K-VARIETIES AND GALOIS REPRESENTATIONS

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Abstract. In a remarkable article Ribet showed how to attach rational 2-dimensional representations to elliptic $\mathbb{Q}$-curves. An abelian variety A is a (weak) K-variety if it is isogenous to all of its Gal_K-conjugates. In this article we study the problem of attaching an absolutely irreducible ℓ-adic representation of Gal_K to an abelian K-variety, which sometimes has smaller dimension than expected. When possible, we also construct a Galois-equivariant pairing, which restricts the image of this representation. As an application of our construction, we prove modularity of abelian surfaces over $\mathbb{Q}$ with potential quaternionic multiplication.

Introduction

Let L be a number field, and let A/L be an abelian variety of dimension g. Given a prime number ℓ, the Tate module $T_\ell A$ is obtained as the inverse limit (over all positive integers n) of the subgroup of ℓ^n-torsion points of A. Since addition is a rational map, the Tate module acquires a structure of $\mathbb{Z}_\ell[\mathrm{Gal}_L]$-module. Let $\mathrm{V}_\ell A := \mathrm{T}_\ell A \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ be the usual extension of scalars, a $\mathbb{Q}_\ell$-vector space of dimension $2g$. The action of Gal_L on $\mathrm{V}_\ell A$ gives the usual $2g$-dimensional Galois representation $\rho_{A,\ell} : \mathrm{Gal}_L \to \mathrm{GL}_{2g}(\mathbb{Q}_\ell)$. Let $\mathrm{End}^0(A)$ denote the $\mathbb{Q}$-algebra of endomorphisms of A defined over L. A well known result states that if $\mathrm{End}^0(A)$ is larger than $\mathbb{Q}$, then there exists a subrepresentation $\rho_{A,\lambda}$ of $\rho_{A,\ell}$ obtained by considering $\mathrm{V}_\ell A$ as an $\mathrm{End}^0(A) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$-module (we recall this construction in § 1).

Let L/K be a Galois extension of number fields. The variety A is called a (weak) K-variety if it is L-isogenous to all of its $\mathrm{Gal}(L/K)$-conjugates, namely for all $\sigma \in \mathrm{Gal}(L/K)$ there exists an isogeny $\mu_\sigma : {}^\sigma A \to A$. The prototypical example of a K-variety is that of an elliptic curve A which is isogenous to all of its Galois conjugates (the so called $\mathbb{Q}$-curves) as studied by Ribet in [Rib04]. The main goal of the present article is to study the following question.

Question: What Galois representations can be naturally attached to a (weak) K-variety?

The reason to study K-varieties is that although the variety A is defined over L, its Galois representation (up to a twist) should extend to an n-dimensional Galois representation of Gal_K for some $n \leq 2\dim(A)$ (this is indeed the case, as proven in Theorem 3.15). In the particular case of $\dim(A) = 1$ (i.e. A being an elliptic curve) without complex multiplication, this result was proven by Ribet in [Rib04]. Existence of endomorphisms makes the situation

2020 *Mathematics Subject Classification.* 11G10,14K15,11F80.

Key words and phrases. K-varieties, Galois representations.

Funded by the Portuguese Foundation for Science and Technology (FCT) Individual Call to Scientific Employment Stimulus (https://doi.org/10.54499/2020.02775.CEECIND/CP1589/CT0032). This work was supported by CIDMA and is funded by the FCT, under grant UIDB/04106/2020 (https://doi.org/10.54499/UIDB/04106/2020). Florit was supported by the Spanish Ministry of Universities (FPU20/05059) and by grants PID2022-137605NB-I00 and 2021 SGR 01468.

more subtle. It is important to remark that as part of the Langlands program, we also expect the resulting Galois representation of Gal_K to match an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_K)$. In the case of $\mathbb{Q}$-curves, this follows from Serre's conjectures (as described in [Rib04]).

An abelian variety A/L is called a *strong* K-variety if furthermore for all $\sigma \in \mathrm{Gal}(L/K)$ the isogeny $\mu_\sigma : {}^\sigma A \to A$ satisfies a natural commutation relation with all endomorphisms defined over L, namely

$$\mu_\sigma {}^\sigma\varphi = \varphi\mu_\sigma \ \forall \sigma \in \mathrm{Gal}(L/K),\ \forall \varphi \in \mathrm{End}(A_L).$$

We want to emphasize that sometimes in the literature, the notion of a K-variety involves the stronger condition that the previous relation holds for all $\varphi \in \mathrm{End}(A_{\overline{\mathbb{Q}}})$. Strong K-varieties (with the more restrictive definition) were studied in [Gui12], in analogy to the $\mathbb{Q}$-curves situation. It is not hard to prove that if A/L is a strong K-variety of dimension g with endomorphism ring $\mathbb{Z}$, then a twist of $\rho_{A,\ell}$ can be extended to a $2g$-dimensional Galois representation of Gal_K (see Theorem 3.3). In some instances (see Remark 4.4), the extended representation also preserves a non-degenerate bilinear form.

In general, the ℓ-adic representation $\rho_{A,\ell}$ of the Galois group Gal_L has an irreducible subrepresentation ρ_λ (coming from the endomorphism ring). Then we can prove (Theorem 3.10) the existence of a maximal subfield $L' \subset L$ such that A is a strong L'-variety. This allows to extend the representation $\rho_{A,\lambda}$ to a representation $\widetilde{\rho_\lambda}$ of $\mathrm{Gal}_{L'}$ of the same dimension (well defined up to twists by characters of $\mathrm{Gal}_{L'}$). To obtain a representation of Gal_K one can just consider its induction from $\mathrm{Gal}_{L'}$ to Gal_K (as done in [FG22] for strong abelian varieties of GL_2-type over $\overline{\mathbb{Q}}$). As an example, let $E/\mathbb{Q}$ be an elliptic curve with complex multiplication by an imaginary quadratic field K. The curve E/K is clearly a K-elliptic curve, though it is not a strong K-elliptic curve (which is consistent with the fact that it does not have a 1-dimensional representation of $\mathrm{Gal}_\mathbb{Q}$ attached to it). The representation attached to E/K decomposes as the sum of two 1-dimensional representations (coming from a Hecke character χ and its complex conjugate), and the induction of χ from Gal_K to $\mathrm{Gal}_\mathbb{Q}$ matches $\rho_{E,\ell}$.

Our construction is well suited to study how the Galois representation of A behaves while extending scalars from L to a field M where it gets extra endomorphisms. In Theorem 3.17 we compare the dimension of the different constructions, which do not always match. We present two interesting applications of our construction: one of them is related to abelian fourfolds whose 4-dimensional Galois representation is contained in $\mathrm{GU}_4(\mathbb{Q}_\ell) \cap \mathrm{GSp}_4(F_\lambda)$, for F an imaginary quadratic field, and λ a prime of F dividing ℓ. Our second application proves that any abelian surface over $\mathbb{Q}$ with potential quaternionic multiplication (QM for short) comes from a Siegel modular form of weight 2.

The article is organized as follows: Section 1 recalls some well known facts on irreducible constituents ρ_λ of Galois representations attached to abelian varieties in terms of its endomorphism ring. Section 2 recalls results on the existence of non-degenerate, bilinear pairings (symplectic, symmetric or hermitian) invariant under the action of ρ_λ. Section 3 contains the main results of the article. We start studying strong K-varieties A/L. Our main result (Theorem 3.3) proves that the representation ρ_λ (after possible a twist) can be extended to a representation $\widetilde{\rho_\lambda}$ of Gal_K of the same dimension. In Section 3.2 we study general K-varieties. We prove that the group $\mathrm{Gal}(L/K)$ has a natural action on the center of the endomorphism algebra. The kernel of the action fixes a subextension L', and it happens that the variety A is a strong L'-variety (see Theorem 3.10). Then the previous result applies and provides an extension of our representation to $\mathrm{Gal}_{L'}$. In Section 3.3 we study how the dimension our constructed representation varies when we enlarge the base field L. In Theorem 3.18 we prove that in some cases the dimension halves (depending on how many new extra endomorphisms the variety gains).

Section 4 gives the necessary (and sufficient) condition for the existence of an invariant pairing for the constructed representation extending that of ρ_λ. The last section contains some applications (including the stated modularity result of abelian surfaces with potential QM).

Notation: for the reader's convenience, we include some notation used during the article.

- L denotes a number field and $\overline{L}$ an algebraic closure of it.
- A denotes a simple abelian variety defined over L.
- $R = \mathrm{End}(A_L)$ and $D = \mathrm{End}^0(A) = R \otimes_{\mathbb{Z}} \mathbb{Q}$. E denotes the center of D and E^+ its maximal totally real subfield.
- d denotes the Schur index of $\mathrm{End}^0(A)$.
- If $\varphi \in D$ and $\sigma \in \mathrm{Gal}_K$, we denote by ${}^\sigma\varphi$ the morphism obtained by applying σ to the coefficients of the endomorphism.

Acknowledgments: We would like to thank Alex Bartel, Francesc Fité and Xevi Guitart for many helpful conversations as well as many suggestions that improved the quality of the article.

1. Endomorphisms and splitting

Let L be a number field, and let A/L be a simple abelian variety of dimension g. Let $\mathrm{End}^0(A) := \mathrm{End}(A_L) \otimes_{\mathbb{Z}} \mathbb{Q}$ be its endomorphism algebra, a division algebra with center a number field E. Then $\dim_E \mathrm{End}^0(A) = d^2$, where the number d is called the Schur index of $\mathrm{End}^0(A)$ (see [Pie82, §13]).

Theorem 1.1. *The endomorphism algebra falls in one of the following types in the Albert classification:*

- *Type* I. $E = \mathrm{End}^0(A)$ *(so $d = 1$); E is a totally real number field and $[E : \mathbb{Q}] \mid g$.*
- *Type* II. *E is a totally real number field,* $\mathrm{End}^0(A)$ *is a totally indefinite quaternion algebra over K (so $d = 2$) and $2[E : \mathbb{Q}] \mid g$.*
- *Type* III. *E is a totally real number field,* $\mathrm{End}^0(A)$ *is a totally definite quaternion algebra over K (so $d = 2$) and $2[E : \mathbb{Q}] \mid g$.*
- *Type* IV. *E is a CM field,* $\mathrm{End}^0(A)$ *is a division algebra and $\frac{[E:\mathbb{Q}]}{2} d^2 \mid g$.*

Proof. See for example Theorem 2 of [Mum08, § 21]. □

Remark 1.2. *Even when $A_{\bar{L}}$ is simple, it is perfectly possible that A_L and $A_{\bar{L}}$ have different Albert types. For example a rational curve with complex multiplication by an imaginary quadratic field K has Albert type* I *over $\mathbb{Q}$, but type* IV *over K. The same phenomenon occurs in higher dimensions, for example with the Jacobian of the genus 2 curve with LMFDB label 20736-a-1 (with complex multiplication over $\mathbb{Q}(\zeta_9)^+$).*

Fix a prime number ℓ. Let $\mathrm{V}_\ell(A) := \mathrm{T}_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ be the rational Tate module. Fix a polarization $\phi : A \to A^\vee$. The Weil pairing and the choice of polarization gives a non-degenerate alternating pairing

$$\Phi : \mathrm{V}_\ell A \times \mathrm{V}_\ell A \to \mathbb{Q}_\ell.$$

This pairing is compatible with the action of Galois: for $\sigma \in \mathrm{Gal}_L$, $u, v \in \mathrm{V}_\ell A$, we have

$$\Phi({}^\sigma u, {}^\sigma v) = {}^\sigma\Phi(u,v) = \chi_\ell(\sigma)\Phi(u,v) \tag{1}$$

with $\chi_\ell : \mathrm{Gal}_L \to \mathbb{Q}_\ell^\times$ the ℓ-adic cyclotomic character. Another way to state property (1) is to say that Φ is Gal_L-equivariant with similitude character χ_ℓ. The pairing (and its Gal_L-invariance) allow to construct a continuous Galois representation

$$\rho_{A,\ell} : \mathrm{Gal}_L \to \mathrm{GSp}(\mathrm{V}_\ell(A), \Phi),$$

with similitude character χ_ℓ, the ℓ-th cyclotomic character. This representation is not irreducible in general. To ease notation, let $R := \mathrm{End}(A_L)$ and let $D := \mathrm{End}^0(A) = R \otimes_{\mathbb{Z}} \mathbb{Q}$. When D is non-commutative define the ramification set

$$\mathrm{Ram}(D) = \{\lambda \text{ prime of } E \,:\, D \otimes_E E_\lambda \not\simeq M_d(E_\lambda)\}. \tag{2}$$

To avoid multiple statements, when $D = E$ set $\mathrm{Ram}(D) = \emptyset$. In both cases, $\mathrm{Ram}(D)$ is a finite set with even cardinality [Pie82, §18.5]. The algebra $\mathrm{End}^0(A)$ acts on $\mathrm{V}_\ell A$, so we can consider it as an $\mathrm{End}^0(A) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$-module, and in particular as an $E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$-module. There is a (categorical product) decomposition

$$E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \simeq \bigtimes_{\lambda|\ell} E_\lambda, \tag{3}$$

where λ runs over the primes of E over ℓ. Let e_λ be the idempotent corresponding to E_λ in (3) and set $\mathrm{V}_\lambda := e_\lambda \cdot \mathrm{V}_\ell A$. Then

$$\mathrm{V}_\ell A \simeq \bigoplus_{\lambda|\ell} \mathrm{V}_\lambda.$$

This is an isomorphism of $E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell[\mathrm{Gal}_L]$-modules, since $\mathrm{End}^0(A)$ is the algebra of endomorphisms defined over L, so they commute with the action of the Galois group Gal_L.

Lemma 1.3. *We have* $\dim_{E_\lambda} \mathrm{V}_\lambda = \frac{2g}{[E:\mathbb{Q}]}$*, which is independent of* ℓ*.*

Proof. This is Theorem 2.1.1 in [Rib76], the argument works the same since V_λ is a free E_λ-module. □

Lemma 1.4. *We have* $\mathrm{End}_{E_\lambda[\mathrm{Gal}_L]}(\mathrm{V}_\lambda) = \mathrm{End}^0(A) \otimes_E E_\lambda$*. Furthermore, if* $\mathrm{End}^0(A) = E$ *is commutative, each* V_λ *is an absolutely irreducible representation of* Gal_L*.*

Proof. By [Fal83, Satz 4], $\mathrm{V}_\ell A$ is a semisimple $\mathbb{Q}_\ell[\mathrm{Gal}_L]$-module and $\mathrm{End}_{\mathrm{Gal}_L}(\mathrm{V}_\ell A) \simeq \mathrm{End}^0(A) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$. Therefore each V_λ is semisimple and satisfies $\mathrm{End}_{E_\lambda[\mathrm{Gal}_L]}(\mathrm{V}_\lambda) = \mathrm{End}^0(A) \otimes_E E_\lambda$. Hence when $\mathrm{End}^0(A) = E$, V_λ is simple over E_λ, and so it is absolutely irreducible. □

When $\mathrm{End}^0(A)$ is non-commutative, the module V_λ cannot be simple. The reason is that $\mathrm{End}^0(A) \otimes_E E_\lambda \simeq M_d(E_\lambda)$ for $\lambda \notin \mathrm{Ram}(D)$, and this ring has d orthogonal idempotents that further break down V_λ. Since we want the simple factors in a decomposition of V_λ to be nondegenerate with respect to a certain pairing, we introduce them in a separate section.

2. Pairings and irreducible constituents

Definition 2.1. *Let K be a field of characteristic different from 2 and let V be a finite dimensional K-vector space. A nondegenerate biadditive form*

$$\Psi : V \times V \to K$$

is called

- symplectic, *if it is K-bilinear and $\Psi(v, v) = 0$ for all $v \in V$.*
- symmetric, *if it is K-bilinear and $\Psi(v, w) = \Psi(w, v)$ for all $v, w \in V$.*
- hermitian, *if K has an automorphism $\bar{\cdot} : K \to K$ which is an involution, and satisfies that Ψ is K-linear in the first entry and*

$$\Psi(v, w) = \overline{\Psi(w, v)}$$

for all $v, w \in V$.

Definition 2.2. *Suppose that A either has Albert type* I, II *or* III. *Let ℓ be a rational prime and let $\lambda \mid \ell$ in $\mathcal{O}_E$. The prime λ has* Property (P) *if $\lambda \notin \mathrm{Ram}(D)$.*

Denote by $x \mapsto x'$ the Rosati involution of $D = \mathrm{End}^0(A)$, which is obtained from the polarization $\phi : A \to A^\vee$.

Lemma 2.3. *Let A have Albert type* IV. *Let E^+ be the maximal totally real subfield of E. Then there exist a finite Galois extension L/E^+ containing E, an element $\gamma \in D$ with $\gamma' = \gamma$ and an L-algebra isomorphism*

$$s : D \otimes_E L \xrightarrow{\sim} M_d(L)$$

such that the positive involution $x \mapsto x^* := \gamma x' \gamma^{-1}$ on $s(D)$ is the restriction of the involution $X \mapsto X^* := \overline{X}^\top$ on $M_d(L)$.

Proof. This is Lemma 2.1 in [BKG21]. ☐

Definition 2.4. *Suppose A has Albert type* IV *with $d > 1$. Let ℓ be a rational prime, let $\lambda \mid \ell$ in $\mathcal{O}_E$, and let $\lambda_0 := \lambda \cap \mathcal{O}_{E^+}$. The prime λ has* Property (P) *if it satisfies the following properties:*

- $\lambda \notin \mathrm{Ram}(D)$.
- *λ is inert over λ_0, and λ splits in $\mathcal{O}_L$.*

Remark 2.5. *As explained in* [BKG21, pg. 1250], *when A has Albert type* IV *there is a positive density of primes λ of E with property (P).*

Theorem 2.6. *Let A/L be a simple abelian variety of dimension g, and let $n = \frac{2g}{d[E:\mathbb{Q}]}$. Let ℓ be a prime number with Property (P), and let $\lambda \mid \ell$ in E. Then there exist an absolutely irreducible $E_\lambda[\mathrm{Gal}_L]$-module W_λ of rank n, and a pairing $\Psi_\lambda : \mathrm{W}_\lambda \times \mathrm{W}_\lambda \to E_\lambda$, satisfying the following properties:*

- *The pairing Ψ_λ is non-degenerate and E_λ-linear in the first coordinate.*
- *The pairing is symplectic if A is of type* I *or* II.
- *The pairing is symmetric if A is of type* III.
- *The pairing is hermitian if A is of type* IV.
- *The form Ψ is invariant (up to a similitude character) under the Galois action: for $u, v \in \mathrm{W}_\lambda$ and $\sigma \in \mathrm{Gal}_L$, $\Psi_\lambda({}^\sigma u, {}^\sigma v) = {}^\sigma\Psi_\lambda(u, v) = \chi_\ell(\sigma)\Psi_\lambda(u, v)$.*

- *There is an isomorphism*
$$V_\lambda \simeq \mathrm{W}_\lambda^{\oplus d}$$
of $E_\lambda[\mathrm{Gal}_L]$*-modules. Furthermore, if every prime ideal* $\lambda \mid \ell$ *of* E *has Property (P), then there is a decomposition*
$$\mathrm{V}_\ell A \simeq \bigoplus_{\lambda' \mid \ell} \mathrm{W}_{\lambda'}^{\oplus d}$$
of $E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell[\mathrm{Gal}_L]$*-modules.*

Proof. This is a summary of [BGK06, Theorem 5.4] and [BGK10, Theorem 3.23]. If A has Albert type I, then this follows from [Chi91, § 2.3] if $d = 1$, and from [BKG21, Theorem 1.1] when $d > 1$. □

3. K-varieties

Definition 3.1. *Let* L/K *be an extension of number fields and let* A *be an abelian variety defined over* L*. The variety* A *is called a* K-variety *if for every* $\sigma \in \mathrm{Gal}_K$ *there exists an isogeny* $\mu_\sigma : {}^\sigma A \to A$*. The variety* A *is called a* strong K-variety *if, in addition, for every* $\varphi \in \mathrm{End}(A_L)$ *the following equality holds*

$$\mu_\sigma {}^\sigma\varphi = \varphi \mu_\sigma. \tag{4}$$

Remark 3.2. *If* L'/L *is a field extension, and* A/L *is a* K*-variety, then the variety* A/L' *is also a* K*-variety. This is not true for strong* K*-varieties. For example if* E *is an elliptic curve over* $\mathbb{Q}$ *with CM by an imaginary quadratic field* K*, then* $E/\mathbb{Q}$ *is clearly a strong* $\mathbb{Q}$*-curve, but* E/K *is not.*

3.1. Strong K-varieties.

Let A be a strong K-variety defined over a number field L. Following the notation of Theorem 2.6, let ρ_λ be the Galois representation attached to A/L.

Theorem 3.3. *Suppose that the extension* L/K *is Galois. Then there exists a finite order character* $\psi : \mathrm{Gal}_L \to \overline{\mathbb{Q}_\ell}^\times$ *such that the twisted representation* $\rho_\lambda \otimes \psi$ *extends to a representation of* Gal_K*.*

Proof. The proof is quite standard (see for example Theorem 11.2 of [Isa06]). For $\tau \in \mathrm{Gal}(L/K)$ let ${}^\tau\rho_\lambda$ denote the representation defined by
$${}^\tau\rho_\lambda(\sigma) = \rho_\lambda(\tau\sigma\tau^{-1}).$$
Use the same notation for the Tate representation $\rho_{A,\ell}$. The fact that A is a K-variety implies that ${}^\tau\rho_{A,\ell}$ is isomorphic to $\rho_{A,\ell}$. The fact that A is a strong K-variety (i.e. the fact that isogenies commute with endomorphisms as in (4)), imply that the same holds for the representation ρ_λ, namely ${}^\tau\rho_\lambda \simeq \rho_\lambda$. Then there exists a matrix $A_\tau \in \mathrm{GL}_n(\overline{\mathbb{Q}_\ell})$ such that for all $\sigma \in \mathrm{Gal}_L$
$${}^\tau\rho_\lambda(\sigma) = A_\tau \rho_\lambda(\sigma) A_\tau^{-1}.$$
Since the representation ρ_λ is irreducible (by Theorem 2.6), Schur's lemma implies that the matrix A_τ is unique up to scalars, i.e. it determines a unique element in $\mathrm{PGL}_n(\overline{\mathbb{Q}_\ell})$.

Consider the projective representation $\mathbb{P}\rho_\lambda : \mathrm{Gal}_L \to \mathrm{PGL}_n(\overline{\mathbb{Q}_\ell})$ (obtained as the composition of ρ_λ with the natural quotient map $\pi : \mathrm{GL}_n(\overline{\mathbb{Q}_\ell}) \to \mathrm{PGL}_n(\overline{\mathbb{Q}_\ell})$). Extend the representation $\mathbb{P}\rho_\ell$ to a map $\widetilde{\mathbb{P}\rho_\ell} : \mathrm{Gal}_K \to \mathrm{PGL}_n(\overline{\mathbb{Q}_\ell})$ by defining $\widetilde{\mathbb{P}\rho_\lambda(\tau)} := A_\tau$ for $\tau \in \mathrm{Gal}_K$. An elementary computation proves that $\widetilde{\mathbb{P}\rho_\lambda}$ is a group morphism (hence a projective representation). Note that if $\tau \in \mathrm{Gal}_L$, then

$$^{\tau}\rho_\lambda(\sigma) = \rho_\lambda(\tau\sigma\tau^{-1}) = \rho_\lambda(\tau)\rho_\lambda(\sigma)\rho_\lambda(\tau)^{-1},$$

hence $A_\tau = \rho_\lambda(\tau)$ (up to a scalar matrix), so the map $\widetilde{\mathbb{P}\rho_\ell}$ really extends the map $\mathbb{P}\rho_\ell$.

The obstruction to lift the projective representation to a continuous homomorphism $\rho : \mathrm{Gal}_K \to \mathrm{GL}_2(\overline{\mathbb{Q}_\ell})$ lies in $H^2(\mathrm{Gal}_K, \overline{\mathbb{Q}_\ell^\times})$, which by a result of Tate (see [Ser77, §6.5]) is trivial. □

Remark 3.4. *It follows from the proof of the previous theorem that the extension is unique up to a twist by a character of* Gal_K. *Note that when L/K is cyclic, a similar argument proves that ρ_λ admits an extension without needing to twist.*

Remark 3.5. *When L/K is not cyclic, the twist might be really needed. The following elementary example was provided to us by Alex Bartel: let Q be the group of quaternions, and let H be the subgroup $H = \{\pm 1\}$. Let $\rho : H \to \mathbb{C}^\times$ the representation sending -1 to -1. It is clear that this representation cannot be extended to a 1-dimensional representation of Q (because the commutator subgroup of Q is H), but it is true that for any $g \in Q$, the representation $^{q}\rho$ is isomorphic to ρ (because H is the center of Q).*

We will denote by $\widetilde{W_\lambda}$ the vector space underlying the extended representation $\widetilde{\rho_\lambda}$, and by $\widetilde{E_\lambda}$ its coefficient field (so $\dim_{E_\lambda} W_\lambda = \dim_{\widetilde{E_\lambda}} \widetilde{W_\lambda}$).

3.2. **General K-varieties.** Let L/K be an extension of number fields, which we assume is Galois (otherwise enlarge L). Let A/L be an isotypic K-variety, so $A \sim B^n$ for some simple abelian variety B. Let $D = \mathrm{End}(A_L) \otimes_{\mathbb{Z}} \mathbb{Q}$ be the endomorphism algebra of A and let E be its center, so E is a field and D is a simple central E-algebra. For $\sigma \in \mathrm{Gal}(L/K)$ and $\varphi \in E$, define the "action"

$$\sigma.\varphi := \mu_\sigma {}^{\sigma}\varphi \mu_\sigma^{-1} \in D. \tag{5}$$

Lemma 3.6. *For every $\sigma \in \mathrm{Gal}(L/K)$ and $\varphi \in E$, $\mu_\sigma {}^{\sigma}\varphi\mu_\sigma^{-1} \in E$.*

Proof. It is clear from its definition that $\mu_\sigma {}^{\sigma}\varphi\mu_\sigma^{-1} \in D$, we just need to prove that it belongs to the center. Let $\alpha \in D$ and define $\beta := \mu_\sigma^{-1}\alpha\mu_\sigma \in \mathrm{End}({}^{\sigma}A) \otimes \mathbb{Q}$. We have $\beta = {}^{\sigma}({}^{\sigma^{-1}}\beta)$, so that

$$\begin{aligned}\mu_\sigma {}^{\sigma}\varphi\mu_\sigma^{-1}\alpha &= \mu_\sigma {}^{\sigma}\varphi\beta\mu_\sigma^{-1} = \mu_\sigma {}^{\sigma}(\varphi({}^{\sigma^{-1}}\beta))\mu_\sigma^{-1} \\ &= \mu_\sigma {}^{\sigma}({}^{\sigma^{-1}}\beta\varphi)\mu_\sigma^{-1} = \alpha\mu_\sigma {}^{\sigma}\varphi\mu_\sigma^{-1}.\end{aligned}$$

Therefore $\mu_\sigma {}^{\sigma}\varphi\mu_\sigma^{-1} \in E$. □

Lemma 3.7. *Under the previous hypothesis, the value $\sigma \cdot \varphi$ does not depend on the choice of the isogeny μ_σ.*

Proof. Let $\widetilde{\mu_\sigma} : {}^{\sigma}A \to A$ be another isogeny. Then

$$\widetilde{\mu_\sigma} {}^{\sigma}\varphi\widetilde{\mu_\sigma}^{-1} = \mu_\sigma(\mu_\sigma^{-1}\widetilde{\mu_\sigma})^{\sigma}\varphi(\mu_\sigma^{-1}\widetilde{\mu_\sigma})^{-1}\mu_\sigma^{-1}.$$

The result follows from the fact that $\mu_\sigma^{-1}\widetilde{\mu_\sigma} \in D$ and φ is in its center. □

Lemma 3.8. *The map $\mathrm{Gal}(L/K) \times E \to E$ defined by (5) gives an action of $\mathrm{Gal}(L/K)$ on E.*

Proof. To easy notation, if $\sigma, \tau \in \mathrm{Gal}(L/K)$, we denote by $c(\sigma,\tau) = \mu_\sigma {}^\sigma\mu_\tau \mu_{\sigma\tau}^{-1} \in D$. Let $\sigma, \tau \in \mathrm{Gal}(L/K)$ and $\varphi \in E$. Then

$$\begin{aligned}\sigma.(\tau.\varphi) &= \mu_\sigma {}^\sigma(\mu_\tau {}^\tau\varphi\mu_\tau^{-1})\mu_\sigma^{-1} = (\mu_\sigma {}^\sigma\mu_\tau)^{\sigma\tau}\varphi(\mu_\sigma {}^\sigma\mu_\tau)^{-1}\\ &= c(\sigma,\tau)\mu_{\sigma\tau} {}^{\sigma\tau}\varphi\mu_{\sigma\tau}^{-1}c(\sigma,\tau)^{-1} = \mu_{\sigma\tau} {}^{\sigma\tau}\varphi\mu_{\sigma\tau}^{-1} = (\sigma\tau).\varphi.\end{aligned}$$

In the second-to-last equality, we have used that $\mu_{\sigma\tau} {}^{\sigma\tau}\varphi\mu_{\sigma\tau}^{-1} \in E$ (the center), and in particular is an element that commutes with $c(\sigma,\tau)$. □

It is easy to verify that $\sigma \cdot (\varphi + \psi) = \sigma \cdot \varphi + \sigma \cdot \psi$, that $\sigma \cdot (\varphi\psi) = (\sigma \cdot \varphi)(\sigma \cdot \psi)$ and that the action on elements of $\mathbb{Q}$ is trivial. In particular, the previous action defines a group homomorphism

$$\psi : \mathrm{Gal}(L/K) \to \mathrm{Aut}(E). \tag{6}$$

Let G_0 be the kernel of ψ, and let E_0 be the subfield of E fixed by the action of $\mathrm{Gal}(L/K)$.

Proposition 3.9. *The field extension E/E_0 is Galois, G_0 is a normal subgroup of $\mathrm{Gal}(L/K)$, and we have an isomorphism*

$$\mathrm{Gal}(L/K)/G_0 \simeq \mathrm{Gal}(E/E_0).$$

Moreover, the isomorphism induces an order-reversing bijection between subgroups H of $\mathrm{Gal}(L/K)$ containing G_0, and subextensions of E/E_0.

Proof. The correspondence associates to a subgroup H of $\mathrm{Gal}(L/K)$ the field

$$E^H = \{\varphi \in E \ : \ \mu_\sigma {}^\sigma\varphi = \varphi\mu_\sigma, \ \forall \sigma \in H\}. \tag{7}$$

The proof that the map gives the stated bijection is standard, and follows the usual correspondence in Galois theory. □

Theorem 3.10. *There exists an intermediate field $K \subseteq L' \subseteq L$ such that L/L' and L'/K are Galois extensions, and such that A is a strong L'-variety.*

Proof. Let $L' := L^{G_0}$. It is clear that L/L' and L'/K are Galois extensions. For each $\sigma \in \mathrm{Gal}(L/L')$ we consider the algebra automorphism $\Psi : D \to D$, defined by $\Psi(\varphi) = \mu_\sigma {}^\sigma\varphi\mu_\sigma^{-1}$. Since $\mathrm{Gal}(L/L') \simeq G_0$, Ψ is an E-algebra homomorphism. Then Skolem-Noether's theorem implies the existence of $\alpha_\sigma \in D^\times$ such that

$$\mu_\sigma {}^\sigma\varphi\mu_\sigma^{-1} = \alpha_\sigma^{-1}\varphi\alpha_\sigma$$

for each $\varphi \in D$. Then A is a strong L'-variety with respect to the system of isogenies $\{\alpha_\sigma\mu_\sigma\}_{\sigma\in G_0}$. □

Corollary 3.11. *Let L/K be a Galois extension, and let A/K be a K-variety. Then A is a strong K-variety if and only if for all elements $\varphi \in E$ relation (4) holds.*

Proof. One implication is clear. For the other, suppose that (4) holds for all elements in E. Then $E = E_0$, so $G_0 = \mathrm{Gal}(L/K)$ and the result follows from last theorem. □

Let $L' := L^{G_0}$ be as in Theorem 3.10, so that A is a strong L'-variety. Let ℓ be a rational prime such that all prime ideals λ of E dividing ℓ satisfy Property (P). Fix one such λ and let W_λ be the E_λ-vector space of dimension $n = \frac{2g}{d[E:\mathbb{Q}]}$ of Theorem 2.6 and let $\rho_\lambda : \mathrm{Gal}_L \to \mathrm{GL}(W_\lambda)$ be the irreducible representation. Let $\sigma \in \mathrm{Gal}(L/K)$, and as in the previous section, denote by ${}^\sigma\rho_\lambda$ the representation defined by ${}^\sigma\rho_\lambda(\tau) = \rho_\lambda(\sigma^{-1}\tau\sigma)$.

Lemma 3.12. *The representation ${}^{\sigma}\rho_\lambda$ is isomorphic to ρ_λ for all $\sigma \in \mathrm{Gal}_{L'}$.*

Proof. The fact that A is a strong L'-variety implies that if $\sigma \in \mathrm{Gal}_{L'}$ then we can assume that μ_σ is a map of E_λ-vector spaces, giving an isomorphism between ρ_λ and ${}^{\sigma}\rho_\lambda$. $\square$

By Theorem 3.3 there exists a character $\theta : \mathrm{Gal}_L \to \overline{\mathbb{Q}}^{\times}$ such that the twisted representation $\rho_\lambda \otimes \theta$ can be extended to a representation $\widetilde{\rho_\lambda}$ of $\mathrm{Gal}_{L'}$.

Remark 3.13. *It is not hard to prove a converse of Lemma 3.12, so the representation ρ_λ cannot be extended any further. However, the representation $\widetilde{\rho_\lambda}$ is defined up to a twist, and it might be the case that a twist of the representation does extend a little further. For example if $E/\mathbb{Q}$ is an elliptic curve, K is a number field, and θ is a quadratic character of K that does not extend to any subfield, then the Galois representation attached to $E \otimes \theta$ does not extend to a Galois group larger than Gal_K, but a twist of it does. To unify statements, we assume from now on that the representation $\widetilde{\rho_\lambda}$ is not isomorphic to ${}^{\sigma}\widetilde{\rho_\lambda}$ for any $\sigma \in \mathrm{Gal}(L'/K)$ different from the identity (we can always take a quadratic twist for this to be true).*

Definition 3.14. *Let A/L be an abelian K-variety. A Galois representation of Gal_K attached to it is $\rho_{A,\lambda} := \mathrm{Ind}_{\mathrm{Gal}_{L'}}^{\mathrm{Gal}_K} \widetilde{\rho_\lambda}$.*

Denote by $\mathcal{V}_\lambda$ the vector space of the induced representation, namely

$$\mathcal{V}_\lambda = \bigoplus_{\sigma \in \mathrm{Gal}(L'/K)} \sigma \widetilde{W_\lambda}. \tag{8}$$

Theorem 3.15. *The Galois representation $\rho_{A,\lambda}$ is absolutely irreducible of dimension $\frac{2g}{d[E_0:\mathbb{Q}]}$.*

Proof. The representation ρ_λ is irreducible by 2.6, so the same holds for its extension $\widetilde{\rho_\lambda}$ to $\mathrm{Gal}_{L'}$. By Remark 3.13, ${}^{\sigma}\widetilde{\rho_\lambda}$ is not isomorphic to $\widetilde{\rho_\lambda}$ if $\sigma \in \mathrm{Gal}(L'/K)$ is not trivial, so irreducibility follows from Mackey's irreducibility criterion.

The dimension of the Galois representation $\rho_{A,\lambda}$ equals $\frac{2g}{d[E:\mathbb{Q}]}[L' : K]$, which by Proposition 3.9 equals $\frac{2g}{d[E:\mathbb{Q}]}[E : E_0] = \frac{2g}{d[E_0:\mathbb{Q}]}$ as claimed. $\square$

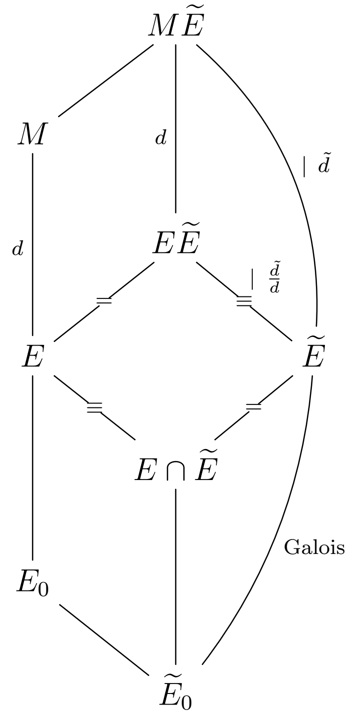
3.3. **Dimension.** A priory the dimension of the representation $\rho_{A,\lambda}$ depends on the extension L/K. A natural problem is to understand its dependence, expecting that the dimension should only decrease when the variety attains extra endomorphisms. Let $\tilde{L}/L$ be a finite Galois extension, let $\tilde{D}$ denote the endomorphism algebra of $A/\tilde{L}$ (tensored with $\mathbb{Q}$), let $\widetilde{E}$ denote its center, and let $\widetilde{E}_0$ denote the subfield of $\widetilde{E}$ fixed by $\mathrm{Gal}(\tilde{L}/K)$. Let d and $\tilde{d}$ be the Schur indices of D and $\tilde{D}$, respectively. By Theorem 1.1, these indices can a priory be any positive integer. A short exercise (or Lemma 3.16 below) shows that $d \mid \tilde{d}$ whenever $A_{\tilde{L}}$ is simple.

Lemma 3.16. *We have the inclusion $\widetilde{E}_0 \subseteq E_0$. If $A_{\tilde{L}}$ is simple, then $[E_0 : \widetilde{E}_0] \mid \frac{\tilde{d}}{d}$.*

Proof. For the inclusion, note that

$$\widetilde{E}_0 = \widetilde{E}^{\mathrm{Gal}(\tilde{L}/K)} = (\widetilde{E}^{\mathrm{Gal}(\tilde{L}/L)})^{\mathrm{Gal}(L/K)} = (\widetilde{E} \cap E)^{\mathrm{Gal}(\tilde{L}/K)} = \widetilde{E}_0 \cap E_0.$$

To bound the degree $[E_0 : \widetilde{E}_0]$, the reader might find the displayed field relations useful. First, we bound $[E : \widetilde{E} \cap E]$. Consider the compositum $E\tilde{E}$ in $\tilde{D}$. By applying the Grunwald-Wang theorem [Pie82, § 18.6] (see e.g. [CMSV19, Lemma 7.4.1]), D contains a maximal subfield M which is E-linearly disjoint from $E\tilde{E}$. Since $[M : E] = d$, we have $[M\tilde{E} : E\tilde{E}] = d$. The degree $[M\tilde{E} : \tilde{E}]$ divides $\tilde{d}$, therefore, $[E\tilde{E} : \tilde{E}]$ divides $\frac{\tilde{d}}{d}$.



Now, the extension $\widetilde{E}/\widetilde{E}_0$ is Galois, therefore, the inclusion $\widetilde{E}_0 \subseteq E \cap \widetilde{E}$ implies we have an equality $[\widetilde{E} : \widetilde{E} \cap E] = [E\widetilde{E} : E]$, and so $[E\widetilde{E} : \widetilde{E}] = [E : E \cap \widetilde{E}]$. It follows that $[E : E \cap \widetilde{E}]$ divides $\frac{\tilde{d}}{d}$.

Apply now a similar argument to the extension $[E_0 : \widetilde{E}_0]$. We have seen above that $\widetilde{E}_0 = (E \cap \widetilde{E})^{\mathrm{Gal}(\widetilde{L}/K)}$, so $(E \cap \widetilde{E})/\widetilde{E}_0$ is Galois, and therefore $[E : E_0] = [E \cap \widetilde{E} : \widetilde{E}_0]$ and $[E : E \cap \widetilde{E}] = [E_0 : \widetilde{E}_0]$. Hence $[E_0 : \widetilde{E}_0]$ divides $\frac{\tilde{d}}{d}$. □

Let $\widetilde{\sigma_{A,\lambda}}$ be the Galois representation of Gal_K attached to $A/\tilde{L}$ as in Definition 3.14.

Theorem 3.17. *If $A_{\tilde{L}}$ is simple, then $\dim(\widetilde{\sigma_{A,\lambda}}) \mid \dim(\rho_{A,\lambda})$. Moreover, equality holds if and only if $[E_0 : \widetilde{E}_0] = \tilde{d}/d$, and in particular whenever $\tilde{d} = d$.*

Proof. By Lemma 3.2 we have $\dim(\rho_{A,\lambda}) = \frac{2g}{d[E_0:\mathbb{Q}]}$ and $\dim(\widetilde{\sigma_{A,\lambda}}) = \frac{2g}{\tilde{d}[\widetilde{E}_0:\mathbb{Q}]}$. Hence the divisibility between dimensions holds by Lemma 3.16. For the second statement, we need three facts.

Fact 1: $\widetilde{E}^{\mathrm{Gal}(\widetilde{L}/L)} = \widetilde{E} \cap D = \widetilde{E} \cap E$.

Since the variety A/L is isogenous to all of its Galois conjugates, we can assume that μ_σ is the identity for all $\sigma \in \mathrm{Gal}(\widetilde{L}/L)$ (recall that the action of $\mathrm{Gal}(\widetilde{L}/K)$ on $\widetilde{E}$ is independent of the choice, by Lemma 3.7). Then if $\sigma \in \mathrm{Gal}(\widetilde{L}/K)$ and $\varphi \in \widetilde{E}$,

$$\sigma \cdot \varphi = {}^{\sigma}\varphi,$$

i.e. the action is given by the Galois action on the coefficients of the morphism φ. By Galois theory, $\varphi \in \widetilde{E}^{\mathrm{Gal}(\widetilde{L}/L)}$ is and only if the coefficients of the morphism can be chosen to be in L, i.e. if $\varphi \in D$, hence the first equality. The second follows from the fact that an element in the center of $\widetilde{D}$ is also in the center of D (because $D \subset \widetilde{D}$).

Fact 2: if $E \subset \widetilde{E}$ then $\dim(\rho_{A,\lambda}) = \dim(\widetilde{\sigma_{A,\lambda}})\frac{\tilde{d}}{d}$.

After choosing an extension of each element in $\mathrm{Gal}(L/K)$ to $\widetilde{L}$ (any will do), $\mathrm{Gal}(\widetilde{L}/K) = \mathrm{Gal}(\widetilde{L}/L)\,\mathrm{Gal}(L/K)$, hence

$$\widetilde{E}_0 = \widetilde{E}^{\mathrm{Gal}(\widetilde{L}/K)} = (\widetilde{E}^{\mathrm{Gal}(\widetilde{L}/L)})^{\mathrm{Gal}(L/K)} = E^{\mathrm{Gal}(L/K)} = E_0,$$

where the middle equality comes from Fact 1 (since $\widetilde{E} \cap E = E$ under our hypothesis). The statement follows from Lemma 3.2.

Fact 3: If $\tilde{d} = d$, then $E \subseteq \tilde{E}$.

This is [CMSV19, Lemma 7.4.2], but the proof is much simpler for division algebras. Consider the subalgebra $D \otimes_E E\tilde{E} \subset \tilde{D}$ obtained by extending scalars to the field compositum $E\tilde{E}$, and note that

$$\dim_{\tilde{E}} D \otimes_E E\tilde{E} = \dim_E D \cdot [E\tilde{E} : \tilde{E}] = d^2[E\tilde{E} : \tilde{E}] \leq \dim_{\tilde{E}} \tilde{D} = \tilde{d}^2.$$

The equality $d = \tilde{d}$ implies $[E\tilde{E} : \tilde{E}] = 1$, hence $E \subseteq \tilde{E}$.

The previous three facts imply that $\dim(\rho_{A,\lambda}) = \dim(\widetilde{\sigma_{A,\lambda}})$ if and only if either $d = \tilde{d}$ or $[E_0 : \widetilde{E}_0] = \tilde{d}/d$: suppose that the equality of dimensions holds, then $[E_0 : \widetilde{E}_0] = \tilde{d}/d$.

Conversely, if $[E_0 : \widetilde{E}_0] = \tilde{d}/d$, then comparing the dimensions again yields the equality. Finally, suppose that $\tilde{d} = d$. Then by Fact 3 we have $E \subseteq \tilde{E}$, and Fact 2 shows $\dim(\rho_{A,\lambda}) = \dim(\widetilde{\sigma_{A,\lambda}})$. ☐

Theorem 3.18. *Suppose that d and $\tilde{d} \leq 2$. Keeping the previous notation, we get the following relation between $\dim(\rho_{A,\lambda})$ and $\dim(\widetilde{\sigma_{A,\lambda}})$:*

- *They are the same if either:* $\dim_{\tilde{E}} \tilde{D} = \dim_E D$ *or* $[E_0 : \widetilde{E}_0] = 2$,
- $\dim(\rho_{A,\lambda}) = 2\dim(\widetilde{\sigma_{A,\lambda}})$ *if* $\dim_{\tilde{E}} \tilde{D} \neq \dim_E D$ *and either* $E \subset \widetilde{E}$ *or* $E_0 = \widetilde{E}_0$.

Proof. If D and $\widetilde{D}$ have the same type, then $E = Z(D) \subset Z(\widetilde{D}) = \widetilde{E}$ and $d = \tilde{d}$ so the statement follows from Fact 2 (of the previous theorem). Otherwise, it must be the case that $\widetilde{D}$ is a quaternion algebra, while D is abelian (in which case $\tilde{d} = 2$ and $d = 1$). If $E \subset \widetilde{E}$ then the statement follows once again from Fact 2. Suppose then that $E \cap \widetilde{E} \subsetneq E$ (so $[E : E \cap \widetilde{E}] = 2$). Then

$$[E_0 : E_0 \cap \widetilde{E}_0] = [E^{\mathrm{Gal}(\widetilde{L}/K)} : E^{\mathrm{Gal}(\widetilde{L}/K)} \cap \widetilde{E}^{\mathrm{Gal}(\widetilde{L}/K)}] \leq [E : E \cap \widetilde{E}] = 2.$$

On the other hand,

$$\widetilde{E}_0 = \widetilde{E}^{\mathrm{Gal}(\widetilde{L}/K)} = (\widetilde{E}^{\mathrm{Gal}(\widetilde{L}/L)})^{\mathrm{Gal}(L/K)} = (\widetilde{E} \cap E)^{\mathrm{Gal}(\widetilde{L}/K)} = \widetilde{E}_0 \cap E_0.$$

In particular, $[E_0 : \widetilde{E}_0] \leq 2$. If it equals 1, then $\dim(\rho_{A,\lambda}) = 2\dim(\widetilde{\sigma_{A,\lambda}})$ by Lemma 3.2, while if it equals 2, then both dimensions are the same as claimed. ☐

4. Equivariant pairings

It is natural to wonder whether/when the constructed representation preserves a bilinear form. Let K be a field. A subgroup $G \subseteq \mathrm{GL}_n(K)$ is called irreducible if there are no non-trivial proper G-invariant subspaces of K^n. We have the following variant of Schur's lemma.

Proposition 4.1. *Let G be an irreducible group, and let $\chi : G \to K^\times$, $^\dagger : G \to GL_n(K)$ be any two maps. Let $M, N \in \mathrm{GL}_n(K)$ be matrices satisfying*

$$\begin{cases} B^\dagger M B &= \chi(B)M \\ B^\dagger N B &= \chi(B)N. \end{cases}$$

for all $B \in G$. Then, there exists a unique $\lambda \in K^\times$ such that $M = \lambda N$.

Proof. Let $p(x) = \det(M - xN) \in K[x]$, a polynomial of degree n and leading coefficient $(-1)^n \det N \neq 0$. Let $\lambda \in \overline{K}$ be a root of $p(x)$. Clearly $\lambda \neq 0$, since M would be singular otherwise. Let $v \in \ker(M - \lambda N)$ be any nonzero vector. Then, for every $B \in G$ we have

$$(M - \lambda N)Bv = \chi(B)(B^\dagger)^{-1}(M - \lambda N)v = 0,$$

hence G stabilizes $\ker(M - \lambda N)$. But this is a nontrivial subspace of K^n, and therefore (since G is irreducible) $\ker(M - \lambda N) = K^n$, so $M = \lambda N$ and $\lambda \in K^\times$. Uniqueness follows from the equality $p(x) = (\lambda - x)^n$. □

The last proposition implies that once a similitude character and a type are fixed, there exists at most one G-equivariant pairing with the given type and similitude character.

Corollary 4.2. *Let G be an irreducible subgroup of $\mathrm{GL}_n(K)$, $V = K^n$, and let $\Psi_i : V \times V \to K$, $i = 1, 2$ be two biadditive form. Suppose that there exists a character $\chi : G \to K^\times$ such that, for every $g \in G$ and all $v, w \in V$,*

$$\Psi_i(gv, gw) = \chi(g)\Psi_i(v, w), \text{ for } i = 1, 2.$$

If both Ψ_1 and Ψ_2 are alternating (resp. symmetric, or hermitian), then there exists $\lambda \in K^\times$ such that $\Psi_1 = \lambda\Psi_2$.

Proof. Follows from the proposition. □

Proposition 4.3. *Let G be a subgroup of $\mathrm{GL}_n(K)$ and let H be a normal irreducible subgroup of G. Let $V = K^n$. Suppose that there exists a biadditive H-invariant form $\Psi : V \times V \to K$ with similitude character $\chi : H \to K^\times$, i.e. for every $h \in H$ and all $v, w \in V$,*

$$\Psi(hv, hw) = \chi(h)\Psi(v, w).$$

Suppose that Ψ is alternating, symmetric or hermitian. Then the following are equivalent:

(1) *The form Ψ is G-invariant (for some similitude character).*
(2) *The character χ extends to a character $\widetilde{\chi} : G \to K^\times$.*
(3) *For all $g \in G$, $h \in H$, $\chi(ghg^{-1}) = \chi(h)$.*

Proof. It is clear that (1) ⇒ (2) ⇒ (3) (the character $\widetilde{\chi}$ being the similitude character). Suppose that the biadditive form Ψ is alternating or symmetric. Then, there exists an antisymmetric matrix $J \in \mathrm{GL}_n(K)$ such that

$$\Psi(v, w) = v^\top Jw.$$

The invariance property translates into $h^\top Jh = \chi(h)J$ for all $h \in H$. Since H is normal in G, for all $g \in G$ and all $h \in H$ we have $ghg^{-1} \in H$, and hence

$$(ghg^{-1})^\top J(ghg^{-1}) = \chi(ghg^{-1})J = \chi(h)J.$$

By direct manipulation we obtain the equality $h^\top(g^\top Jg)h = \chi(h)g^\top Jg$. Proposition 4.1 (with $h^\dagger = h^\top$, $M = g^\top Jg$ and $N = J$) implies that there is a unique $\tilde{\chi}(g) \in K^\times$ such that

$$g^\top Jg = \tilde{\chi}(g)J.$$

The similitude map $g \mapsto \tilde{\chi}(g)$ is actually a character, since $(gg')^\top Jgg' = \tilde{\chi}(gg')J = \tilde{\chi}(g)\tilde{\chi}(g')J$. By uniqueness of each $\tilde{\chi}(g)$, we obtain $\tilde{\chi}|_H = \chi$.

The hermitian case works the same replacing $g^\top$ by $\overline{g}^\top$, where $\bar{\cdot}$ is the involution of K. □

Remark 4.4. *The proposition implies that the representation $\widetilde{\rho_\lambda}$ attached to a strong K-variety preserves the bilinear form Ψ from Theorem 2.6 precisely when the twisting character ψ (as in Theorem 3.3) satisfies that $\psi^2(ghg^{-1}) = \psi^2(h)$ for all $h \in \mathrm{Gal}_L$, $k \in \mathrm{Gal}_K$.*

The same remark implies that when L/K is cyclic, ψ can be taken to be trivial, so in this case the representation always preserves a bilinear form.

A similar result holds while studying existence of an invariant bilinear form for the induction of a representation.

Proposition 4.5. *Let G be a group and $H \lhd G$ be a finite index normal subgroup. Let K be a field and let W be a finite dimensional K-vector space. Let $\rho : H \to \mathrm{GL}(W)$ be a representation and let $\Psi : W \times W \to K$ be an additive H-invariant form with similitude character χ. Let*

$$V = \sum_{h \in G/H} hW, \tag{9}$$

be the underlying K-vector space of the induced representation $\mathrm{Ind}_H^G \rho$. Then the following are equivalent:

1. *There exists a K-bilinear, G-equivariant non-degenerate pairing $\widetilde{\Psi}$ on V whose restriction to W equals Ψ.*
2. *The character χ extends to a character of G.*

Proof. Start supposing that there exists a G-equivariant pairing $\widetilde{\Psi} : V \times V \to K$ with similitude character $\widetilde{\chi}$ such that $\widetilde{\Psi}|_{W\times W} = \Psi$ (identifying W with the susbpace corresponding to $h = 1$ in (9)). The equivariance property implies that, for all $u, v \in W$ and all $g \in G$,

$$\widetilde{\Psi}(g \cdot u, g \cdot v) = \widetilde{\chi}(g)\widetilde{\Psi}(u, v) = \widetilde{\chi}(g)\Psi(u, v).$$

If $g \in H$ then the left hand side also equals $\chi(g)\Psi(u, v)$ so (since Ψ is non-degenerate) $\chi(g) = \widetilde{\chi}(g)$ for $g \in H$; $\widetilde{\chi}$ is the desired extension.

Conversely, suppose that χ extends to a character $\widetilde{\chi}$ of G. Let $\{h_1, \ldots, h_n\}$ be representatives for G/H with $h_1 = 1$. Define a bilinear pairing $\widetilde{\Psi} : V \times V \to K$ by decreeing that:

1. $\widetilde{\Psi}(h_iW, h_jW) = 0$ if $i \neq j$,
2. $\widetilde{\Psi}(h_iu, h_iv) := \widetilde{\chi}(h_i)\Psi(u, v)$ for $i = 1, \ldots, n$.

The pairing $\widetilde{\Psi}$ thus defined satisfies the expected properties, namely: it is G-equivariant, non-degenerate and its restriction to $W \times W$ coincides with Ψ.

□

Remark 4.6. *If the similitude character of χ extends to G, its extension need not be unique. There are as many extensions as characters of the group G/H, and each extension gives a different pairing.*

Propositions 4.3 and 4.5 imply that the representation $\rho_{A,\lambda}$ preserves an extension of the pairing obtained in Theorem 2.6 if and only if the character ψ of Theorem 3.3 (or a multiple of it by the square of a character of $\mathrm{Gal}_{L'}$) extends to Gal_K.

5. Applications

5.1. **Some atypical images of Galois representations.** Let A be an abelian fourfold defined over a number field K. Let L/K be a finite Galois extension such that $\mathrm{End}^0(A_{\overline{K}}) = \mathrm{End}^0(A_L)$. Assume the following two conditions:

(1) $F = \mathrm{End}^0(A)$ is an imaginary quadratic field, and
(2) $\mathrm{End}^0(A_L)$ is a (possibly split) indefinite quaternion algebra over $\mathbb{Q}$.

Let ℓ be a rational prime that remains inert in F and let λ be the prime of F over ℓ, so that F_λ is a quadratic extension of $\mathbb{Q}_\ell$. Let γ be the nontrivial automorphism of $F_\lambda/\mathbb{Q}_\ell$. The space $V_\ell(A)$ has an F_λ-module structure; denote by W_λ the vector space $V_\ell(A)$ with this structure. By Theorem 2.6, W_λ is irreducible as an $F_\lambda[\mathrm{Gal}_K]$-module and has F_λ-dimension 4. Denote by $\rho_{A,\lambda}$ the corresponding λ-adic representation.

The space W_λ comes with a non-degenerate F_λ-hermitian pairing $\Psi_\lambda : W_\lambda \times W_\lambda \to F_\lambda$, with the property that for all $v, w \in W_\lambda$ and all $\sigma \in \mathrm{Gal}_K$,

$$\Psi_\lambda(\rho_{A,\lambda}(\sigma)v, \rho_{A,\lambda}(\sigma)w) = \chi_\ell(\sigma) \cdot \Psi_\lambda(v, w).$$

Hence the image of $\rho_{A,\lambda}$ is contained in $\mathrm{GU}_4(\mathbb{Q}_\ell)$.

On the other hand, the same theorem can be applied to A_L, but now the center of $\mathrm{End}^0(A_L)$ is $\mathbb{Q}$. Hence there exists an absolutely irreducible $\mathbb{Q}_\ell[\mathrm{Gal}_L]$-submodule $\mathcal{W}_\ell$ of $V_\ell(A_L)$, such that $V_\ell(A_L) \simeq \mathcal{W}_\ell \times \mathcal{W}_\ell$, together with a non-degenerate $\mathbb{Q}_\ell$-bilinear alternating pairing $\Phi_\ell : \mathcal{W}_\ell \times \mathcal{W}_\ell \to \mathbb{Q}_\ell$. This pairing satisfies that, for all $v, w \in \mathcal{W}_\lambda$ and each $\tau \in \mathrm{Gal}_L$,

$$\Phi_\ell(\rho_{A,\lambda}(\tau)v, \rho_{A,\lambda}(\tau)w) = \chi_\ell(\tau) \cdot \Phi_\ell(v, w).$$

Now there is an isomorphism of $F_\lambda[\mathrm{Gal}_L]$-modules $W_\lambda \simeq \mathcal{W}_\ell \otimes_{\mathbb{Q}_\ell} F_\lambda$. To see this, note that W_λ (seen as a $\mathbb{Q}_\ell[\mathrm{Gal}_L]$-module) is $V_\ell(A_L)$, and hence tensoring with F_λ we obtain

$$V_\ell(A_L) \otimes_{\mathbb{Q}_\ell} F_\lambda \simeq W_\lambda \times {}^\gamma W_\lambda.$$

Here ${}^\gamma W_\lambda$ is W_λ with a γ-linear action of F_λ. On the other hand we have $V_\ell(A_L) \otimes_{\mathbb{Q}_\ell} F_\lambda \simeq (\mathcal{W}_\ell \times \mathcal{W}_\ell) \otimes_{\mathbb{Q}_\ell} F_\lambda$, and since all the modules are absolutely irreducible, we obtain the isomorphism $W_\lambda \simeq \mathcal{W}_\ell \otimes_{\mathbb{Q}_\ell} F_\lambda$. Hence W_λ inherits (by extension of scalars to F_λ) the alternating pairing Φ_ℓ. By Proposition 4.3 the pairing Φ_ℓ is also Gal_K-equivariant up to a similitude character. More explicitly, there exists a finite order character $\theta : \mathrm{Gal}_K \to \bar{\mathbb{Q}}^\times$ which factors through $\mathrm{Gal}(L/K)$, and such that for all $v, w \in W_\lambda$ and all $\sigma \in \mathrm{Gal}_K$, we have

$$\Phi_\ell(\rho_{A,\lambda}(\sigma)v, \rho_{A,\lambda}(\sigma)w) = \theta(\sigma)\chi_\ell(\sigma) \cdot \Phi_\ell(v, w).$$

This shows that the image of $\rho_{A,\lambda}$ is contained in $\mathrm{GSp}_4(F_\lambda)$ and has similitude character $\theta\chi_\ell$. We have thus seen that $\rho_{A,\lambda}(\mathrm{Gal}_L) \subseteq \mathrm{GU}_4(\mathbb{Q}_\ell) \cap \mathrm{GSp}_4(F_\lambda)$, the intersection taking place in $\mathrm{GL}_4(F_\lambda)$. Particular examples can be found in the forthcoming preprint [FFG25].

Observe that A_L is a strong abelian K-variety, by Corollary 3.11 and the fact that $Z(\mathrm{End}^0(A_L)) = \mathbb{Q} \subset F = \mathrm{End}^0(A)$. Therefore the representation $\rho_{A,\lambda}|_{\mathrm{Gal}_L}$ extends (after a twist) to a representation of Gal_K by Theorem 3.3, but the twist is in fact trivial since $\rho_{A,\lambda}$ is already a representation of Gal_K.

If $\mathrm{End}^0(A_L)$ is a *definite* quaternion algebra (instead of an indefinite one), the same argument proves that $\rho_{A,\lambda}(\mathrm{Gal}_L) \subseteq \mathrm{GU}_4(\mathbb{Q}_\ell) \cap \mathrm{GO}_4(F_\lambda)$. An example of this case is given in [CFLV23], where A is the Jacobian of the genus 4 curve

$$C : y^2 = x(x^4 + 1)(x^4 + x^2 + 1).$$

5.2. **Abelian surfaces with potential QM are modular.** Let $A/\mathbb{Q}$ be an abelian surface such that $\mathrm{End}^0(A_{\overline{\mathbb{Q}}})$ is an indefinite (possibly split) quaternion algebra. If $\mathrm{End}^0(A)$ is a quadratic field, the surface A is modular by Ribet's results ([Rib04]), together with Serre's modularity conjecture ([KW09a, KW09b]. We assume that $\mathrm{End}^0(A) = \mathbb{Q}$, and we claim that A is also modular in this case.

Lemma 5.1. *A_L is a strong $\mathbb{Q}$-variety.*

Proof. The relation ${}^{\sigma}\varphi = \varphi$ holds trivially for all $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$ and all $\varphi \in Z(\mathrm{End}^0(A_L)) = \mathbb{Q}$. Then Corollary 3.11 implies there are isogenies $\mu_\sigma : A_L \to A_L$ such that A_L is a strong $\mathbb{Q}$-variety. □

By [DR04, Proposition 2.1], there exists a smallest Galois extension $L/\mathbb{Q}$ such that $D := \mathrm{End}^0(A_L) = \mathrm{End}^0(A_{\bar{\mathbb{Q}}})$, and such that $\mathrm{Gal}(L/\mathbb{Q})$ is either C_n or D_n with $n = 2, 3, 4$ or 6.

Lemma 5.2. *For each field $K \subseteq L$ such that L/K is cyclic, $\mathrm{End}^0(A_K)$ contains a quadratic field. In particular, the assumption $\mathrm{End}^0(A) = \mathbb{Q}$ implies that $L/\mathbb{Q}$ is a dihedral extension, and there exists a quadratic extension $K/\mathbb{Q}$ such that $\mathrm{End}^0(A_K)$ is a quadratic field.*

Proof. By the Skolem-Noether theorem, for each $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$ there exists some $\alpha_\sigma \in D^\times$ such that ${}^{\sigma}\varphi = \alpha_\sigma \varphi \alpha_\sigma^{-1}$ for each $\varphi \in D$. In particular, for each $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$ the subalgebra $\mathrm{End}^0(A_{L^{\langle\sigma\rangle}})$ equals the centralizer in D of the field $\mathbb{Q}(\alpha_\sigma)$.

With this in mind, we let $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$ be any element and let $K := L^{\langle\sigma\rangle}$. If $\alpha_\sigma \in \mathbb{Q}^\times$, then the centralizer of $\mathbb{Q}$ in D is D, and in fact $K = L$. Otherwise, α_σ generates a (maximal) quadratic subfield of D, and the centralizer of $\mathbb{Q}(\alpha_\sigma)$ is $\mathbb{Q}(\alpha_\sigma)$ itself. This proves the claim. □

Theorem 5.3. *The abelian surface A is Siegel modular, i.e. there exists a Siegel newform of $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ whose L-series matches that of A.*

Proof. Since all endormorphisms of A are defined over L, there exists an irreducible Galois representation $\rho_{\lambda,L} : \mathrm{Gal}_L \to \mathrm{GL}_2(\overline{\mathbb{Q}_\ell})$ attached to A/L. By Lemma 5.1 A/L is a strong $\mathbb{Q}$-variety, so Theorem 3.3 implies that there exists a character θ of Gal_L and a representation $\widetilde{\rho_\lambda} : \mathrm{Gal}_\mathbb{Q} \to \mathrm{GL}_2(\overline{\mathbb{Q}_\ell})$ (with Hodge-Tate weights $\{0, 1\}$) such that $\widetilde{\rho_\lambda}|_{\mathrm{Gal}_L} \simeq (\rho_{\lambda,L}) \otimes \theta$.

Let ℓ be a prime number such that the residual representation $\overline{\widetilde{\rho_\lambda}}$ is absolutely irreducible (this is always the case if ℓ is large enough). Serre's modularity conjecture ([KW09a, KW09b]) implies that the residual representation $\overline{\widetilde{\rho_\lambda}}$ is modular so (by [Kis09]) $\widetilde{\rho_\lambda}$ itself is modular. By solvable base change for GL_2 (as proved in [Lan80]), the representation $\widetilde{\rho_\lambda}$ restricted to Gal_L is modular, and then the same is true for ρ_λ (as twisting preserves modularity).

Fix a quadratic extension $K/\mathbb{Q}$ as given in Lemma 5.2 and let $E = \mathrm{End}^0(A_K)$ be the corresponding quadratic field. Let ℓ be a rational prime splitting in E. Consider the ℓ-adic representation of A, $\rho_\ell : \mathrm{Gal}_\mathbb{Q} \to \mathrm{GSp}_4(\mathbb{Q}_\ell)$. Fix λ a prime of E over ℓ, then there is a subrepresentation (abusing notation) $\rho_\lambda : \mathrm{Gal}_K \to \mathrm{GL}_2(E_\lambda)$ of $\rho_\ell|_{\mathrm{Gal}_K}$, and $\rho_\ell = \mathrm{Ind}_{\mathrm{Gal}_\mathbb{Q}}^{\mathrm{Gal}_K} \rho_\lambda$. The representation ρ_λ must be a twist of $\widetilde{\rho_\lambda}$ restricted to Gal_K (as they have the same projective representation), so ρ_λ is also modular. Automorphic induction (as proven in [Hen12, Théorème 3]) then implies that $\mathrm{Ind}_{\mathrm{Gal}_\mathbb{Q}}^{\mathrm{Gal}_K} \rho'_\lambda \simeq \rho_\ell$ is modular itself.

We are led to prove that the automorphic representation Π of $\mathrm{GL}_4(\mathbb{A}_\mathbb{Q})$ is actually a transfer from $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$. The fact that A preserves the Weil pairing implies that Π has symplectic type (as in [BCGP21] §2.9) and the result then follows from [BCGP21, Theorem 2.9.3]. ☐

In general we do not know how to prove that the automorphic form of the last theorem has a paramodular fixed vector. Following the previous notation, if ρ_λ has trivial Nebentypus (i.e. its similitude character equals the cyclotomic one), then we can show the paramodularity of its induction to $\mathrm{Gal}_{\mathbb{Q}}$. We do that as follows.

Proposition 5.4. *Suppose* $\mathrm{Gal}(L/\mathbb{Q}) \simeq D_4 \simeq C_2 \times C_2$*. Then, there is at most one quadratic subfield* K *of* L *such that* $E = \mathrm{End}^0(A_K)$ *is an imaginary quadratic field. If such* K *exists, the determinant of the representation* $\rho_\lambda : \mathrm{Gal}_K \to \mathrm{GL}_2(E_\lambda)$ *equals* $\eta_{L/K} \cdot \chi_\ell$*, with* $\eta_{L/K}$ *the nontrivial character of* $\mathrm{Gal}(L/K)$*. For any other quadratic subfield* $K' \subset L$*,* $\mathrm{End}^0(A_{K'})$ *is real quadratic, and the determinant of the corresponding representation is* χ_ℓ*.*

Proof. Let K_1 and K_2 be any two different quadratic subfields of L. We will show that either $E_1 = \mathrm{End}^0(A_{K_1})$ or $E_2 = \mathrm{End}^0(A_{K_2})$ is a real quadratic field.

Suppose for a contradiction that E_1 and E_2 are both imaginary quadratic fields. We first show that $E_1 \neq E_2$. For that, let $\sigma_i \in \mathrm{Gal}(L/\mathbb{Q})$ be the nontrivial automorphism fixing K_i for $i = 1, 2$. Then $E_i = \mathrm{End}^0(A_{L^{\langle\sigma_i\rangle}})$ for $i = 1, 2$ by Lemma 5.2. Hence if $E_1 = E_2$, we would have $E_1 \subset \mathrm{End}^0(A_{L^{\langle\sigma_1,\sigma_2\rangle}}) = \mathrm{End}^0(A)$. But this is a contradiction, since we are assuming that $\mathrm{End}^0(A) = \mathbb{Q}$. Hence $E_1 \neq E_2$.

It follows that the quaternion algebra $D = \mathrm{End}^0(A_L)$ is generated (as an algebra over $\mathbb{Q}$) by two quadratic elements $\sqrt{a_1}$, $\sqrt{a_2}$ with $a_1, a_2 \in \mathbb{Q}$ that generate E_1 and E_2, respectively. These elements do not commute, since $E_1 \neq E_2$ are maximal fields and D is noncommutative. Since $\mathrm{End}^0(A_L)$ is an indefinite quaternion algebra, either a_1 or a_2 must be positive: otherwise, $D \otimes_{\mathbb{Q}} \mathbb{R}$ would be isomorphic to the Hamilton quaternions, which contradicts the fact that D is indefinite. Therefore either $E_1 = \mathbb{Q}(\sqrt{a_1})$ or $E_2 = \mathbb{Q}(\sqrt{a_2})$ is a real quadratic field.

Let $K \subset L$ be any quadratic subfield and let $E = \mathrm{End}^0(A_K)$ be the corresponding quadratic field of endomorphisms. Given $\sigma \in \mathrm{Gal}_K$, by Skolem-Noether we have that there exists some $\alpha_\sigma \in E^\times$ such that ${}^\sigma\varphi = \alpha_\sigma \varphi \alpha_\sigma^{-1}$ for all $\varphi \in D$. Each α_σ is determined up to multiplication by elements in $\mathbb{Q}^\times$, the center of D. It can be checked that the determinant of the representation $\rho_\lambda : \mathrm{Gal}_K \to \mathrm{GL}_2(E_\lambda)$ is the cyclotomic character times

$$\chi : \mathrm{Gal}(L/K) \to E^\times$$
$$\sigma \mapsto \alpha_\sigma / \overline{\alpha_\sigma}$$

where $\bar{\cdot}$ denotes complex conjugation (cf. [Pyl04], Lemma 5.11). In particular, when E is a real field, then complex conjugation restricts to the identity on E, and so χ is trivial.

When E is an imaginary field, we use [Rib04, Proposition 3.5] to find a trace of Frobenius $a_{\mathfrak{p}} \neq 0$ with $E = \mathbb{Q}(a_{\mathfrak{p}})$, and then by [Rib04, Proposition 3.4] we have $a_{\mathfrak{p}} = \chi(\mathrm{Frob}_{\mathfrak{p}})\overline{a_{\mathfrak{p}}}$. This shows $\chi(\mathrm{Frob}_{\mathfrak{p}}) = -1$, and so χ is nontrivial when E is imaginary quadratic. ☐

Remark 5.5. *Keeping the last proposition hypothesis, if there exists a quadratic field* K *such that* $\mathrm{End}^0(A_K)$ *is imaginary quadratic, then the representation* $\rho_{A,\ell} : \mathrm{Gal}_{\mathbb{Q}} \to \mathrm{GL}_4(\mathbb{Q}_\ell)$ *preserves two alternating pairings. The first of them is obtained from* K *by taking the skew-symmetric form on a 2-dimensional vector space, whose similitude character matches*

$\det \rho_\lambda = \eta_{L/K} \cdot \chi_\ell$. *Since* $\mathrm{Gal}(L/\mathbb{Q})$ *is abelian, the character* $\eta_{L/K}$ *extends to the full Galois group, and Proposition 4.5 produces a non-degenerate alternating pairing on the induced representation* $\rho_{A,\ell} = \mathrm{Ind}_{\mathrm{Gal}_\mathbb{Q}}^{\mathrm{Gal}_K} \rho_\lambda$.

On the other hand, the representation $\rho_{A,\ell}$ *preserves the Weil pairing, which has similitude character* χ_ℓ. *The statement is consistent with the fact that many traces of* $\rho_{A,\ell}$ *are zero (otherwise, the two similitude characters would coincide).*

Here is an example of this phenomenon from [DR04] *(page 620). Consider the genus* 2 *curve*

$$\mathcal{C} : y^2 = (x^2+7)(83/30x^4 + 14x^3 - 1519/30x^2 + 49x - 1813/120),$$

and let A *denote its Jacobian. Then the following holds:*

- $\mathrm{End}_L(A)$ *is a maximal order in* B_6 *(the indefinite quaternion algebra of discriminant* 6*), for* $L = \mathbb{Q}(\sqrt{-6}, \sqrt{-14})$.
- $\mathrm{End}_\mathbb{Q}(A) = \mathbb{Z}$.
- $\mathrm{End}_{\mathbb{Q}(\sqrt{-6})}(A) \otimes_\mathbb{Z} \mathbb{Q} = \mathbb{Q}(\sqrt{-6})$.

Corollary 5.6. *Suppose that* A *is principally polarizable and that* $\mathrm{End}(A_L)$ *is an hereditary order in* $\mathrm{End}^0(A_L)$. *Then* A *is paramodular.*

Proof. By [DR04, Theorem 3.4], our hypotheses imply that $\mathrm{Gal}(L/\mathbb{Q}) \simeq C_2 \times C_2$, so the Proposition 5.4 implies that we can take K such that $\mathrm{End}^0(A_K)$ is real quadratic in the proof of Theorem 5.3, so the Nebentypus of $\rho_{A,\lambda}$ is trivial. Now the result follows from [JLR12] (main theorem) when K is real quadratic and from [BDPc15, Theorem 4.1] when K is imaginary quadratic. □

References

[BCGP21] George Boxer, Frank Calegari, Toby Gee, and Vincent Pilloni. Abelian surfaces over totally real fields are potentially modular. *Publ. Math. Inst. Hautes Études Sci.*, 134:153–501, 2021.

[BDPc15] Tobias Berger, Lassina Dembélé, Ariel Pacetti, and Mehmet Haluk Şengün. Theta lifts of Bianchi modular forms and applications to paramodularity. *J. Lond. Math. Soc. (2)*, 92(2):353–370, 2015.

[BGK06] G. Banaszak, W. Gajda, and P. Krasoń. On the image of l-adic Galois representations for abelian varieties of type I and II. *Doc. Math.*, pages 35–75, 2006.

[BGK10] Grzegorz Banaszak, Wojciech Gajda, and Piotr Krasoń. On the image of Galois l-adic representations for abelian varieties of type III. *Tohoku Math. J. (2)*, 62(2):163–189, 2010.

[BKG21] Grzegorz Banaszak and Aleksandra Kaim-Garnek. The Tate module of a simple abelian variety of type IV. *New York J. Math.*, 27:1240–1257, 2021.

[CFLV23] Victoria Cantoral-Farfán, Davide Lombardo, and John Voight. Monodromy groups of jacobians with definite quaternionic multiplication, 2023.

[Chi91] Wên Chên Chi. On the l-adic representations attached to simple abelian varieties of type IV. *Bull. Austral. Math. Soc.*, 44(1):71–78, 1991.

[CMSV19] Edgar Costa, Nicolas Mascot, Jeroen Sijsling, and John Voight. Rigorous computation of the endomorphism ring of a Jacobian. *Math. Comp.*, 88(317):1303–1339, 2019.

[DR04] Luis V. Dieulefait and Victor Rotger. The arithmetic of QM-abelian surfaces through their Galois representations. *J. Algebra*, 281(1):124–143, 2004.

[Fal83] G. Faltings. Endlichkeitssätze für abelsche Varietäten über Zahlkörpern. *Invent. Math.*, 73(3):349–366, 1983.

[FFG25] Francesc Fité, Enric Florit, and Xavier Guitart. Abelian varieties genuinely of GL(n) type, 2025. In preparation.

[FG22] Francesc Fité and Xavier Guitart. Tate module tensor decompositions and the Sato-Tate conjecture for certain abelian varieties potentially of GL_2-type. *Math. Z.*, 300(3):2975–2995, 2022.

[Gui12] Xavier Guitart. Abelian varieties with many endomorphisms and their absolutely simple factors. *Rev. Mat. Iberoam.*, 28(2):591–601, 2012.

[Hen12] Guy Henniart. Induction automorphe globale pour les corps de nombres. *Bull. Soc. Math. France*, 140(1):1–17, 2012.

[Isa06] I. Martin Isaacs. *Character theory of finite groups*. AMS Chelsea Publishing, Providence, RI, 2006. Corrected reprint of the 1976 original [Academic Press, New York; MR0460423].

[JLR12] Jennifer Johnson-Leung and Brooks Roberts. Siegel modular forms of degree two attached to Hilbert modular forms. *J. Number Theory*, 132(4):543–564, 2012.

[Kis09] Mark Kisin. Moduli of finite flat group schemes, and modularity. *Ann. of Math. (2)*, 170(3):1085–1180, 2009.

[KW09a] Chandrashekhar Khare and Jean-Pierre Wintenberger. Serre's modularity conjecture. I. *Invent. Math.*, 178(3):485–504, 2009.

[KW09b] Chandrashekhar Khare and Jean-Pierre Wintenberger. Serre's modularity conjecture. II. *Invent. Math.*, 178(3):505–586, 2009.

[Lan80] Robert P. Langlands. *Base change for* $\mathrm{GL}(2)$, volume No. 96 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1980.

[Mum08] David Mumford. *Abelian varieties*, volume 5 of *Tata Institute of Fundamental Research Studies in Mathematics*. Tata Institute of Fundamental Research, Bombay; by Hindustan Book Agency, New Delhi, 2008. With appendices by C. P. Ramanujam and Yuri Manin, Corrected reprint of the second (1974) edition.

[Pie82] Richard S. Pierce. *Associative algebras*, volume 9 of *Studies in the History of Modern Science*. Springer-Verlag, New York-Berlin, 1982. Graduate Texts in Mathematics, 88.

[Pyl04] Elisabeth E. Pyle. Abelian varieties over $\mathbb{Q}$ with large endomorphism algebras and their simple components over $\overline{\mathbb{Q}}$. In *Modular curves and abelian varieties*, volume 224 of *Progr. Math.*, pages 189–239. Birkhäuser, Basel, 2004.

[Rib76] Kenneth A. Ribet. Galois action on division points of Abelian varieties with real multiplications. *Amer. J. Math.*, 98(3):751–804, 1976.

[Rib04] Kenneth A. Ribet. Abelian varieties over $\mathbb{Q}$ and modular forms. In *Modular curves and abelian varieties*, volume 224 of *Progr. Math.*, pages 241–261. Birkhäuser, Basel, 2004.

[Ser77] J.-P. Serre. Modular forms of weight one and Galois representations. In *Algebraic number fields: L-functions and Galois properties (Proc. Sympos., Univ. Durham, Durham, 1975)*, pages 193–268. Academic Press, London-New York, 1977.

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