B-D} y^N = x^A(\pi - x)^B(1-x)^C. \tag{39}$$

Up to a twist (coming from the change of variables $y \to \pi^{(A+B-D)/N} y$), it is enough to study the curve

$$\widetilde{\mathcal{C}} : y^N = x^A(\pi - x)^B(1-x)^C. \tag{40}$$

The defining polynomial has now roots $0, 1, \pi$, so its cluster picture looks like (see [13]). Then the decorated graph of $(\mathcal{X}, \mathcal{D})$ equals



Define the following quantities:

- $d_1 := \gcd(N, C, A+B)$, $d_2 := \gcd(N, A, B)$.
- $e := \gcd(N, A+B)$, $t := \frac{e}{d_1 d_2}$.

It is clear that $d_1 \mid e$ and $d_2 \mid e$. Condition (i) implies that d_1 and d_2 are coprime, hence $t \in \mathbb{Z}$. If $d \mid N$, we denote by ζ_d the root d-th root of unity $\zeta^{N/d}$.

The special fiber $\overline{\mathcal{Y}^{\mathrm{st}}}$ has d_1 components over $\overline{X_1}$ (setting $x_1 = x$, $y_1 = y$) given by

$$\mathcal{C}_1^{(i)} : y_1^{\frac{N}{d_1}} = \zeta_{2d_1}^{B+2i} x_1^{\frac{A+B}{d_1}} (1-x_1)^{\frac{C}{d_1}}, \qquad i = 0, \dots, d_1 - 1, \tag{41}$$

corresponding to the irreducible components of the curve

$$\mathcal{C}_1 : y_1^N = (-1)^B x_1^{A+B} (1-x_1)^C.$$

Over $\overline{X_2}$ the special fiber has d_2 components, setting $x_2 = \pi x$ and $y = \pi^{\frac{A+B}{N}} y_2$, they are given by the equation

$$\mathcal{C}_2^{(j)} : y_2^{\frac{N}{d_2}} = \zeta_{d_2}^{i} x_2^{\frac{A}{d_2}} (1-x_2)^{\frac{B}{d_2}}, \qquad j = 0, \dots, d_2 - 1, \tag{42}$$

corresponding to the irreducible components of the curve

$$\mathcal{C}_2 : y_2^N = x_2^A (1-x_2)^B.$$

The components $\mathcal{C}_1^{(i)}$ and $\mathcal{C}_2^{(j)}$ intersect at e different points as in Figure 2.

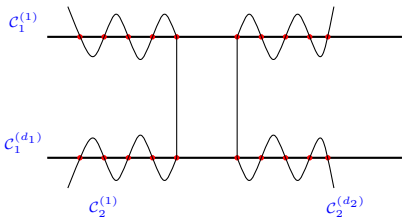


FIGURE 2. Intersection of Special Fiber of $\mathcal{Y}^{\mathrm{st}}$

Denote by $\widetilde{\mathcal{A}}$ the Jacobian of $\widetilde{\mathcal{C}}$ and by $\widetilde{\mathcal{A}}^{\mathrm{new}}$ its new part. Let η denote the character of Gal_K corresponding to the extension $K(\sqrt[N]{t})/K$, and let θ be the character corresponding to the (at most quadratic) extension $K(\sqrt[N]{-1})/K$.

Theorem A.1. *Let ℓ be a prime number different from the characteristic of K.*

- *If $N \nmid A+B$, set $r_i = \frac{\varphi(N)}{\varphi(N/d_i)}$ for $i = 1, 2$. Then*

$$T_\ell(\widetilde{\mathcal{A}}^{new}) \simeq T_\ell(\mathcal{A}_1)^{r_1} \oplus T_\ell(\mathcal{A}_2)^{r_2} \otimes \eta^{-(A+B)},$$

  where $\mathcal{A}_1$ and $\mathcal{A}_2$ are abelian varieties with complex multiplication over K.
- *Otherwise, as $\mathbb{Q}_\ell[I_K]$-modules*

$$T_\ell(\widetilde{\mathcal{A}}^{new}) \simeq \bigoplus_{i=1}^{\varphi(N)} \mathrm{Sp}_2 \otimes \theta^B.$$

Proof. Start assuming that $N \nmid A+B$, so the curves $\mathcal{C}_1^{(i)}$ and $\mathcal{C}_2^{(j)}$ have positive genus. Let μ be a generator of $\boldsymbol{\mu}_N$. The action of μ gives an isomorphism between points of $\mathcal{C}_1^{(i)}$ and of $\mathcal{C}_1^{(i+1)}$ (for i an element of $\mathbb{Z}/d_1$). Then

$$T_\ell(\mathrm{Jac}(\mathcal{C}_1^1)) \simeq T_\ell(\mathrm{Jac}(\mathcal{C}_1^i)) \qquad \text{for all } i.$$

The same holds for $\mathcal{C}_2^{(j)}$. Let $\mathcal{A}_1$ (respectively $\mathcal{A}_2$) be the new part of $\mathrm{Jac}(\mathcal{C}_1^{(1)})$ (resp. $\mathrm{Jac}(\mathcal{C}_2^{(1)})$); $\mathcal{A}_1$ consists of the eigenspaces for the action of $\boldsymbol{\mu}_{\frac{N}{d_1}}$ whose eigenvalues are primitive N/d_1-th roots of unity.

Let v be an element of $\mathrm{Jac}(\mathcal{C}_1^{(1)})$ on the ζ^{d_1}-eigenspace for the action of μ^{d_1}. Let $i \in (\mathbb{Z}/N)^\times$ be an element congruent to 1 modulo N/d_1 and consider the element in $\mathrm{Jac}(\mathcal{C}_1) \simeq \bigoplus_j \mathrm{Jac}(\mathcal{C}_1^{(j)})$ defined by

$$w_i = \zeta^{i(d_1-1)} v + \mu \cdot \zeta^{i(d_1-2)} v + \cdots + \mu^{d_1-2} \cdot \zeta^i v + \mu^{d_1-1} \cdot v.$$

It is clear from its definition that $\mu \cdot w_i = \zeta^i w_i$ (since $\mu^{d_1} \cdot v = \zeta^{d_1} v$ and $i \equiv 1 \pmod{\frac{N}{d_1}}$), so w_i belongs to the ζ^i-eigenspace for the action of μ on $\mathrm{Jac}(\mathcal{C}_1)$. Furthermore, the map $v \to w_i$ gives an isomorphism between the ζ^{d_1}-eigenspace for the action of μ^{d_1} on $\mathcal{A}_1$ and the ζ^i-eigenspace for the action of μ on $\mathrm{Jac}(\mathcal{C}_1)$, so as Galois modules

$$T_\ell(\mathrm{Jac}(\mathcal{C}_1)^{\mathrm{new}}) \simeq T_\ell(\mathcal{A}_1)^{r_1},$$

where $r_1 = \frac{\varphi(N)}{\varphi(N/d_1)}$ (the number of possible values of i).

A similar result holds for the curve $\mathcal{C}_2$ and its components, but note that $\mathcal{C}_2^{(j)}$ is not a special fiber of $\widetilde{\mathcal{C}}$ but a twist of it by $\eta^{-(A+B)}$ is (due to the change of variables $y = \pi^{\frac{A+B}{N}} y_2$).

Condition (ii) (and the hypothesis $N \nmid A+B$) implies that the curve $\mathcal{C}_1^{(i)}$ (respectively $\mathcal{C}_2^{(j)}$) has genus $\varphi(N/d_1)/2$ (respectively $\varphi(N/d_2)/2$) and has an action of $\boldsymbol{\mu}_{\frac{N}{d_1}}$ (respectively $\boldsymbol{\mu}_{\frac{N}{d_2}}$), so its Jacobian has complex multiplication (see [33]).

Suppose then that $N \mid A+B$. Then $e = N$ and the components $\mathcal{C}_1^{(i)}$ and $\mathcal{C}_2^{(j)}$ have all genus zero. By (38) the Galois representation is a twist of the representation Sp_2 by a character coming from the action of the inertia group I_K on the graph cohomology. The edges of the graph $\mathrm{H}^1(\Upsilon, \mathbb{Z})$ correspond to the components $\mathcal{C}_1^{(i)}$ and $\mathcal{C}_2^{(j)}$, which are defined over K, so the inertia group I_K acts trivially on them. The intersection points of $\mathcal{C}_1^{(i)}$ and $\mathcal{C}_2^{(j)}$ correspond to the desingularizations points at $(0,0)$ of the first curve and at ∞ of the second. By §3.1.1 and §3.1.2 of [3], the intersection points of the desingularized curves are defined over the extension $K(\sqrt[N]{(-1)^B})$, giving the extra twist by θ^B. □

A.1. **The hypergeometric motive.** Keeping the previous notation, let $\mathcal{A}^\zeta$ denote the submotive of $\mathcal{A}$ given by the ζ-eigenspace for the action of μ. Then (as done in 6.8)

$$\mathcal{H}((a,b),(c,d)|z) = \mathcal{A}^\zeta(-1) \otimes \mathbf{J}((-a,-b,c,d),(c-b,d-a))^{-1} \varkappa_K^{(d-b)N}.$$

The motive $\mathbf{J}((-a,-b,c,d),(c-b,d-a))^{-1}$ is a Hecke character (independent of the variable z), so it is unramified while restricted to $\mathrm{Gal}(\overline{\mathbb{Q}(z)}/\overline{\mathbb{Q}}(z))$ (the same holds for $\varkappa_K$).

Let $\ell \equiv 1 \pmod N$ be a prime and let

$$\rho_\ell : \mathrm{Gal}(\overline{\mathbb{Q}(z)}/\overline{\mathbb{Q}}(z)) \to \mathrm{GL}_2(\mathbb{Q}_\ell)$$

denote the 2-dimensional Galois representation attached to $\mathcal{H}((a,b),(c,d)|z)$. Let M_0 denote the monodromy matrix around 0 as defined in Proposition 2.3.

Theorem A.2. *The image of the inertia group at $z = 0$ under ρ_ℓ is spanned (up to GL_2-conjugation) by the matrix M_0. Furthermore, if $s \in \mathbb{Q}[z]$ and we consider the Euler curve with parameter s, then the same holds, but replacing M_0 by $M_0^{v(s)}$, where $v(s)$ denote the valuation of s at 0.*

Proof. Let $\mathcal{O} = \mathbb{Q}[[z]]$, the completion of $\mathbb{Q}[z]$ at the prime ideal (z). It is a Henselian complete valuation ring with local uniformizer z, so we can apply the results of the previous section to the curve $\widetilde{\mathcal{C}}$. To relate the curve $\mathcal{C}$ to $\widetilde{\mathcal{C}}$ we need to twist by $\eta^{A+B-D} = \eta^{-cN}$. Since $A+B = (c-d)N$, if $c-d \notin \mathbb{Z}$ we are in the first case of Theorem A.1, so inertia acts on the ζ-eigenspace like the sum of the two characters

$$\eta^{-c} \oplus \eta^{-d}.$$

When $c-d \in \mathbb{Z}$, we are in the second case of Theorem A.1, but note that the character θ is trivial on $\mathrm{Gal}(\overline{\mathbb{Q}(z)}/\overline{\mathbb{Q}}(z))$. □

Corollary A.3. *Let L be a number field containing F and let $\xi \in L$, $\xi \neq 0, 1$. Let $\mathfrak{p}$ be a prime ideal of L not dividing N with $v_\mathfrak{p}(\xi) > 0$. Then the image of the inertia group $I_\mathfrak{p}$ on $\mathcal{H}((a,b),(c,d)|\xi)$ is spanned (up to conjugation) by $M_0^{v_\mathfrak{p}(\xi)}$.*

Proof. It mimics the proof of the previous theorem, noting that the Hecke character $\mathbf{J}((-a,-b,c,d),(c-b,d-a))^{-1}$ and the quadratic character $\varkappa_N$ are only ramified at primes dividing N. □

Analogous results hold for the inertia group at 1 and at ∞.

Theorem A.4. *The image of the inertia group at $z=\infty$ (respectively at $z=1$) under ρ_ℓ is spanned (up to GL_2-conjugation) by the matrix M_∞ (respectively M_1).*

Proof. It mimics the proof of the last theorem. For the first statement, the local parameter is $1/z$. The change of variables $x'=zx$, $y'=(1/z)^b y$ (on the equation for $\mathcal{C}$) gives the component

$$\mathcal{C}_1 : y'^N = x'^A(1-x')^C.$$

Similarly, the second component is obtained by setting $y'=(1/z)^a y$, with equation

$$\mathcal{C}_2 : y'^N = (-1)^C x^{A+B}(1-x)^B.$$

The result then follows from the same arguments as in the last theorem. For the second statement, consider the two components of $\widetilde{\mathcal{C}}$ given by

$$\mathcal{C}_1 : y^N = x^A(1-x)^{B+C}, \qquad \mathcal{C}_2 : y'^N = x'^B(1-x')^C,$$

where the second curve is obtained by looking at the special fiber of the curve obtained via the change of variables $x'=\frac{(x-1)}{(z-1)}$, $y'=(1-z)^{-\frac{B+C}{N}}y$. The result follows from the fact that $B+C=\delta$ and that the twist needed to relate $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ is unramified at $z-1$. □

Appendix B. Stable models and the trace of Frobenius

The goal of the present appendix is to give a formula for the trace of a Frobenius element $\mathrm{Frob}_{\mathfrak{p}}$ when the motive has good reduction at $\mathfrak{p}$ but the equation used to define the Euler curve is singular at $\mathfrak{p}$. There are three different cases, depending on whether the specialization ξ reduces to 0, 1 or ∞ modulo $\mathfrak{p}$. More concretely, let $(a,b),(c,d)$ be generic rational parameters and let N be their least common denominator. Let $F=\mathbb{Q}(\zeta_N)$ and let $\mathfrak{p}$ be a prime ideal of F not dividing N. Consider the following three cases:

(1) The monodromy matrix M_0 has finite order r_0, $v_{\mathfrak{p}}(\xi)$ is positive and divisible by r_0.
(2) The monodromy matrix M_1 has finite order r_1, $v_{\mathfrak{p}}(\xi-1)$ is positive and divisible by r_1.
(3) The monodromy matrix M_∞ has finite order r_∞, $v_{\mathfrak{p}}(\xi)$ is negative and divisible by r_∞.

The following three theorems describe the value of the trace of $\mathrm{Frob}_{\mathfrak{p}}$ in each case (assuming an extra technical condition that holds in most instances).

Theorem B.1. *Let $(a,b),(c,d)$ be rational generic parameters and let N be their least common denominator. Let $\xi \in \mathbb{Q}\setminus\{0,1\}$. Suppose that condition (1) is satisfied. Write $\xi = p^{v_{\mathfrak{p}}(\xi)}\tilde{\xi}$. Suppose that the following two extra hypotheses hold:*

$$\gcd(N,(d-b)N,(b-c)N)=1, \quad \textit{and} \quad \gcd(N,(d-c)N,(a-d)N)=1.$$

Then the trace of $\mathrm{Frob}_{\mathfrak{p}}$ *acting on* $\mathcal{H}((a,b),(c,d)|\xi)$ *equals*

$$-\chi_{\mathfrak{p}}(-1)^{(d-b)(\mathcal{N}\mathfrak{p}-1)}\,\mathbf{J}((-a,-b,c,d),(c-b,d-a))(\mathfrak{p})^{-1}\cdot \tag{43}$$
$$\left(\chi_{\mathfrak{p}}(\tilde{\xi})^{dN}J(\chi_{\mathfrak{p}}^{(d-b)N},\chi_{\mathfrak{p}}^{(b-c)N})+\chi_{\mathfrak{p}}(-1)^{(b-c)N}\chi_{\mathfrak{p}}(\tilde{\xi})^{cN}J(\chi_{\mathfrak{p}}^{(d-c)N},\chi_{\mathfrak{p}}^{(a-d)N})\right),$$

where $J(\chi_1,\chi_2)$ denotes the usual Jacobi sum.

Proof. The proof follows standard arguments (see [9], [13] and [22]). Let $A=(d-b)N$, $B=(b-c)N$, $C=(a-d)N$ and $D=dN$. Then the Euler curve is defined by the equation

$$\mathcal{C}: y^N = x^A(1-x)^B(1-\xi x)^C p^{v_{\mathfrak{p}}(\xi)D}\tilde{\xi}^D. \tag{44}$$

The order of M_0 is $\mathrm{lcm}\{\mathrm{den}(c),\mathrm{den}(d)\}$, hence the hypothesis of being in case (1) implies that both $v_{\mathfrak{p}}(\xi)c$ and $v_{\mathfrak{p}}(\xi)d$ are integers. The change of variables $y\to yp^{-v_{\mathfrak{p}}(\xi)d}$ is defined over F. The new model reduces modulo $\mathfrak{p}$ to the curve

$$\mathcal{C}_1: y^N = x^A(1-x)^B\tilde{\xi}^D.$$

The first extra hypothesis implies that $\mathcal{C}_1$ is irreducible. Since the parameters are generic, $N\nmid A$ and $N\nmid B$. Furthermore, $N\nmid A+B=N(d-c)$ because the matrix M_0 has finite order. Then $\dim(\mathrm{Jac}(\mathcal{C}_1)^{\mathrm{new}})=\frac{\phi(N)}{2}>0$, so $\mathcal{C}_1$ is a component of the stable model of $\mathcal{C}$ (see [9] for more details).

To find the other component, consider the change of variables $x\to xp^{-v_{\mathfrak{p}}(\xi)}$, which transforms (44) into

$$y^N p^{v_{\mathfrak{p}}(\xi)(A+B-D)} = x^A(p^{v_{\mathfrak{p}}(\xi)}-x)^B(1-\tilde{\xi}x)^C\tilde{\xi}^D.$$

From its definition, $A+B-D=-cN$ so after the change of variables $y\to yp^{v_{\mathfrak{p}}(\xi)c}$ we get an equation whose reduction modulo $\mathfrak{p}$ equals

$$\mathcal{C}_2: y^N = (-1)^B x^{A+B}(1-\tilde{\xi}x)^C\tilde{\xi}^D.$$

The second extra hypothesis implies that $\mathcal{C}_2$ is irreducible. The genericity condition on the parameters implies that $N\nmid C$, $N\nmid A+B$ and $N\nmid A+B+C=N(a-c)$ so $\dim(\mathrm{Jac}(\mathcal{C}_2)^{\mathrm{new}})=\frac{\phi(N)}{2}$ and $\mathcal{C}_2$ is another component of the stable reduction of $\mathcal{C}$. Then the Galois representation of the decomposition group at $\mathfrak{p}$ acting on $H^1_{\text{ét},\chi}(\widehat{\mathcal{C}},\mathbb{Q}_\ell)$ is the direct sum of the contribution from $\mathcal{C}_1$ and that from $\mathcal{C}_2$ (both being abelian varieties with complex multiplication by $\mathbb{Z}[\zeta_N]$).

To compute the trace of $\mathrm{Frob}_{\mathfrak{p}}$ acting on $\mathcal{C}_1$ (respectively $\mathcal{C}_2$), let $\chi_{\mathfrak{p}}$ be as in (7). Then we can apply Theorem 5.4 to $\mathcal{C}_1$ to get that the trace of $\mathrm{Frob}_{\mathfrak{p}}$

acting on $H^{1,\zeta_N}_{et}(\widehat{\mathcal{C}}_1,\mathbb{Q}_\ell)$ equals

$$-\sum_{x\in\mathbb{F}_q}\chi_{\mathfrak{p}}(x^A(1-x)^B\tilde{\xi}^D)=-\chi_{\mathfrak{p}}(\tilde{\xi})^D J(\chi_{\mathfrak{p}}^A,\chi_{\mathfrak{p}}^B).$$

The same computation for $\mathcal{C}_2$ gives that the trace of $\mathrm{Frob}_{\mathfrak{p}}$ on the (new part of its) étale cohomology equals

$$-\chi_{\mathfrak{p}}(-1)^B\chi_{\mathfrak{p}}(\tilde{\xi})^{D-A-B}J(\chi_{\mathfrak{p}}^{A+B},\chi_{\mathfrak{p}}^C).$$

The result follows from the definition of A,B,C and D together with the relation between the hypergeometric motive and the Euler curve given in (26). □

With the same technique one can prove the following results for the other two cases.

Theorem B.2. *Let $(a,b),(c,d)$ be rational generic parameters and let N be their least common denominator. Let $\xi\in\mathbb{Q}$. Suppose that condition (2) is satisfied. Let $v=v_{\mathfrak{p}}(\xi-1)$ and write $\xi=1+p^v\tilde{\xi}$. Suppose that the following two extra hypotheses hold:*

$$\gcd(N,(d-b)N,(a+b-c-d)N)=1,\quad \textit{and}\quad \gcd(N,(b-c)N,(a-d)N)=1.$$

Then the trace of $\mathrm{Frob}_{\mathfrak{p}}$ acting on $\mathcal{H}((a,b),(c,d)|\xi)$ equals

$$-\chi_{\mathfrak{p}}(-1)^{(d-b)(\mathcal{N}\mathfrak{p}-1)}\,\mathbf{J}((-a,-b,c,d),(c-b,d-a))(\mathfrak{p})^{-1}\cdot \tag{45}$$
$$\left(J(\chi_{\mathfrak{p}}^{(d-b)N},\chi_{\mathfrak{p}}^{(a+b-c-d)N})+\chi_{\mathfrak{p}}(-1)^{(a-d)N}\chi_{\mathfrak{p}}(\tilde{\xi})^{(a-b)N}J(\chi_{\mathfrak{p}}^{(d-b)N},\chi_{\mathfrak{p}}^{(a-d)N})\right).$$

Theorem B.3. *Let $(a,b),(c,d)$ be rational generic parameters and let N be their least common denominator. Let $\xi\in\mathbb{Q}$. Suppose that condition (3) is satisfied. Write $\xi=p^{v_{\mathfrak{p}}(\xi)}\tilde{\xi}$. Suppose that the following two extra hypotheses hold:*

$$\gcd(N,(a-b)N,(b-c)N)=1,\quad \textit{and}\quad \gcd(N,(d-b)N,(a-d)N)=1.$$

Then the trace of $\mathrm{Frob}_{\mathfrak{p}}$ acting on $\mathcal{H}((a,b),(c,d)|\xi)$ equals

$$-\chi_{\mathfrak{p}}(-1)^{(d-b)(\mathcal{N}\mathfrak{p}-1)}\,\mathbf{J}((-a,-b,c,d),(c-b,d-a))(\mathfrak{p})^{-1}\cdot \tag{46}$$
$$\left(\omega(\tilde{\xi})^{bN}J(\omega^{(d-b)N},\omega^{(a-d)N})+\omega(-1)^{(a-d)N}\omega(\tilde{\xi})^{aN}J(\omega^{(a-b)N},\omega^{(b-c)N})\right).$$

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