ON RANK 2 HYPERGEOMETRIC MOTIVES

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Abstract. Hypergeoemtric motives have a combinatorial nature that makes them more tractable than general motives. In the present article we prove most of their expected properties in the rank 2 case.

1. Introduction

The theory of hypergeometric motives has its origins in the study of the complex hypergeometric series

$$ {}_2F_1(a,b;c|z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n, $$

for z a complex number satisfying $|z| < 1$ (as done by Gauss [23], Kummer [34], Goursat [25] et al). Riemann (in [38], "Contribution a la théorie des fonctions représentables par la série de Gauss $F(a,b,c,x)$") studied the differential equation satisfied by this series. This differential equation yields a *rank two local system* on $\mathbb{P}^1 \setminus \{0,1,\infty\}$ which in turn determines a 2-dimensional *monodromy representation*

$$ \rho : \pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\}, z_0) \to \mathrm{GL}_2(\mathbb{C}), $$

describing how a chosen basis of local solutions at a regular point z_0 changes while looping around the missing points. Riemann realized that the monodromy representation is a natural way to encode information of the differential equation and its solutions.

We denote by M_0, M_1 and M_∞ the local monodromy matrices obtained by looping around the point 0, 1 and ∞ respectively, chosen so that

$$ M_0 M_1 M_\infty = I_2, $$

where I_2 is the identity matrix.

We are interested in algebraic encarnations of the hypergeometric series, so for the rest of the introduction we let a, b, c be rational numbers and let N be their common denominator. Under this assumption, a result of Levelt implies that the monodromy representation is *algebraic*: it can be chosen to take values in the number field $F := \mathbb{Q}(\zeta_N)$, with ζ_N a primitive N-th root of unity. Given a prime ideal $\mathfrak{p}$ of $\mathbb{Z}[\zeta_N]$ one can extend the monodromy representation to a *geometric* continuous representation

$$ \rho_{\mathfrak{p}} : \mathrm{Gal}(\overline{\mathbb{Q}(z)}/\overline{\mathbb{Q}}(z)) \to \mathrm{GL}_2(F_{\mathfrak{p}}), \tag{1} $$

where $F_{\mathfrak{p}}$ denotes the $\mathfrak{p}$-adic completion of F at $\mathfrak{p}$ (as explained in §2) and z is a variable.

We expect the representation (1) to be *motivic*, namely to match the restriction of a representation of $\mathrm{Gal}(\overline{\mathbb{Q}(z)}/F(z))$ to $\mathrm{Gal}(\overline{\mathbb{Q}(z)}/\overline{\mathbb{Q}}(z))$ arising from a motive defined over a number field F. In

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particular and more precisely, for every specialization $z \in L$ of the parameter in a finite extension L/F there should exist a pure motive $\mathcal{H}(a, b; c \,|\, z)$ defined over L related to Gauss's hypergeometric series. Concretely, we should be able to determine from the defining hypergeometric data the characteristic polynomial of the Frobenius automorphism under the compatible system of Galois representations determined by $\mathcal{H}(a, b; c \,|\, z)$, as well as obtain information on the action of inertia.

In more detail, let H denote the set of elements $i \in (\mathbb{Z}/N)^\times$ fixing c and the set $\{a, b\}$ modulo $\mathbb{Z}$; i.e., $ic \equiv c \bmod \mathbb{Z}$ and $i\{a, b\} \equiv \{a, b\} \bmod \mathbb{Z}$. Let $K = F^H$, after identifying H with a subgroup of $\mathrm{Gal}(F/\mathbb{Q})$ in a natural way. Then the following should hold:

(i) *The motive $\mathcal{H}(a, b; c \,|\, z)$ has a model defined over its field of moduli K.*

(ii) *For generic values of $z \in K$ the motive $\mathcal{H}(a, b; c \,|\, z)$ defined over K resulting from specializing (i) has coefficient field K.*
For $z \neq 0, 1 \in F$ we have the following.

(iii) *The prime ideals $\mathfrak{q}$ of F not diving N, z, z^{-1} and $z - 1$ are primes of good reduction for $\mathcal{H}(a, b; c \,|\, z)$.*

(iv) *Let $\mathfrak{q}$ be a prime ideal of F of good reduction for $\mathcal{H}(a, b; c \,|\, z)$. Then the trace of* $\mathrm{Frob}_\mathfrak{q}$ *on $\mathcal{H}(a, b; c \,|\, z)$ matches the finite field analogue of the function ${}_2F_1(a, b; c \,|\, z)$ (defined in §4).*

(v) *The primes $\mathfrak{q}$ of F not dividing N but dividing z, z^{-1} or $z-1$ are (at worst)* tame *primes for $\rho_\mathfrak{p}$. Moreover, the conjugacy class of a generator of the image of inertia $\rho_\mathfrak{p}(I_\mathfrak{q})$ (for any prime ideal $\mathfrak{p}$ whose residual characteristic is prime to that of $\mathfrak{q}$) is that of $M_s^{k_s}$, where $s = 0, 1, \infty$ and*

$$k_0 := v_\mathfrak{q}(z), \qquad k_1 := v_\mathfrak{q}(z - 1), \qquad k_\infty := v_\mathfrak{q}(z^{-1}).$$

It is expected that all of the above properties actually hold for hypergeometric motives of any rank (see §3). In general if the rank is n the motive can be described as an isotypical component of the middle cohomology of a corresponding Euler variety

$$V: \qquad y^N = \prod_{i=1}^{n-1} x_i^{A_i}(1 - x_i)^{B_i}(1 - zx_1 \cdots x_{n-1})^{A_n} \tag{2}$$

under the action of the automorphsim $y \mapsto \zeta_N y$ (see [33] for results on compactifications of these varieties). For $K = \mathbb{Q}$ the motive may also be seen as the top weight piece of the middle cohomology of a hypersurface in a torus (see [6], [39]). Partial results on properties (i)-(v) were obtained by Katz in [32] (see also [21]).

Our main focus in the present article is the case of rank two, where we prove all properties (i)-(v). For the general rank n case we prove a slightly weaker form of (ii) and (iv). More precisely, let $\boldsymbol{\alpha} = (a, b), \boldsymbol{\beta} = (c, d)$ be two pairs of generic rational numbers (see Definition 3.4 for the precise meaning of generic) and let N denote their common denominator. Define integers

$$A = (d - b)N, \qquad B = (b + 1 - c)N, \qquad C = (1 + a - d)N, \qquad D = (d - 1)N \tag{3}$$

to be representatives of their classes modulo N in the interval $[0, N)$.

To simplify the exposition let us assume throughout this introduction that $\gcd(A, B, C, D) = 1$. Then for any divisor m of N let $\mathcal{C}_m$ denote the Euler curve defined over $\mathbb{Q}$ given by the equation

$$\mathcal{C}_m: \qquad y^m = x^A(1 - x)^B(1 - zx)^C z^D. \tag{4}$$

By the gcd assumption $\mathcal{C}_m$ is absolutely irreducible. We have a natural map $\iota_m : \mathcal{C}_N \to \mathcal{C}_m$ sending $(x, y) \to (x, y^{N/m})$. Let $\mathcal{J}_N^{\mathrm{new}}$ be the quotient of the Jacobian $\mathrm{Jac}(\mathcal{C}_N)$ by the sum of the images $\iota_m(\mathrm{Jac}(\mathcal{C}_m))$ of all proper divisors m of N. It carries a natural action of the group $\boldsymbol{\mu}_N$ of N-th roots of unity via $\zeta_N \cdot (x, y) = (x, \zeta_N y)$ on $\mathcal{C}_N$, which in turn yields a corresponding action on $\mathcal{J}_N^{\mathrm{new}}$. For $\zeta_N \in \boldsymbol{\mu}_N$ a primitive root of unity let $\mathcal{J}_N^{\zeta_N,\mathrm{new}}$ denote the ζ_N-eigenspace for this action.

To define our hypergeometric motive we reason heuristically as follows. Assume first that $d = 1$. The hypergeometric series ${}_2F_1$ has the integral presentation

$$\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\, {}_2F_1(a, b; c \,|\, z) = \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}dx. \tag{5}$$

We may interpret this formula as an identity of periods. The right hand side (the integral) is a period of the holomorphic differential dx/y of the Euler curve $\mathcal{C}_N$; the first factor on the left hand side (a product of Gamma values) is a period of a Jacobi motive $J((a, b), (c, 1))$ (defined in §5) and the second factor (the series ${}_2F_1$ itself) should correspond to a period of our hypergeometric motive. We obtain the general case with parameters $(a, b), (c, d)$ as a twist of that with parameters $(a - d, b - d), (c - d, 1)$.

Concretely, we define for any specialization $z \in F$

$$\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta} \,|\, z) := \mathcal{J}_N^{\zeta_N,\mathrm{new}} \otimes J(\boldsymbol{\alpha}, \boldsymbol{\beta})^{-1}. \tag{6}$$

For $\mathfrak{p}$ a prime ideal of F, let $H_{\mathfrak{p}}(\boldsymbol{\alpha}, \boldsymbol{\beta} \,|\, z)$ be the finite hypergeometric sum in Definition 6.4 (see also Definition 6.1). We have the following (see Theorem 7.23)

Theorem 1. *Let $\mathfrak{p}$ be a prime ideal of $\mathbb{Q}(\zeta_N)$ satisfying (iii). Then $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta} \,|\, z)$ has good reduction at $\mathfrak{p}$ and the trace of the Frobenius automorphism $\mathrm{Frob}_{\mathfrak{p}}$ acting on $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta} \,|\, z)$ equals $H_{\mathfrak{p}}(\boldsymbol{\alpha}, \boldsymbol{\beta} \,|\, z)$.*

We may be more precise.

Definition 1.1. *Let H be the subgroup of elements in $(\mathbb{Z}/N)^{\times}$ that fix (under multiplication) the multisets $\{\alpha_1, \ldots, \alpha_n\}$ and $\{\beta_1, \ldots, \beta_n\}$ modulo $\mathbb{Z}$.*

The group H is naturally identified with a subgroup of $\mathrm{Gal}(F/\mathbb{Q})$. Since $-1 \in (\mathbb{Z}/N\mathbb{Z})^{\times}$ corresponds to complex conjugation, $K := F^H \subseteq \mathbb{Q}(\zeta_N)^+$ (or equivalently, it is totally real) if and only if $-1 \in H$.

Theorems 8.15 and 8.2 imply the following (see also Corollary 8.19).

Theorem 2. *If $-1 \in H$ then $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta} \,|\, z)$ maybe defined over the totally real field K.*

Under some extra hypothesis (see Proposition 8.21) we can prove that for generic values of the parameter $z_0 \in \mathbb{Q}$, the motive $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta} \,|\, z)$ can be defined over K and has coefficient field also K. It seems to be true that always the field of definition of the motive is the same as the coefficient field (even when it is smaller than K, like in Example 9).

Let p be a rational prime. For a, b rational numbers, denote by $a \sim_p b$ the relation defined by the condition that the denominator of $a - b$ is a p-th power. Extend this relation to vectors component-wise.

Theorem 3. *Let $\boldsymbol{\alpha}, \boldsymbol{\beta}$ and $\boldsymbol{\alpha}', \boldsymbol{\beta}'$ be generic pairs of rational numbers (see 3.4) such that $\boldsymbol{\alpha} \sim_p \boldsymbol{\alpha}'$ and $\boldsymbol{\beta} \sim_p \boldsymbol{\beta}'$ for some p prime. Then $\mathcal{H}((a, b), (c, d)|z)$ is congruent to $\mathcal{H}((a', b'), (c', d')|z)$ modulo any prime ideal $\mathfrak{p}$ of a common field of definition dividing p.*

The previous results has two interesting applications: on the one hand, it provides a "lowering the level" result, given by removing a prime ideal p from the denominator of any hypergeometric motive's parameters. This is very useful for example in the study of Diophantine equations (as used in [24] to study solutions of the generalized Fermat equation).

On the other hand, it gives a "raising the level" result, since given a pair of rational parameters $\boldsymbol{\alpha}, \boldsymbol{\beta}$ we can add a rational number r/p^s to each α_i and β_i, with r, s positive integers and p a prime not dividing their denominator. Note that the resulting motive will a priori have wild ramification at primes dividing p.

Example 1. Consider the hypergeometric motive with parameters $(1/2, 1/2), (1, 1)$, corresponding to Legendre's family of elliptic curves

$$E_z : y^2 = x(x-1)(1-zx).$$

Adding/substracting $1/3$ to the first parameters, we obtain the rational motive with parameters $(1/6, -1/6), (1, 1)$ corresponding (as proved in [28]) to the family of elliptic curves with equation

$$\widetilde{E_z} : y^2 = x^3 - \frac{x^2}{4} + \frac{z}{432}.$$

Then for any specialization z_0 of the parameter z the elliptic curves are congruent modulo 3 as implied by Theorem 2.

We mostly restricted to rank two hypergeometric motives because of the applications to the study of solutions to the generalized Fermat's equation, following the program described by Darmon in [16], as explained in the article [24]. We expect, however, that our proofs extend to the arbitrary rank case by considering the general Euler variety (2).

The article is organized as follows: section §2 is devoted to rank 1 hypergeometric motives. Although the theory of rank 1 motives is quite elementary (solutions to the differential equation are algebraic), it gives a flavor of the general theory.

Section §3 starts recalling the definition of the rank n hypergeometric series, the differential equation it satisfies, and some of its well known properties. It includes a detailed description of the construction of the geometric representation (1) extending the monodromy one.

Section §4 contains a discussion of arbitrary rank n hypergeometric motives. Following computations of David Roberts and the third author, analogues of properties (i) to (v) are expected to hold. The section includes a the proof of a weaker version of statement (ii) for general rank n hypergeometric motives (based on [4]). We end the section by recalling an algorithm to compute the Hodge numbers of the hypergeometric motive of parameters $\boldsymbol{\alpha}, \boldsymbol{\beta}$ and include different examples.

Section §5 is a short survey on Jacobi motives (following [49] and [51]). It includes results needed to interpret the product/quotient of Gamma values in (5) as a Jacobi motive. Section §6 gives the definition of the finite field analogue $H_q(\boldsymbol{\alpha}, \boldsymbol{\beta}|z)$ of the hypergeometric series ${}_nF_{n-1}$ (as studied in [26] and [31]). Proposition 6.3 summarizes the main properties satisfied by finite hypergeometric sums needed in the present article. There is an alternate p-adic definition of finite hypergeometric sums in terms of the p-adic Gamma functions (as defined by Morita in [35]). Theorem 6.18 proves that both definitions coincide (using the Gross-Koblitz formula). The section ends (Theorem 6.25) with a relation between hypergeometric character sums and finite hypergeometric series. The result plays a crucial role while computing the zeta function of Euler's curve (4).

In Section §7 we focus on the case $n = 2$. Previous results on rank 2 motives (like [2] and [21]) always assumed that $d = 1$. This a priori "innocent" assumption not only excludes many interesting motives, but also has deep consequences since it implies that Euler's curve is absolutely irreducible. For general values of d this is no longer true (as proved in Example 9); Euler's curve can have any number of irreducible components. To circumvent this problem, we start studying the case of irreducible curves, and use them to define a general rank 2 hypergeometric motive in Definition 7.10. In §7.2 we study the possible Hodge numbers of a rank 2 hypergeometric motive, a very reach theory. In Example 11 we compute the Hodge numbers of some curves studied by Shimura (in [44]). The last part of the section is devoted to computing the zeta function of the Euler curve $\mathcal{C}$ (and its factorization according to the action of μ_N). The main result (Theorem 7.23) gives a precise relation between $H_{\mathfrak{p}}(\boldsymbol{\alpha}, \boldsymbol{\beta}|z)$ and the trace of a Frobenius element $\mathrm{Frob}_{\mathfrak{p}}$ acting on $H_{et}^{1,\zeta_N}(\widehat{\mathcal{C}}, \mathbb{Q}_\ell)$ (for $\mathfrak{p}$ a prime ideal of F where $\mathcal{C}$ has good reduction), where $\widehat{\mathcal{C}}$ denotes the desingularization of the Euler curve, and the superscript ζ_N denotes the ζ_N isotypical component. The result is an arithmetic counterpart to (5) and a manifestation of (6).

Section §8 is devoted to prove results on the field of definition of the motive (property (i) of the introduction). Our main result reads.

Theorem 8.2. *Let $(a,b),(c,d)$ be generic parameters such that $a+b$ and $c+d$ are integers and* (**Irr**) *holds. Then $\mathcal{H}((a,b),(c,d)|z)$ is defined over F^H and its coefficient field is contained in F^H.*

Its proof is interesting on its own, since it consists on constructing involutions of Euler's curve. The section includes a study of the Galois representation attached to specializations of the hypergeometric motive. We prove irreducibility results (Theorem 8.16), extension results (Theorem 8.15 and Corollary 8.19) and results on the coefficient field of the motive (Proposition 8.21).

Section §9 is devoted to the study of hypergeometric motives defined over totally real fields (meaning that $-1 \in H$). In this case we can prove a stronger irreducibility result (Theorem 9.1). The last section §10 contains the explicit statement and proof of congruences between hypergeometric motives. Although the proof is elementary, the result has deep implications in Darmon's program (as exploited in [24]). The present article ends with an appendix addressing the following problem: keeping the previous notation, let $\mathfrak{p}$ be a prime ideal of F not dividing N but (say) dividing the numerator of z. Then it is a priori a tame prime for the motive, but it may actually be of good reduction (as follows from (4), since the matrix M_0 might have finite order). In this case the stated formula (proven in Theorem 7.23) does not apply. One can still compute the trace of the Frobenius element geometrically, by computing the stable model of Euler's equation (as explained in the appendix). It would be interesting to have an explicit description for wild primes.

The third author wrote a script to compute $H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|z)$ p-adically in terms of the p-adic Gamma function (a different algorithm to compute the L-series of a hypergeometric motive is given in [12] and [13]). The script can be downloaded from the github repository

https://github.com/frvillegas/frvmath.

Example 2. Here is an example of how it works. Consider the hypergeometric motive $\mathcal{H}_z$ with parameters $\boldsymbol{\alpha} = (1/8,-1/8), \boldsymbol{\beta} = (3/8,-3/8)$ Tate twisted so that it has weight one. A priori $\mathcal{H}_z$ is defined over $\mathbb{Q}(\sqrt{2})$ as explained in Example 15.

First we take $z = 9$ and compute the action of Frobenius for the prime $p = 7$.

```
? hgm(9,[1/8,-1/8],[3/8,-3/8],7)
3*7^-1 + 6 + 6*7 + 6*7^2 + 6*7^3 + 6*7^4 + 6*7^5 + 6*7^6 + 6*7^7 + 6*7^8 +
6*7^9 + 6*7^10 + 6*7^11 + 6*7^12 + 6*7^13 + 6*7^14 + 6*7^15 + 6*7^16 + 6*7^17 +
6*7^18 + O(7^19)
? recognizep(hgm(9,[1/8,-1/8],[3/8,-3/8],7))
-4/7
```

We conclude that the trace of Frobenius on $\mathcal{H}_9$ is -4. To get the full characteristic polynomial $L_7(T)$ of Frobenius we can proceed as follows

```
? hgmfrob(9,[1/8,-1/8],[3/8,-3/8],7)
(7^-1 + O(7^18))*x^2 + (4*7^-1 + O(7^19))*x + 1
? polrecognizep(hgmfrob(9,[1/8,-1/8],[3/8,-3/8],7))
1/7*x^2 + 4/7*x + 1
```

This case is special however since the specialization parameter is a square, which results in the field of definition being $\mathbb{Q}$. The characteristic polynomial of Frobenius at $p = 7$ is then

$$L_7(T) = 7T^2 + 4T + 1.$$

For a further discussion of this example see Example 9.

Example 3. For $z = 3$ on the other hand, we have

```
? L1=hgmfrob(3,[1/8,-1/8],[3/8,-3/8],7);
? L2=hgmfrob(3,[3/8,-3/8],[1/8,-1/8],7);
? polrecognizep(L1*L2)
1/49*x^4 + 6/49*x^2 + 1
```

In fact, looking at the polynomials L1,L2 individually we find that the characteristic polynomial $L_{\mathcal{P}}(T)$ of Frobenius for primes $\mathcal{P}$ of $\mathbb{Q}(\sqrt{2})$ dividing 7 acting on $\mathcal{H}_3$ is

$$L_{\mathcal{P}}(T) = 7T^2 \pm 2\sqrt{2}T + 1.$$

Let χ be the quadratic character of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{2})$ corresponding to the extension $\mathbb{Q}(\sqrt{\sqrt{2}+2})$ over $\mathbb{Q}(\sqrt{2})$. Computing for more primes, it is not hard to verify that the twist by χ of the Frobenius polynomials of the hypergeometric motive $\mathcal{H}_3$ match those of a Hilbert newform over the real quadratic field $\mathbb{Q}(\sqrt{2})$ of parallel weight 2 and level $3^2 \cdot \sqrt{2}^5$ for all prime ideals of $\mathbb{Q}(\sqrt{2})$ less than 200. This form is not in the LMFDB, but we can nevertheless uniquely identify it using its standarized labeling/ordering of forms. To this end, we compute the trace of the Hecke eigenvalue of each (Galois orbit) eigenform for primes up to norm 31 (sorted as explained in [15]). The form corresponds to the 11-th one in the resulting ordering. Here is the code in magma to compute the space of all relevant Hilbert modular forms.

```
> R<x> := PolynomialRing(IntegerRing());
> F := NumberField(x^2-2);  OF := Integers(F);
> P2:=Factorisation(2*OF)[1][1];
> M := HilbertCuspForms(F, 9*P2^5);
> decomp := NewformDecomposition(NewSubspace(M));
```

It seems plausible that the Faltings-Serre's method can be used to prove modularity of the corresponding motive and hence the equality of Frobenius polynomials for all p.

Remark. One important advantage of working p-adically is that there is typically no need to pass to an algebraic extension to compute the corresponding Gauss sums. For the above example, we would a priori need to compute in the field $\mathbb{Q}(\zeta_8)$ or possibly $\mathbb{Q}(\sqrt{2})$ if set up appropriately. As it stands we simply compute the trace of Frobenius in $\mathbb{Z}_p$ and then recognize it as an (algebraic) integer.

We finish this introduction with a few challenges that we expect to address in the near future:

- Can the inertial information be used to prove large image results for hypergeometric motives? (We do know (see [5]) that the Zariski closure of the geometric monodromy is big.)
- Extend the analogue of property (iii) to primes of the field of definition of $\mathcal{H}((a,b),(c,d)|z)$.
- Prove properties (i) to (v) for rank n hypergeometric motives.

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2. Rank one hypergeometric motives

We start by considering a baby example of the theory, that of rank 1 motives. The definitions used in the present section will be explained in detail in subsequent sections for the general case of rank n hypergeometric motives.

Let α be a rational number of denominator $N > 1$. Consider the differential equation (see (11) for its general definition)

$$D(\alpha, 1) := z\frac{d}{dz} - z(z\frac{d}{dz} + \alpha) = z(1-z)\frac{d}{dz} - z\alpha = 0. \tag{7}$$

Its one-dimensional space of holomorphic local solutions around some base point $z_0 \neq 0, 1$ in the unit disk is spanned by the algebraic function $H(z) = (1-z)^{-\alpha}$ (for any fixed branch determination).

Analytic continuation along closed loops around z_0 determines the *monodromy representation*

$$\rho : \pi_1(\mathbb{C} \setminus \{1\}, z_0) \to \mathbb{C}^\times.$$

It is easy to verify that if γ is a simple loop around 0 (in counterclockwise direction) then $\rho(\gamma) = \exp\left(\frac{2\pi i}{N}\right)$. Let $F = \mathbb{Q}(\zeta_N)$. Algebraically, ρ corresponds to the Galois representation

$$\widetilde{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}(z)}/F(z)) \to \mathbb{C}^\times,$$

given by

$$\widetilde{\rho}(\sigma) = \frac{\sigma(\sqrt[N]{1-z})}{\sqrt[N]{1-z}}.$$

The representation is ramified only at 1 and ∞. Taking $z_0 \in \mathbb{Q}$, $z_0 \neq 1$, we can study the specialization of the representation at z_0, which corresponds to the extension of number fields $F(\sqrt[N]{1-z_0})/F$. By class field theory, the extension corresponds to a (conjugacy class of a) character of order N of the ray class group of F whose conductor is supported at primes dividing N, $1-z_0$ and its denominator.

Let $\mathcal{O}_F$ denote the ring of integers of F, let $\mathfrak{q}$ a prime ideal of $\mathcal{O}_F$ unramified in the extension $F(\sqrt[N]{1-z_0})/F$ and let $\mathbb{F}_q$ denote the finite field $\mathcal{O}_F/\mathfrak{q}$ of $q = \mathcal{N}\mathfrak{q}$ elements. Since $N \mid q-1$, $\mathbb{F}_q$ contains the N-th roots of unity. Fix ϖ a generator of the character group of $\mathbb{F}_q^\times$ and ψ an additive character of $\mathbb{F}_q$. For $z \in \mathbb{F}_q$, define (see §6)

$$H_q(\alpha, 1|z_0) := \frac{1}{1-q} \sum_{\varphi} \frac{\mathcal{J}(\alpha\varphi, \varphi)}{\mathcal{J}(\alpha, 1)} \varphi(z_0), \tag{8}$$

where the sum runs over characters φ of $\mathbb{F}_q^\times$ and

$$\mathcal{J}(\alpha\varphi, \varphi) := g(\psi, \varpi^{(q-1)\alpha}\varphi) g(\psi^{-1}, \overline{\varphi}),$$

where $g(\psi, \varphi)$ denotes the standard Gauss sum (as recalled in (32)). Denote by ε the character $\varpi^{(q-1)\alpha}$ (a character of order N). Since $g(\psi^a, \varphi) = \overline{\varphi}(a) g(\psi, \varphi)$, we have

$$\frac{\mathcal{J}(\varepsilon\varphi, \varphi)}{\mathcal{J}(\varepsilon, 1)} = -\varphi(-1) \frac{g(\psi, \varepsilon\varphi) g(\psi, \overline{\varphi})}{g(\psi, \varepsilon)} = -\varphi(-1) J(\varepsilon\varphi, \overline{\varphi}),$$

where for φ, η characters, $J(\varphi, \eta)$ denotes the standard Jacobi sum (whose definition is recalled in Definition 6.23). Then (8) becomes

$$\frac{1}{q-1} \sum_{\varphi} \sum_{t \neq 1} \varphi(-z_0 t) \varepsilon(t) \overline{\varphi}(1-t) = \frac{1}{q-1} \sum_{t \neq 1} \varepsilon(t) \sum_{\varphi} \varphi\left(\frac{-z_0 t}{1-t}\right).$$

The last sum equals zero unless $-z_0 t = 1 - t$ and $q-1$ in that case. Therefore

$$H_q(\alpha, 1|z_0) = \varepsilon(1-z_0)^{-1},$$

showing that our finite hypergeometric series $H_q(\alpha, 1|z_0)$ at a prime ideal $\mathfrak{q}$ matches the Dirichlet character of the extension $F(\sqrt[N]{1-z_0})/F$.

Consider now the same problem for parameters $1, -\alpha$. The differential equation becomes

$$D(1, -\alpha) = z(1-z)\frac{d}{dz} - \alpha = 0.$$

Its space of solutions is spanned by the function $G(z) = \left(\frac{z}{1-z}\right)^\alpha$. If N denotes the denominator of α, the solution gives an abelian Galois extension $F\left(\sqrt[N]{\frac{z}{1-z}}\right)/F(z)$, which is unramified outside $\{0, 1\}$. An elementary computation proves that both the differential equations and the field extensions are related to the previous case by the change of variables $z \to 1/z$. This is consistent with the finite hypergeometric series behavior

$$H_q(1,-\alpha|z_0) = \frac{1}{1-q}\sum_{\varphi}\frac{J(\varphi,\varepsilon^{-1}\varphi)}{J(1,\varepsilon^{-1})}\varphi(z_0) = \frac{1}{1-q}\sum_{\varphi}\frac{J(\varepsilon\varphi^{-1},\varphi^{-1})}{J(\varepsilon,1)}\varphi(z_0) = H_q(\alpha,1|z_0^{-1}).$$

3. Hypergeometric Series and Representations

For $\alpha \in \mathbb{C}$ and k a positive integer, let $(\alpha)_k$ denote the Pochammer symbol, defined by

$$(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1) = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}.$$

Definition 3.1. *Let $\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_{n-1}$ be complex numbers, and $z \in \mathbb{C}$ with $|z|<1$. The hypergeometric series with parameters $\boldsymbol{\alpha},\boldsymbol{\beta}$ evaluated at z is defined as*

$$(9)\qquad {}_nF_{n-1}(\alpha_1,\dots,\alpha_n;\beta_1,\dots,\beta_{n-1}|z) = \sum_{k=0}^{\infty}\frac{(\alpha_1)_k\cdots(\alpha_n)_k z^k}{(\beta_1)_k\cdots(\beta_{n-1})_k k!}.$$

We will briefly recall some the properties of this classical function and refer the reader to [5] and the references therein for details. For $\mathrm{Re}(\beta_i) > \mathrm{Re}(\alpha_i) > 0$, for $i = 1,\dots n$, we have the following integral presentation

$$(10)\qquad {}_nF_{n-1}(\boldsymbol{\alpha};\boldsymbol{\beta}|z) = \prod_{i=1}^{n-1}\frac{\Gamma(\beta_i)}{\Gamma(\alpha_i)\Gamma(\beta_i-\alpha_i)}\int_0^1\cdots\int_0^1\frac{\prod_{i=1}^{n-1}(x_i^{\alpha_i-1}(1-x_i)^{\beta_i-\alpha_i-1}dx_i)}{(1-zx_1\cdots x_{n-1})^{\alpha_n}},\quad |z|<1.$$

As it is well-known the hypergeometric series ${}_nF_{n-1}$ satisfies the differential equation

$$(11)\qquad D(\boldsymbol{\alpha};\boldsymbol{\beta})u = 0,\qquad D(\boldsymbol{\alpha};\boldsymbol{\beta}) := (\theta+\beta_1-1)\cdots(\theta+\beta_n-1) - z(\theta+\alpha_1)\cdots(\theta+\alpha_n),$$

where $\theta = z\frac{d}{dz}$. This equation has regular singularities at the points $z = 0,1,\infty$. We have ([5, §2]).

Proposition 3.2. *If the number $\beta_1,\dots,\beta_n$ are distinct modulo $\mathbb{Z}$, n independent solutions of (11) are given by*

$$(12)\qquad u_i(z) := z^{1-\beta_i}{}_nF_{n-1}(1+\alpha_1-\beta_i,\dots,1+\alpha_n-\beta_i;1+\beta_1-\beta_i,\overset{\vee}{\dots},1+\beta_n-\beta_i|z),$$

where $\vee$ denotes omission of $1+\beta_i-\beta_i$.

Lemma 3.3. *Let $\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n \in \mathbb{C}$. Then the change of variables $z \to u = 1/z$ sends solutions of the equation $D(\boldsymbol{\alpha};\boldsymbol{\beta})$ to solutions of the equation $D(-\boldsymbol{\beta}-1;-\boldsymbol{\alpha}-1)$.*

Proof. Set $u = 1/z$, so $z\frac{d}{dz} = -u\frac{d}{du}$. In the variable u, equation (11) becomes

$$\frac{(-1)^{n+1}}{u}\left((\theta-\alpha_1)\cdots(\theta-\alpha_n) - u(\theta-\beta_1-1)\cdots(\theta-\beta_n-1)\right).$$

□

We are interested in the case when the parameters α_i,β_i are rational numbers, where we can expect a geometric origin for the differential equation. We restrict to this case from now on. Set N to be their least common denominator and define the quantities:

$$(13)\quad A_i := N(\beta_n-\alpha_i),\quad B_i := N(\alpha_i-\beta_i),\ \text{for } i=1,\dots,n-1,\quad A_n := N(\alpha_n-\beta_n),\quad B_n := N\beta_n.$$

Then u_n is a period of the middle cohomology of the twisted Euler's variety

$$(14)\qquad V_n:\quad y^N = z^{B_n}\prod_{i=1}^{n-1}x_i^{A_i}(1-x_i)^{B_i}(1-zx_1\cdots x_{n-1})^{A_n}.$$

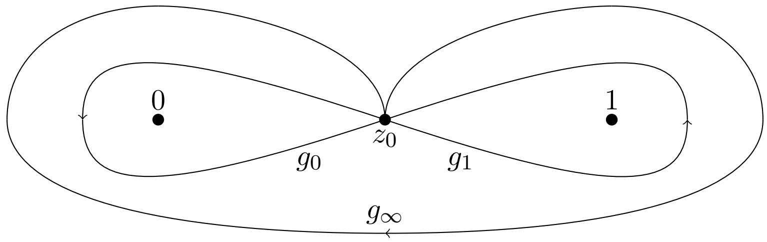


FIGURE 1. Fundamental group of $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\}, z_0)$

Definition 3.4. *Let* $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$ *and* $\boldsymbol{\beta} = (\beta_1, \dots, \beta_n)$ *be* n*-tuples of complex numbers. We say that* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$ *are* generic *if the sets* $\{\exp(2\pi i \alpha_j)\}, \{\exp(2\pi i \beta_k)\}$ *are disjoint. We will also say that such a pair determines a* hypergeometric data.

3.1. **The monodromy representation of the differential equation.** Keep the notation of the previous section. For X a topological space, and $x_0 \in X$, let $\pi_1(X, x_0)$ denote the fundamental group of X based at the point x_0.

Definition 3.5. *Let* z_0 *be a complex number different from* $0, 1$*. The differential equation* $D(\boldsymbol{\alpha}; \boldsymbol{\beta})$ *has attached a natural* monodromy representation

$$\rho : \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\}, z_0) \to \mathrm{GL}_n(\mathbb{C}) \tag{15}$$

obtained as follows: pick a basis $\{f_1, \dots, f_n\}$ *of* n *independent solutions to* $D(\boldsymbol{\alpha}; \boldsymbol{\beta})$ *around the point* z_0*. Given a loop* $\alpha \in \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\}, z_0)$ *extend each solution* f_i *along* α*, to obtain a new basis* $\{\tilde{f}_1, \dots, \tilde{f}_n\}$ *of linearly independent solutions at* z_0*. The matrix* $\rho(\alpha)$ *is defined as the change of basis matrix.*

Proposition 3.6. *The monodromy representation attached to the parameters* $\boldsymbol{\alpha}, \boldsymbol{\beta}$ *is irreducible if and only if the parameters are generic.*

Proof. See Propositions 2.7 and 3.3 of [5]. ☐

Proposition 3.7. *If the parameters* $\boldsymbol{\alpha}$*,* $\boldsymbol{\beta}$ *are generic then the monodromy representation for the parameters* $\boldsymbol{\alpha}, \boldsymbol{\beta}$ *is isomorphic to the one corresponding to the parameters* $\boldsymbol{\alpha} + \boldsymbol{n}$*,* $\boldsymbol{\beta} + \boldsymbol{m}$ *for any* $\boldsymbol{n}, \boldsymbol{m} \in \mathbb{Z}^n$*.*

Proof. See Corollary 2.6 of [5]. ☐

Let g_0, g_1, g_∞ denote paths going through $0, 1, \infty$ respectively in counterclockwise direction as in Figure 1. It follows from Van Kampen's theorem that

$$\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\}, z_0) = \langle g_0, g_1, g_\infty \ : \ g_0 g_1 g_\infty = 1 \rangle.$$

In particular, the monodromy representation is determined by the matrices

$$M_0 := \rho(g_0), \qquad M_1 := \rho(g_1), \qquad M_\infty := \rho(g_\infty), \tag{16}$$

corresponding to the monodromy matrices around $0, 1$ and ∞ respectively.

Definition 3.8. *A* pseudo-reflection *is a matrix* M *satisfying that* $M - 1$ *has rank one. Its determinant is called the special eigenvalue of* M*.*

Proposition 3.9. *Let* $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$ *and* $\boldsymbol{\beta} = (\beta_1, \dots, \beta_n)$ *be generic. Let* $a_j = \exp(2\pi i \alpha_j)$ *and* $b_j = \exp(2\pi i \beta_j)$ *for* $j = 1, \dots, n$*. Then*

$$\det(t - M_\infty) = \prod_{j=1}^{n} (t - a_j), \qquad \det(t - M_0^{-1}) = \prod_{j=1}^{n} (t - b_j), \tag{17}$$

and M_1 is a pseudo-reflection with special eigenvalue $\exp(2\pi i\gamma)$*, where* $\gamma = \sum_{j=1}^{n}(\beta_j - \alpha_j)$*.*

Proof. See Proposition 2.10 and Proposition 3.2 of [5]. $\square$

Remark 3.10. Two monodromy representations ρ_1, ρ_2 are isomorphic if there exists a change of basis matrix that sends one to the other. Equivalently, if the monodromy matrices $\rho_1(g_0), \rho_1(g_1), \rho_1(g_\infty)$ are conjugate (by some matrix M) to $\rho_2(g_0), \rho_2(g_1), \rho_2(g_\infty)$. The generic case correspond to what is called a *rigid* case, and more is true, namely the representations ρ_1 and ρ_2 are isomorphic if each monodromy matrix $\rho_1(g_i)$ is conjugate to $\rho_2(g_i)$ (for $i = 0, 1, \infty$) for a matrix M_i that might depend in i (as proven in Theorem 3.5 of [5]).

Let $\boldsymbol{\alpha}, \boldsymbol{\beta}$ be generic rational parameters and let N be their least common denominator. Then a construction due to Levelt (see Theorem 3.5 of [5]) allows to give explicit matrices M_0, M_1, M_∞ in $\mathrm{GL}_n(\mathbb{Q}(\zeta_N))$ for the monodromy representation; then if $F = \mathbb{Q}(\zeta_N)$, the image of the monodromy representation lies in $\mathrm{GL}_N(F)$.

3.2. **Geometric representations.** A good reference for the statements of the present section is §6.3 of [43]. Let $\mathfrak{K}$ be an algebraic closed field, and X a projective smooth curve of genus g over $\mathfrak{K}$ (in our case $X = \mathbb{P}^1$, so $g = 0$). Let $P_1, \ldots, P_k$ be distinct points in $X(\mathfrak{K})$. Let $\overline{\mathfrak{K}(X)}$ be an algebraic closure of $\mathfrak{K}(X)$ and let $\Omega \subset \overline{\mathfrak{K}(X)}$ be the maximal extension of $\mathfrak{K}(X)$ unramified outside the points $P_1, \ldots, P_k$. The algebraic fundamental group is defined by

$$\pi_1^{\mathrm{alg}}(X \setminus \{P_1, \ldots, P_k\}) := \mathrm{Gal}(\Omega/\mathfrak{K}(X)). \tag{18}$$

Definition 3.11. *Let G be a discrete group. The* profinite completion *of G is the topological group*

$$\widehat{G} := \varprojlim G/H,$$

where the inverse limit is with respect to finite index normal subgroups H of G.

Given a discrete group G, we denote by $\widehat{G}$ its profinite completion.

Definition 3.12. *Let g, k be positive integers. The group $\pi_1(g, k)$ is defined as the group with generators and relations*

$$\pi_1(g,k) = \langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_k \ : \ a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} c_1 \cdots c_k = 1\rangle. \tag{19}$$

Theorem 3.13. *The algebraic fundamental group $\pi_1^{alg}(X \setminus \{P_1, \ldots, P_k\})$ is isomorphic to $\widehat{\pi_1(g,k)}$, the profinite completion of the group $\pi_1(g, k)$.*

Proof. See [43, Theorem 6.3.1]. This is a particular case of the well known general result: the algebraic fundamental group of a complex variety is isomorphic to the profinite completion of the topological fundamental group. $\square$

Remark 3.14. As noted in [43], the canonical map $\pi_1(g, k) \to \widehat{\pi_1(g,k)}$ is in fact injective.

Let F be a number field, and let $\mathfrak{p}$ be a prime ideal of its ring of integers. We denote by $F_\mathfrak{p}$ the completion of F at $\mathfrak{p}$.

Corollary 3.15. *Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be generic rational parameters. Then for each prime ideal $\mathfrak{p}$ of $F = \mathbb{Q}[\zeta_N]$, there exists a (unique up to isomorphism) continuous representation*

$$\rho_\mathfrak{p} : \mathrm{Gal}_{\mathfrak{K}(z)} \to \mathrm{GL}_n(F_\mathfrak{p}), \tag{20}$$

unramified outside $\{0, 1, \infty\}$, whose restriction to $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}, z_0)$ is isomorphic to ρ.

Proof. Let $\mathfrak{p}$ be a prime ideal of F and $F \hookrightarrow F_{\mathfrak{p}}$ the natural field homomorphism. Consider $\pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\}, z_0)$ as a topology group with the discrete topology. By Van Kampen's theorem $\pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\}, z_0) \simeq \pi_1(0,3)$, so we can think of ρ as a continuous representation of $\pi_1(0,3)$, which can be extended (by continuity) to a representation

$$\rho_{\mathfrak{p}} : \widehat{\pi_1(0,3)} \to \mathrm{GL}_n(F_{\mathfrak{p}}).$$

Theorem 3.13 (with $X = \mathbb{P}^1$ and $\{P_1, P_2, P_3\} = \{0,1,\infty\}$) gives an isomorphism $\widehat{\pi_1(0,3)} \simeq \mathrm{Gal}(\Omega/\mathfrak{R}(z))$, the later being a quotient of $\mathrm{Gal}_{\mathfrak{R}(z)}$ (corresponding to extensions unramified outside $\{0,1,\infty\}$). This proves the existence statement. Uniqueness follows from the fact that any discrete group G is dense in its profinite completion $\widehat{G}$. □

It follows from the last corollary that for rational generic parameters, the differential equation satisfies by the hypergeometric series induces a family of Galois representations

$$\rho_{\mathfrak{p}} : \mathrm{Gal}(\overline{\overline{\mathbb{Q}}(z)}/\overline{\mathbb{Q}}(z)) \to \mathrm{GL}_N(F_{\mathfrak{p}}).$$

A realization of the motive incorporates arithmetic information, allowing to extend the representation to $\mathrm{Gal}_{K(z)}$, for K some number field.

4. Generalities on hypergeometric motives

We refer the reader to the survey [39] for an introduction to the subject (see also Katz's article [32]). Let $\boldsymbol{\alpha}, \boldsymbol{\beta}$ be vectors of generic rational numbers (as in Definition 3.4), and let N be their least common denominator. Keeping the previous notation, let $F = \mathbb{Q}(\zeta_N)$. Then for $z_0 \in F$, $z_0 \neq 0, 1$, there should exist a motive $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}|z_0)$, a *hypergeometric motive or HGM*, attached to the parameters $\boldsymbol{\alpha}, \boldsymbol{\beta}$ satisfying the following properties

(i) It is a pure motive of degree n defined over F.
(ii) Its ℓ-ádic étale realization is related to the monodromy representation (see Theorem 4.8).
(iii) The primes of F of bad reduction belong to $S_{\text{pt}} \cup S_{\text{pw}}$ (*potentially tame* and *potentially wild* primes respectively), where

$$S_{\text{pw}} := \{\mathfrak{p} \,|\, v_{\mathfrak{p}}(N) > 0\}, \qquad S_{\text{pt}} := (S_0 \cup S_1 \cup S_\infty) \setminus S_{\text{pw}},$$

and

$$S_0 := \{\mathfrak{p} \,|\, v_{\mathfrak{p}}(z_0) > 0\}, \qquad S_1 := \{\mathfrak{p} \,|\, v_{\mathfrak{p}}(z_0 - 1) > 0\}, \qquad S_\infty := \{\mathfrak{p} \,|\, v_{\mathfrak{p}}(z_0) < 0\}.$$

(iv) For a prime $\mathfrak{p}$ not in $S_{\text{pt}} \cup S_{\text{pw}}$ the trace of powers of the Frobenius automorphism at $\mathfrak{p}$ acting on $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}|z_0)$ are given by an explicit finite hypergeometric sum.

Remark 4.1. As is well-known (iv) yields a way to compute the characteristic polynomial of the Frobenius automorphism at $\mathfrak{p}$ acting on $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}|z_0)$ by means of Newton's formulas.

Remark 4.2. The existence of a motive over a number field for general rigid local systems was proved by Katz in [32].

Among the extensive literature on topics related to HGMs we may highlight the following as most relevant for our purposes: for *rational* HGMs (see Definition 4.7) [6] and [39] and for the specific case of rank $n = 2$ with $\beta_2 = 1$ [2] and [21].

Let H be the subgroup of $\mathrm{Gal}(F/\mathbb{Q})$ that fixes the set $\{\exp(2\pi i\boldsymbol{\alpha})\}$ and the set $\{\exp(2\pi i\boldsymbol{\beta})\}$. Via the natural identification of $\mathrm{Gal}(F/\mathbb{Q})$ with $(\mathbb{Z}/N)^\times$, the group H can be defined as the set of elements $j \in (\mathbb{Z}/N)^\times$ satisfying

$$(21) \qquad j \cdot \{\alpha_1, \ldots, \alpha_n\} \equiv \{\alpha_1, \ldots, \alpha_n\} \bmod \mathbb{Z}, \quad \text{and} \quad j \cdot \{\beta_1, \ldots, \beta_n\} \equiv \{\beta_1, \ldots, \beta_n\} \bmod \mathbb{Z}.$$

Definition 4.3. *We define the* base field $K := \mathbb{Q}(\zeta_N)^H \subseteq \mathbb{Q}(\zeta_N)$ *of the hypergeometric data* $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ *as the field fixed by* H.

Definition 4.4. *Given a motive we will call the field generated by the traces of all Frobenius elements the* coefficient field *of the motive.*

We have (see Proposition 6.3)

$$\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}|z_0)^\sigma = \mathcal{H}(j\boldsymbol{\alpha}, j\boldsymbol{\beta}|z_0^\sigma), \tag{22}$$

where $1 \leq j \leq N-1$ coprime to N corresponds to σ. Hence it is natural to expect (and all numerical evidence supports) the following

Conjecture 4.5. *Let K be the base field of the hypergeometric data $\boldsymbol{\alpha}, \boldsymbol{\beta}$. Then*

(1) The motive $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}|z)$ has a model $\mathcal{H}(z)$ defined over K.
(2) The specialization $\mathcal{H}(z_0)$ for generic $z_0 \in K$ has coefficient field also K.
(3) For primes $\mathfrak{p}$ of K of good reduction the trace of Frobenius on $\mathcal{H}(z_0)$ is given by the finite hypergeometric sum $H_{\mathfrak{p}}(\boldsymbol{\alpha}, \boldsymbol{\beta} \,|\, z_0)$.

Remark 4.6. For particular values of the parameter z, we might by able to define the motive over a proper subfield of $L \subseteq K$. For example the motive of Example 15 has base field $\mathbb{Q}(\sqrt{2})$, but evaluated at squares is defined over $\mathbb{Q}$. The coefficient field of all motives we have studied is also L.

Example 4. A generic twist of a HGM typically increases the degree of its base field. But in some special cases it does not. For example, the following parameter set

$$\boldsymbol{\alpha} = (1/24, 11/24, 17/24, 19/24), \qquad \boldsymbol{\beta} = (1/4, 1/2, 3/4, 1)$$

has base field $\mathbb{Q}(\sqrt{-8})$. However, its twist by $-1/8$

$$\boldsymbol{\alpha}' = (1/3, 7/12, 2/3, 11/12) \qquad \boldsymbol{\beta}' = (1/8, 3/8, 5/8, 7/8)$$

has base field $\mathbb{Q}(i)$. This HGM is number 38 in the Beukers-Heckman [5] list of algebraic hypergeometric functions; it has weight zero and corresponds to an Artin motive. The geometric monodromy group for $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}|z)$ is of order 2304 and is a central extension by C_2 of $S_4 \wr S_2$.

Note that the pair $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ does not actually appear in the Beukers-Heckman list since they mod out by twisting. There are only three cases of this kind for rank $n > 2$, all with base field $\mathbb{Q}(\sqrt{-8})$. Namely,

$$\begin{array}{cc} (1/24, 11/24, 17/24, 19/24) & (1/4, 1/2, 3/4, 1) \\ (1/24, 11/24, 17/24, 19/24) & (1/4, 5/8, 3/4, 7/8) \\ (1/8, 1/4, 3/8, 3/4) & (1/6, 1/3, 2/3, 5/6) \end{array}$$

Definition 4.7. *A pair $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ of vectors with rational entries is* rational *if its base field is $\mathbb{Q}$ (equivalently, $H = \mathrm{Gal}(F/\mathbb{Q})$).*

Suppose for the rest of the section that the motive $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}|z)$ for generic rational parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ can be defined over $K(z)$ for some number field K. Let L denote its coefficient field and $\mathcal{O}_L$ its ring of integers. Define

$$S(n, L) := \gcd\{\#\mathrm{GL}_n(\mathcal{O}_L/\mathfrak{p}) : \mathfrak{p} \text{ is a prime ideal of } \mathcal{O}_L\}. \tag{23}$$

The following result follows the lines of [16, Lemma 1.2].

Theorem 4.8. *Keep the previous notation and assumptions. Let $z_0 \in \mathbb{P}^1(K) \setminus \{0, 1, \infty\}$ and let $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}|z_0)$ be the specialization at z_0. Let p, q, r denote the order of the monodromy representation at the points $0, 1, \infty$ respectively (they need not be finite). Let $\mathfrak{n}$ be a prime ideal in $\mathcal{O}_K$ not dividing $S(n, L)$ and satisfying one of the following properties:*

- $\mathfrak{n} \mid z_0$, *p is finite and* $p \mid v_{\mathfrak{n}}(z_0)$,
- $\mathfrak{n} \mid z_0 - 1$, *$q$ is finite and* $q \mid v_{\mathfrak{n}}(z_0 - 1)$,
- $v_{\mathfrak{n}}(z_0) < 0$, *$r$ is finite and* $r \mid v_{\mathfrak{n}}(z_0)$.
- $v_{\mathfrak{n}}(z_0) = v_{\mathfrak{n}}(z_0 - 1) = 0$.

Then the compatible family of Galois representations attached to $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}|z_0)$ is unramified at $\mathfrak{n}$.

Proof. Let $\mathfrak{n}$ be a prime ideal of K not dividing $S(n, L)$ and satisfying the stated hypothesis. Then there exists a prime ideal λ of L such that $\mathfrak{n} \nmid \#\mathrm{GL}_n(\mathcal{O}_L/\lambda)$ (in particular, $\mathfrak{n}$ does not divide the norm of λ). Let

$$\rho_\lambda : \mathrm{Gal}_{K(z)} \to \mathrm{GL}_n(L_\lambda)$$

be the λ-adic representation attached to the motive $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}|z)$. Start considering the *geometric* part of the representation, namely its restriction to $G^{\mathrm{geom}} := \mathrm{Gal}(\overline{\mathbb{Q}(z)}/\overline{\mathbb{Q}}(z))$.

By Corollary 3.15, the monodromy representation determines a geometric one. By a result of Katz (Theorem 5.4.4 of [31]) such a representation is isomorphic to the de Rham cohomology $H^{n-1}_{dR}(V_z)$. By the well known cohomology compatibility, the de Rham cohomology is isomorphic to the étale one, i.e. $H^{n-1}_{dR}(V_z, \mathbb{C}) \simeq H^{n-1}_{\text{ét}}(V_z, \overline{\mathbb{Q}_\ell})$ (after an isomorphism between $\mathbb{C}$ and $\overline{\mathbb{Q}_\ell}$ is chosen). In particular, the image of inertia at $z = 0$, $z = 1$ and $z = \infty$ under (the restriction to G^{geom} of) ρ_λ has order p, q and r respectively. Since the extension $\overline{\mathbb{Q}}(z)/K(z)$ is unramified at $z = 0$, $z = 1$ and $z = \infty$, the image of inertia at these three points under ρ_λ also has order p, q and r respectively.

After choosing a lattice fixed by ρ_λ, we can assume that our representation actually takes values in $\mathrm{GL}_n(\mathcal{O}_\lambda)$ (the completion of $\mathcal{O}_K$ at λ). Let m be a positive integer, and let $\overline{\rho_{\lambda,m}}$ be the reduction of ρ_λ modulo λ^m. The kernel of $\overline{\rho_{\lambda,m}}$ corresponds to a curve $X_{\lambda,m}$ which is a finite cover of $\mathbb{P}^1$ unramified outside $\{0, 1, \infty\}$. Then Theorem 1.2 of [4] implies that if one of our hypothesis is satisfied and if $\mathfrak{n}$ does not divide the order of the image of $\overline{\rho_{\lambda,m}}$, then the image of $I_{\mathfrak{n}}$ under the specialization map is trivial. But the order of the image of $\overline{\rho_{\lambda,m}}$ divides the order of $\mathrm{GL}_n(\mathcal{O}_L/\lambda^m)$, whose order is supported at the same primes as $\mathrm{GL}_n(\mathcal{O}_L/\lambda)$, so if $\mathfrak{n} \nmid \#\mathrm{GL}_n(\mathcal{O}_L/\lambda)$, $\overline{\rho_{\lambda,m}}(I_{\mathfrak{n}}) = 1$ for all m, hence $\rho_\lambda(I_{\mathfrak{n}}) = 1$, and the family is unramified at $\mathfrak{n}$. □

We may relate $S(n, \mathbb{Q})$ and S_{pw} in the case the motive is rational.

Lemma 4.9. *If $L = K = \mathbb{Q}$ and $n > 1$, then $p \mid S(n, \mathbb{Q})$ if and only if $p \leq n + 1$.*

Proof. Let p be a prime number such that $p - 1 = \phi(p) \leq n$. Then there is an injective map $\psi : \mathbb{Z}[\zeta_p] \to \mathrm{GL}_n(\mathbb{Z})$; for example, let M be the $n \times n$ matrix made up of two blocks on the diagonal (and zero elsewhere). The first block (of size $(p-1) \times (p-1)$) being the companion matrix of $(x^p - 1)/(x - 1)$ and the second one being the identity. Then $\mathbb{Z}[\zeta_p] \simeq \mathbb{Z}[M] \subset \mathrm{GL}_n(\mathbb{Z})$.

Since $n \geq 2$, it is always the case that $p \mid |\mathrm{GL}_n(\mathbb{Z}/p)|$. Let q be a rational prime number different from p. Since q does not ramify in $\mathbb{Z}[\zeta_p]/\mathbb{Z}$, the group $\mathrm{GL}_n(\mathbb{Z}/q)$ contains an element of order p (the image under the reduction map of $\psi(\zeta_p)$), so $p \mid \#\mathrm{GL}_n(\mathbb{Z}/q)$ hence $p \mid S(n, \mathbb{Q})$.

Reciprocally, let $1 \leq x \leq p-1$ be any integer and let r be its order in $\mathbb{F}_p^\times$. By Dirichlet's theorem on arithmetic progressions, there exists a rational prime q congruent to x modulo p. By hypothesis, $p \mid \#\mathrm{GL}_n(\mathbb{Z}/q) = q^*(q-1)(q^2-1)\cdots(q^n-1)$, so $x^i \equiv q^i \equiv 1 \pmod{p}$ for some $i \leq n$, i.e. any element modulo p has multiplicative order at most n, hence $p - 1 \leq n$. □

Corollary 4.10. *For rational motives, if $p \in S_{\mathrm{pw}}$ then $p \mid S(n, \mathbb{Q})$.*

Remark 4.11. We expect Theorem 4.8 to hold for prime ideals $\mathfrak{n}$ dividing $S(n, L)$ but not dividing N (as they are *potentially tame primes* according to (iii)), but we do not know how to prove this stronger statement in general. The rank 2 case is proved in Appendix A when one of the first three properties holds and in Theorem 7.23 when the last one holds (see also [22] for a description of inertia at tame primes).

4.1. **Hodge numbers and normalization.** The Hodge numbers of a rational HGM can be computed using a formula conjectured by Corti and Golyshev (proved in general in [11, 20, 48]). The formula can be applied to arbitrary hypergeometric parameters by means of the *zig-zag diagram* (see §5 of [39]), giving the dimensions of the associated graded of the Hodge filtration (see [39, Figure 5.1] for a nice rank 5 example). Let us recall the procedure. Assume $\boldsymbol{\alpha}, \boldsymbol{\beta}$ are generic.

Algorithm 4.12. The zig-zag procedure

Set S the sequence of ordered parameters of $\boldsymbol{\alpha}$ as elements in $(0, 1]$ and $\boldsymbol{\beta}$ as elements in $[0, 1)$.
To each $s \in S$ associate the color red if $s \in \boldsymbol{\alpha}$ and blue if $s \in \boldsymbol{\beta}$.
$P \leftarrow 0$
for $i = 1 \dots 2n$ ***do***
    Draw a point with the color $S[i]$ at (i, P).
    if $S[i]$ *is blue* ***then***
        $P \leftarrow P - 1$
    else
        $P \leftarrow P + 1$
    end if
end for

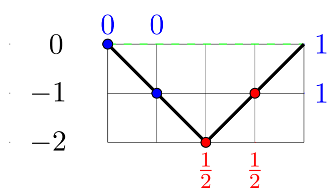
Remark 4.13. In fact, this procedure can be extended to give the full mixed Hodge numbers of the middle cohomology of the corresponding toric model in the rational case, see [48].

In §7.2 we will prove the relation between the Hodge numbers of Euler's curve and the output of the zig-zag procedure for rank two hypergeometric motives.

Example 5. As a first example, consider the rational HGM with parameters $\boldsymbol{\alpha} = (1/2, 1/2)$, $\boldsymbol{\beta} = (0, 0)$ of the Legendre family of elliptic curves

$$E_t : y^2 = x(x - 1)(1 - zx).$$

The zig-zag procedure gives the following picture



The Hodge number $h^{-i,i}$ corresponding to $\boldsymbol{\alpha}, \boldsymbol{\beta}$ is obtained by counting the number of blue (or shifting up one step red!) points at level i of the output. In our example the Hodge vector equals $(1, 1)$, with Hodge numbers $h^{0,0} = h^{1,-1} = 1$.

Example 2.(continued) Consider the hypergeometric motive $\mathcal{H}_z$ of the introduction, with parameters $\boldsymbol{\alpha} = (1/8, -1/8), \boldsymbol{\beta} = (3/8, -3/8)$ defined over its base field, the real quadratic field $K = \mathbb{Q}(\sqrt{2}) \subset \mathbb{Q}(\zeta_8)$. For $j \in (\mathbb{Z}/8)^\times$ the Hodge numbers of $\mathcal{H}_z^{\sigma_j}$ are obtained by applying the zig-zag procedure to the parameters $\boldsymbol{\alpha} = (j/8, -j/8), \boldsymbol{\beta} = (3j/8, -3j/8)$. The result is given in Figure 2. The Hodge numbers in both cases are $h^{-1,0} = h^{0,-1} = 1$.

Example 6. As a further example of a motive not defined over $\mathbb{Q}$, let $\boldsymbol{\alpha} = (\frac{1}{2}, \frac{1}{2})$, $\boldsymbol{\beta} = (0, \frac{1}{4})$. Here $N = 4$ and $K = \mathbb{Q}(i)$. We find the Hodge numbers $h^{1,-1} = h^{0,0} = 1$ and $h^{0,0} = h^{-1,1} = 1$ for the respective complex embeddings K, as illustrated in Figure 3. The Hodge vector of the restriction of scalars $\mathcal{H}(z)$ to $\mathbb{Q}$ (the sum of $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}|z)^\sigma$ over the two complex embeddings σ of K) equals $(1, 2, 1)$.

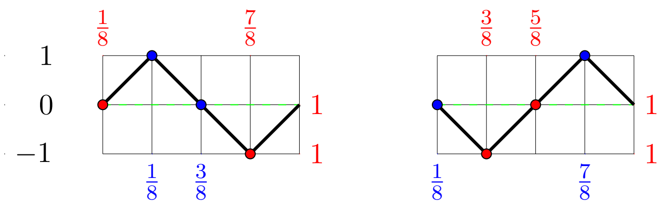


FIGURE 2. The Hodge numbers of $(\frac{1}{8}, \frac{7}{8}), (\frac{3}{8}, \frac{5}{8})$ and $(\frac{3}{8}, \frac{5}{8}), (\frac{1}{8}, \frac{7}{8})$

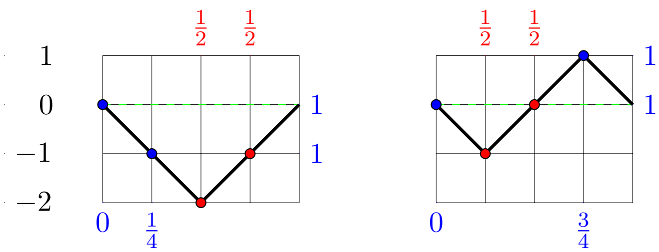


FIGURE 3. The Hodge numbers of $(\frac{1}{2}, \frac{1}{2}), (0, \frac{1}{4})$ and $(\frac{1}{2}, \frac{1}{2}), (0, \frac{3}{4})$

We can check this calculation numerically in Pari/GP using the third's author package. Taking for example $z = 3$ and $p = 17$, we get:

```
? polrecognizep(hgmfrob(3,[1/2,1/2],[3/4,1],17)*hgmfrob(3,[1/2,1/2],[1/4,1],17))
x^4 - 10/17*x^3 + 18/17*x^2 - 10/17*x + 1
```

This is the Euler factor $L_{17}(T)$ of $\mathcal{H}(3)$. Its Newton polygon at $p = 17$ should lie above the Hodge polynomial. In fact they are actually equal in this case. Concretely, L_{17} has roots of absolute value $\{-1, 0, 0, 1\}$ with multiplicities matching the Hodge vector $(1, 2, 1)$ of $\mathcal{H}(z)$.

Remark 4.14. Taking a Tate twist has the effect of shifting the zig-zag diagram up or down. When working with Galois representations, it is customary to take the minimal Tate twist that makes the Hodge numbers of all Galois conjugates non-negative (making them the Hodge numbers of an *effective* motive) because this is the standard normalization for automorphic forms. In particular, the resulting Euler factors have integral coefficients. We will call this process the *effective normalization*.

In the present article (and also in the Pari/GP code written by the third author) we do not make any additional Tate twist when considering HGM's unless explicitly mentioned. The reader should bear this in mind when matching a HGM to an automorphic form.

The effective normalization is the Tate twist that shifts the minimum value of the zig-zag diagrams over all embeddings to zero. As an illustration, we have the following.

Example 6.(continued) The minimum value of the zig-zag function over the two diagrams in Fig. 3 is -1. Therefore the effective normalization is the Tate twist

$$\mathcal{H}((\tfrac{1}{2}, \tfrac{1}{2}), (\tfrac{1}{4}, 1)|z)(1)$$

In terms of Euler factors (like $L_{17}(T)$ above) it amounts to replacing T by qT.

Remark 4.15. Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be generic rational parameters and let N be their least common denominator. Let $F = \mathbb{Q}(\zeta_N)$ and let H be the subgroup of $(\mathbb{Z}/N)^\times$ defined in 1.1. Then for $z_0 \in \mathbb{Q}$, the motive $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}|z_0)$ is expected to be defined over F^H and hence has $\phi(N)/|H|$ Hodge vectors (indexed by the embeddings of F^H into $\mathbb{C}$) instead of $\phi(N)$. This is consistent with the fact that if $j \in H$ then the Hodge vector of $\boldsymbol{\alpha}, \boldsymbol{\beta}$ is the same as the one of $j\boldsymbol{\alpha}, j\boldsymbol{\beta}$.

5. Jacobi motives

In this section we review some basic facts about Jacobi motives and set up the notation we will use. The main reference are the articles [50, 51], [1], [40] and [49] (the software package Magma [7] contains an implementation of Jacobi motives due to Mark Watkins).

Fix an integer $N > 1$. For a prime ideal $\mathfrak{p}$ in $F := \mathbb{Q}(\zeta_N) \subseteq \mathbb{C}$ of norm q, where ζ_N is a primitive N-st root of unity, let $\chi_{\mathfrak{p}}$ be the character

$$\chi_{\mathfrak{p}} : (\mathcal{O}_F/\mathfrak{p})^\times \to \mathbb{C}^\times. \tag{24}$$

of order N satisfying

$$\chi_{\mathfrak{p}}(x) \equiv x^{(q-1)/N} \bmod \mathfrak{p}.$$

Extend the definition by setting $\chi_{\mathfrak{p}}(x) = 0$ if $\mathfrak{p} \mid x$. Here $\mathcal{O}_F$ is the ring of integers of F.

We will define Gauss sums as functions on $\frac{1}{N}\mathbb{Z}/\mathbb{Z}$ as follows. Fix a non-trivial additive character ψ on $\mathbb{F}_q$ and let

$$g(\psi, a, \mathfrak{p}) := \sum_{x \in \mathbb{F}_q^\times} \psi(x)\chi_{\mathfrak{p}}^{Na}(x), \qquad a \in \frac{1}{N}\mathbb{Z}/\mathbb{Z}. \tag{25}$$

The dependence on the choice of additive character is straightforward. For any non-zero $y \in \mathcal{O}_F$ coprime to $\mathfrak{p}$ if $\psi'(x) := \psi(y^{-1}x)$ then

$$g(\psi', a, \mathfrak{p}) = \chi_{\mathfrak{p}}^{Na}(y) g(\psi, a, \mathfrak{p}). \tag{26}$$

Definition 5.1. *Let* $\boldsymbol{\theta} = \sum_i n_i \langle \theta_i \rangle \in \mathbb{Z}[\frac{1}{N}\mathbb{Z}/\mathbb{Z}]$ *satisfying*

$$\sum_i n_i \theta_i \equiv 0 \bmod \mathbb{Z}. \tag{27}$$

The Jacobi sum *attached to* $\boldsymbol{\theta}$ *for the prime ideal* $\mathfrak{p}$ *is defined as*

$$\mathbf{J}(\boldsymbol{\theta})(\mathfrak{p}) := (-1)^{\sum_i n_i + 1} \prod_i g(\psi, \theta_i, \mathfrak{p})^{n_i} \tag{28}$$

It is easy to verify, thanks to (27) and (26), that the definition is independent of the choice of the additive character ψ and therefore there is no need to include it in the notation.

By the main result of [51] the map

$$\mathfrak{p} \mapsto \mathbf{J}(\boldsymbol{\theta})(\mathfrak{p})$$

determines a Grössencharacter $\mathbf{J}(\boldsymbol{\theta})$ of F. In particular, it has an associated compatible system of l-adic Galois representations where $\mathbf{J}(\boldsymbol{\theta})(\mathfrak{p})$ equals the trace of $\mathrm{Frob}_{\mathfrak{p}}$. This is an incarnation of the *Jacobi motive* associated to $\boldsymbol{\theta}$ (see [1, §5]). We will denote this pure motive also by $\mathbf{J}(\boldsymbol{\theta})$ if there is no risk of confusion.

It will sometimes be convenient to use an alternative notation for Jacobi motives as a pair of tuples of rational numbers $\boldsymbol{a} = (a_1, \ldots, a_r)$ and $\boldsymbol{b} = (b_1, \ldots, b_s)$ with common denominator N such that

$$\sum_{i=1}^r a_i \equiv \sum_{i=1}^s b_i \bmod \mathbb{Z},$$

corresponding to

$$\boldsymbol{\theta} := \sum_{i=1}^r \langle a_i \rangle - \sum_{i=1}^s \langle b_i \rangle,$$

We will then simply write $\mathbf{J}(\boldsymbol{a}, \boldsymbol{b})$ instead of $\mathbf{J}(\boldsymbol{\theta})$.

By [1, Theorem 1] (see also Remark 2.3.2 of loc. cit) the infinity type of the Grössencharacter associated to $\mathbf{J}(\boldsymbol{\theta})$ is given by

$$\sigma_j \mapsto \sum_i n_i\{j\theta_i\}, \qquad \gcd(j,N)=1, \tag{29}$$

where $\{\cdot\}$ denotes the fractional part of real numbers and $\sigma_j \in \mathrm{Gal}(F/\mathbb{Q})$ is the automorphism satisfying $\sigma_j(\zeta_N) = \zeta_N^j$. The (motivic) weight of $\mathbf{J}(\boldsymbol{\theta})$ equals

$$w := \sum_i n_i,$$

where the sum only includes indexes i for which $\theta_i \notin \mathbb{Z}$. Define the Hodge values

$$p := \sum_i n_i\{\theta_i\}, \qquad q := \sum_i n_i\{-\theta_i\}, \tag{30}$$

so the motive appears in $H^{(p,q)}(X)$, for X a Fermat hypersurface (as described in §10 of [1]). Our convention follows [49] and it is a Tate twist of Anderson's one.

Example 7. Consider the Jacobi motive $\mathbf{J}(\boldsymbol{a},\boldsymbol{b})$ where

$$\begin{aligned} \boldsymbol{a} &:= (1/10, 1/10, 1/10, 3/10, 13/30, 7/10, 23/30, 9/10) \\ \boldsymbol{b} &:= (1/5, 1/3, 2/5, 2/3, 4/5, 1/5, 3/10, 1/2). \end{aligned}$$

It is perhaps not immediately obvious but this motive is a Tate twist of a motive of weight zero and therefore the Tate twist of an Artin motive. Indeed, we can quickly check that we have

$$\sum_i \{ja_i\} - \sum_i \{jb_i\} = 8,$$

for all j coprime to 10.

Computations with MAGMA show that the associated Artin motive is given by a Dirichlet character of $\mathbb{Q}(\zeta_5)$ of order 10 and conductor $2^2 \cdot 3 \cdot (1-\zeta_5)^2$.

The Jacobi sums $\mathbf{J}(\boldsymbol{\theta})$ belong to F and for $\sigma_j \in \mathrm{Gal}(F/\mathbb{Q})$ as before we have

$$\sigma_j(\mathbf{J}(\boldsymbol{\theta}))(\mathfrak{p}) = \mathbf{J}(j\boldsymbol{\theta})(\mathfrak{p}),$$

where

$$j\boldsymbol{\theta} := \sum_i n_i \langle j\theta_i \rangle.$$

Let $T = \{\sigma \in \mathrm{Gal}(F/\mathbb{Q}) \ : \ \sigma(\mathbf{J}(\boldsymbol{\theta})) = \mathbf{J}(\boldsymbol{\theta})\}$. Then the base field of the motive $\mathbf{J}(\boldsymbol{\theta})$ is F^T (see §2.3 of [49]). Note its resemblance with Conjecture 4.5. In some very particular instances, the Grössencharacter $\mathbf{J}(\boldsymbol{\theta})$ can be defined over a proper subextension of F^T.

Example 8. Consider the Jacobi motive $\mathbf{J}(\boldsymbol{\theta})$ where

$$\boldsymbol{\theta} = \left\langle \frac{1}{3} \right\rangle + \left\langle \frac{2}{3} \right\rangle + \left\langle \frac{1}{5} \right\rangle + \left\langle \frac{4}{5} \right\rangle + \left\langle \frac{7}{15} \right\rangle + \left\langle \frac{8}{15} \right\rangle.$$

The set T of elements in $(\mathbb{Z}/15)^\times$ fixing θ equals $\{\pm 1\}$, so the base field of $\mathbf{J}(\boldsymbol{\theta})$ is $\mathbb{Q}(\zeta_{15})^+$ (the maximal totally real subfield of $\mathbb{Q}(\zeta_{15})$). However the values of the Jacobi sum attached to our choice of parameters at a prime ideal $\mathfrak{p}$ of $\mathbb{Q}(\zeta_{15})$ equals $\mathcal{N}\mathfrak{p}^3$ (see Lemma 8.3). Then there is a second extension of the Grössencharakter to $\mathrm{Gal}_{\mathbb{Q}}$ given by the cubic power of the cyclotomic character.

6. Finite Hypergeometric Sums

Let us recall Katz's definition (given in [31, p.258]) of the finite version of a hypergeometric series. Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n), \boldsymbol{\beta} = (\beta_1, \ldots, \beta_n) \in \mathbb{Q}^n$ be arbitrary vectors of rational numbers and let N be the common denominator of the α's and β's. Fix $q \equiv 1 \bmod N$ be a prime power and a generator ϖ of $\widehat{\mathbb{F}_q^\times}$. Following [31, p. 258] (but note that our parameter z is the inverse of Katz's though it matches the normalization in [6]), define for $z \in \mathbb{F}_q$ the exponential sum (here $x_i, y_j \in \mathbb{F}_q^\times$):

$$Hyp_q(\boldsymbol{\alpha}, \boldsymbol{\beta}|z) = \sum_{zx_1\cdots x_n = y_1 \cdots y_n} \psi(x_1 + \cdots + x_n - y_1 - \cdots - y_n)\varpi(\boldsymbol{x})^{\boldsymbol{\alpha}(q-1)}\overline{\varpi}(\boldsymbol{y})^{\boldsymbol{\beta}(q-1)}, \tag{31}$$

where $\varpi(\boldsymbol{x})^{\boldsymbol{\alpha}(q-1)} = \varpi(x_1)^{\alpha_1(q-1)} \cdots \varpi(x_n)^{\alpha_n(q-1)}$ and similarly with β. Katz relates (using the Lefschetz trace formula) such finite hypergeometric sum to the trace of Frobenius acting on some concrete hypergeometric $\mathcal{D}$-module [31, Chap. 8.2].

It will be convenient to expand Hyp_q in terms of characters of $\mathbb{F}_q^\times$. The calculation is straightforward (see [31, §8.2.8]); we sketch it here for the reader's convenience. The coefficient c_φ of $\varphi \in \widehat{\mathbb{F}_q^\times}$ equals

$$c_\varphi = \frac{1}{q-1} \sum_{z \in \mathbb{F}_q^\times} \overline{\varphi}(z) Hyp_q(\boldsymbol{\alpha}, \boldsymbol{\beta}|z).$$

Interchanging order of summation we get

$$c_\varphi = \frac{1}{q-1} \sum_{(\boldsymbol{x},\boldsymbol{y}) \in \mathcal{T}_z} \psi(x_1 + \cdots + x_n - y_1 - \cdots - y_n)\varpi(\boldsymbol{x})^{\boldsymbol{\alpha}(q-1)}\overline{\varpi}(\boldsymbol{y})^{\boldsymbol{\beta}(q-1)}\varphi(x_1 \cdots x_n y_1^{-1} \cdots y_n^{-1})$$

and hence

$$c_\varphi = \frac{1}{q-1} \prod_{i=1}^n \sum_{x_i \in \mathbb{F}_q^\times} \psi(x_i)\varpi(x_i)^{(q-1)\alpha_i}\varphi(x_i) \prod_{i=1}^n \sum_{y_i \in \mathbb{F}_q^\times} \psi(-y_i)\overline{\varpi}(y_i)^{(q-1)\beta_i}\overline{\varphi}(y_i).$$

In terms of Gauss sums

$$g(\psi, \varphi) := \sum_{x \in \mathbb{F}_q^\times} \varphi(x)\psi(x), \tag{32}$$

we finally have

$$c_\varphi = \frac{1}{q-1} \prod_{i=1}^n g(\psi, \varpi^{(q-1)\alpha_i}\varphi) \prod_{i=1}^n g(\psi^{-1}, \overline{\varpi}^{(q-1)\beta_i}\overline{\varphi})$$

and therefore

$$Hyp_q(\boldsymbol{\alpha}, \boldsymbol{\beta}|z) = \frac{1}{q-1} \sum_{\varphi} \prod_{i=1}^n g(\psi, \varpi^{(q-1)\alpha_i}\varphi) \prod_{i=1}^n g(\psi^{-1}, \overline{\varpi}^{(q-1)\beta_i}\overline{\varphi})\varphi(z).$$

It is instructive to check the simplest non-trivial case where $n = 1$, $\alpha_1 = 1/2, \beta_1 = 1$ and $q = p$ is an odd prime. In this case, a quick calculation shows that

$$Hyp(1/2, 1 \,|\, z) = g(\psi_{z-1}, \epsilon),$$

where $\psi_u(x) := \psi(ux)$ and ϵ is the quadratic character of $\mathbb{F}_p^\times$. It follows that

$$Hyp(1/2, 1 \,|\, z) = \epsilon(1-z)\sqrt{p^*}, \qquad p^* := (-1)^{(p-1)/2}p,$$

as is well known. In particular, the values $Hyp(1/2, 1 \,|\, z_0)$ as p varies, for fixed $z_0 \in \mathbb{Q}$ say, do not lie in any given number field. We therefore cannot expect $Hyp(1/2, 1 \,|\, z_0)$ to be the trace of Frobenius of a motive. To achieve this we normalize the hypergeometric sum $Hyp_q(\boldsymbol{\alpha}, \boldsymbol{\beta}|z)$ by dividing by the appropriate constant so that the sum of its values over all $z \in \mathbb{F}_q^\times$ equals -1. Concretely, we consider the following.

Definition 6.1. *For $z \in \mathbb{F}_q$, define the finite hypergeometric sum $H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|z)$ by*

$$H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|z) := \frac{1}{1-q}\sum_{\varphi}\frac{\mathfrak{J}(\boldsymbol{\alpha}\varphi,\boldsymbol{\beta}\varphi)}{\mathfrak{J}(\boldsymbol{\alpha},\boldsymbol{\beta})}\varphi(z), \tag{33}$$

where

$$\mathfrak{J}(\boldsymbol{\alpha}\varphi,\boldsymbol{\beta}\varphi) := \prod_{i=1}^{n} g(\psi,\varpi^{(q-1)\alpha_i}\varphi)\prod_{i=1}^{n} g(\psi^{-1},\overline{\varpi}^{(q-1)\beta_i}\overline{\varphi}).$$

For any integer a coprime with p we have

$$g(\psi^a,\varphi) = \overline{\varphi}(a)g(\psi,\varphi), \tag{34}$$

so one can alternatively define

$$\mathfrak{J}(\boldsymbol{\alpha}\varphi,\boldsymbol{\beta}\varphi) = \varphi(-1)^n\prod_{i=1}^{n} g(\psi,\varpi^{(q-1)\alpha_i}\varphi)\prod_{i=1}^{n} g(\psi,\overline{\varpi}^{(q-1)\beta_i}\overline{\varphi}), \tag{35}$$

and the definition of $H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|z)$ in (33) does not change. To simplify the notation set $g(m) := g(\psi,\varpi^m)$ for $m \in \mathbb{Z}$. Then, explicitly,

$$H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|z) = \frac{1}{1-q}\sum_{m=0}^{q-2}\prod_{i=1}^{n}\left(\frac{g(m+\alpha_i(q-1))g(-m-\beta_i(q-1))}{g(\alpha_i(q-1))g(-\beta_i(q-1))}\right)\varpi((-1)^n z)^m. \tag{36}$$

The finite hypergeometric sum $H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|z)$ does not depend on the choice of the additive character ψ, but it does (in general) depend on the choice of the character ϖ. For our previous rank one example we have (see §2)

$$H_p(1/2,1\,|z_0) = \epsilon(1-z_0),$$

where ϵ is the quadratic character of $\mathbb{F}_q$, matching

$$\sum_{n\geq 0}\frac{(1/2)_n}{(1)_n}z_0^n = \frac{1}{\sqrt{1-z_0}}.$$

Note that $\sum_{p\in\mathbb{F}_p^\times} H_p(1/2,1\,|z_0) = -\epsilon(1) = -1$ as desired.

Remark 6.2. It follows from its definition that $H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|z)$ is independent of the ordering of $\{\alpha_1,\dots,\alpha_n\}$ and $\{\beta_1,\dots,\beta_n\}$.

6.1. Properties of $H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|z)$.

We record the basic properties of the hypergeoemetric sum $H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|z)$.

Proposition 6.3. *(1) (Inversion) For $z \in \mathbb{F}_q^\times$ we have*

$$H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|z) = H_q(-\boldsymbol{\beta},-\boldsymbol{\alpha}|z^{-1}).$$

(2) (Galois action) Let $z \in \mathbb{F}_q^\times$ and $\sigma \in \mathrm{Gal}_{\mathbb{Q}}$ with $\sigma(\zeta_N) = \zeta_N^j$ for some j coprime with N then

$$H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|z)^\sigma = H_q(j\boldsymbol{\alpha},j\boldsymbol{\beta}|z)$$

(3) (Field of moduli) Let $z \in \mathbb{F}_q^\times$ and let H be the subgroup of $(\mathbb{Z}/N)^\times$ defined in 1.1. We have

$$H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|z) \in \mathbb{Q}(\zeta_N)^H.$$

(4) (Twists) Let $\rho \in \mathbb{Q}$ and $\rho\boldsymbol{\alpha} := (\alpha_1+\rho,\dots,\alpha_n+\rho)$ and $\rho\boldsymbol{\beta} := (\beta_1+\rho,\dots,\beta_n+\rho)$ then

$$H_q(\rho\boldsymbol{\alpha},\rho\boldsymbol{\beta}|z) = (-1)^{n(q-1)\rho}\overline{\varpi}(z)^{(q-1)\rho}\frac{\mathfrak{J}(\boldsymbol{\alpha},\boldsymbol{\beta})}{\mathfrak{J}(\rho\boldsymbol{\alpha},\rho\boldsymbol{\beta})}H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|z),$$

whenever both sides are well defined.

(5) *(Non-generic) If* $\alpha_1 \equiv \beta_1 \bmod \mathbb{Z}$ *let* $\boldsymbol{\gamma} = (\alpha_2, \dots, \alpha_n)$ *and* $\boldsymbol{\delta} = (\beta_2, \dots, \beta_n)$. *Then*

$$H_q(\boldsymbol{\alpha}, \boldsymbol{\beta}|z) = q^{\delta} \left(\frac{\mathcal{J}((-\alpha_1)\boldsymbol{\gamma}, (-\alpha_1)\boldsymbol{\delta})}{\mathcal{J}(\boldsymbol{\gamma}, \boldsymbol{\delta})} + qH_q(\boldsymbol{\gamma}, \boldsymbol{\delta}) \right),$$

where

$$\delta = \begin{cases} 0 & \text{if } \alpha_1 \in \mathbb{Z} \\ -1 & \text{otherwise} \end{cases}$$

Proof. (1) It is clear from (35) that

$$\mathcal{J}(\boldsymbol{\alpha}\varphi, \boldsymbol{\beta}\varphi) = \mathcal{J}(-\boldsymbol{\beta}\,\overline{\varphi}, -\boldsymbol{\alpha}\,\overline{\varphi}).$$

The result follows from the change of variables $\varphi \to \overline{\varphi}$ in (33).
(2) Since σ is a ring morphism,

$$\sigma(H_q(\boldsymbol{\alpha}, \boldsymbol{\beta}|z)) = \frac{1}{1-q} \sum_{\varphi} \prod_{i=1}^{n} \left(\frac{\sigma(g(\psi, \varpi^{(q-1)\alpha_i}\varphi))\sigma(g(\psi^{-1}, \varpi^{-(q-1)\beta_i}\varphi^{-1})}{\sigma(g(\psi, \varpi^{(q-1)\alpha_i}))\sigma(g(\psi^{-1}, \varpi^{-(q-1)\beta_i}))} \right) \sigma(\varphi(z)). \tag{37}$$

If $\sigma \in \mathrm{Gal}_{\mathbb{Q}}$, φ is a multiplicative character of $\mathbb{F}_q^{\times}$ and ψ is an additive character, then

$$\sigma(g(\psi, \varphi)) = g(\psi^{\sigma}, \varphi^{\sigma}),$$

where $\varphi^{\sigma}(x) = \sigma(\varphi(x))$ (similarly for ψ). Let b be an integer (prime to $q-1$) satisfying $\sigma(\zeta_{q-1}) = \zeta_{q-1}^{b}$, for ζ_{q-1} a primitive $(q-1)$-th root of unity. Then

$$\sigma(\varpi^{m+a(q-1)}) = \varpi^{bm+ba(q-1)} = \varpi^{bm}\varpi^{ba(q-1)}.$$

The rational number ba equals ja up to translation by an integer. Then changing the summation order (replacing φ by $\sigma(\varphi)$), the left hand side of (37) becomes

$$\frac{1}{1-q} \sum_{\varphi} \prod_{i=1}^{n} \left(\frac{g(\psi, \varpi^{(q-1)j\alpha_i}\varphi))g(\psi^{-1}, \varpi^{-(q-1)j\beta_i}\varphi^{-1})}{g(\psi, \varpi^{(q-1)j\alpha_i})g(\psi^{-1}, \varpi^{-(q-1)j\beta_i})} \right) \varphi(z).$$

(3) For $z \in \mathbb{F}_q$, the standard properties of Gauss sums imply that the value $H_q(\boldsymbol{\alpha}, \boldsymbol{\beta}|z)$ is an element of $\mathbb{Q}(\zeta_{q-1})$ (see also [6, Proposition 3.2]). Note that all elements of $(\mathbb{Z}/(q-1))^{\times}$ that are congruent to 1 modulo N leave the sets $\{\alpha_1, \dots, \alpha_n\}$ and $\{\beta_1, \dots, \beta_n\}$ stable under multiplication (up to translation by integers), hence the second statement (and Galois theory) imply that actually $H_q(\boldsymbol{\alpha}, \boldsymbol{\beta}|z) \in F = \mathbb{Q}(\zeta_N)$. Since multiplication by elements of H also fix the sets $\{\alpha_1, \dots, \alpha_n\}$ and $\{\beta_1, \dots, \beta_n\}$, the result follows.
(4) It follows from (35) that

$$\mathcal{J}(\rho\boldsymbol{\alpha}\varphi, \rho\boldsymbol{\beta}\varphi) = \mathcal{J}(\boldsymbol{\alpha}\varphi\varpi^{(q-1)\rho}, \boldsymbol{\beta}\varphi\varpi^{(q-1)\rho})(-1)^{n(q-1)\rho}.$$

Then

$$H_q(\rho\boldsymbol{\alpha}, \rho\boldsymbol{\beta}|z) = \frac{1}{1-q} \sum_{\varphi} \frac{\mathcal{J}(\rho\boldsymbol{\alpha}\varphi, \rho\boldsymbol{\beta}\varphi)}{\mathcal{J}(\rho\boldsymbol{\alpha}, \rho\boldsymbol{\beta})} \varphi(z) = (-1)^{n(q-1)\rho} \frac{\mathcal{J}(\boldsymbol{\alpha}, \boldsymbol{\beta})}{\mathcal{J}(\rho\boldsymbol{\alpha}, \rho\boldsymbol{\beta})} \overline{\varpi}(z)^{(q-1)\rho} H_q(\boldsymbol{\alpha}, \boldsymbol{\beta}|z).$$

(5) If ψ is an additive character of $\mathbb{F}_q$ and χ is a multiplicative character, then

$$g(\psi, \chi)g(\psi^{-1}, \chi^{-1}) = \begin{cases} q & \text{if } \chi = 1, \\ 1 & \text{otherwise.} \end{cases}$$

If α_1 and β_1 are integers, then

$$H_q(\boldsymbol{\alpha}, \boldsymbol{\beta}|z) = \frac{1}{1-q} \left(\sum_{\varphi \neq 1} \frac{q\mathcal{J}(\boldsymbol{\gamma}\varphi, \boldsymbol{\delta}\varphi)}{\mathcal{J}(\boldsymbol{\gamma}, \boldsymbol{\delta})} + 1 \right) = qH_q(\boldsymbol{\gamma}, \boldsymbol{\delta}) + 1.$$

Similarly, if $\alpha_1 \equiv \beta_1 \bmod \mathbb{Z}$ but they are not integers, let $\varphi_0 = \varpi^{-(q-1)\alpha_1}$. Then

$$H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|z) = \frac{1}{1-q}\left(\sum_{\varphi\neq\varphi_0} \frac{\mathcal{J}(\boldsymbol{\gamma}\varphi,\boldsymbol{\delta}\varphi)}{\mathcal{J}(\boldsymbol{\gamma},\boldsymbol{\delta})} + \frac{\mathcal{J}(\boldsymbol{\gamma}\varphi_0,\boldsymbol{\delta}\varphi_0)}{q\mathcal{J}(\boldsymbol{\gamma},\boldsymbol{\delta})}\right) = H_q(\boldsymbol{\gamma},\boldsymbol{\delta}) + \frac{\mathcal{J}((-\alpha_1)\boldsymbol{\gamma},(-\alpha_1)\boldsymbol{\delta})}{q\mathcal{J}(\boldsymbol{\gamma},\boldsymbol{\delta})}.$$

□

Definition 6.4. *Let $\boldsymbol{\alpha},\boldsymbol{\beta} \in \mathbb{Q}^n$ be generic parameters and let N be their least common denominator. Let $F = \mathbb{Q}(\zeta_N)$ and let $\mathfrak{p}$ be a prime ideal of F of norm q not dividing N. Let $z_0 \in \mathbb{Q}$ satisfy $v_{\mathfrak{p}}(z_0) = v_{\mathfrak{p}}(z_0-1) = 0$. The* $\mathfrak{p}$-hypergeometric sum, *denoted $H_{\mathfrak{p}}(\boldsymbol{\alpha},\boldsymbol{\beta}|z_0)$ is the value $H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|z_0)$ for ϖ a generator of $\widehat{\mathbb{F}_q^\times}$ satisfying $\varpi^{\frac{q-1}{N}} = \chi_{\mathfrak{p}}^{-1}$.*

The independence of ϖ follows from the following result.

Lemma 6.5. *Let $\boldsymbol{\alpha},\boldsymbol{\beta} \in \mathbb{Q}^n$ be generic parameters and let N be their least common denominator. Let ϖ_1 and ϖ_2 be two generators of $\widehat{\mathbb{F}_q^\times}$ such that $\varpi_1^{\frac{q-1}{N}} = \varpi_2^{\frac{q-1}{N}}$. Let $z_0 \in \mathbb{F}_q^\times$, $z_0 \neq 1$. Then the value $H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|z_0)$ taking ϖ_1 as a generator of $\widehat{\mathbb{F}_q^\times}$ is the same as the value obtained by taking ϖ_2.*

Proof. Let $j \in (\mathbb{Z}/(q-1))^\times$ be such that $\varpi_2 = \varpi_1^j$. The hypothesis $\varpi_1^{\frac{q-1}{N}} = \varpi_2^{\frac{q-1}{N}}$ implies that $j \equiv 1 \pmod N$. The value $H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|z)$ taking ϖ_2 as a generator equals

$$\frac{1}{1-q}\sum_{\varphi} \frac{\prod_{i=1}^n g(\psi,\varpi_1^{(q-1)j\alpha_i}\varphi) g(\psi^{-1},\overline{\varpi_1}^{(q-1)j\beta_i}\overline{\varphi})}{\prod_{i=1}^n g(\psi,\varpi_1^{(q-1)j\alpha_i}) g(\psi^{-1},\overline{\varpi_1}^{(q-1)j\beta_i})}\varphi(z_0),$$

which matches the value of the finite hypergeometric series for the generator ϖ_1 with parameters $j\boldsymbol{\alpha}$, $j\boldsymbol{\beta}$. Since $j \equiv 1 \pmod N$, $\boldsymbol{\alpha} = j\boldsymbol{\alpha} \pmod{\mathbb{Z}^n}$ and the same is true for $\boldsymbol{\beta}$ hence the result. □

6.2. Relation with the p-adic Gamma function.

For the reader's convenience, we will use bold letters for p-adic functions. Let p be a prime number. Recall the following definition (see [35]).

Definition 6.6. *The* p-adic Gamma function *is the continuous function $\boldsymbol{\Gamma} : \mathbb{Z}_p \to \mathbb{Q}_p$ that at a positive integer n takes the value*

$$\boldsymbol{\Gamma}(n) = (-1)^n \prod_{\substack{i=1\\ p\nmid i}}^{n-1} i. \tag{38}$$

Remark 6.7. We used the nowadays standard definition for the p-adic Gamma function. It does not match Morita's definition ([27]), but they are related by $\boldsymbol{\Gamma}(n) = (-1)^z\Gamma_p(z)$.

As proved in [35, Lemma 1], the p-adic Gamma function satisfies the relation

$$\boldsymbol{\Gamma}(n + p^r m) \equiv \boldsymbol{\Gamma}(n) \pmod{p^r},$$

hence condition (38) determines it uniquely. An important property of the p-adic Gamma function is that it determines an analytic function.

Theorem 6.8 (Morita)**.** *Set $Q = 8$ if $p = 2$ and $Q = 1$ otherwise. Then the p-adic $\boldsymbol{\Gamma}$ function is an analytic function from $Q\mathbb{Z}_p \to \mathbb{Q}_p$.*

Proof. See [35, Theorem 3]. □

Let $q = p^f$ for a positive integer f. The following functions play an important role.

Definition 6.9. *Let $\star \in \{0, \infty\}$. The function $\{\,\cdot\,\}^\star : \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}$ is defined by*

$$\{x\}^\infty := x - [x], \qquad \textit{and} \qquad \{x\}^0 := 1 - \{-x\}^\infty, \tag{39}$$

where $[x]$ denotes the floor of x (so $\{x\}^\infty \in [0,1)$ while $\{x\}^0 \in (0,1]$). Consider the following functions

$$\mathbf{\Gamma}_q^\star : \mathbb{Z}_{(p)}/\mathbb{Z} \to \mathbb{Q}_p, \qquad \mathbf{\Gamma}_q^\star(x) := \prod_{i=0}^{f-1} \mathbf{\Gamma}(\{p^i x\}^\star). \tag{40}$$

$$\eta_q^\star : \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}, \qquad \eta_q^\star(x) := \sum_{i=0}^{f-1} \{p^i x\}^\star. \tag{41}$$

$$\eta_{q,m}^\star : \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}, \qquad \eta_{q,m}^\star(x) := \eta_q^\star\left(x + \frac{m}{1-q}\right) - \eta_q^\star(x). \tag{42}$$

If $\boldsymbol{x} \in (\mathbb{Q}/\mathbb{Z})^n$, we extend the last map linearly component-wise, namely

$$\eta_{q,m}^\star(\boldsymbol{x}) = \sum_{i=1}^{n} \eta_{q,m}^\star(x_i).$$

Definition 6.10. *Let $x \in \mathbb{Q}/\mathbb{Z}$ be such that its denominator is not divisible by p. The q-*orbit *of x is the set*

$$\mathcal{O}(x) := \{x, qx, q^2 x, \ldots\} \subset \mathbb{Q}/\mathbb{Z}.$$

We denote by $\operatorname{len}(x)$ *its number of elements.*

Remark 6.11. If $x \in \mathbb{Q}/\mathbb{Z}$ and its q-orbit has r elements, then $(q^r - 1)x \in \mathbb{Z}$. In particular, the q-orbit of x has a unique element precisely when $\operatorname{den}(x) \mid q - 1$.

Lemma 6.12. *Let $x \in \mathbb{Q}/\mathbb{Z}$ be such that its denominator is not divisible by p. Then*

$$\prod_{u \in \mathcal{O}(x)} \mathbf{\Gamma}_q^\star(u) = \mathbf{\Gamma}_{q^{\operatorname{len}(x)}}^\star(x). \tag{43}$$

Similarly,

$$\sum_{u \in \mathcal{O}(x)} \eta_{q,m}^\star(u) = \eta_{q^{\operatorname{len}(x)},m}^\star(x). \tag{44}$$

Proof. By definition of the q-Gamma function

$$\prod_{u \in \mathcal{O}(x)} \mathbf{\Gamma}_q^\star(u) = \prod_{i=0}^{\operatorname{len}(x)-1} \prod_{j=0}^{f-1} \mathbf{\Gamma}^\star(\{p^j q^i x\}) = \prod_{i=0}^{f\operatorname{len}(x)-1} \mathbf{\Gamma}^\star(\{p^i x\}) = \mathbf{\Gamma}_{q^{\operatorname{len}(x)}}^\star(x).$$

The second statement is an additive version of the same proof. $\square$

Definition 6.13. *Let p be a prime number, let $q = p^f$ and let n be a positive integer. Let $x \in \mathbb{Z}_{(p)}/\mathbb{Z}$. For $\star \in \{0, \infty\}$, define the Pochammer symbol*

$$(x)_{q,n}^\star := \frac{\mathbf{\Gamma}_q^\star(x + \frac{n}{1-q})}{\mathbf{\Gamma}_q^\star(x)}. \tag{45}$$

Let $\mathcal{O}$ denote the ring of integers of $\mathbb{Q}_p(\zeta_{q-1})$ and let $\mathfrak{p}$ be its maximal ideal. Let Teich be the character

$$\text{Teich} : \mathbb{F}_q^\times \to \mathcal{O}^\times \tag{46}$$

that while composed with the quotient map $\pi : \mathcal{O} \to \mathbb{F}_q$ is the identity (it is the p-adic version of the character $\chi_\mathfrak{p}$ of (24) for $N = q-1$).

Definition 6.14. *Let $\boldsymbol{\alpha}, \boldsymbol{\beta} \in (\mathbb{Q}/\mathbb{Z})^n$. Let p be a prime number not dividing the denominator of $\boldsymbol{\alpha}$ nor $\boldsymbol{\beta}$ and let $q = p^f$. The* q-adic finite hypergeometric sum *is the function defined at $z_0 \in \mathbb{F}_q$ by*

$$\boldsymbol{H}_q(\boldsymbol{\alpha}, \boldsymbol{\beta}|z_0) := \frac{1}{1-q} \sum_{m=0}^{q-2} \frac{(\alpha_1)_{q,m}^\infty \cdots (\alpha_n)_{q,m}^\infty}{(\beta_1)_{q,m}^0 \cdots (\beta_n)_{q,m}^0} (-p)^{\eta_{q,m}^\infty(\boldsymbol{\alpha}) - \eta_{q,m}^0(\boldsymbol{\beta})} \,\text{Teich}(z_0)^m. \tag{47}$$

Remark 6.15. The Pari/GP package described in the introduction (developed by the third author) computes the value (47). The routine "hgm" takes as input the specialization z_0 (a rational number), the parameters $\boldsymbol{\alpha}, \boldsymbol{\beta}$ and a prime number p and outputs the p-adic number $\boldsymbol{H}_q(\boldsymbol{\alpha}, \boldsymbol{\beta}|z_0)$. A priori the q-adic finite hypergeometric sum is just an element of the p-adic field $\mathbb{Q}_p$ (with a default precision of 20 digits). See Example 2 in the introduction.

Definition 6.16. *Let $\boldsymbol{\alpha} \in (\mathbb{Q}/\mathbb{Z})^n$ and let $q = p^f$, with p not dividing the denominator of the coordinates of $\boldsymbol{\alpha}$. The vector $\boldsymbol{\alpha}$ is called* q-stable *if the set $\{q\alpha_1, \dots, q\alpha_n\} = \{\alpha_1, \dots, \alpha_n\}$.*

We will prove in Theorem 6.18 that $\boldsymbol{H}_q(\boldsymbol{\alpha}, \boldsymbol{\beta}|z)$ is algebraic for q-stable parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. We start with some preliminaries.

Let $\boldsymbol{\psi} : \mathbb{F}_p \to \mathbb{Z}_p[\zeta_p]^\times$ be a non-trivial additive character. For $a \in \frac{1}{q-1}\mathbb{Z}/\mathbb{Z}$, define the *$p$-adic Gauss sum*

$$\boldsymbol{g}(\boldsymbol{\psi}, a, q) := -\sum_{u \in \mathbb{F}_q^\times} \text{Teich}(u)^{-a(q-1)} \boldsymbol{\psi}(\text{Tr}_{\mathbb{F}_q/\mathbb{F}_p} u). \tag{48}$$

Let $M = \mathbb{Q}(\zeta_{q-1})$ and let $L = M(\zeta_p)$. Let $\mathfrak{p}$ be a prime ideal of M dividing p and let $\mathfrak{q}$ be a prime ideal of L dividing $\mathfrak{p}$. Let $\iota : L \hookrightarrow L_\mathfrak{q}$ be the natural inclusion, where $L_\mathfrak{q}$ denotes the completion of L at $\mathfrak{q}$. Let $\psi : \mathbb{F}_q \to \mathbb{Z}[\zeta_p]^\times$ be the additive character satisfying $\iota \circ \psi = \boldsymbol{\psi} \circ \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}$. Let ϖ be a generator of $\widehat{\mathbb{F}_q^\times}$ such that $\iota \circ \varpi = \text{Teich}^{-1}$. Then for $a \in \frac{1}{q-1}\mathbb{Z}/\mathbb{Z}$

$$\boldsymbol{g}(\boldsymbol{\psi}, a, q) = -\iota(g(\psi, a, \mathfrak{p})), \tag{49}$$

where $g(\psi, a, \mathfrak{p}) \in L$ is the complex Gauss sum defined in (25).

Set $\zeta_p := \psi(1)$ and let $\pi \in \overline{\mathbb{Q}}$ satisfy

$$\pi^{p-1} = -p, \qquad \pi \equiv (\zeta_p - 1) \pmod{(\zeta_p - 1)^2}.$$

Theorem 6.17 (Gross-Koblitz)**.** *Let $a \in \frac{1}{q-1}\mathbb{Z}$. Then*

$$\boldsymbol{g}(\boldsymbol{\psi}, a, q) = \pi^{(p-1)\eta_q^\infty(a)} \boldsymbol{\Gamma}_q^\infty(a), \tag{50}$$

and

$$\frac{q}{\boldsymbol{g}(\boldsymbol{\psi}', -a, q)} = \pi^{(p-1)\eta_q^0(a)} \boldsymbol{\Gamma}_q^0(a), \tag{51}$$

where $\boldsymbol{\psi}'(a) := \boldsymbol{\psi}(-a)$.

Proof. The first result follows from [27, Theorem 1.7] and (49), taking (in Gross-Koblitz's notation) $N = q-1$. If we replace $\boldsymbol{\Gamma}_q^\infty$ and η_q^∞ by their $\boldsymbol{\Gamma}_q^0$ and η_q^0 counterparts, we find that

$$\pi^{(p-1)\eta_q^0(a)} \boldsymbol{\Gamma}_q^0(a) = \begin{cases} q & \text{if } a \in \mathbb{Z}, \\ \boldsymbol{g}(\boldsymbol{\psi}, a, q) & \text{if } a \notin \mathbb{Z}. \end{cases}$$

The formula follows from the well known relation

$$\boldsymbol{g}(\boldsymbol{\psi}, a, q) \cdot \boldsymbol{g}(\boldsymbol{\psi}', -a, q) = \begin{cases} 1 & \text{if } a \in \mathbb{Z}, \\ q & \text{if } a \notin \mathbb{Z}. \end{cases}$$

□

Theorem 6.18. *Let $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{Q}^n$ be q-stable vectors. Consider the decomposition of each set as a disjoint union of orbits (after relabeling the parameters)*

$$\{\alpha_1, \dots, \alpha_n\} = \mathcal{O}(\alpha_1) \cup \cdots \cup \mathcal{O}(\alpha_u), \qquad \{\beta_1, \dots, \beta_n\} = \mathcal{O}(\beta_1) \cup \cdots \cup \mathcal{O}(\beta_v).$$

To ease notation, let $l_i = \operatorname{len}(\alpha_i)$ and $l_i' = \operatorname{len}(\beta_i)$. Then for $z_0 \in \mathbb{F}_q$,

$$\boldsymbol{H}_q(\boldsymbol{\alpha}, \boldsymbol{\beta}|z_0) = \frac{1}{1-q} \sum_{m=0}^{q-2} \prod_{i=1}^{u} \frac{\boldsymbol{g}(\boldsymbol{\psi}, \alpha_i + \frac{m}{1-q}, q^{l_i})}{\boldsymbol{g}(\boldsymbol{\psi}, \alpha_i, q^{l_i})} \prod_{i=1}^{v} \frac{\boldsymbol{g}(\boldsymbol{\psi}', -\beta_i - \frac{m}{1-q}, q^{l_i'})}{\boldsymbol{g}(\boldsymbol{\psi}', -\beta_i, q^{l_i'})} \operatorname{Teich}(z_0)^m. \tag{52}$$

In particular $\boldsymbol{H}_q(\boldsymbol{\alpha}, \boldsymbol{\beta}|z_0) \in \overline{\mathbb{Q}}$.

Proof. Since $q(\alpha_i + \frac{m}{1-q}) = q\alpha_i + \frac{m}{1-q}$ in $\mathbb{Q}/\mathbb{Z}$, the decomposition in q-orbits of $\{\alpha_1 + \frac{m}{1-q}, \dots, \alpha_n + \frac{m}{1-q}\}$ mimics that of $\{\alpha_1, \dots, \alpha_n\}$. Then using Lemma 6.12

$$(\alpha_1)_{q,m}^{\infty} \cdots (\alpha_n)_{q,m}^{\infty} = \prod_{i=1}^{n} \frac{\boldsymbol{\Gamma}_q^{\infty}(\alpha_i + \frac{m}{1-q})}{\boldsymbol{\Gamma}_q^{\infty}(\alpha_i)} = \prod_{i=1}^{u} \frac{\boldsymbol{\Gamma}_{q^{l_i}}^{\infty}(\alpha_i + \frac{m}{1-q})}{\boldsymbol{\Gamma}_{q^{l_i}}^{\infty}(\alpha_i)}.$$

By Remark 6.11, both $(q^{l_i} - 1)\alpha_i$ and $(q^{l_i} - 1)(\alpha_i + \frac{m}{1-q})$ are integers, so Theorem 6.17 implies that the product equals

$$\prod_{i=1}^{u} \frac{\boldsymbol{g}(\boldsymbol{\psi}, \alpha_i + \frac{m}{1-q}, q^{l_i})}{\boldsymbol{g}(\boldsymbol{\psi}, \alpha_i, q^{l_i})} \pi^{(1-p)\left(\eta_{q^{l_i}}^{\infty}(\alpha_i + \frac{m}{1-q}) - \eta_{q^{l_i}}^{\infty}(\alpha_i)\right)}.$$

The same argument applies to the values $(\beta_i)_{q,m}^{0}$ using (51). Then the result follows from Lemma 6.12 and the fact that $\pi^{1-p} = (-p)$. The algebricity statement follows from (49), because the right hand side of the equality belongs to the number field L. □

Corollary 6.19. *Let $\boldsymbol{\alpha}, \boldsymbol{\beta} \in (\mathbb{Q}/\mathbb{Z})^n$ and let N be their least common denominator. Let $\mathfrak{p}$ be a prime ideal of $F = \mathbb{Q}(\zeta_N)$ of norm q. Let $\iota : \mathbb{Q}(\zeta_N) \hookrightarrow \mathbb{Q}(\zeta_N)_{\mathfrak{p}}$ be the natural map. Then for $z_0 \in \mathbb{F}_q^{\times}$*

$$\iota(H_{\mathfrak{p}}(\boldsymbol{\alpha}, \boldsymbol{\beta}|z_0)) = \boldsymbol{H}_q(\boldsymbol{\alpha}, \boldsymbol{\beta}|z_0). \tag{53}$$

Proof. Since $N \mid q-1$, the sets $\{\alpha_1, \dots, \alpha_n\}$ and $\{\beta_1, \dots, \beta_n\}$ decompose as n-disjoint q-orbits of length 1 (so $l_i = l_i' = 1$). Since $\iota(\chi_{\mathfrak{p}}) = \operatorname{Teich}^{(q-1)/N}$, $\iota(\chi_{\mathfrak{p}}^{N\alpha_i}(x)) = \operatorname{Teich}^{(q-1)\alpha_i}(x)$, hence the right hand side of (52) matches the definition of $H_{\mathfrak{p}}(\boldsymbol{\alpha}, \boldsymbol{\beta}|z_0)$. □

Remark 6.20. In the hypothesis of the last lemma, if $\mathfrak{p}$ is a prime ideal of F, we can define the value $H_{\mathfrak{p}}(\boldsymbol{\alpha}, \boldsymbol{\beta}|z_0)$ as the algebraic number $\alpha \in F$ such that $\iota(\alpha) = \boldsymbol{H}_q(\boldsymbol{\alpha}, \boldsymbol{\beta}|z_0)$, for $\iota : F \to F_{\mathfrak{p}}$.

6.3. **Finite hypergeometric sums over K.** Let $\boldsymbol{\alpha}, \boldsymbol{\beta}$ be generic parameters and N be their least common denominator. Set $K = F^H$. We extend the definition of $\mathfrak{p}$-hypergeometric sums to prime ideals of K following the proof of Theorem 6.18. Let $\mathfrak{p}$ be a prime ideal of K of norm q prime to N.

Definition 6.21. *Keeping the previous hypothesis, let $z_0 \in \mathbb{Q}$ be such that $v_{\mathfrak{p}}(z_0) = v_{\mathfrak{p}}(z_0 - 1) = 0$. Consider the q-orbits decomposition*

$$\{\alpha_1, \dots, \alpha_n\} = \mathcal{O}(\alpha_1) \cup \dots \cup \mathcal{O}(\alpha_u), \qquad \{\beta_1, \dots, \beta_n\} = \mathcal{O}(\beta_1) \cup \dots \cup \mathcal{O}(\beta_v).$$

Let $\mathfrak{q}$ be a prime ideal of K dividing $\mathfrak{p}$. The $\mathfrak{p}$-hypergeometric sum *is the value*

$$H_{\mathfrak{p}}(\boldsymbol{\alpha}, \boldsymbol{\beta}|z_0) = \frac{1}{1-q} \sum_{\varphi} \prod_{i=1}^{u} \frac{g(\psi, \chi_{\mathfrak{q}}^{-N\alpha_i}\varphi, \mathfrak{q})}{g(\psi, \chi_{\mathfrak{q}}^{-N\alpha_i}, \mathfrak{q})} \prod_{i=1}^{v} \frac{g(\psi^{-1}, \chi_{\mathfrak{q}}^{N\beta_i}\overline{\varphi}, \mathfrak{q})}{g(\psi^{-1}, \chi_{\mathfrak{q}}^{N\beta_i}, \mathfrak{q})} \varphi(z_0),$$

where the sum runs over characters of $\mathbb{F}_q^\times$.

Our definition is the algebraic analogue of the respective q-adic hypergeometric sum. At a first glance it looks like the definition depends on a particular choice of an element on each orbit. However this value is well defined as the following lemma shows.

Lemma 6.22. *Let $q = p^r$ be a prime power. Let ψ be an additive character of $\mathbb{F}_q$ obtained as the composition of the trace map $\mathrm{Tr} : \mathbb{F}_q \to \mathbb{F}_p$ with an additive character of $\mathbb{F}_q$. Let φ be a multiplicative character of $\mathbb{F}_q^\times$. Then*

$$g(\psi, \varphi) = g(\psi, \varphi^p). \tag{54}$$

Proof. By definition

$$g(\psi, \varphi^p) = \sum_{x \in \mathbb{F}_q^\times} \varphi(x^p)\psi(x) = \sum_{y \in \mathbb{F}_q^\times} \varphi(y)\psi(y^{p^{f-1}}).$$

The result follows from the choice of our additive character, because $\mathrm{Tr}(y^{p^r}) = \mathrm{Tr}(y)$. ☐

The extended hypergeometric sum satisfies all the expected properties. For example, if $z_0 \in \mathbb{Q}$, $H_{\mathfrak{p}}(\boldsymbol{\alpha}, \boldsymbol{\beta}|z_0) \in \mathbb{Q}(\zeta_N)^H$ (the proof mimics that given in Proposition 6.3).

6.4. **Hypergeometric character sums.** The result of the present section will be used to relate the number of points of a variety to a finite hypergeometric sum. Although the definition of Gauss sums depend on the choice of an additive character, the products/quotients considered in the present section will not, so we omit writing the dependence to easy notation. Recall the following well known definition.

Definition 6.23. *Let φ, η be two multiplicative characters on $\mathbb{F}_q^\times$. The Jacobi sum attached to them is defined by*

$$J(\varphi, \eta) = \sum_{x \in \mathbb{F}_q} \varphi(x)\eta(1-x),$$

where $\varphi(0) = \eta(0) = 0$.

Lemma 6.24. *Let φ, η be characters of $\mathbb{F}_q$, with η non-trivial. Then*

$$J(\varphi, \eta) = \varphi(-1)\frac{g(\varphi^{-1}\eta^{-1})g(\varphi)}{g(\eta^{-1})}.$$

Proof. If φ and $\varphi\eta$ are non-trivial, then the left hand side is the usual Jacobi sum, and its value equals

$$J(\varphi, \eta) = \frac{g(\varphi)g(\eta)}{g(\varphi\eta)}.$$

For φ a non-trivial character, the equality $g(\varphi) = \varphi(-1)\frac{q}{g(\varphi^{-1})}$ applied to $\varphi = \varphi\eta$ and η implies that

$$J(\varphi, \eta) = \varphi(-1)\frac{g(\varphi)g(\varphi^{-1}\eta^{-1})}{g(\eta^{-1})}.$$

If $\varphi = 1$, the left hand side equals -1 (because $\varphi(0) = 0$), which equals the right hand side as well. At last, if $\varphi\eta = 1$, then the left hand side equals $J(\varphi, \varphi^{-1}) = -\varphi(-1)$, which also equals the value of the right hand side, since $g(1) = -1$. □

Let $\varepsilon_1, \eta_1, \dots, \varepsilon_{n-1}, \eta_{n-1}, \chi$ be characters of $\mathbb{F}_q^\times$ (extended to $\mathbb{F}_q$ by setting their value at 0 to be 0). For $z \in \mathbb{F}_q$ let

$$H(z) = \sum_{x_1,\dots,x_{n-1}} \prod_{i=1}^{n-1} \varepsilon_i(x_i)\eta_i(1-x_i)\chi^{-1}(1 - zx_1\cdots x_{n-1}). \tag{55}$$

Theorem 6.25. *Keep the previous notation, and set $\varepsilon_n = \chi$ and $\varepsilon_n\eta_n = 1$. Let $\boldsymbol{\alpha}, \boldsymbol{\beta}$ be rational numbers such that $\varpi^{(q-1)\alpha_i} = \varepsilon_i$ and $\varpi^{(q-1)\beta_i} = \varepsilon_i\eta_i$ for $i = 1, \dots, n$. Then $H(z)$ equals*

$$H(z) = (-1)^{n-1}\left(\prod_{i=1}^{n-1}\varepsilon_i(-1)\right)\mathbf{J}((\boldsymbol{\alpha}, -\boldsymbol{\beta}), (\boldsymbol{\alpha}-\boldsymbol{\beta}))H_q(\boldsymbol{\alpha}, \boldsymbol{\beta}|z).$$

Proof. Add one more variable x_n to the definition of $H(z)$ defined by

$$x_n := zx_1\cdots x_{n-1}.$$

Then

$$H(z) = \frac{1}{q-1}\sum_{\varphi}\sum_{x_1,\dots,x_n}\prod_{i=1}^{n-1}\varepsilon_i(x_i)\eta_i(1-x_i)\chi^{-1}(1-x_n)\varphi(zx_n^{-1}x_1\cdots x_{n-1}),$$

where the first sum ranges over all characters φ of $\mathbb{F}_q^\times$ (we are abusing notation, declaring x_n^{-1} to be 0 if x_n equals 0). Interchanging sum and product, we get

$$H(z) = \frac{1}{q-1}\sum_{\varphi}\left(\prod_{i=1}^{n-1}\left(\sum_{x_i}\varepsilon_i\varphi(x_i)\eta_i(1-x_i)\right)\cdot\sum_{x_n}\varphi^{-1}(x_n)\chi^{-1}(1-x_n)\right)\varphi(z). \tag{56}$$

Applying Lemma 6.24 we get that

$$\begin{aligned} H(z) &= \frac{1}{q-1}\sum_{\varphi}\left(\prod_{i=1}^{n-1}\varepsilon_i(-1)\varphi(-1)\frac{g(\varepsilon_i\varphi)g(\varepsilon_i^{-1}\eta_i^{-1}\varphi^{-1})}{g(\eta_i^{-1})}\right)\cdot\frac{g(\varphi\chi)g(\varphi^{-1})}{g(\chi)}\varphi(-1)\varphi(z)\\ &= \prod_{i=1}^{n-1}\frac{\varepsilon_i(-1)g(\varepsilon_i)g(\varepsilon_i^{-1}\eta_i^{-1})}{g(\eta_i^{-1})}\left(\frac{(-1)}{(q-1)}\sum_{\varphi}\left(\prod_{i=1}^{n}\frac{g(\varepsilon_i\varphi)g(\varepsilon_i^{-1}\eta_i^{-1}\varphi^{-1})}{g(\varepsilon_i)g(\varepsilon_i^{-1}\eta_i^{-1})}\right)\varphi(-1)^n\varphi(z)\right),\end{aligned}$$

as claimed (with the convention $\varepsilon_n = \chi$ and $\varepsilon_n\eta_n = 1$), the extra -1 coming from the fact that $g(1) = -1$. □

7. Rank two Hypergeometric motives

The main result of [6] provides an explicit formula relating the function $H_p(\vec{\alpha}, \vec{\beta}|z)$ to the number of points of a non-singular projective variety V when the parameters are rational. Such relations allow to realize the motive $\mathcal{H}(\vec{\alpha}, \vec{\beta}|z)$ in V. A nice instance of their result is the following result of Ono (see [36]).

Theorem 7.1. *Let p be an odd prime power and $\lambda \in \mathbb{F}_p$ and $\lambda \neq 0, 1$. Let E_λ be the projective elliptic curve given by the Legendre affine equation*

$$E_\lambda : y^2 = x(x-1)(x-\lambda)$$

Then the set of $\mathbb{F}_p$-rational points (including the one at infinity) equals

$$|E_\lambda(\mathbb{F}_p)| = p + 1 - (-1)^{(p-1)/2}H_p((1/2, 1/2), (1, 1)|\lambda).$$

Equivalently,

$$H_p((1/2,1/2),(1,1)|\lambda) = (-1)^{(p-1)/2} a_p(E_\lambda), \tag{57}$$

where $a_p(E_\lambda)$ is the trace of the Frobenius endomorphism acting on E_λ.

Our definition of the geometric realization of the hypergeometric motive (Definition 7.7) is analogue to (57). The hypergeometric motive is a twist by a finite order character of a motive appearing in Euler's curve times a Jacobi motive. For $\boldsymbol{\alpha} = (1/2, 1/2)$ and $\boldsymbol{\beta} = (1,1)$, Euler's curve is the elliptic curve E_λ, and the Jacobi motive $\mathbf{J}((-1/2,-1/2,1,1),(1-1/2,1-1/2))$ is trivial.

The explicit relation between the action of Frobenius and the finite hypergeometric sum is a particular instance of Theorem 7.23.

Definition 7.2. *Let (a,b),(c,d) be rational numbers, and let N be their least common denominator. Define the quantities*

$$A = (d-b)N, \qquad B = (b+1-c)N, \qquad C = (1+a-d)N, \qquad D = (d-1)N. \tag{58}$$

Then Euler's curve attached to the parameters $(a,b),(c,d)$ is the curve with equation

$$\mathcal{C} : y^N = x^A (1-x)^B (1-zx)^C z^D. \tag{59}$$

Remark 7.3. If we translate any of the parameters $(a,b),(c,d)$ by an integer, we get different equations that are related by a simple change of variables.

Definition 7.4. *Let $(a,b),(c,d)$ be a pair of generic rationals parameters. The parameters satisfy condition* (**Irr**) *if Euler's curve (59) is irreducible over $\overline{\mathbb{Q}(z)}$.*

Lemma 7.5. *Let $(a,b),(c,d)$ be a pair of generic rationals parameters. Then condition* (**Irr**) *holds if and only if*

$$\operatorname{lcm}\{\operatorname{den}(a), \operatorname{den}(b), \operatorname{den}(c), \operatorname{den}(d)\} = \operatorname{lcm}\{\operatorname{den}(d-b), \operatorname{den}(b-c), \operatorname{den}(a-d)\}. \tag{60}$$

Proof. By making the change of variables (over $\overline{\mathbb{Q}(z)}$) $y = \sqrt[N]{z^D} y$, it is enough to study the curve

$$y^N = x^A (1-x)^B (1-zx)^C,$$

which is irreducible if and only of $\gcd(A,B,C,N) = 1 = \gcd((d-b)N, (b-c)N, (a-d)N, N)$. The later equality holds if and only if

$$\operatorname{lcm}\{\operatorname{den}(d-b), \operatorname{den}(b-c), \operatorname{den}(a-d)\} = N = \operatorname{lcm}\{\operatorname{den}(a), \operatorname{den}(b), \operatorname{den}(c), \operatorname{den}(d)\}.$$

□

Remark 7.6. If r denotes the quotient of N by $\operatorname{lcm}\{\operatorname{den}(d-b), \operatorname{den}(b-c), \operatorname{den}(a-d)\}$, then the curve $\mathcal{C}$ decomposes as the union of r irreducible components defined over $\mathbb{Q}(\zeta_r)[\sqrt[r]{z}]$. This phenomena did not appear before in the literature because when $d = 1$, condition (**Irr**) is always satisfied.

Example 9. Continuing with Example 2, let $a = 1/8, b = 7/8, c = 3/8, d = 5/8$ so $N = 8$. Euler's curve is defined by

$$\mathcal{C}: \quad y^8 = x^6 (1-x)^4 (1-zx)^4 z^5.$$

Over $\mathbb{Q}(w)$, where $w^2 = z$, it becomes the union of the curves

$$\mathcal{C}_\pm: \quad y^4 = \pm w x^3 (1-x)^2 (1-w^2 x)^2.$$

These curves have genus two with hyperelliptic model

$$y^2 = wx(x^2 \mp w)(x^2 \mp w^{-1})$$

The map

$$\iota: \quad (x,y) \mapsto (x^{-1}, yx^{-3})$$

yields an involution of the curve and it is not to hard to see that the quotient of $\mathcal{C}_\pm$ by ι equals

$$E_\pm : y^2 = x^3 - 4wx^2 \mp w(w-1)^2 x.$$

The point $P := (0,0)$ is a 2-torsion point on $E_\pm(w)$ and

$$E_\pm(w)/\langle P\rangle \simeq E_\mp(w) \otimes \sqrt{-2}.$$

It can be verified (using the involution $(x,y) \to (x^{-1}, -yx^{-3})$) that actually

$$\mathrm{Jac}(\mathcal{C}_\pm) \simeq E_\pm(w) \oplus E_\pm(w) \otimes \sqrt{-1}$$

If we now take $w = 3$, so $z = 9$ as in Example 2, the curve $E_+(w)$ specializes to the elliptic curve

$$E : \quad y^2 = x^3 - 60x + 176,$$

of conductor 576 and CM by $\mathbb{Q}(\sqrt{-12})$. We can quickly verify for small primes p split in $\mathbb{Q}(\sqrt{2})$ that the trace of Frobenius Frob_p on $H^1(E)$ and on $\mathcal{H}_9$ indeed agree (the latter may be computed in $\mathbb{Z}_p$ as

```
? e=ellinit([0,0,0,-60,176]);
? forprime(p=5,100,if(kronecker(2,p)==1,
    print(p,"  ",ellap(e,p),"  ",recognizep(hgm(9,[1/8,-1/8],[3/8,-3/8],p)*p))))

7  -4  -4
17  0  0
23  0  0
31  -4  -4
41  0  0
47  0  0
71  0  0
73  -10  -10
79  -4  -4
89  0  0
97  14  14
```

with the GP code already mentioned).

We will come back to this example later.

Many problems appear when Euler's curve is not irreducible. For this reason we start studying the irreducible case, and in the reducible one, we define our hypergeometric motive as a twist of the motive attached to an irreducible Euler curve (similar to formula (12)).

7.1. **Hypergeometric motive definition.** Let $\mathcal{J}_N^{\text{new}}$ denote the new part of (the Jacobian of) Euler's curve (see §7.4). It has an action of μ_N (the N-th roots of unity); denote by $\mathcal{J}_N^{\zeta_N,\text{new}}$ its ζ_N-eigenspace.

Definition 7.7. *Let $(a,b),(c,d)$ be a pair of generic rational numbers satisfying condition* (**Irr**) *and let N be their least common denominator. The hypergeometric motive with parameters $(a,b),(c,d)$ equals*

$$\mathcal{H}((a,b),(c,d)|z) := \mathcal{J}_N^{\zeta_N,\text{new}} \otimes \mathbf{J}((-a,-b,c,d),(c-b,d-a))^{-1}(-1)^{d-b}, \tag{61}$$

where $(-1)^{d-b}$ denotes the quadratic character that at an odd prime ideal $\mathfrak{p}$ of F not dividing N takes the value

$$\omega(-1)^{(d-b)(\mathcal{N}\mathfrak{p}-1)}, \tag{62}$$

for ω any generator of the group of characters of $(\mathbb{Z}[\zeta_N]/\mathfrak{p})^\times$.

Formula (61) involves three different motives, and a priori the field of definition of each motive might be different (as shown in Example 10). This is precisely the reason why we can only prove Conjecture 4.5 under extra hypotheses (for example conditions making the Jacobi motive to be rational). To prove cases not covered by our assumptions, one probably needs to study other geometric varieties where the HGM appears. Let us illustrate the situation with some examples.

Example 10. Let $a = \frac{1}{15}, b = \frac{4}{15}, c = \frac{1}{8}$ and $d = \frac{3}{8}$. The Jacobi motive for these parameters is $\mathbf{J}(\boldsymbol{\theta})$, where

$$\boldsymbol{\theta} = \left\langle \frac{14}{15} \right\rangle + \left\langle \frac{11}{15} \right\rangle + \left\langle \frac{1}{8} \right\rangle + \left\langle \frac{3}{8} \right\rangle - \left\langle \frac{103}{120} \right\rangle - \left\langle \frac{37}{120} \right\rangle.$$

Since 19 is the only element in $(\mathbb{Z}/120)^\times$ that fixes (under multiplication) the sets $\{\frac{14}{15}, \frac{11}{15}, \frac{1}{8}, \frac{3}{8}\}$ and $\{\frac{103}{120}, \frac{37}{120}\}$ (in $\mathbb{Q}/\mathbb{Z}$), the Jacobi motive is defined over $\mathbb{Q}(\zeta_{120})^{\sigma_{19}}$. The field of definition of the motive $\mathcal{H}((\frac{1}{15}, \frac{4}{15}), (\frac{1}{8}, \frac{3}{8})|z)$ corresponds to the field fixed by the group $H = \langle 49, 91 \rangle$. In particular, the hypergeometric motive is defined over a field properly contained in the Jacobi motive's one.

On the other hand, if we take $a = \frac{1}{8}, b = \frac{7}{8}, c = \frac{3}{8}$ and $d = \frac{5}{8}$ then the Jacobi motive is rational (since multiplication by any odd integer fixes the sets $\{\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}\}$ and $\{\frac{1}{2}\}$), while the hypergeometric motive is defined over $\mathbb{Q}(\sqrt{2})$.

Definition 7.8. *Let $\alpha = \frac{a}{N}$ be a rational number, with a, N coprime integers. Define the character*

$$\eta_\alpha : \mathrm{Gal}(\overline{\mathbb{Q}(z)}/\mathbb{Q}(z, \zeta_N)) \to \overline{\mathbb{Q}}^\times,$$

to be the character that factors through $\mathrm{Gal}(\mathbb{Q}(\sqrt[N]{z}, \zeta_N)/\mathbb{Q}(z, \zeta_N))$ and whose value at σ equals

$$\eta_\alpha(\sigma) = \left(\frac{\sigma(\sqrt[N]{z})}{\sqrt[N]{z}} \right)^a.$$

Remark 7.9. The character η_α equals the hypergeometric motive $\mathcal{H}(\alpha, 1|1 - z)$ defined in §2.

Let $z_0 \in \mathbb{Q} \setminus \{1\}$, and consider the specialization of η_α at z_0. Let $\mathfrak{p}$ be a prime ideal of $\mathbb{Q}(\zeta_N)$ of norm q, not dividing N and such that $v_\mathfrak{p}(z_0) = 0$. Let $\widetilde{\mathfrak{p}}$ be a prime ideal of $\mathbb{Q}(\sqrt[N]{z_0}, \zeta_N)$ dividing $\mathfrak{p}$. It follows from its definition that

$$\mathrm{Frob}_{\widetilde{\mathfrak{p}}}(\sqrt[N]{z_0}) \equiv (\sqrt[N]{z_0})^q \pmod{\widetilde{\mathfrak{p}}},$$

hence

$$\frac{\mathrm{Frob}_{\widetilde{\mathfrak{p}}}(\sqrt[N]{z_0})}{\sqrt[N]{z_0}} \equiv z_0^{\frac{q-1}{N}} \pmod{\mathfrak{p}}.$$

This implies that the specialization of η_α at z_0 satisfies

$$\eta_\alpha(\mathrm{Frob}_\mathfrak{p}) = \chi_\mathfrak{p}(z_0)^a, \tag{63}$$

where $\chi_\mathfrak{p}$ is the character defined in (24).

Definition 7.10. *Let $(a, b), (c, d)$ be generic rational numbers, and let N be their least common denominator. Set $F = \mathbb{Q}(\zeta_N)$. The hypergeometric motive $\mathcal{H}((a, b), (c, d)|z)$ is defined by*

$$\mathcal{H}((a - d, b - d), (c - d, 1)|z) \otimes \mathbf{J}((-a, -b, c, d), (d - a, d - b, c - d))^{-1} \eta_d(z), \tag{64}$$

where the hypergeometric motive $\mathcal{H}((a - d, b - d), (c - d, 1); z)$ is considered as a motive over F.

Remark 7.11. A priori it is not clear why the two given definitions coincides when (**Irr**) holds. We will prove that this is indeed the case in Theorem 7.23.

Lemma 7.12. *Let A, B, C, N be positive integers such that $\gcd(A,B,C,N) = 1$. Let $\kappa(N)$ be the number of elements in the set $\{A, B, C, A+B+C\}$ divisible by N. Then if $N > 1$*

$$\dim(\mathcal{J}_N^{new}) = \frac{2-\kappa(N)}{2}\phi(N)$$

Proof. See formula (16) of [2]. □

Then the abelian variety $\mathcal{J}_N^{\text{new}}$ has dimension $\phi(N)$ and its endomorphism ring contains $\mathbb{Z}[\zeta_N]$, hence it has attached a strong compatible family of 2-dimensional Galois representations (as defined by Serre in [42])

$$\{\rho_{z,\mathfrak{p}} : \text{Gal}_F \to \text{GL}_2(\mathbb{Z}[\zeta_N]_{\mathfrak{p}})\}_{\mathfrak{p}}, \tag{65}$$

indexed by prime ideals $\mathfrak{p}$ of F. By definition the same is true for the motive $\mathcal{H}((a,b),(c,d)|z)$; denote the family by $\{\rho_{\mathcal{H}((a,b),(c,d)|z),\mathfrak{p}} : \text{Gal}_{F(z)} \to \text{GL}_2(F_{\mathfrak{p}})\}_{\mathfrak{p}}$.

Theorem 7.13. *The Galois representation $\rho_{\mathcal{H}((a,b),(c,d)|z),\mathfrak{p}}$ restricted to $\text{Gal}_{\overline{\mathbb{Q}}(z)}$ matches the representation given in Corollary 3.15.*

Proof. The Galois representation attached to the Jacobi motive does not depend on z, its restriction to $\text{Gal}_{\overline{\mathbb{Q}}(z)}$ is trivial. Then to prove the result we need to understand the reduction type of Euler's curve at $0, 1$ and ∞. The main result of [22] proves that the image of inertia at all three points is generated by the matrices M_0, M_1 and M_∞ respectively. □

7.2. **Hodge numbers.** For rank two motives, we can give an explicit relation between the Hodge vector of $\mathcal{H}((a,b),(c,d)|z)$ and the output of the zig-zag procedure, since the Hodge numbers of Euler's curve (and of its ζ_N-eigenvalue) as well as the ones of the Jacobi motive are well known.

Theorem 7.14. *Let $(a,b),(c,d)$ be generic rational parameters. Let r be the number of parameters in $\mathbb{Z}$. Let $\sigma_j \in \text{Gal}(F/\mathbb{Q})$ be the automorphism sending $\zeta_N \to \zeta_N^j$. Then the Hodge vector of the motive $\mathcal{H}((a,b),(c,d)|z)$ attached to σ_j equals*

$$\sum_P h^{-P[2],P[2]+r-1},$$

where the sum runs over the blue points P of the zig-zag procedure applied to the parameters $(aj,bj),(cj,dj)$.

Proof. By definition, the hypergeometric motive $\mathcal{H}((a,b),(c,d)|z)$ satisfies

$$\mathcal{H}((a,b),(c,d)|z) \otimes \mathbf{J}((-a,-b,c,d),(c-b,d-a)) = \mathcal{J}_N^{\zeta_N,\text{new}}(-1)^{d-b}.$$

Let $\mathcal{C}$ be Euler's curve as in (59). The space of differentials of the first kind on $\mathcal{C}$ has an action of μ_N (the N-th roots of unity); the dimension of the ζ_N^j eigenspace (for $\gcd(j,N) = 1$) is given by the formula

$$\left\{\frac{jA}{N}\right\} + \left\{\frac{jB}{N}\right\} + \left\{\frac{jC}{N}\right\} - \left\{\frac{j(A+B+C)}{N}\right\}, \tag{66}$$

where $\{a\}$ denotes the fractional part of a (see for example §4, formula (21) of [53]).

The motivic weight of the Jacobi motive equals $n = 2 - r$. The Hodge number (corresponding to the embedding σ_j) of the Jacobi motive is given (see (30)) by

$$p := \{-aj\} + \{-bj\} + \{cj\} + \{dj\} - \{(c-b)j\} - \{(d-a)j\}, \quad p + q = n.$$

Table 7.2 contains, for $a < b \in (0,1]$ and $c < d \in [0,1)$, all possible outcomes of the zig-zag procedure together with the hodge number of the Jacobi motive and of Euler's curve for the embedding $\sigma(\zeta_N) = \zeta_N$ in the case $r = 0$. The statement follows by comparing the last two columns.

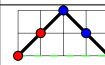
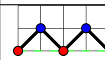
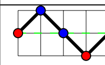
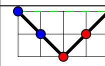
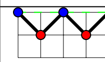
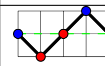
Case	Condition	$\mathbf{J}$	Euler	$\mathcal{H}((a,b),(c,d)\|z)$	ZigZag
I	$a<b<c<d$	$(2,0)$	$(1,0)\oplus(0,1)$	$(-1,0)\oplus(-2,1)$	
II	$a<c<b<d$	$(1,1)$	$2(0,1)$	$2(-1,0)$	
III	$a<c<d<b$	$(1,1)$	$(1,0)\oplus(0,1)$	$(0,-1)\oplus(-1,0)$	
IV	$c<d<a<b$	$(0,2)$	$(1,0)\oplus(0,1)$	$(1,-2)\oplus(0,-1)$	
V	$c<a<d<b$	$(1,1)$	$2(1,0)$	$2(0,-1)$	
VI	$c<a<b<d$	$(1,1)$	$(1,0)\oplus(0,1)$	$(0,-1)\oplus(-1,0)$	

TABLE 7.1. Possible Hodge diagrams in the rank 2 case

It is easy to verify that the values of the columns (namely $\mathbf{J}$, Euler and $\mathcal{H}((a,b),(c,d)|z)$) remain the same while permuting (a,b) and (c,d) in cases I, III, IV and VI. In case II, when $b<c<a<d$ or when $a<d<b<c$, Euler's curve has Hodge numbers $2(1,0)$ while $\mathbf{J}$ has Hodge number $(2,0)$, so the value for $\mathcal{H}$ stays the same. A similar phenomena occurs in case V.

When r is non-zero, the p-value of the Jacobi motive remains the same, but the motivic weight decreases by r hence the same occurs to q. □

Example 11. Let $\mathcal{C}$ be the Euler curve with equation

$$\mathcal{C} : y^5 = x(1-x)(1-tx),$$

as studied by Shimura in [44] (case (4) of the table in the first page of the paper). It corresponds to the parameters $\left(\frac{1}{5},\frac{4}{5}\right),\left(\frac{3}{5},1\right)$ (using formula (58)). The field $\mathbb{Q}(\zeta_5)$ has four embeddings into $\mathbb{C}$, parametrized (once we choose a fifth root of unity in $\mathbb{C}$) by the elements $i \in (\mathbb{Z}/5)^\times$. It is easy to verify that the values $i = 1, 4$ give type V, the value $i = 2$ gives type IV, while the value $i = 3$ gives type VI (see Figure 7.1). Its Hodge values are given in Table 7.2. In particular, for $i = 1$ we gave

i	Case	$\mathcal{H}^{j,-j}$					Euler$^{j,1-j}$					Jacobi$^{j,1-j}$				
j		-2	-1	0	1	2	-2	-1	0	1	2	-2	-1	0	1	2
1	V			2						2					1	
2	IV			1	1				1	1				1		
3	VI		1	1					1	1					1	
4	V			2					2					1		
Total		0	1	6	1	0	0	0	4	4	0			2	2	

TABLE 7.2. Hodge values

$h^{0,0} = 2$; for $i = 2$, $h^{0,0} = h^{1,-1} = 1$; for $i = 3$, $h^{-1,1} = h^{0,0} = 1$ and for $i = 4$, $h^{0,0} = 2$. Note that all Hodge vectors but two have $h^{0,0} = 2$ (this type of examples are the ones considered in Shimura's article). It is not hard to verify (using (30)) that the Jacobi motive (as defined in (28))

$$\mathbf{J}((-i/5, i/5, 3i/5), (4i/5, -i/5))$$

has Hodge value $h^{1,0} = 1$ for $i = 1, 3$ and $h^{0,1} = 1$ for $i = 3, 4$. From Definition 7.7 it follows the consistency of the values given in Table 7.2.

We see that the HGM $\mathcal{H}$ is the *half-twist* of H^1 of the Euler curve $\mathcal{C}$, in the notation of van Geemen[47], with respect to the CM type $\Sigma := \{\sigma_1, \sigma_2\}$ of $\mathbb{Q}(\zeta_5)$.

*** This is not very clear

Similarly, we can study the case (6) of Shimura's table (in [44]). It is given by Euler's curve with equation

$$\mathcal{C} : y^7 = x(1 - x)(1 - tx),$$

corresponding to the parameters $\left(\frac{1}{7}, \frac{6}{7}\right), \left(\frac{5}{7}, 1\right)$. Now the embeddings are parametrized by the elements $i \in (\mathbb{Z}/7)^\times$. The values $i = 1, 2, 5, 6$ yield zig-zag diagrams of type V, with Hodge number $h^{0,0} = 2$ while for $i = 3$ we get type IV and hence $h^{1,-1} = h^{0,0} = 1$ and for $i = 4$ type VI and hence $h^{0,0} = h^{-1,1} = 1$. Summarizing, the Hodge vectors are

$$2(0, 0), 2(0, 0), (0, 0) \oplus (1, -1), (-1, 1) \oplus (0, 0), 2(0, 0), 2(0, 0).$$

7.3. **Superelliptic curves.** To study properties of the hypergeometric motive we recall how Frobenius acts on Euler's curve as described in [2] and in [21] (Theorem 1.1) with a few little modifications to include the case $d \notin \mathbb{Z}$ (assuming that (**Irr**) holds). Our contribution is the use of Lefschetz's trace formula to pin down the trace of Frobenii acting at the μ_N-eigenspace of a superelliptic curve. The results obtained (specially Theorem 7.22) might be of independent interest.

In this section we let N be a positive integer and L be a number field or a local field containing the N-th roots of unity. Let $f(x) \in L[x]$. Let $\mathcal{C}$ be the superelliptic curve with equation

$$\mathcal{C} : y^N = f(x). \tag{67}$$

Assume that the curve $\mathcal{C}$ is irreducible. The group of N-th roots of unity acts on the L-rational points $\mathcal{C}(L)$ of $\mathcal{C}$ by

$$\zeta_N^i \cdot (x, y) = (x, \zeta_N^i y).$$

Let $\mathcal{J} := \mathrm{Jac}(\mathcal{C})$ denote the Jacobian of the (smooth model of the) curve $\mathcal{C}$. The ring $\mathbb{Z}[\zeta_N]$ is contained in the endomorphism ring of $\mathcal{J}$ over L.

7.4. **The new part of $\mathcal{J}$.** Let us recall the contribution of the so called *old parts* for cyclic covers of curves (as explained in [30]). For a positive integer d, let $\widehat{\mathcal{C}}_d$ denote the smooth model of the curve

$$\mathcal{C}_d : y^d = f(x).$$

The Jacobian variety $\mathrm{Jac}(\widehat{\mathcal{C}}_N)$ has a contribution coming from the curves $\widehat{\mathcal{C}}_d$ for divisors d of N. More concretely, let d be a proper divisor of N and consider the natural map $\pi_d : \mathcal{C}_N \to \mathcal{C}_d$ given by $\pi_d(x, y) = (x, y^{N/d})$ (and a similar map for their desingularizations). The map π_d induces two morphisms between $\mathrm{Jac}(\widehat{\mathcal{C}}_N)$ and $\mathrm{Jac}(\widehat{\mathcal{C}}_d)$ namely the push-forward

$$\pi_* : \mathrm{Jac}(\widehat{\mathcal{C}}_N) \to \mathrm{Jac}(\widehat{\mathcal{C}}_d),$$

and the pullback

$$\pi_d^* : \mathrm{Jac}(\widehat{\mathcal{C}}_d) \to \mathrm{Jac}(\widehat{\mathcal{C}}_N).$$

Let $A_{N/d}$ denote the connected component of $\ker(\pi_*)$. Then there exists an abelian subvariety A_d (isomorphic to $\mathrm{Jac}(\widehat{\mathcal{C}}_d)$) of $\mathrm{Jac}(\widehat{\mathcal{C}}_N)$ so that

$$\mathrm{Jac}(\widehat{\mathcal{C}}_N) \sim A_d \times A_{N/d},$$

where we use the symbol $\sim$ to denote that the two varieties are isogenous. Then $\mathrm{Jac}(\widehat{\mathcal{C}}_N)$ contains what is called an N-*new* part which is a complement to the contribution of all proper divisors of N. This gives a decomposition

$$\mathrm{Jac}(\widehat{\mathcal{C}}_N) \sim \bigoplus_{d|N} \mathrm{Jac}(\widehat{\mathcal{C}}_d)^{d\text{-new}}. \tag{68}$$

Let ℓ be a prime number and let $V_\ell(\mathrm{Jac}(\widehat{\mathcal{C}}))$ denote the Tate module of the Jacobian variety of $\widehat{\mathcal{C}}$. The decomposition (68) gives a similar decomposition of Gal_L-modules

$$V_\ell(\mathrm{Jac}(\widehat{\mathcal{C}})) = \bigoplus_{d|n} V_\ell(\mathrm{Jac}(\widehat{\mathcal{C}}_d))^{d\text{-new}}. \tag{69}$$

The action of the group μ_N on the curve $\mathcal{C}$ induces an action of μ_N on the cohomology group $H^1_{\text{ét}}(\widehat{\mathcal{C}}, \mathbb{Q}_\ell)$ and a decomposition

$$H^1_{\text{ét}}(\widehat{\mathcal{C}}, \mathbb{Q}_\ell) = \bigoplus_{d|n} H^1_{\text{ét}}(\widehat{\mathcal{C}}, \mathbb{Q}_\ell)^{d\text{-new}} = \bigoplus_{d|n} \left(\bigoplus_{i \in (\mathbb{Z}/d)^\times} H^{1,\zeta_d^i}_{\text{ét}}(\widehat{\mathcal{C}}, \mathbb{Q}_\ell) \right), \tag{70}$$

where $H^{1,\zeta_d^i}_{\text{ét}}(\widehat{\mathcal{C}}, \mathbb{Q}_\ell)$ is the eigenspace for the eigenvalue ζ_d^i. The well known isomorphism of Gal_L-modules

$$H^1_{\text{ét}}(\widehat{\mathcal{C}}, \mathbb{Q}_\ell) \xrightarrow{\sim} V_\ell(\mathrm{Jac}(\widehat{\mathcal{C}}))(-1) \otimes \mathbb{Q}_\ell,$$

preserves d-new subspaces. We denote by $H^1_{\text{ét}}(\widehat{\mathcal{C}}, \mathbb{Q}_\ell)^{\text{new}}$ the N-new vector space (over $\mathbb{Q}_\ell$) and by $\mathcal{J}_N^{\text{new}}$ the N-new part of the Jacobian.

7.5. **Counting points: the zeta function.** Let $\mathcal{O}$ be the discrete valuation ring consisting of the ring of integers of L (when L is a local field) or a completion of its ring of integers at a prime ideal (when L is a global field). Let $\mathfrak{p}$ be its maximal ideal and k its residue field, a finite field of characteristic p with q elements (so $N \mid q-1$). Let $f(x) \in \mathcal{O}[x]$ and $\mathcal{C}$ be the superelliptic curve (67). For the rest of the section we make the following assumptions.

Assumption 1. *Keeping the previous notation, the polynomial f, the integer N and the residual characteristic p satisfy the following properties:*

(1) $p \nmid N$,
(2) $p \nmid \mathrm{disc}\left(\frac{f}{\gcd(f,f')}\right)$,
(3) *The leading coefficient c of f is a unit in $\mathcal{O}$,*
(4) *The curve $\mathcal{C}$ is irreducible over $\overline{L}$.*

The assumptions assure that the reduction of a smooth model of $\mathcal{C}$ will be smooth.

Remark 7.15. When $\mathcal{C}$ matches Euler's curve as defined in (59), a prime ideal p satisfies Assumption 1 precisely when $p \nmid N$ and $v_p(z_0(z_0-1)) = 0$. Equivalently, the primes satisfying the assumption are precisely the primes of good reduction of the motive (as described by property (iii) in the introduction).

Let us fix some notations: by $\widetilde{\mathcal{C}}$ we will denote the desingularization of $\mathcal{C}$, and by $\widehat{\mathcal{C}}$ a projective non-singular model. The following result is standard.

Lemma 7.16. *Keeping the previous assumptions*

$$\#\mathcal{C}(k) = \sum_{\substack{\omega \in \widehat{k^\times} \\ \omega^N = 1}} \sum_{x \in k} \omega(f(x)). \tag{71}$$

Proof. Clearly $\#\mathcal{C}(k) = \sum_{x \in k} \delta(f(x))$, where $\delta(x) = \#\{y \in k : y^N = x\}$ whose value equals

(72)
$$\delta(x) = \begin{cases} 1 & \text{if } x = 0, \\ N & \text{if } x \text{ is an } N\text{-th power}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathfrak{C}_N := \{\omega \in \widehat{k^\times} : \omega^N = 1\}$ (the set of characters whose order divides N). I follows from [29, Proposition 8.1.5] that for any $x \in k$

(73)
$$\delta(x) = \sum_{\omega \in \mathfrak{C}_N} \omega(x).$$

□

Lemma 7.17. *Let $\mathcal{C}$ be the superelliptic curve (67). Suppose that $f(x) = x^a g(x)$, where $g(0) \neq 0$. Then the desingularized curve $\widetilde{\mathcal{C}}$ has $d = \gcd(a, N)$ points over the point $(0,0)$, all of them defined over the field $K(\sqrt[d]{g(0)})$.*

Proof. See §3.1.1 of [2]. □

Since $\mathcal{C}$ is an affine curve, its "points at infinity" are missing (and the missing points might be singular for the model).

Lemma 7.18. *Let $\mathcal{C}$ be the hyperelliptic curve (67). Let $r = \deg(f(x))$ and consider the projective curve*

$$\mathcal{C}_p : y^N z^{\max\{N,r\}-N} = f(x/z) z^{\max\{N,r\}}.$$

Let $d = \gcd(N, r)$. Then there are d points on $\widehat{\mathcal{C}}$, the desingularization of the projective curve $\mathcal{C}_p$, lying above the points at the infinity line. Furthermore, they are defined over the extension $K(\sqrt[d]{c})$, where c is the leading coefficient of $f(x)$.

Proof. See §3.1.2 of [2]. □

Keeping the previous notation, let $\mathcal{C}$ be a superelliptic curve given by (67) satisfying Assumption 1. Let $\widehat{\mathcal{C}}$ denote the projective desingularization of $\mathcal{C}$ and let ℓ be any prime number different from the characteristic of k. Let Frob denote the geometric Frobenius endomorphism. Then Lefschetz trace formula implies that

(74)
$$\#\{x \in \widehat{\mathcal{C}}(\overline{k}) : \mathrm{Frob}(x) = x\} = \sum_{i=0}^{2} (-1)^i \operatorname{Tr}(\mathrm{Frob})|_{H^i_{\text{ét}}(\widehat{\mathcal{C}}, \mathbb{Q}_\ell)}.$$

The contribution of the sum on the left hand side is 1 when $i = 0$ and q when $i = 2$, providing the well known formula

(75)
$$\operatorname{Tr}(\mathrm{Frob})|_{H^1_{\text{ét}}(\widehat{\mathcal{C}}, \mathbb{Q}_\ell)} = q + 1 - \#\widehat{\mathcal{C}}(k).$$

Let $\mathfrak{C}^*_N$ be the set of characters of precise order N.

Theorem 7.19. *Let $\mathcal{C}$ be the superelliptic curve given by (67) for $N > 1$. Suppose it satisfies the following hypothesis:*

- *Assumption 1 holds,*
- *For each root α (defined over $\overline{K}$) of $f(x)$, the order of vanishing of $f(x)$ at α is not divisible by N.*

Let Frob *denote a geometric Frobenius endomorphism. Then*

$$\operatorname{Tr}(\mathrm{Frob})|_{H^1_{ét}(\widehat{\mathcal{C}},\mathbb{Q}_\ell)^{new}} = -\sum_{\omega\in\mathcal{C}_N^*}\sum_{x\in k}\omega(f(x)) - \begin{cases}\sum_{\omega\in\mathcal{C}_N^*}\omega(c) & \text{if } N\mid\deg(f),\\ 0 & \text{otherwise},\end{cases} \tag{76}$$

where, as before, c is the leading coefficient of f.

Proof. Start supposing that N is a prime number. The hypothesis on the vanishing order of the roots of $f(x)$ implies (by Lemma 7.17) that the desingularization $\widetilde{\mathcal{C}}$ of $\mathcal{C}$ has the same number of points as $\mathcal{C}$. By Lemma 7.18, the projective desingularization has $g=\gcd(N,\deg(f))$ points at infinity defined over the field $K(\sqrt[g]{c})$. The hypothesis on c being a unit in $\mathcal{O}$ and the fact $p\nmid N$ imply that the reduction of the points are defined over k if and only if c is a g-th power in k. If $g=1$ (i.e. $N\nmid\deg(f)$) there is a unique point at the infinity line, while if $N\mid\deg(f)$, the number of points is either N or 0. Then

$$\#\widehat{\mathcal{C}}(k)=\#\mathcal{C}(k)+\sum_{\omega\in\mathcal{C}_g}\omega(c).$$

Replacing in (75) gives

$$\operatorname{Tr}(\mathrm{Frob})|_{H^1_{\text{ét}}(\widehat{\mathcal{C}},\mathbb{Q}_\ell)} = q+1-\#\widehat{\mathcal{C}}(k) = q+1-\left(\#\mathcal{C}(k)+\sum_{\omega\in\mathcal{C}_g}\omega(c)\right).$$

By Lemma 7.16

$$\#\mathcal{C}(k)=\sum_{\omega\in\mathcal{C}_N}\sum_{x\in k}\omega(f(x)) = q+\sum_{\omega\in\mathcal{C}_N^*}\sum_{x\in k}\omega(f(x)),$$

proving the formula when N is a prime number.

The general case follows from an inductive argument: the case $N=2$ follows from the prime case. Let N be a positive integer larger than 2, and suppose that the result is proven for all values smaller than N. Since the prime case was already proved, we can assume that N is composite. The decomposition (68) implies that

$$\operatorname{Tr}(\mathrm{Frob})|_{H^1_{\text{ét}}(\widehat{\mathcal{C}},\mathbb{Q}_\ell)} = \sum_{d\mid n}\operatorname{Tr}(\mathrm{Frob})|_{H^1_{\text{ét}}(\widehat{\mathcal{C}},\mathbb{Q}_\ell)^{d-\text{new}}}.$$

For $d\mid N$ a proper divisor, let $\widetilde{f}_d$ be the polynomial (in $L[x]$) obtained by removing from f all roots whose multiplicity is divisible by d, and let $\mathcal{C}(d)$ be the curve

$$\mathcal{C}(d): y^N=\widetilde{f}_d.$$

Keeping the previous notation, let $\widehat{\mathcal{C}(d)}$ denote the desingularization of the projectivization of $\mathcal{C}(d)$. Let $\mu:\mathbb{Z}\to\mathbb{Z}$ be the function given by

$$\mu(d)=\begin{cases}1 & \text{if } d\mid\deg(f),\\ 0 & \text{otherwise.}\end{cases} \tag{77}$$

The inductive hypothesis for $d\mid N$ a proper divisor not equal to 1 implies that

$$\operatorname{Tr}(\mathrm{Frob})|_{H^1_{\text{ét}}(\widehat{\mathcal{C}},\mathbb{Q}_\ell)^{d-\text{new}}} = \operatorname{Tr}(\mathrm{Frob})|_{H^1_{\text{ét}}(\widehat{\mathcal{C}(d)},\mathbb{Q}_\ell)^{\text{new}}} = -\sum_{\omega\in\mathcal{C}_d^*}\sum_{x\in k}\omega(\widetilde{f}_d(x)) - \mu(d)\sum_{\omega\in\mathcal{C}_d^*}\omega(c).$$

Using (75) we get the relation

$$q+1-\#\widehat{\mathcal{C}}(k) = \operatorname{Tr}(\mathrm{Frob})|_{H^1_{\text{ét}}(\widehat{\mathcal{C}},\mathbb{Q}_\ell)^{\text{new}}} - \sum_{\substack{d\mid N\\ d\neq 1,N}}\sum_{\omega\in\mathcal{C}_d^*}\left(\sum_{x\in k}\omega(\widetilde{f}_d(x))+\mu(d)\omega(c)\right).$$

Let $g = \gcd(\deg(f), N)$, so that

$$\#\widehat{\mathcal{C}}(k) = \#\widetilde{\mathcal{C}}(k) + \sum_{\omega \in \mathfrak{C}_g} \omega(c).$$

Lemma 7.16 gives the value of $\#\mathcal{C}(k)$, proving the equality

$$(78)\quad q + 1 - \sum_{\omega \in \mathfrak{C}_g} \omega(c) - \sum_{\omega \in \mathfrak{C}_N} \sum_{x \in k} \omega(f(x)) - (\#\widetilde{\mathcal{C}}(k) - \#\mathcal{C}(k)) =$$
$$\operatorname{Tr}(\operatorname{Frob})|_{H^1_{\text{ét}}(\widehat{\mathcal{C}}, \mathbb{Q}_\ell)^{\text{new}}} - \sum_{\substack{d \mid N \\ d \neq 1, N}} \sum_{\omega \in \mathfrak{C}_d^*} \left(\sum_{x \in k} \omega(\widetilde{f}_d(x)) + \mu(d)\omega(c) \right).$$

The sum on the middle left hand side equals q when $\omega = 1$, canceling the first summand. Hence we are led to prove the equalities

$$(79)\quad \sum_{x \in k} \sum_{\substack{\omega \in \mathfrak{C}_N \\ \operatorname{ord}(\omega) \neq 1, N}} \omega(f(x)) + (\#\widetilde{\mathcal{C}}(k) - \#\mathcal{C}(k)) = \sum_{x \in k} \sum_{\substack{d \mid N \\ d \neq 1, N}} \sum_{\omega \in \mathfrak{C}_d^*} \omega(\widetilde{f}_d(x)).$$

and

$$(80)\quad \sum_{\omega \in \mathfrak{C}_g} \omega(c) = \sum_{d \mid N} \sum_{\omega \in \mathfrak{C}_d^*} \mu(d)\omega(c).$$

The second equality is trivial, since $\mu(d)$ is zero when $d \nmid \deg(f)$ (or equivalently when $d \nmid g$).

To prove (79) define S to be the set of singular points of $\mathcal{C}$ defined over k. If $P = (x(P), y(P)) \notin S$, there is a unique point in $\widetilde{\mathcal{C}}$ above P (defined over the same field) hence it does not contribute to the term $\#\widetilde{\mathcal{C}}(k) - \#\mathcal{C}(k)$. To compute the contribution of $x(P)$ to the other terms of both sides of (79) consider the following two cases:

- If $f(x(P)) = 0$ (so $x(P)$ is a single root of f), all other terms involving $x(P)$ in (79) are zero, so the contribution of $x(P)$ is the same on both sides of the equality.
- Suppose that $x(P)$ is not a root of f. The difference between the left hand side and the right hand side corresponds to a character $\omega \in \mathfrak{C}_d^*$ evaluated at $f(x(P))$ (on the left hand side) minus the character evaluated at $\widetilde{f}_d(x(P))$ (on the right hand side). Both values are non-zero, and they differ by a d-th power, so ω evaluates the same at them.

We are led to study the contribution of points $P \in S$ (where $f(x) = 0$ but $\widetilde{f}_d(x)$ might be non-zero) to both sides. For d a proper divisor of N (not equal to 1) let S_d be the set of points in S satisfying that $\gcd(N, \operatorname{ord}_P(f)) = d$, so

$$S = \bigsqcup_{d \mid N} S_d.$$

For $d \mid N$ a proper divisor not equal to 1 define

$$g_d = \frac{f}{\prod_{P \in S_d} (x - x(P))^{\operatorname{val}_P(d)}},$$

a polynomial in $K[x]$ (which does not vanish at any point of S_d). Since all exponents in the denominator are divisible by d, it is clear that for any $P \in S_d$, the points of $\widetilde{\mathcal{C}}$ above P are defined over K if and only if $g_d(x(P))$ is a d-th power in K. The second assumption implies that $g_d(x(P))$ is a unit in $\mathcal{O}$ (since different roots of f do not reduce to the same value in k), and since $p \nmid N$, $p \nmid d$, so $g_d(x(P))$ is a d-th power if and only if its reduction is (by Hensel's lemma). Let $\delta_d : k^\times \to \mathbb{Z}$ be the function defined by

$$\delta_d(x) = \begin{cases} d & \text{if } x \text{ is a } d\text{-th power}, \\ 0 & \text{otherwise}. \end{cases} \tag{81}$$

Then

$$\#\widetilde{\mathcal{C}}(k) - \#\mathcal{C}(k) = \sum_{\substack{d \mid N \\ d \neq 1}} \sum_{P \in S_d} (\delta_d(\overline{g_d(x(P))}) - 1). \tag{82}$$

The first sum on the left hand side of (79) evaluated at $x(P)$ for $P \in S$ is zero. The contribution of the sum on the right hand side for $t \mid N$ and $\omega \in \mathcal{C}_d^*$ at a point $P \in S_d$ equals $\omega(\widetilde{f}_t(x(P)))$ which is non-zero if and only if $t \mid d$. Then (recalling that $\widetilde{f}_t$ and g_t differ by a perfect t-th power) the contribution of $P \in S_d$ equals

$$\sum_{\substack{t \mid d \\ t \neq 1}} \sum_{\omega \in \mathcal{C}_t^*} \omega(\overline{g_t(x(P))}). \tag{83}$$

Once again, $g_t(x(P))$ differs from $g_d(x(P))$ by a t-th power, so the proof follows from the equality (proved in Lemma 7.16)

$$\sum_{\omega \in \mathcal{C}_d} \omega(\overline{g_d(x(P))}) = \delta_d(\overline{g_d(x(P))})$$

□

7.6. **Factoring the zeta function.** Keeping the previous section notation, let $f(x) \in \mathcal{O}[x]$ and N be a positive integer satisfying both Assumption 1 and that no root of $f(x)$ (over an algebraic closure) has multiplicity divisible by N. Let ω denote a character of $(\mathcal{O}/\mathfrak{p})^\times$.

Definition 7.20. *The* counting function N *is defined by*

$$N(\omega; z) := \sum_{x \in \mathbb{F}_q} \omega(f(x)) + \begin{cases} \omega(x) & \text{if } N \mid \deg(f), \\ 0 & \text{otherwise}. \end{cases} \tag{84}$$

Let $k \in \mathbb{F}_q$ be the reduction of ζ_N modulo $\mathfrak{p}$, an N-th root of unity in $\mathbb{F}_q$. Let $\alpha \in \overline{\mathbb{F}_q}$ be such that $\alpha^{q-1} = k$. Then $\varepsilon := \alpha^N \in \mathbb{F}_q$ (since it is a root of $x^{q-1} - 1$), and it generates $\mathbb{F}_q^\times$.

Lemma 7.21. *The character $\chi_\mathfrak{p}$ (defined in (24)) satisfies*

$$\chi_\mathfrak{p}(\varepsilon) = \zeta_N. \tag{85}$$

Proof. By definition $\chi_\mathfrak{p}(\varepsilon) = \chi_\mathfrak{p}(\alpha^N) \equiv (\alpha^N)^{(q-1)/N} = \alpha^{q-1} = \kappa \equiv \zeta_N \pmod{\mathfrak{p}}$. □

Theorem 7.22. *Let ℓ be a prime number different from p. Then the trace of* Frob *on* $\mathrm{H}^{1,\zeta_N}_{\mathrm{et}}(\widehat{\mathcal{C}}, \mathbb{Q}_\ell)$ *equals $-N(\chi_\mathfrak{p}; z)$.*

Proof. Recall the decomposition $H^1_{\text{ét}}(\widehat{\mathcal{C}}, \mathbb{Q}_\ell)^{\text{new}} = \bigoplus_{j \in (\mathbb{Z}/N)^\times} H^{1,\zeta_N^j}_{\text{ét}}(\widehat{\mathcal{C}}, \mathbb{Q}_\ell)$. For $j \in (\mathbb{Z}/N)^\times$, let A_j be the matrix of Frob acting on $H^{1,\zeta_N^j}_{et}(\widehat{\mathcal{C}}, \mathbb{Q}_\ell)$. By Theorem 7.19

$$-\sum_{j=1}^{N-1} \mathrm{Tr}(A_j) = \sum_{j=1}^{N-1} N(\chi_\mathfrak{p}^j; z).$$

Let $X/\mathbb{F}_q$ be a non-singular variety and let $\sigma \in \mathrm{Aut}_{\mathbb{F}_q}(X)$. By Lefschetz's trace formula

$$\#\{x \in X(\overline{\mathbb{F}_q}) : \mathrm{Frob}_q(\sigma(x)) = x\} = \sum_{i=1}^{2\dim(X)} (-1)^i \mathrm{Tr}(\mathrm{Frob}_q \circ \sigma)|_{H^i_{et}(X,\mathbb{Q}_\ell)}. \tag{86}$$

Let σ_i be the automorphism of $\widehat{\mathcal{C}}$ defined by $(x,y) \mapsto (x, \zeta_N^i y)$. Then (86) with $\sigma = \sigma_i$ gives

$$\#\{(x,y) \in \widehat{\mathcal{C}}(\mathbb{F}_q) : x \in \mathbb{F}_q, y^{q-1} = k^{-iq}\} = 1 + q - \sum_{j=1}^{N-1} \zeta_N^{ji} \mathrm{Tr}(A_j) + \text{"old contribution"}$$

Since $k \in \mathbb{F}_q$, $k^q = k$, hence

$$\{(x,y) \in \widehat{\mathcal{C}}(\mathbb{F}_q) : x \in \mathbb{F}_q, y^{q-1} = k^{-iq}\} = \{(x,y) \in \widehat{\mathcal{C}}(\mathbb{F}_q) : x \in \mathbb{F}_q, y^{q-1} = k^{-i}\}.$$

Recall the notation introduced before: $\alpha \in \overline{\mathbb{F}_q}$ satisfies $\alpha^{q-1} = k$ and $\varepsilon := \alpha^N$. Then the map $y \leftrightarrow \tilde{y} := y\alpha^i$ gives a bijection between the sets

$$\{(x,y) \in \widehat{\mathcal{C}}(\mathbb{F}_q) \text{ fixed by } \mathrm{Frob}_q \circ \sigma_i\} \leftrightarrow \{(x,\tilde{y}) \in \mathbb{F}_q^2 : \varepsilon^{-i}\tilde{y}^N = f(x)\}, \tag{87}$$

Indeed,

$$y^{q-1} = k^{-i} \Leftrightarrow \frac{\tilde{y}^{q-1}}{(\alpha^{q-1})^i} = k^{-i} \Leftrightarrow \tilde{y}^q = \tilde{y}.$$

The elements of the right hand side of (87) are the $\mathbb{F}_q$-points of the twisted curve

$$\mathcal{C}_i : y^N = \varepsilon^i f(x).$$

Theorem 7.19 applied to the twisted curve imply the relation

$$\#\widehat{\mathcal{C}}_i(\mathbb{F}_q) = 1 + q + \sum_{j=1}^{N-1} \chi_{\mathfrak{p}}^j(\varepsilon^i) N(\chi_{\mathfrak{p}}^j; z) + \text{"old contribution"}$$

By Lemma 7.21, $\chi_{\mathfrak{p}}(\varepsilon) = \zeta_N$, hence

$$-\sum_{j=1}^{N} \mathrm{Tr}(A_j) = \sum_{j=1}^{N} N(\chi_{\mathfrak{p}}^j; z)$$

$$-\sum_{j=1}^{N} \mathrm{Tr}(A_j)\zeta_N^{ij} = \sum_{j=1}^{N} \zeta_N^{ij} N(\chi_{\mathfrak{p}}^j; z), \text{ for } 1 \le i \le N-1.$$

Consider the linear system $Mx = b$ where M is the $N \times N$ matrix given by $M_{ij} = \zeta_N^{(i-1)j}$ and b is the $N \times 1$ vector given by $b_i = \sum_{j=1}^N \zeta_N^{(i-1)j} N(\chi_{\mathfrak{p}}^j; z)$, for $i = 1, \ldots, N$. The previous equalities imply that the system has at least two solutions, namely $x_j = -\mathrm{Tr}(A_j)$ and $x_j = N(\chi_{\mathfrak{p}}^j; z)$. Since M is invertible (as it is a Vandermonde matrix), $\mathrm{Tr}(A_1) = -N(\chi_{\mathfrak{p}}; z)$. □

Theorem 7.23. *Let $(a,b),(c,d)$ be generic rational parameters and let N be their least common denominator. Let $z_0 \in \mathbb{Q}^\times$, $z_0 \neq 1$. Let $\mathfrak{p}$ be a prime ideal of F satisfying that $\mathfrak{p} \nmid N$ and $v_{\mathfrak{p}}(z_0(z_0-1)) = 0$. Then the trace of the Frobenius element* $\mathrm{Frob}_{\mathfrak{p}}$ *acting on $\mathcal{H}((a,b),(c,d)|z_0)$ equals $H_{\mathfrak{p}}((a,b),(c,d)|z_0)$.*

Before proving the statement, recall that we gave two different definitions of $\mathcal{H}((a,b),(c,d)|z_0)$ depending on whether (**Irr**) holds or not. Part of the statement is that the trace of a Frobenius element is the same for both definitions. We need the following auxiliary result.

Lemma 7.24. *Let $(a,b),(c,d)$ be rational generic parameters and let N be their least common denominator. Let q be a rational prime number such that $N \mid q-1$. Then*

$$H_{\mathfrak{p}}((a-d,b-d),(c-d,1)|z_0) = \chi_{\mathfrak{p}}(z_0)^{-dN}\,\mathbf{J}((-a,-b,c,d),(d-a,d-b,c-d))H_{\mathfrak{p}}((a,b),(c,d)|z_0).$$

Proof. Recall that we chose a generator ϖ of the character group $\widehat{\mathbb{F}_q^\times}$ so that $\varpi^{(q-1)/N} = \chi_{\mathfrak{p}}^{-1}$. Then [6, Theorem 3.4] implies that

$$H_{\mathfrak{p}}((a-d,b-d),(c-d,1)|z_0) = \varpi(z_0)^{d(q-1)}\frac{g(\chi_{\mathfrak{p}}^{-aN})g(\chi_{\mathfrak{p}}^{cN})}{g(\chi_{\mathfrak{p}}^{(d-a)N})g(\chi_{\mathfrak{p}}^{(c-d)N})}\frac{g(\chi_{\mathfrak{p}}^{-bN})g(\chi_{\mathfrak{p}}^{dN})}{g(\chi_{\mathfrak{p}}^{(d-b)N})g(1)}H_{\mathfrak{p}}((a,b),(c,d)|z_0).$$

From its definition (Definition 5.1) the middle factor equals the stated Jacobi motive. As a side remark, the first statement of [6, Theorem 3.4] is correct, but the second one is not. The numerator should be the denominator and vice-versa (since the function S_q is the product of H_q with a Gauss sum involving the same parameters, as follows from (3.1) of loc. cit.) ☐

Proof of Theorem 7.23. Suppose that (**Irr**) holds and let $\mathcal{C}$ denote Euler's curve as defined in (59). The hypothesis on $\mathfrak{p}$ implies (by Remark 7.15) that Assumption 1 holds. The genericity condition on the parameters imply that $a-c, b-c, a-d, b-d \notin \mathbb{Z}$, so $N \nmid A$, $N \nmid B$, $N \nmid C$ and $N \nmid \deg(f(x)) = A+B+C = (a-c)N$. By Theorem 7.22 the trace of Frobenius on $\mathrm{H}^{1,\zeta_N}_{\mathrm{et}}(\widehat{\mathcal{C}},\mathbb{Q}_\ell)$ equals

$$-N(\chi_{\mathfrak{p}};z_0) = -\chi_{\mathfrak{p}}(z_0)^D\sum_{x\in\mathbb{F}_q}\chi_{\mathfrak{p}}(x^A(1-x)^B(1-z_0x)^C),$$

where $A=(d-b)N$, $B=(b-c)N$, $C=(a-d)N$ and $D=dN$. Set $\alpha_1=(d-b)N$, $\alpha_2=(d-a)N$, $\beta_1=(d-c)N$ and $\beta_2=0$. Then Theorem 6.25 implies that

$$-N(\chi_{\mathfrak{p}};z_0) = \chi_{\mathfrak{p}}(z_0)^D\chi_{\mathfrak{p}}(-1)^A\,\mathbf{J}([\alpha_1,-\beta_1],[\alpha_1-\beta_1])H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|z_0). \tag{88}$$

By definition (6.4) the value $H_{\mathfrak{p}}(\boldsymbol{\alpha},\boldsymbol{\beta}|z)$ equals the value $H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|z_0)$ choosing a generator ϖ satisfying $\varpi^{(q-1)/N} = \chi_{\mathfrak{p}}^{-1}$, hence

$$H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|z_0) = H_{\mathfrak{p}}(((a-d)N,(b-d)N),((c-d)N,1)|z_0).$$

Then (88) equals

$$\chi_{\mathfrak{p}}(z_0)^{dN}\chi_{\mathfrak{p}}^{(d-b)N}(-1)\,\mathbf{J}((d-b,c-d),(c-b))H_{\mathfrak{p}}\left((a-d,b-d),(c-d,1)|z_0\right).$$

Using Lemma 7.24 this value equals

$$\chi_{\mathfrak{p}}^{(d-b)N}(-1)\,\mathbf{J}((-a,-b,c,d),(c-b,d-a))H_{\mathfrak{p}}\left((a,b),(c,d)|z_0\right),$$

which matches the motive of Definition 7.7.

To prove that $\mathcal{H}((a,b),(c,d)|z_0)$ also matches (64), we apply the proven result to the parameters $(a-d,b-d),(c-d,0)$ and are led to prove the equality

$$H_q((a-d,b-d),(c-d,1)|z_0)\,\mathbf{J}((-a,-b,c,d),(d-a,d-b,c-d))^{-1}\eta_d(z_0) = H_q((a,b),(c,d)|z_0),$$

which follows from Lemma 7.24 since $\eta_d(\mathrm{Frob}_{\mathfrak{p}})(z_0) = \chi_{\mathfrak{p}}(z_0)^{dN}$. ☐

Remark 7.25. Let $(a,b),(c,d)$ be generic rational parameters satisfying (**Irr**). The change of variables $y \to yz^{-D/N}$, defined over the extension $L = \mathbb{Q}(\zeta_N, \sqrt[N]{z^D})$, gives an isomorphism between Euler's curves with parameters $(a,b),(c,d)$ and $(a-d,b-d),(c-d,1)$. Since L/F is abelian, the two curves are *twist* of each other. This is the underlying reason why Definition 7.7 and Definition 7.10 coincide.

It is not clear to us if there is a definition of the motive that does not involve twists when condition (**Irr**) is not satisfied. Such a description for example would be useful while proving properties regarding the field of definition of the motive.

Example 12. Continuing with Example 2, take $a = 1/8, b = 7/8, c = 3/8, d = 5/8$, so $N = 8$. The values $\tilde{a} = a - d = 1/2$, $\tilde{b} = b - d = 1/4$ and $\tilde{c} = c - d = 3/4$ have denominator 4, so the curve $\mathcal{C}$ is reducible (as studied in detail in Example 9). Euler's curve for the parameters $(\tilde{a}, \tilde{b}), (\tilde{c}, 1)$ correspond to the curve with equation

$$\widetilde{\mathcal{C}} : y^4 = x^3(1-x)^2(1-zx)^2,$$

an irreducible curve which is a twist of an irreducible component of $\mathcal{C}$. According to Conjecture 4.5, the motive $\mathcal{H}((1/2, 1/4), (3/4, 1)|z)$ is defined over $\mathbb{Q}(\zeta_4) = \mathbb{Q}(i)$ (which is clearly a field of definition for the motive), but the motive $\mathcal{H}((1/8, 7/8), (3/8, 5/8)|z)$ (that is defined by restricting the motive $\mathcal{H}((1/2, 1/4), (3/4, 1)|z)$ to $\mathbb{Q}(\zeta_8)$ and twisting by $\sqrt[8]{z^5}$) should be defined over $\mathbb{Q}(\zeta_8)^+ = \mathbb{Q}(\sqrt{2})$. We do not know how to give a "geometric" proof of this result.

Lemma 7.26. *If $z_0 \in F^H$ then the coefficient field of the motive $\mathcal{H}((a, b), (c, d)|z_0)$ (i.e. the field extension of $\mathbb{Q}$ generated by the trace of Frobenius' elements) of is contained in F^H.*

Proof. Follows from Theorem 7.23. ☐

Remark 7.27. Let $(a, b), (c, d)$ be generic parameters with denominator N. Let $z_0 \in \mathbb{Q}$ and let $\mathfrak{p}$ be a prime ideal of F not dividing N, but dividing z_0. Suppose furthermore that $v_{\mathfrak{p}}(z_0)$ is divisible by the order of the monodromy matrix of the motive at 0. Then Theorem 4.8 implies that the motive $\mathcal{H}((a, b), (c, d)|z_0)$ is unramified at $\mathfrak{p}$, so the trace of the action of $\mathrm{Frob}_{\mathfrak{p}}$ on our motive is well defined. Since Assumption 1 is not satisfied, there is no reason for it to match the value $H_{\mathfrak{p}}((a, b), (c, d)|z_0)$ (and it does not). Still, a formula can be explicitly computed as done in Appendix A.

Proposition 7.28. *The motives $\mathcal{H}((a, b), (c, d)|z)$ and $\mathcal{H}((-c, -d), (-a, -b)|z^{-1})$ are isomorphic.*

Proof. Let N be the least common multiple of the denominators of a, b, c, d and assume (for simplicity) that condition (**Irr**) holds. Keeping the previous notation, the Euler curve for the parameters $(-c, -d), (-a, -b)$ at the variable z^{-1} has equation

$$y^N = x^A(1-x)^C(1-x/z)^B z^{D-A}.$$

The change of variables $x = zx'$, $y = y'$ provide an isomorphism between this curve and Euler's curve with parameters $(a, b), (c, d)$. Clearly the quadratic character $(-1)^{d-b}$ is invariant under the substitution $(a, b), (c, d) \to (-c, -d), (-a, -b)$ and the same holds for the Jacobi motive $\mathbf{J}((-a, -b, c, d), (c - b, d - a))$ appearing in the motive definition (61). ☐

The last result was expected from the similar result regarding solutions to the differential equation satisfied by the hypergeometric series proven in Lemma 3.3.

7.7. **On the quadratic character** $(-1)^{d-b}$**.** The factor corresponding to the quadratic character (if non-trivial) matches the restriction of an explicit rational character of $\mathrm{Gal}_{\mathbb{Q}}$.

Lemma 7.29. *Let $N = 2^t M$, where $2 \nmid M$. If $t = 0$ then $(-1)^{1/N}$ is trivial. If $t \geq 1$, let κ be any primitive character of conductor 2^{t+1} thought as a character of $\mathrm{Gal}_{\mathbb{Q}}$. Then the quadratic character $(-1)^{1/N}$ of $F = \mathrm{Gal}_{\mathbb{Q}(\zeta_N)}$ corresponds to the restriction of κ to Gal_F.*

Proof. Let p be an odd rational prime not dividing N and let $\mathfrak{p}$ a prime idea in $\mathbb{Z}[\zeta_N]$ dividing it. By construction, the value of the character $(-1)^{1/N}$ evaluated at the prime ideal $\mathfrak{p}$ equals $\chi_{\mathfrak{p}}(-1) = \varpi(-1)^{(q-1)/N}$, where ϖ is a generator of the character group $\widehat{\mathbb{F}_q^\times}$ and q is the norm of $\mathfrak{p}$. In particular, $\chi_{\mathfrak{p}}(-1) = 1$ if $(q-1)/N$ is even, and -1 otherwise; equivalently,

$$\chi_{\mathfrak{p}}(-1) = \begin{cases} 1 & \text{if } q \equiv 1 \pmod{2^{t+1}}, \\ -1 & \text{otherwise.} \end{cases}$$

Clearly $\chi_{\mathfrak{p}}(-1) = 1$ if $t = 0$ proving the first statement. Suppose then that $t \geq 1$. Recall that if we denote by $r = \mathrm{ord}_N(p)$ the order of p modulo N, then $q = p^r$. In particular, $p^r \equiv 1 \pmod{2^t}$ and $p^{2r} \equiv 1 \pmod{2^{t+1}}$. If κ is a primitive character of conductor 2^{t+1},

$$\kappa^2(\mathfrak{p}) = \kappa^2(q) = \kappa(p^{2r}) = 1.$$

Then $\kappa(\mathfrak{p}) = \pm 1$. Recall that a character of conductor 2^{t+1} (for $t \geq 1$) is primitive if and only if its value at $1 + 2^t$ is not 1. Then $\kappa(\mathfrak{p}) = 1$ if and only if $p^r \equiv 1 \pmod{2^{t+1}}$ if and only if $\chi_{\mathfrak{p}}(-1) = 1$. When the equivalent statements do not hold, both values are -1, proving the result. □

8. Extension to K

Let H be the group (21) for generic parameters $(a, b), (c, d)$.

Lemma 8.1. *The group H is a subgroup of $\mathbb{Z}/2 \times \mathbb{Z}/2$.*

Proof. For S a finite set, let $\mathrm{Perm}(S)$ denote the group of bijective maps on S. Set $S = \{a, b\} \times \{c, d\}$ (we are not assuming $a \neq b$ nor $c \neq d$). Then there is a group morphism

$$\psi : H \to \mathrm{Perm}(\{a, b\}) \times \mathrm{Perm}(\{c, d\}),$$

sending an element h to the bijective map given by multiplication by h on the sets $\{a, b\}$ and $\{c, d\}$. Since $\mathrm{Perm}(\{a, b\}) \times \mathrm{Perm}(\{c, d\}) \hookrightarrow \mathbb{Z}/2 \times \mathbb{Z}/2$, it is enough to prove that ψ is injective. Let N be the least common denominator of a, b, c, d. Let $i \in (\mathbb{Z}/N)^\times$ be an element in the kernel of ψ, so i satisfies

$$ia = a, \qquad ib = b, \qquad ic = c, \qquad id = d,$$

as elements of $\mathbb{Q}/\mathbb{Z}$. Write $a = \frac{\alpha}{N}$, $b = \frac{\beta}{N}$, $c = \frac{\gamma}{N}$, $d = \frac{\delta}{N}$ with $\gcd(\alpha, \beta, \gamma, \delta, N) = 1$. Then there are integers x_1, x_2, x_3, x_4 such that the following congruence holds

$$\alpha x_1 + \beta x_2 + \gamma x_3 + \delta x_4 \equiv 1 \pmod{N}.$$

Multiplying by i we get that

$$i \equiv i\alpha x_1 + i\beta x_2 + i\gamma x_3 + i\delta x_4 \equiv \alpha x_1 + \beta x_2 + \gamma x_3 + \delta x_4 \equiv 1 \pmod{N}.$$

□

As stated in Conjecture 4.5, it is expected that $K = F^H$ is both the field of definition and the coefficient field of the motive $\mathcal{H}((a, b), (c, d)|z)$.

Theorem 8.2. *Let $(a, b), (c, d)$ be generic parameters such that $a + b$ and $c + d$ are integers and (**Irr**) holds. Then $\mathcal{H}((a, b), (c, d)|z)$ is defined over F^H and its coefficient field is contained in F^H.*

We start proving some preliminary results.

Lemma 8.3. *Let $(a, b), (c, d)$ be generic rational parameters satisfying that $a + b$ and $c + d$ are integers. Let N be their least common denominator. Set*

$$\delta = \begin{cases} 1 & \text{if } a, c \notin \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathfrak{p}$ be a prime ideal of $\mathbb{Q}(\zeta_N)$ of norm q, not dividing N. Then

$$\mathbf{J}((-a, -b, c, d), (c - b, d - a))(\mathfrak{p}) = q^\delta.$$

In particular, we can assume that the Jacobi motive is rational.

Proof. The hypothesis $a+b \in \mathbb{Z}$ (respectively $c+d \in \mathbb{Z}$) imply that we can replace b by $-a$ (respectively d by $-c$) in the Jacobi motive

$$\mathbf{J}((-a,-b,c,d),(c-b,d-a))(\mathfrak{p}) = \mathbf{J}((-a,a,c,-c),(a+c,-a-c))(\mathfrak{p}).$$

If χ is a character of $\mathbb{F}_q^\times$,

$$g(\chi)g(\overline{\chi}) = \chi(-1)\begin{cases} q & \text{if } \chi \neq 1, \\ 1 & \text{if } \chi = 1. \end{cases}$$

The genericity condition on the coefficients imply that $a+c = a-d \notin \mathbb{Z}$ and at most one of a, c is an integer. ☐

Then when $a+b$ and $c+d$ are integers the hypergeometric motive $\mathcal{H}((a,b),(c,d)|z)$ is (up to a Tate twist) a twist by κ of the motive coming from Euler's curve.

Lemma 8.4. *If $(a,b),(c,d)$ are generic parameters such that $a+b$ and $c+d$ are integers, then the character κ constructed in Lemma 7.29 is at most quadratic while restricted to Gal_K.*

Proof. By Lemma 7.29 κ is a primitive character of conductor 2^{t+1}, where $t = \mathrm{val}_2(N)$ (which we can assume is positive, as otherwise κ is trivial). Equivalently, it is a character of the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_{2^{t+1}})/\mathbb{Q})$. Since $-1 \in H$ (complex conjugation), a simple Galois theory exercise shows that $\mathbb{Q}(\zeta_{2^{t+1}}) \cap F^H = \mathbb{Q}(\zeta_{2^t})^+$. Then κ restricted to Gal_K factors through $\mathrm{Gal}(\mathbb{Q}(\zeta_{2^{t+1}})/\mathbb{Q}(\zeta_{2^t})^+)$, a group isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$ if $t > 1$ and $\mathbb{Z}/2$ if $t = 1$, hence κ is at most quadratic. ☐

Our goal is to construct a motive $\mathfrak{M}$ defined over K whose base change to F is isomorphic to Euler's motive and define $\mathcal{H}((a,b),(c,d)|z)$ as in (61). Since κ is (at most) quadratic and the Jacobi motive corresponds to a Tate twist, the coefficient field of $\mathcal{H}((a,b),(c,d)|z)$ is the same as that of $\mathfrak{M}$.

The motive $\mathfrak{M}$ appears in the quotient of Euler's curve $\mathcal{C}$ by involutions that we now describe.

Lemma 8.5. *The map $(x,y) \to \left(\frac{zx-1}{z(x-1)}, \frac{1}{y}\right)$ gives an isomorphism between Euler's curve with parameters $(b,a),(c,d)$ and a twist by $z^{(c+d)N}(z-1)^{(a+b-c-d)N}$ of Euler's curve with parameters $(a,b),(c,d)$.*

Proof. If we denote by A', B', C', D' the exponents of Euler's curve with parameters $(b,a),(c,d)$, then an elementary computation proves that

$$A' = -C, \qquad B' = A+B+C, \qquad C' = -A, \qquad D' = D.$$

The values $A, B, C, A+B+C$ correspond to the ramification degrees of the cover $\mathcal{C} \to \mathbb{P}^1$ sending $(x,y) \to x$ at the ramified points $\{0, 1, 1/t, \infty\}$ respectively, so up to a sign we are just permuting the ramification points. The change of variables $x = \frac{zx'-1}{z(x'-1)}$, $y = \frac{1}{y'}$ sends the equation

$$y^N = x^{A'}(1-x)^{B'}(1-zx)^{C'}z^{D'}$$

to the equation

$$\frac{1}{y'^N} = \frac{z^C(x'-1)^C}{(zx'-1)^C}\frac{(1-z)^{A+B+C}}{z^{A+B+C}(x'-1)^{A+B+C}}\frac{(x'-1)^A z^D}{(1-z)^A x'^A},$$

which satisfies the equivalent equation

$$z^{2D-A-B}(z-1)^{B+C}y'^N = x'^A(1-x')^B(1-zx')^C z^D.$$

The result follows from the equalities $2D-A-B = (c+d)N$ and $B+C = (a+b-c-d)N$. ☐

Lemma 8.6. *The map $(x,y) \to \left(\frac{x-1}{zx-1}, \frac{1}{y}\right)$ gives an isomorphism between Euler's curve with parameters $(a,b),(d,c)$ and a twist by $z^{(c+d)N}(1-z)^{(a+b-c-d)N}$ of Euler's curve with parameters $(a,b),(c,d)$.*

Proof. Mimics the previous case one, noting that in this case the relation between the exponents reads

$$A' = -B, \qquad B' = -A, \qquad C' = A + B + C, \qquad D' = D - A - B.$$

□

Lemma 8.7. *The map $(x, y) \to \left(\frac{1}{zx}, y\right)$ gives an isomorphism between Euler's curve with parameters $(b, a), (d, c)$ and a twist by $(-1)^{(a+b-c-d)N}$ of Euler's curve with parameters $(a, b), (c, d)$.*

Proof. Let A', B', C', D' denote the exponents of Euler's curve with parameters $(b, a), (d, c)$. They are related to A, B, C, D by

$$A = -A' - B' - C', \quad B = C', \quad C = B', \quad D = D' - A' - B'.$$

Then the change of variables $x = \frac{1}{zx'}$, $y = y'$ gives the curve

$$y'^N = (-1)^{B'+C'} x'^{-(A'+B'+C')} (1 - x')^{C'} (1 - zx')^{B'} z^{D'-A'-B'}.$$

□

When both $a + b$ and $c + d$ are integers after a simple change of variables (as explained in Remark 7.3) we can (and do for simplicity) assume that actually $a+b = 0 = c+d$. Then no twist is needed in any of the last three lemmas and permutations of the parameters give isomorphic curves.

If j is prime to N, there is an elementary morphism ψ_j from Euler's curve with parameters $(a, b), (c, d)$ to Euler's curve with parameters $(ja, jb), (jc, jd)$, defined by

$$\psi_j(x, y) = (x, y^j).$$

Recall from the proof of Lemma 8.1 that the group H might be thought of as a subgroup of $\mathrm{Perm}(\{a, b\}) \times \mathrm{Perm}(\{c, d\})$, so if $j \in H$ there is an isomorphism ϕ_j between Euler's curves for the parameters $(aj, bj), (cj, dj)$ and $(a, b), (c, d)$.

Definition 8.8. *For each $j \in H$ let $\iota_j : \mathcal{C} \to \mathcal{C}$ be the rational map defined as the composition*

$$\iota_j(x, y) := \phi_j \circ \psi_j(x, y). \tag{89}$$

The next lemmas prove that ι_j is an involution for any $j \in H$.

Lemma 8.9. *Let a, b, c, d be generic parameters such that $a + b = c + d = 0$. The map $\iota_{-1} : \mathcal{C} \to \mathcal{C}$ is an involution given by*

$$\iota_{-1}(x, y) = \left(\frac{1}{zx}, \frac{1}{y}\right).$$

Proof. Since $j = -1$, it corresponds to the map that permutes both (a, b) and (c, d), so by Lemma 8.7, $\varphi_j(x) = \frac{1}{zx}$. The hypothesis $a + b = c + d = 0$ imply that the exponents of Euler's curve are $A = (a - c)N$, $B = -(a + c)N$, $C = -B$ and $D = -cN$, hence Euler's curve equals

$$y^N = x^A (1 - x)^B (1 - zx)^{-B} z^{-cN}. \tag{90}$$

Let (x, y) is a point of $\mathcal{C}$. To prove that $\left(\frac{1}{zx}, \frac{1}{y}\right)$ also belongs to $\mathcal{C}$, we substitute on (90), and need to verify the equality

$$y^{-N} = x^{-A} (zx - 1)^B (x - 1)^{-B} z^{-cN-A-B},$$

which holds because $-cN - A - B = cN$. Clearly $(\iota_{-1})^2$ is the identity map. □

Lemma 8.10. *Let a, b, c, d be generic parameters such that $a + b = c + d = 0$. Let $j \in H$ be such that $ja = a + r$ and $jc = -c + s$, for $r, s \in \mathbb{Z}$. Then map $\iota_j : \mathcal{C} \to \mathcal{C}$ is an involution defined by*

$$\iota_j(x, y) = \left(\frac{x - 1}{zx - 1}, \frac{x^{r-s}(1 - x)^{-r-s}(1 - xz)^{r+s} z^{-s}}{y^j}\right). \tag{91}$$

Proof. From its definition, $\psi_j(x,y) = (x, y^j)$ is an isomorphism between Euler's curve with parameters $(a,-a),(c,-c)$ and the one with parameters $(ja,-ja),(jc,-jc)$. By definition,

$$(ja,-ja) = (a+r,-a-r), \qquad (jc,-jc) = (-c+s, c-s).$$

Let A,B,C,D be exponents of Euler's curve with parameters $(a,-a),(-c,c)$ and A',B',C',D' those for $(ja,-ja),(jc,-jc)$. They satisfy the relations

$$A' = -B+(r-s)N, \quad B' = -A-(r+s)N, \quad C' = A+(r+s)N, \quad D' = D-sN.$$

Then the map ϕ_j is obtained by composing the map $(x,y) \to (x, \frac{y}{x^{r-s}(1-x)^{-r-s}(1-xz)^{r+s}z^{-s}})$ with the map $(x,y) \to (\frac{x-1}{zx-1}, \frac{1}{y})$ of Lemma 8.6, giving the stated formula.

The proof that ι_j is an involution is straightforward. Applying the formula twice we get

$$\iota_j^2(x,y) = (x, y^{j^2}x^{-r(j+1)+s(j-1)}(1-x)^{-r(j+1)-s(j-1)}(1-xz)^{r(j+1)+s(j-1)}z^{s(j-1)}).$$

From its definition, $r(j+1) = (j^2-1)a$ and $s(j-1) = (j^2-1)c$. Let $t = \frac{j^2-1}{N}$ (an integer). Then the second coordinate equals

$$y \cdot (y^{tN}x^{-At}(1-x)^{-Bt}(1-xz)^{Bt}z^{ctN}) = y.$$

□

Lemma 8.11. *Let a,b,c,d be generic parameters such that $a+b = c+d = 0$. Let $j \in H$ be such that $ja = -a+r$ and $jc = c+s$, for $r,s \in \mathbb{Z}$. Then map $\iota_j : \mathcal{C} \to \mathcal{C}$ is an involution defined by*

$$\iota_j(x,y) = \left(\frac{zx-1}{z(x-1)}, \frac{x^{r-s}(1-x)^{-r-s}(1-xz)^{r+s}z^{-s}}{y^j}\right). \tag{92}$$

Proof. Mimics the previous ones, noting that if A,B,C,D are the exponents for the parameters $(-a,a),(c,-c)$ and A',B',C',D' those for $(ja,-ja),(jc,-jc)$. They satisfy the relations

$$A' = B+(r-s)N, \qquad B' = A-(r+s)N, \qquad C' = -A+(r+s)N, \qquad D' = D-sN.$$

Then the formula follows from Lemma 8.5. □

As explained in §7.4, for each $\delta \mid N$ Euler's curve $\mathcal{C}$ has an old contribution coming from the curve denoted C_δ, which in turns is Euler's curve for the parameters $(\frac{Na}{\delta},\frac{Nb}{\delta}),(\frac{Nc}{\delta},\frac{Nd}{\delta})$. If $j \in H$ we can think of it as an element of $(\mathbb{Z}/\delta)^\times$ which also preserves (by multiplication) the sets $\{\frac{Na}{\delta},\frac{Nb}{\delta}\}$ and $\{\frac{Nc}{\delta},\frac{Nd}{\delta}\}$, so for each $j \in H$, the curve C_δ also has its respective involution ι_j.

Proposition 8.12. *Let a,b,c,d be generic rational parameters satisfying $a+b = c+d = 0$ such that* (**Irr**) *holds. Let $j \in H$. If $N > 2$ then the new part of the quotient of Euler's curve $\mathcal{C}$ by the involution ι_j has genus $\phi(N)/2$.*

Proof. For ease of notation, denote by ι the involution. By Riemann-Hurwitz's formula, if $2\mathfrak{f}_\iota$ denotes the number of points fixed by ι, then

$$g(\mathcal{C}/\iota) = \frac{g(\mathcal{C})+1-\mathfrak{f}_\iota}{2}. \tag{93}$$

Fact 1: The number of points fixed by ι equals:

- 2 (i.e. $\mathfrak{f}_\iota = 1$) if $j = -1$ and N is odd.
- 0 (i.e. $\mathfrak{f}_\iota = 0$) if $j = -1$ and N is even.
- $2\gcd(j+1,N)$ (i.e. $\mathfrak{f}_\iota = \gcd(j+1,N)$) if $j \neq -1$.

Let (x_0, y_0) be such a fixed point. By Lemmas 8.5, 8.6 and 8.7, $x_0 = \varphi_j(x_0)$ for φ_j one of $\frac{zx-1}{x-1}, \frac{x-1}{zx-1}$ or $\frac{1}{zx}$. In all cases there are precisely two different solutions (note that the singular points $\{0, 1, \infty\}$ are not preserved by the functions φ_j, so we can count the number of fixed points on the singular model).

The condition satisfied by y_0 depends on whether $j = -1$ (i.e. it permutes both coordinates) or not. Start considering the case $j = -1$. Then $x_0 = \frac{\varepsilon}{\sqrt{z}}$, where $\varepsilon = \pm 1$. Substituting in the equation for $\mathcal{C}$ we get the relation

$$y_0^N = (-1)^{(a+c)N}(\varepsilon)^{2aN} = (-1)^{(a+c)N}. \tag{94}$$

On the other hand, by Lemma 8.9, $y_0^2 = 1$. If N is odd, only one of ± 1 can satisfy (94), so there are two fixed points (and $\mathfrak{f}_\iota = 1$). If N is even, the hypothesis (**Irr**) implies that $(a+c)N$ has to be odd (the hypothesis $a + b$ and $c + d$ integers imply that the exponents satisfy $A \equiv B \equiv C \pmod 2$); if they are even, Euler's curve is not irreducible). Then the involution has no fixed points (so $\mathfrak{f} = 0$).

Suppose that multiplication by j corresponds to the permutation that fixes a and b (and permutes c and d) and let (x_0, y_0) be a fixed point. Then x_0 satisfies the quadratic equation $(1 - x_0 z)x_0 = (1 - x_0)$ (which has two solutions). Replacing in the equation defining $\mathcal{C}$ and in (91), the value y_0 satisfies the equations

$$\begin{cases} y_0^{j+1} = (x_0^2 z)^{-s}, \\ y_0^N = (x_0^2 z)^{-cN}. \end{cases}$$

But the relation $(j+1)c = s$ implies that both equations are "compatible", so we have $\gcd(j+1, N)$ solutions for y_0. A similar computation proves that the same formula holds when j fixes c and d (and permutes a and b).

To prove the statement we need a more general formula that holds for all old contributions (where the hypotheses $N \nmid A$, $N \nmid B$, $N \nmid C$ might not hold). For $M \mid N$, let $\mathcal{C}_M$ denote the curve (as in (4))

$$\mathcal{C}_M : y^M = x^A(1-x)^B(1-zx)^C z^D.$$

Let $\kappa(M)$ denote the number of elements in the set $\{A, B, C, A+B+C\}$ divisible by M.

Fact 2: If $M > 2$,

$$\dim(\mathrm{Jac}(\mathcal{C}_M/\iota)^{\mathrm{new}}) = \frac{(2-\kappa(M))}{4} \cdot \begin{cases} \phi(M) & \text{if } M \nmid \mathfrak{f}_\iota, \\ 0 & \text{if } M \mid \mathfrak{f}_\iota. \end{cases} \tag{95}$$

When $M = 2$, the curve $\mathcal{C}_M/\iota$ has genus 1 if $j = -1$ and 0 otherwise. For $M = 1$ the curve always has genus 0.

When $M = N$, $\kappa(N) = 0$ and $N \nmid \mathfrak{f}_\iota$, so we recover the original statement. In any case, the generic hypothesis on the parameters implies that $\kappa(M) \in \{0, 1, 2\}$. The Riemann-Hurwitz formula ([2, Theorem 4.1]) implies that

$$g(\mathcal{C}_M) = M - \left(\frac{\gcd(A, M) + \gcd(B, M) + \gcd(C, M) + \gcd(A+B+C, M)}{2}\right) + 1. \tag{96}$$

Case $j = -1$. Fact 1 implies that $\mathfrak{f}_\iota$ is 0 if M is even and 1 if M is odd (in particular $M \nmid \mathfrak{f}_\iota$ when $M > 2$). If M is an odd prime, the formula reads

$$g(\mathcal{C}_M) = M - \frac{M\kappa(M) + (4 - \kappa(M))}{2} + 1 = \frac{(2-\kappa(M))(M-1)}{2}.$$

The statement follows from (93) (since Fact 1 implies that there are precisely two fixed points). When $M = 2$ the curve $\mathcal{C}$ has genus 1 (since A, B, C are odd) and the involution has no fixed points, so the quotient also has genus 1.

Let M be any positive integer, and suppose that the statement is true for all proper divisors δ of M. On the one hand equation (93) gives

$$g(\mathcal{C}_M/\iota) = \frac{M - (\gcd(A,M) + \gcd(B,M) + \gcd(C,M) + \gcd(A+B+C,M))/2 + 2 - \mathfrak{f}_\iota}{2}. \tag{97}$$

On the other hand, the inductive hypothesis implies

$$g(\mathcal{C}_M/\iota) = g(\mathcal{C}_M/\iota)^{\text{new}} + \sum_{\substack{\delta \mid M \\ \delta \neq 1,2,M}} \phi(\delta)\frac{(2-\kappa(\delta))}{4} + \begin{cases} 1 & \text{if } 2 \mid M, \\ 0 & \text{otherwise.} \end{cases} \tag{98}$$

Note that the middle sum can be written like

$$\frac{1}{2}\sum_{\substack{\delta \mid M \\ \delta \neq 1,2,M}} \phi(\delta) - \frac{1}{4}\sum_{\substack{\delta \mid \gcd(A,M) \\ \delta \neq 1,2,M}} \phi(\delta) - \frac{1}{4}\sum_{\substack{\delta \mid \gcd(B,M) \\ \delta \neq 1,2,M}} \phi(\delta) - \frac{1}{4}\sum_{\substack{\delta \mid \gcd(C,M) \\ \delta \neq 1,2,M}} \phi(\delta) - \frac{1}{4}\sum_{\substack{\delta \mid \gcd(A+B+C,M) \\ \delta \neq 1,2,M}} \phi(\delta).$$

The sum of the missing terms for $\delta = 1$ equals $-\frac{1}{2}$, while the sum for $\delta = 2$ equals $\frac{1}{2}$ if M is even and 0 otherwise (since $2 \nmid \gcd(A,M)$ nor the other similar terms). Then Möbius inversion formula implies that it equals

$$\frac{M - \phi(M) - (\gcd(A,M) + \gcd(B,M) + \gcd(C,M) + \gcd(A+B+C,M))/2}{2} + \frac{\kappa(M)\phi(M)}{4}$$

Substracting (98) to (97) we obtain

$$0 = g(\mathcal{C}_M/\iota)^{\text{new}} - \frac{\phi(M)}{2} + \frac{\mathfrak{f}_\iota}{2} - 1 + \frac{\kappa(M)\phi(M)}{4} + \begin{cases} 1 & \text{if } 2 \mid M, \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Since $\mathfrak{f}_\iota = 0$ if M is even and $\mathfrak{f}_\iota = 1$ if M is odd, the result follows.

Case $j \neq -1$: Let $\eta = \gcd(\mathfrak{f}_\iota, A)$.

Fact 3: $\eta \mid 2$ (the same is true for $\gcd(\mathfrak{f}_\iota, B)$ and $\gcd(\mathfrak{f}_\iota, C)$).

Write $a = \frac{\alpha}{r}$ and $c = \frac{\gamma}{s}$ with $\gcd(\alpha, r) = \gcd(\gamma, s) = 1$. Suppose that multiplication by j, up to integer translation, fixes a and sends c to $-c$ (the other case being similar). The hypothesis $ja - a \in \mathbb{Z}$ and $jc + c \in \mathbb{Z}$ imply that $j \equiv 1 \pmod{r}$ and $j \equiv -1 \pmod{s}$. In particular $\gcd(r,s) \mid 2$. If $4 \mid \eta$ then $4 \mid N = \operatorname{lcm}(r,s)$ so $4 \mid r$ or $4 \mid s$, but not both (as otherwise both congruences cannot hold). Also $4 \mid A = (\alpha s - \gamma r)\frac{N}{rs}$. If both r, s are even, then $v_2(\alpha s - \gamma r) = 1$ and $v_2(\frac{N}{rs}) = -1$ so $v_2(A) = 0$. If only one of r, s is even, then $v_2(\alpha s - \gamma r) = 0$, so A is also odd, contradicting the fact that $\eta \mid A$.

Suppose then that p is an odd prime dividing η, so $p \mid r$ or $p \mid s$ (but not both). If $p \mid r$, since $p \mid A \mid \alpha s - \gamma r$, then $p \mid \alpha$ contradicting the fact that $\gcd(\alpha, r) = 1$. A similar contradiction is obtained when $p \mid s$, proving Fact 3.

Going back to the proof of Fact 2 (for $j \neq -1$), when $\kappa(M) = 2$ the formula is clearly true since in this case $\mathcal{C}_M$ has genus zero (by (96)). Assume then that $\kappa(M) = 0$.

If M is an odd prime, then either $\mathfrak{f}_\iota = 1$ (when $j \not\equiv -1 \pmod{M}$) or $\mathfrak{f}_\iota = M$ (otherwise). The proof when $\mathfrak{f}_\iota = 1$ mimics the $j = -1$ case one given before. Suppose then that $\mathfrak{f}_\iota = M$. Since $\kappa(M) = 0$ and there is no old contribution, we are led to prove that $\mathcal{C}_M/\iota$ has genus 0, which follows from (93) (since $\mathcal{C}_M$ has genus $M - 1$). When $M = 1$, $\mathcal{C}_M$ has genus zero, while when $M = 2$, $\mathcal{C}_M$ has genus 1 if $\kappa(2) = 0$ and 0 otherwise. But $M = 2$ implies that $j + 1$ is even, so the involution has 4 fixed points and $\mathcal{C}_2/\iota$ has genus 0 as claimed.

Assume that $M > 2$ and that the result holds for all proper divisors of M. Following the proof of the previous case, now (98) becomes (using the inductive hypothesis)

$$g(\mathcal{C}_M/\iota) = g(\mathcal{C}_M/\iota)^{\text{new}} + \sum_{\substack{\delta \mid M \\ \delta \neq 2, M}} \phi(\delta)\frac{(2-\kappa(\delta))}{4} - \sum_{\substack{\delta \mid \mathfrak{f}_\iota \\ \delta \neq 2, M}} \phi(\delta)\frac{(2-\kappa(\delta))}{4}.$$

Note that $\mathfrak{f}_\iota$ is even if and only if M is even, so we can add the term with $\delta = 2$ to both sums (when $2 \mid M$). Fact 2 implies that if $\delta \mid \mathfrak{f}_\iota$ and $\delta > 2$ then $\kappa(\delta) = 0$. If $2 \mid \mathfrak{f}_\iota$ (so N is even) then $\kappa(2) = 0$ (since A, B and C are odd in this case). Since $\kappa(1) = 4$ we get

$$\sum_{\substack{\delta \mid \mathfrak{f}_\iota \\ \delta \neq M}} \phi(\delta)\frac{(2-\kappa(\delta))}{4} = \frac{\mathfrak{f}_\iota}{2} - 1 - \begin{cases} \frac{\phi(M)}{2} & \text{if } M \mid \mathfrak{f}_\iota, \\ 0 & \text{otherwise.} \end{cases}$$

Then using Möbius inversion formula

$$(99)\quad g(\mathcal{C}_M/\iota) = g(\mathcal{C}_M/\iota)^{\text{new}} + \frac{M}{2} - \frac{\phi(M)}{2} - \frac{\gcd(A,M)}{4} - \frac{\gcd(B,M)}{4} - \frac{\gcd(C,M)}{4} - \\ - \frac{\gcd(A+B+C,M)}{4} - \frac{\mathfrak{f}_\iota}{2} + 1 + \begin{cases} \frac{\phi(M)}{2} & \text{if } M \mid \mathfrak{f}_\iota, \\ 0 & \text{otherwise.} \end{cases}$$

Substracting (99) to (97) we get the equality

$$g(\mathcal{C}_M/\iota)^{\text{new}} = \frac{\phi(M)}{2} - \begin{cases} \frac{\phi(M)}{2} & \text{if } M \mid \mathfrak{f}_\iota, \\ 0 & \text{otherwise.} \end{cases}$$

□

Proof of Theorem 8.2. Let N be the least common denominator of the parameters a, b, c, d. Let $j \in H$ and let ι_j be the involution defined in (89). The last proposition states that the new part of $\mathcal{C}/\iota$ has genus $\phi(N)/2$. Abusing notation, denote by ζ_N the map on $\mathcal{C}$ given by $\zeta_N(x,y) = (x, \zeta_N y)$. It follows easily from its definition that

$$(100)\qquad \iota_j \circ \zeta_N = \begin{cases} \zeta_N^{-j} \circ \iota_j & \text{if } j \neq -1, \\ \zeta_N^{-1} \circ \iota_j & \text{if } j = -1. \end{cases}$$

Then over the field $F^{\langle j \rangle}$ there is an action of the ring $\mathbb{Z}[\zeta_N + \zeta_N^{-j}]$ (respectively $\mathbb{Z}[\zeta_N + \zeta_N^{-1}]$) on $\text{Jac}(\mathcal{C}/\iota)$. Looking at the new part, we get a 2-dimensional representation with coefficient field $F^{\langle j \rangle}$. If H is cyclic, this concludes the proof; when H is not cyclic, take the quotient by two different isogenies attached to elements on H (getting a representation over F^H). □

The irreducibility assumption (**Irr**) in the theorem is not a strong one.

Lemma 8.13. *Let $(a,b), (c,d)$ be generic parameters such that $a+b$ and $c+d$ are integers. If* (**Irr**) *does not hold then $H = \{\pm 1\}$.*

Proof. By Lemma 7.5 the values $N := \text{lcm}\{\text{den}(a), \text{den}(c)\}$ and $N' := \text{lcm}\{\text{den}(a+c), \text{den}(a-c)\}$ are different so $N = 2N'$. This implies that $v_2(\text{den}(a)) = v_2(\text{den}(c))$. If $j \in H$ and $j \neq \pm 1$ then $ja = a \pmod{\mathbb{Z}}$ and $jc = -c \pmod{\mathbb{Z}}$ or vice-versa. But then $v_2(j-1) = v_2(j+1) = v_2(\text{den}(a))$, a contradiction. □

A result similar to Theorem 8.2 can be proved under the following hypothesis.

Theorem 8.14. *Let $(a,b), (c,d)$ be generic parameters such that* (**Irr**) *holds. Suppose furthermore that $H = \langle i \rangle$ where the action of multiplication by i is not trivial on both sets $\{a,b\}$ and $\{c,d\}$. Then $\mathcal{H}((a,b),(c,d)|z)$ is defined over F^H and its coefficient field is contained in F^H.*

Proof. Consider the *twisted* Euler curve

$$\widetilde{\mathcal{C}} : y^N = (-1)^{(b-d)N} x^A (1-x)^B (1-xz)^C z^D, \tag{101}$$

where the exponents are as in (58). Then the motive $\mathcal{H}((a,b),(c,d)|z)$ can be defined using the twisted Euler curve as in (61) but without the quadratic twist, namely

$$\mathcal{H}((a,b),(c,d)|z) = \widetilde{\mathcal{J}}_N^{\zeta_N,\mathrm{new}} \otimes \mathbf{J}((-a,-b,c,d),(c-b,d-a))^{-1},$$

where $\widetilde{\mathcal{J}}$ denotes the Jacobian of $\widetilde{\mathcal{C}}$. The action of multiplication by i fixes the Jacobi motive, and with the new definition, it also fixes the twisted Euler curve (by Lemma 8.7). Then we can define an involution ι_i and the proof follows mutatis mutandis that of Theorem 8.2. ☐

When the group H is cyclic, we can prove a weaker general version of the result.

Theorem 8.15. *Let $(a,b),(c,d)$ be generic parameters such that $|H| \le 2$. Then for all $z_0 \in \mathbb{Q} \setminus \{0,1\}$, the Galois representation of $\mathcal{H}((a,b),(c,d)|z_0)$ extends to Gal_{F^H} and its coefficient field is contained in a quadratic extension of F^H.*

Let us start with some auxiliary needed results.

Theorem 8.16. *Let $(a,b),(c,d)$ be generic rational parameters. For z_0 outside a thin set of $\mathbb{Q}$, the Galois representation $\rho_{\mathcal{H}((a,b),(c,d)|z),\mathfrak{p}}$ of Gal_F is absolutely irreducible.*

Proof. By Theorem 7.13 the Galois representation $\rho_{\mathcal{H}((a,b),(c,d)|z),\mathfrak{p}}$ extends the monodromy representation ρ, a representation which is absolutely irreducible representation (by [5, Proposition 3.3]). After the choice of a stable lattice, if $\mathcal{O}_\mathfrak{p}$ denotes the ring of integers of $F_\mathfrak{p}$, we can assume that our representation $\rho_{\mathcal{H}((a,b),(c,d)|z),\mathfrak{p}}$ takes values in $\mathrm{GL}_2(\mathcal{O}_\mathfrak{p})$. Then there exists a positive integer n such that the reduction

$$\rho_{\mathcal{H}((a,b),(c,d)|z),\mathfrak{p},n} : \mathrm{Gal}(\overline{\mathbb{Q}(z)}/F(z)) \to \mathrm{GL}_2(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^n)$$

is irreducible. The field fixed by its kernel is a finite extension $L(z)/F(z)$. Then Hilbert's irreducibility theorem (see for example §3.4 of [43]) implies that for z_0 outside a thin set, $\mathrm{Gal}(L(z_0)/F) = \mathrm{Gal}(L(z)/F(z))$, so the representation $\rho_{\mathcal{H}((a,b),(c,d)|z_0),\mathfrak{p}}$ is irreducible. ☐

Example 13. Let $F = \mathbb{Q}(\zeta_3)$, where ζ_3 is a third root of unity. The motive $\mathcal{H}((1/3,2/3),(1,1)|z)$ matches (as discovered in [28]) the motive attached to the rational elliptic curve with equation

$$E_t : y^2 + xy + \frac{z}{27} y = x^3.$$

For $z_0 = 9/8$ the elliptic curve is defined over $F = \mathbb{Q}(\sqrt{-3})$ and has complex multiplication by $\mathbb{Z}[\zeta_3]$ (as can be verified for example using Sage [46]). The map: $x = u^2x' + r$, $y = u^3y' + u^2sx' + t$, with

$$u = \zeta_3, \quad r = -\frac{\zeta_3+2}{12}, \quad s = \frac{\zeta_3-1}{2}, \quad t = \frac{\zeta_3+2}{24}.$$

is an endomorphism of the curve defined over F. Then its Galois representation is not irreducible.

Example 14 (Shimura's Example 11 continued)**.** Consider Shimura's example at $z_0 = 1/2$, corresponding to the curve

$$\mathcal{C} : y^5 = x(1-x)(1-x/2).$$

The map $\iota(x,y) = (2-x,-y)$ is an involution of $\mathcal{C}$. The quotient curve can be computed using [7].

```
A<x,y>:=AffineSpace(Rationals(),2);
C:=Curve(A,y^5-x*(1-x)*(1-x/2));
G:=AutomorphismGroup(C);
CG,prj := CurveQuotient(G);
```

It is given by the hyperelliptic model

$$\mathcal{C}_1 : y^2 = x(x^5 - 8).$$

This implies that the motive at $z_0 = 1/2$ is reducible. To obtain a complement to the Jacobian of $\mathcal{C}_1$ in the Jacobian of $\mathcal{C}$ we use the code developed in [14] and we find the hyperelliptic curve

$$\mathcal{C}_2 : y^2 = x(x^5 - 16).$$

The two curves are isomorphic over the field $\mathbb{Q}(\sqrt[5]{2})$, the isomorphism being $(x, y) \to (\sqrt[5]{2}x, \sqrt[5]{8}y)$. Their Jacobians are not isogenous over $F = \mathbb{Q}(\zeta_5)$; this can be verified by computing their Frobenius polynomial at 11

```
P<x> := PolynomialRing(RationalField());
C := HyperellipticCurve(x*(x^5-8));
D := HyperellipticCurve(x*(x^5-16));
LPolynomial(ChangeRing(C,GF(11)));
> 121*x^4 + 121*x^3 + 51*x^2 + 11*x + 1
LPolynomial(ChangeRing(D,GF(11)));
> 121*x^4 - 99*x^3 + 41*x^2 - 9*x + 1
```

Both curves have an order 5 automorphism sending $(x, y) \to (\zeta_5^2 x, \zeta_5 y)$ so correspond to abelian surfaces with complex multiplication. Then the motive $\mathcal{H}((\frac{1}{5}, \frac{4}{5}), (\frac{3}{5}, 1)|\frac{1}{2})$ corresponds to the sum of two (distinct) Hecke characters.

A result of Bailey (see [3]) gives the following identity

$${}_2F_1\left(a, 1-a; c|\frac{1}{2}\right) = \frac{\Gamma(\frac{c}{2})\Gamma(\frac{c+1}{2})}{\Gamma(\frac{c+a}{2})\Gamma(\frac{1+c-a}{2})}.$$

A similar formula is expected to hold for the hypergeometric motive (where the right hand side has to be understood as a Jacobi motive). We expect $\mathrm{Jac}((\frac{c}{2}, \frac{c+1}{2}), (\frac{a+c}{2}, \frac{1+c-a}{2})$ for $a = 1/5$, $c = 2/5$ and $a = 6/5$, $c = 2/5$ (taking $c = 7/5$ gives the same values) to be part of the motive, i.e. that

$$\mathcal{H}\left(\left(\frac{1}{5}, \frac{4}{5}\right), \left(\frac{3}{5}, 1\right) \middle| \frac{1}{2}\right) = \mathbf{J}\left(\left(\frac{1}{5}, \frac{7}{10}\right), \left(\frac{4}{5}, \frac{1}{10}\right)\right) \oplus \mathbf{J}\left(\left(\frac{1}{5}, \frac{7}{10}\right), \left(\frac{3}{10}, \frac{3}{5}\right)\right). \tag{102}$$

Equivalently, the curve $\mathcal{C}_1$ should correspond to the Jacobi motive

$$\mathbf{J}\left(\left(\frac{1}{5}, \frac{1}{5}, \frac{3}{5}, \frac{7}{10}\right), \left(\frac{1}{10}, \frac{4}{5}, \frac{4}{5}\right)\right),$$

while the curve $\mathcal{C}_2$ should correspond to the Jacobi motive

$$\mathbf{J}\left(\left(\frac{1}{5}, \frac{1}{5}, \frac{7}{10}\right), \left(\frac{3}{10}, \frac{4}{5}\right)\right).$$

We verified that their Euler factors coincide for primes up to 500 (it seems like a nice exercise to prove the stated equality).

Question: Let S denote the set of values where the specialization of the Galois representation is reducible. Besides the proven statement on S being thin, can more be said? Is it finite?

Lemma 8.17. *Let L/K be a cyclic extensions of number fields, and let $\rho : \mathrm{Gal}_L \to \mathrm{GL}_d(\overline{\mathbb{Q}_p})$ be an irreducible continuous Galois representation. Then the representation ρ extends to a representation $\tilde{\rho} : \mathrm{Gal}_K \to \mathrm{GL}_d(\overline{\mathbb{Q}_p})$ if and only if for all $\tau \in \mathrm{Gal}(L/K)$, the representation $\rho^\tau : \mathrm{Gal}_L \to \mathrm{GL}_d(\overline{\mathbb{Q}_p})$ defined by $\rho^\tau(\sigma) := \rho(\tau\sigma\tau^{-1})$ is isomorphic to ρ.*

Proof. See [17, Lemma 3.4] (even when the result is stated for L/K quadratic, the proof works the same for cyclic extensions). □

Remark 8.18. The result is only valid for cyclic extensions; it is not hard to construct examples of abelian extensions L/K with Galois group $\mathbb{Z}/2 \times \mathbb{Z}/2)$ where the last result does not hold. It is however true in general that a twist of our representation does extend.

Proof Theorem 8.15. Start assuming that $\rho_{\mathcal{H}((a,b),(c,d)|z_0),\mathfrak{p}}$ is absolutely irreducible (by Theorem 8.16 and Remark 8.18 this occurs for z_0 outside a thin set). Since H is cyclic, Lemma 8.17 implies that it is enough to prove that for all $\sigma \in H$, the representation $\rho^{\sigma}_{\mathcal{H}((a,b),(c,d)|z_0),\mathfrak{p}}$ is isomorphic to $\rho_{\mathcal{H}((a,b),(c,d)|z_0),\mathfrak{p}}$, or equivalently (since $\rho_{\mathcal{H}((a,b),(c,d)|z_0),\mathfrak{p}}$ is irreducible) that for all prime ideals $\mathfrak{q}$ of F on a density one set both representations have the same trace at $\mathrm{Frob}_{\mathfrak{q}}$. By Theorem 7.23 and Proposition 6.3, if $\sigma(\zeta_N) = \zeta_N^j$, it amounts to verify that

$$H_{\mathfrak{q}}((a,b),(c,d)|z_0) = H_{\mathfrak{q}}((ja,jb),(jc,jd)|z_0),$$

whose veracity follows from the definition of H.

If the representation is reducible, then there exists Hecke characters χ_1, χ_2 of Gal_F such that (up to semisimplification)

$$\rho^{\mathrm{ss}}_{\mathcal{H}((a,b),(c,d)|z_0),\mathfrak{p}} \simeq \chi_{1,\mathfrak{p}} \oplus \chi_{2,\mathfrak{p}}.$$

Let σ be the non-trivial element of $\mathrm{Gal}(F/K)$. Either ${}^{\sigma}\chi_1 = \chi_1$ or ${}^{\sigma}\chi_1 = \chi_2$. In the first case, both characters χ_i, $i = 1, 2$, extend to Gal_K, so the extension is defined as the sum of the two extended characters. In the second case, the extension is defined as the induced representation $\mathrm{Ind}_{\mathrm{Gal}_F}^{\mathrm{Gal}_K} \chi_1$ (which is isomorphic to the induction of χ_2). ☐

Corollary 8.19. *If $a+b$ and $c+d$ are integers, then for $z_0 \in \mathbb{Q}$, the Galois representation attached to $\mathcal{H}((a,b),(c,d)|z_0)$ extends to Gal_K.*

Proof. If (**Irr**) holds, the result follow from Theorem 8.2. Otherwise, Lemma 8.13 implies that H is cyclic, so the result follows from Theorem 8.15. ☐

8.1. **On the coefficient field of the motive.**

Lemma 8.20. *Let a, b, c, d be generic parameters such that $a \neq b$. Let N be their least common denominator. Let $z_0 \in \mathbb{Q}$ be such that there exists a prime $p \nmid N$ with $v_p(z_0) > 0$ prime to N. Then both ζ_{c+d} and $\zeta_c + \zeta_d$ belong to the coefficient field of the hypergeometric motive. Similarly, if $c \neq d$ and there exists a prime $p \nmid N$ with $v_p(z_0) < 0$ prime to N then both ζ_{a+b} and $\zeta_a + \zeta_b$ belong to the coefficient field.*

Proof. Recall that the geometric representation attached to $\mathcal{H}((a,b),(c,d)|z)$ has monodromy matrices $M_\infty := \left(\begin{smallmatrix} \zeta_a & 0 \\ 0 & \zeta_b \end{smallmatrix}\right)$ at ∞ if $a \neq b$ and $M_0 := \left(\begin{smallmatrix} \zeta_{-c} & 0 \\ 0 & \zeta_{-d} \end{smallmatrix}\right)$ at 0 if $c \neq d$.

Let p be a prime number not dividing N, such that $v_p(z_0) = r$, where r is positive and prime to N. Let $\mathfrak{p}$ be a prime ideal of F divisible by p. Then by [4, Theorem 1.2] (see also [22]) the image of inertia $I_{\mathfrak{p}}$ is generated by M_0^r, so [18, Lemma 15] implies that the coefficient field contains both $\zeta_{r(c+d)}$ and $\zeta_{-rc} + \zeta_{-rd}$, which is equivalent to say that it contains ζ_{c+d} and $\zeta_{-c} + \zeta_{-d}$ (because the extension $\mathbb{Q}(\zeta_c, \zeta_d)/\mathbb{Q}$ is abelian). The second statement follows from a similar argument. ☐

Proposition 8.21. *Let $z_0 \in \mathbb{Q}$ be such that it has a prime dividing its numerator and a prime dividing its denominator satisfying the hypothesis of Lemma 8.20. Then if (**Irr**) holds and $|H| = 4$, the coefficient field equals F^H.*

Proof. Let M denote the coefficient field. By Theorem 8.2, M is contained in F^H hence $[F : M] \geq |H|$. The hypothesis on z_0 (and last lemma) imply that $\mathbb{Q}(\zeta_{a+b}, \zeta_a + \zeta_b, \zeta_{c+d}, \zeta_c + \zeta_d) \subset M \subset F^H \subset F$, so $[F : M] \leq 4$ (because $F = \mathbb{Q}(\zeta_a, \zeta_b, \zeta_c, \zeta_d)$). Since $|H| = 4$, $M = F^H$. ☐

Example 15. Last proposition suggests that some genericity condition on the parameters is needed for the coefficient field to match F^H. Consider the case of parameters $(3/8, -3/8), (1/8, -1/8)$ (studied in Example 2 and Example 9). The group $H = \langle -1 \rangle$ in $(\mathbb{Z}/8)^\times$, so for any specialization z_0 of the parameter, the motive is defined over $\mathbb{Q}(\sqrt{2})$ (by Theorem 8.2). But in Example 9 we showed that when z_0 is a square, the motive actually corresponds to a rational elliptic curve.

Remark 8.22. It is not true in general that if $\rho : \mathrm{Gal}_K \to \mathrm{GL}_N(A)$ is a continuous Galois representation and B is a subring of A that contains the trace of all elements of Gal_K, then the representation can be defined over $\mathrm{GL}_N(B)$. This is true under the extra hypothesis that ρ is absolutely irreducible (see for example Mazur's article in [10], Corollary in page 256). Then although we can (in many instances) prove properties of the coefficient field of the motive (like being contained in K), we cannot guarantee that there exists a representation into $\mathrm{GL}_2(K_\mathfrak{p})$, since the Galois representation attached to $\mathcal{H}((a, b), (c, d)|z_0)$ is not always irreducible (as shown in Example 13 and Example 14).

9. Hypergeometric motives defined over totally real fields

Our understanding of L-series coming from Galois representations is still very limited. There are a few instances where some general results are known, including odd 2-dimensional representations of weight two over totally real fields. In this case, it is known (see for example [45, Theorem 1.1.1]) that the L-series extends meromorphically to the whole complex plane and satisfies a functional equation (it is furthermore expected the extension to be holomorphic). For this reason, these section is dedicated to hypergeometric motives defined over totally real fields.

According to Conjecture 4.5 the motive $\mathcal{H}((a, b), (c, d)|z)$ is defined over a totally real field precisely when $-1 \in H$; i.e. when

$$\{a, b\} = \{-a, -b\}, \qquad \{c, d\} = \{-c, -d\}.$$

Equivalently, the motive is defined over a totally real field when one of the following (non-disjoint) cases occurs:

- $a = -b$ and $c = -d$, or equivalently, $a + b \in \mathbb{Z}$ and $c + d \in \mathbb{Z}$,
- $a = -b$ (respectively $c = -d$), $c = -c$ and $d = -d$ (respectively $a = -a$ and $b = -b$).
- $a = -a$, $b = -b$, $c = -c$ and $d = -d$.

The last case (for generic parameters) corresponds to the motive with parameters $(1/2, 1/2), (1, 1)$ or $(1, 1)(1/2, 1/2)$. It is a rational motive corresponding to Legendre's family of elliptic curves

$$E_z : y^2 = x(1 - x)(1 - zx).$$

In this case modularity of the motive is known (by [52] and [9]), so we study the other two cases. For a rational number a, the relation $a \equiv -a \pmod{\mathbb{Z}}$ implies that either $a = 1$ or $a = \frac{1}{2}$, so we can separate (up to a quadratic twist obtained by adding $1/2$ to all parameters) totally real motives into two disjoint cases (that we study separately):

(1) Motives with parameters $(a, b)(c, d)$ with $a + b \in \mathbb{Z}$ and $c + d \in \mathbb{Z}$,
(2) Motives with parameters $(a, -a)(1/2, 1)$ or $(1/2, 1)(c, -c)$.

9.1. **Case (1).** We start computing the Hodge vectors as explained in §4.1 (using Theorem 7.14). If two of the parameters are integers, then for all embeddings $\sigma : K \hookrightarrow \mathbb{C}$, the zigzag procedure gives

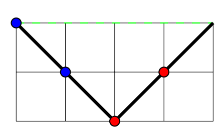


Figure 4. Hodge vector when $a, b \in \mathbb{Z}$ or $c, d \in \mathbb{Z}$

the diagram in Figure 4, so $h^{0,1} = h^{1,0} = 1$. Then for any $z_0 \in K \setminus \{0,1\}$, the Galois representation of Gal_K attached to $\mathcal{H}((a,b),(c,d)|z_0)$ has Hodge-Tate weights $\{0,1\}$.

When no parameter is an integer, the zig-zag procedure gives the diagram in Figure 5. In this

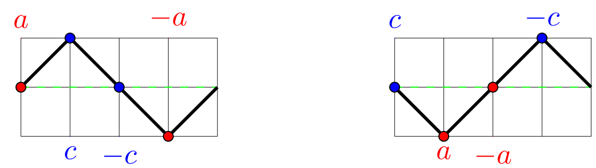


FIGURE 5. The zigzag procedure in case (1) for $a < c$ and $a > c$

case, the Hodge vector has values $h^{0,-1} = h^{-1,0} = 1$. Then for any $z_0 \in K \setminus \{0,1\}$ the Galois representation of Gal_K attached to $\mathcal{H}((a,b),(c,d)|z_0)$ has Hodge-Tate weights $\{-1,0\}$.

In both cases, the L-series attached to $\mathcal{H}((a,b),(c,d)|z_0)$ extends meromorphically to the whole complex plane and satisfies a functional equation (by [45, Theorem 1.1.1]). As already mentioned, it is expected that the L-series is actually holomorphic, and comes (up to a Tate twist) from a parallel weight 2 Hilbert modular form defined over K. In some cases one can use modularity lifting theorems and congruences to prove modularity of all specializations of the motive (as done in [24] for the parameters $(\frac{1}{2p}, -\frac{1}{2p}), (1,1)$ for example). In other cases (like the motive corresponding to the parameters $(3/8,-3/8),(1/8,-1/8)$ described in the introduction) one can prove modularity of the motive for some particular specializations (like $z_0 = 3$) using Faltings-Serre method.

9.2. **Case (2).** The zig-zag procedure gives (for all embeddings $\sigma : K \hookrightarrow \mathbb{C}$) the diagram shown in Figure 6. Its Hodge vector has $h^{0,0} = 2$, so it should correspond to a parallel weight 1 modular form. This is indeed the case, since such motives have finite monodromy as studied by Schwarz

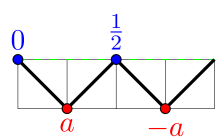


FIGURE 6. The zigzag procedure in case (2)

(in [41]), corresponding to dihedral projective image. A detailed description will be given in a subsequent article by the last two authors.

In both cases, we can prove a stronger version of Theorem 8.16.

Theorem 9.1. *Let $(a,b),(c,d)$ be rational generic parameters, and let $z_0 \in \mathbb{Q} \setminus \{0,1\}$. Suppose that $a+b$ and $c+d$ are integers. Then the Galois representation of the motive $\mathcal{H}((a,b),(c,d)|z_0)$ extended to Gal_K is irreducible.*

Proof. Recall that any Hecke character of the idèle group of a totally real field is (up to a finite order character) a multiple of the norm character (the reason is that if $d = [K : \mathbb{Q}]$, the compatibility relation at the units of K impose $d-1$ relations among the d possible characters at the archimedean places). Suppose then that the representation is reducible, namely

$$\rho_\lambda = \begin{pmatrix} \chi_\ell^a \varepsilon_1 & * \\ 0 & \chi_\ell^b \varepsilon_2 \end{pmatrix}, \tag{103}$$

for some $a, b \in \mathbb{Z}$ and $\varepsilon_1, \varepsilon_2$ finite order characters of the idèle group of K, where χ_ℓ denotes the cyclotomic character (the precise values of a and b can be given using the zig-zag procedure as previously explained). Then ρ_λ is also reducible while restricted to Gal_F and the same holds for any twist, so the 2-dimensional representation coming from $H^{1,\zeta_N^i}_{\text{ét}}(\widehat{\mathcal{C}}, \mathbb{Q}_\ell)$ also looks like (103) (since by Lemma 8.3, the Jacobi motive is also a power of the cyclotomic character).

The Hodge-Tate weights of the Galois representation attached to the new part of Euler's curve are $\{0, \ldots, 0, 1, \ldots, 1\}$, each one with multiplicity $\frac{\phi(N)}{2}$. From the decomposition

$$H^1_{\text{ét}}(\widehat{\mathcal{C}}, \mathbb{Q}_\ell)^{\text{new}} = \bigoplus_{\substack{i=1 \\ \gcd(i,N)=1}}^{N} H^{1,\zeta_N^i}_{\text{ét}}(\widehat{\mathcal{C}}, \mathbb{Q}_\ell),$$

it follows that for each i prime to N the representation of $H^{1,\zeta_N^i}_{\text{ét}}(\widehat{\mathcal{C}}, \mathbb{Q}_\ell)$ has a decomposition like (103) for some characters $\varepsilon_1^{(i)}, \varepsilon_2^{(i)}$ (but the exponents a, b, are independent of i, since they are Galois conjugate of each other). In particular, $a = 0$ and $b = 1$ or vice-versa.

Let $\mathfrak{q}$ be a prime ideal of F not dividing $N\ell$ such that $v_{\mathfrak{q}}(z_0(z_0 - 1)) = 0$ and also $\varepsilon_1^{(i)}(\mathfrak{q}) = \varepsilon_2^{(i)}(\mathfrak{q}) = 1$ for all i prime to N (there are infinitely many such prime ideals by Chebotarev density theorem). Then the trace of $\mathrm{Frob}_{\mathfrak{q}}$ acting on the new part of the Jacobian equals $(\mathcal{N}(q) + 1)\phi(N)$.

On the other hand, Weil's conjectures imply that such trace cannot be larger than $2\phi(N)\sqrt{\mathcal{N}(q)}$, which is a contradiction, since $x + 1 > 2\sqrt{x}$ if $x > 1$. □

10. Congruences between hypergeometric motives

Congruences between Galois representation have many applications in number theory. In Langlands' program they play a crucial role in the proof of results like base change, Serre's conjectures, potential modularity, the Shimura-Taniyama conjecture, etc. In the study of Diophantine equations, they also play a crucial role, like in Wiles' proof of Fermat's last theorem. Hypergeometric motives have the property that they satisfy many congruences.

Definition 10.1. *Let ℓ be a rational prime. Define on $\mathbb{Q}/\mathbb{Z}$ the relation defined by the condition that the denominator of $a - b$ is an ℓ-th power. Denote it by $a \sim_\ell b$.*

The following result is easy to verify.

Lemma 10.2. *The relation $\sim_\ell$ is an equivalence relation.*

Extend the definition to $(\mathbb{Q}/\mathbb{Z})^n$ component-wise.

Theorem 10.3. *Let ℓ be a prime number, and let $(a, b), (c, d)$ and $(a', b'), (c', d')$ be two pairs of generic rational parameters. Let F be the composite of the fields of definition of the motives attached to both parameters. Then if $(a, b) \sim_\ell (a', b')$ and $(c, d) \sim_\ell (c', d')$, the base change of both motives to F are congruent modulo $\mathfrak{l}$, for $\mathfrak{l}$ any prime ideal of their coefficient field dividing ℓ.*

Proof. By Theorem 7.23 the trace of the hypergeometric motive at a prime $\mathfrak{p}$ of good reduction matches a finite hypergeometric sum. From its very definition (equation (36)) the finite hypergeometric sum is a sum and product/quotients of Gauss sums of the form $g(m + \alpha(q - 1), \mathfrak{p})$, where α is one of the parameters (and $q = p^r$ is the order of the residue field $\mathcal{O}_F/\mathfrak{p}$). To prove the statement it is enough to verify that if $\alpha \sim_\ell \beta$ then $g(m + \alpha(q - 1), \mathfrak{p}) \equiv g(m + \beta(q - 1), \mathfrak{p}) \pmod{\mathfrak{l}}$. Recall that if ω denotes a generator of the multiplicative group $\mathbb{F}_q^\times$, then

$$g(m + \alpha(q - 1)) = \sum_{x \in \mathcal{O}_F/\mathfrak{p}} \omega^m(x)\omega^{\alpha(q-1)}(x)\psi(x), \tag{104}$$

where ψ is an additive character of $\mathbb{F}_q^\times$. The last sum is an element of $\mathbb{Z}[\zeta_p, \zeta_{q-1}]$. Let $\mathfrak{l}$ be a prime ideal in this ring dividing ℓ. Then

$$\frac{\omega^{\alpha(q-1)}(x)}{\omega^{\beta(q-1)}(x)} = \omega^{(\alpha-\beta)(q-1)}(x) = \omega^{(q-1)/\ell^r}(x),$$

for some non-negative integer r. In particular, it is a root of unity whose order divides ℓ^r, so congruent to 1 modulo $\mathfrak{l}$. This implies that $g(m+\alpha(q-1)) \equiv g(m+\beta(q-1)) \pmod{\mathfrak{l}}$ as claimed. □

Some explicit examples of lowering/raising the level using congruences between hypergeometric motives were given at the introduction.

Appendix A. Stable models and Frobenius' trace

The goal of the present appendix is to give a formula for the trace of a Frobenius element $\mathrm{Frob}_\mathfrak{p}$ when the motive has good reduction at $\mathfrak{p}$ but the equation used to define Euler's curve is singular at $\mathfrak{p}$. There are three different cases, depending on whether the specialization z_0 reduces to 0, 1 or ∞ modulo $\mathfrak{p}$. More concretely, let $(a,b),(c,d)$ be generic rational parameters and let N be their least common denominator. Let $F = \mathbb{Q}(\zeta_N)$ and let $\mathfrak{p}$ be a prime ideal of F not dividing N. Consider the following three cases:

(1) The monodromy matrix M_0 has finite order r_0, $v_\mathfrak{p}(z_0)$ is positive and divisible by r_0.
(2) The monodromy matrix M_1 has finite order r_1, $v_\mathfrak{p}(z_0-1)$ is positive and divisible by r_1.
(3) The monodromy matrix M_∞ has finite order r_∞, $v_\mathfrak{p}(z_0)$ is negative and divisible by r_∞.

The following three theorems describe the value of the trace of $\mathrm{Frob}_\mathfrak{p}$ in each case (assuming an extra technical condition that holds in most instances).

Theorem A.1. *Let $(a,b),(c,d)$ be rational generic parameters and let N be their least common denominator. Let $z_0 \in \mathbb{Q}$. Suppose that condition (1) is satisfied. Write $z_0 = p^{v_\mathfrak{p}(z_0)}\tilde{z_0}$. Suppose that the following two extra hypotheses hold:*

$$\gcd(N,(d-b)N,(b-c)N) = 1, \quad \textit{and} \quad \gcd(N,(d-c)N,(a-d)N) = 1.$$

Then the trace of $\mathrm{Frob}_\mathfrak{p}$ acting on $\mathcal{H}((a,b),(c,d)|z_0)$ equals

$$(105) \qquad -\chi_\mathfrak{p}(-1)^{(d-b)(\mathcal{N}\mathfrak{p}-1)}\,\mathbf{J}((-a,-b,c,d),(c-b,d-a))(\mathfrak{p})^{-1}\cdot \left(\chi_\mathfrak{p}(\tilde{z_0})^{dN}J(\chi_\mathfrak{p}^{(d-b)N},\chi_\mathfrak{p}^{(b-c)N}) + \chi_\mathfrak{p}(-1)^{(b-c)N}\chi_\mathfrak{p}(\tilde{z_0})^{cN}J(\chi_\mathfrak{p}^{(d-c)N},\chi_\mathfrak{p}^{(a-d)N})\right),$$

where $J(\chi_1,\chi_2)$ denotes the usual Jacobi sum.

Proof. The proof follows the arguments on [22] (see also [8], [19] and [37]). Let $A = (d-b)N$, $B = (b-c)N$, $C = (a-d)N$ and $D = dN$. Then Euler's curve is defined by the equation

$$(106) \qquad \mathcal{C} : y^N = x^A(1-x)^B(1-z_0x)^C p^{v_\mathfrak{p}(z_0)D}\tilde{z_0}^D.$$

The order of M_0 is $\operatorname{lcm}\{\operatorname{den}(c),\operatorname{den}(d)\}$, hence the hypothesis of being in case (1) implies that both $v_\mathfrak{p}(z_0)c$ and $v_\mathfrak{p}(z_0)d$ are integers. The change of variables $y \to yp^{-v_\mathfrak{p}(z_0)d}$ is defined over F. The new model reduces modulo $\mathfrak{p}$ to the curve

$$\mathcal{C}_1 : y^N = x^A(1-x)^B\tilde{z_0}^D.$$

The first extra hypothesis implies that $\mathcal{C}_1$ is irreducible. Since the parameters are generic, $N \nmid A$ and $N \nmid B$. Furthermore, $N \nmid A+B = N(d-c)$ because the matrix M_0 has finite order. Then Lemma 7.12 implies that $\dim(\mathrm{Jac}(\mathcal{C}_1)^{\mathrm{new}}) = \frac{\phi(N)}{2} > 0$, so $\mathcal{C}_1$ is a component of the stable model of $\mathcal{C}$ (see [8] for more details).

To find another component, consider the change of variables $x \to xp^{-v_{\mathfrak{p}}(z_0)}$, which transforms (106) into

$$y^N p^{v_{\mathfrak{p}}(z_0)(A+B-D)} = x^A (p^{v_{\mathfrak{p}}(z_0)} - x)^B (1-\tilde{z}_0 x)^C \tilde{z}_0{}^D.$$

From its definition, $A+B-D = -cN$ so after the change of variables $y \to yp^{v_{\mathfrak{p}}(z_0)c}$ we get an equation whose reduction modulo $\mathfrak{p}$ equals

$$\mathcal{C}_2 : y^N = (-1)^B x^{A+B}(1-\tilde{z}_0 x)^C \tilde{z}_0{}^D.$$

The second extra hypothesis implies that $\mathcal{C}_2$ is irreducible. The genericity condition on the parameters implies that $N \nmid C$, $N \nmid A+B$ and $N \nmid A+B+C = N(a-c)$ so Lemma 7.12 implies that $\dim(\mathrm{Jac}(\mathcal{C}_2)^{\mathrm{new}}) = \frac{\phi(N)}{2}$, so $\mathcal{C}_2$ is another component of the stable reduction of $\mathcal{C}$. Then the Galois representation of the decomposition group at $\mathfrak{p}$ acting on $H^{1,\zeta_N}_{et}(\widehat{\mathcal{C}}, \mathbb{Q}_\ell)$ is the direct sum of the contribution from $\mathcal{C}_1$ and that from $\mathcal{C}_2$ (both being abelian varieties with complex multiplication by $\mathbb{Z}[\zeta_N]$).

To compute the trace of $\mathrm{Frob}_{\mathfrak{p}}$ acting on $\mathcal{C}_1$ (respectively $\mathcal{C}_2$), let $\chi_{\mathfrak{p}}$ be as in (24). Then we can apply Theorem 7.22 to $\mathcal{C}_1$ to get that the trace of $\mathrm{Frob}_{\mathfrak{p}}$ acting on $H^{1,\zeta_N}_{et}(\widehat{\mathcal{C}}_1, \mathbb{Q}_\ell)$ equals

$$-\sum_{x \in \mathbb{F}_q} \chi_{\mathfrak{p}}(x^A(1-x)^B \tilde{z}_0{}^D) = -\chi_{\mathfrak{p}}(\tilde{z}_0)^D J(\chi_{\mathfrak{p}}^A, \chi_{\mathfrak{p}}^B).$$

The same computation for $\mathcal{C}_2$ gives that the trace of $\mathrm{Frob}_{\mathfrak{p}}$ on the (new part of its) étale cohomology equals

$$-\chi_{\mathfrak{p}}(-1)^B \chi_{\mathfrak{p}}(\tilde{z}_0)^{D-A-B} J(\chi_{\mathfrak{p}}^{A+B}, \chi_{\mathfrak{p}}^C).$$

The result follows from the definition of A, B, C and D together with the relation between the hypergeometric motive and Euler's curve given in (61). ☐

Similar results hold in the other two cases.

Theorem A.2. *Let $(a,b),(c,d)$ be rational generic parameters and let N be their least common denominator. Let $z_0 \in \mathbb{Q}$. Suppose that condition (2) is satisfied. Let $v = v_{\mathfrak{p}}(z_0-1)$ and write $z_0 = 1 + p^v \tilde{z}_0$. Suppose that the following two extra hypotheses hold:*

$$\gcd(N, (d-b)N, (a+b-c-d)N) = 1, \quad \textit{and} \quad \gcd(N, (b-c)N, (a-d)N) = 1.$$

Then the trace of $\mathrm{Frob}_{\mathfrak{p}}$ *acting on* $\mathcal{H}((a,b),(c,d)|z_0)$ *equals*

$$(107) \quad -\chi_{\mathfrak{p}}(-1)^{(d-b)(N\mathfrak{p}-1)} \mathbf{J}((-a,-b,c,d),(c-b,d-a))(\mathfrak{p})^{-1} \cdot \left(J(\chi_{\mathfrak{p}}^{(d-b)N}, \chi_{\mathfrak{p}}^{(a+b-c-d)N}) + \chi_{\mathfrak{p}}(-1)^{(a-d)N} \chi_{\mathfrak{p}}(\tilde{z}_0)^{(a-b)N} J(\chi_{\mathfrak{p}}^{(d-b)N}, \chi_{\mathfrak{p}}^{(a-d)N})\right).$$

Proof. The reduction modulo $\mathfrak{p}$ of (106) gives the curve

$$\mathcal{C}_1 : y^N = x^A(1-x)^{B+C}.$$

From their definition, $B + C = (a+b-c-d)N$, hence the curve $\mathcal{C}_1$ is irreducible by the first extra hypothesis. Since M_1 has finite order, $a+b-c-d$ is not an integer, so $B+C$ is not divisible by N. The genericity hypothesis on the parameters implies that $N \nmid A$ and $N \nmid A+B+C$. Then Lemma 7.12 implies that the new part of the Jacobian of $\mathcal{C}_1$ has dimension $\frac{\phi(N)}{2}$.

Set $x' = (1-x)p^{-v}$, so that $1 - z_0 x = p^v(x' - \tilde{z}_0 + p^v \tilde{z}_0 x')$ and $(1-x) = p^v x'$. Then (106) becomes

$$p^{-v(B+C)} y^N = (1-p^v x')^A x'^B (x' - \tilde{z}_0 + +p^v \tilde{z}_0 x')^C z_0^d.$$

Since condition (2) is satisfied, $v(a+b-c-d)$ is an integer, then the change of variables $y \to yp^{v(a+b-c-d)}$ gives a model whose reduction modulo $\mathfrak{p}$ equals

$$\mathcal{C}_2 : y^N = x^B(x-\tilde{z}_0)^C \tilde{z}_0{}^d.$$

The second extra hypothesis implies that $\mathcal{C}_2$ is irreducible. The genericity condition on the parameters implies that $N \nmid B$ and $N \nmid C$ (it was proved before that $N \nmid B + C$). Lemma 7.12 implies that the new part of the Jacobian of $\mathcal{C}_2$ has dimension $\frac{\phi(N)}{2}$. Then the Galois representation giving the action of the decomposition group at $\mathfrak{p}$ on the new part of $H^{1,\zeta_N}_{et}(\widehat{\mathcal{C}}, \mathbb{Q}_\ell)$ equals the sum of the contribution from $\mathcal{C}_1$ plus the contribution from $\mathcal{C}_2$. Theorem 7.22 applied to both $\mathcal{C}_1$ and $\mathcal{C}_2$ proves the statement. ☐

Theorem A.3. *Let $(a, b), (c, d)$ be rational generic parameters and let N be their least common denominator. Let $z_0 \in \mathbb{Q}$. Suppose that condition (3) is satisfied. Write $z_0 = p^{v_\mathfrak{p}(z_0)}\tilde{z_0}$. Suppose that the following two extra hypotheses hold:*

$$\gcd(N, (a-b)N, (b-c)N) = 1, \quad \textit{and} \quad \gcd(N, (d-b)N, (a-d)N) = 1.$$

Then the trace of $\mathrm{Frob}_\mathfrak{p}$ *acting on* $\mathcal{H}((a, b), (c, d)|z_0)$ *equals*

$$(108) \quad -\chi_\mathfrak{p}(-1)^{(d-b)(\mathcal{N}\mathfrak{p}-1)}\,\mathbf{J}((-a, -b, c, d), (c-b, d-a))(\mathfrak{p})^{-1} \cdot \left(\omega(\tilde{z_0})^{bN} J(\omega^{(d-b)N}, \omega^{(a-d)N}) + \omega(-1)^{(a-d)N}\omega(\tilde{z_0})^{aN} J(\omega^{(a-b)N}, \omega^{(b-c)N})\right).$$

Proof. The proof is similar to the previous ones. Condition (3) implies that both $av_\mathfrak{p}(z_0)$ and $bv_\mathfrak{p}(z_0)$ are integers. From the equality $z_0 = p^{v_\mathfrak{p}(z_0)}\tilde{z_0}$ and a little manipulation we get that (106) defines the same curve as

$$p^{-v_\mathfrak{p}(z_0)(C+D)}y^N = x^A(1-x)^B(p^{-v_\mathfrak{p}(z_0)} - \tilde{z_0}x)^C\tilde{z_0}^D.$$

Since $C + D = aN$ the change of variables $y \to yp^{v_\mathfrak{p}(z_0)}$ gives a model with reduction

$$\mathcal{C}_1 : y^N = (-1)^C x^{A+C}(1-x)^B\tilde{z_0}^{C+D}.$$

The first extra hypothesis implies that $\mathcal{C}_1$ is irreducible. Since M_∞ has finite order, $N(a-b) \notin \mathbb{Z}$ hence $N \nmid A + C$. The genericity condition implies that $N \nmid B$ and that $N \nmid A + B + C$. Then Lemma 7.12 implies that the new part of the Jacobian of $\mathcal{C}_1$ has dimension $\frac{\phi(N)}{2}$. Replacing in (106) x by $p^{v_\mathfrak{p}(z_0)}x$ and a little manipulation gives the equation

$$p^{-v_\mathfrak{p}(z_0)(D-A)}y^N = x^A(1 - p^{-v_\mathfrak{p}(z_0)}x)^B(1-\tilde{z_0}x)^C\tilde{z_0}^D.$$

Since $D - A = -bN$, the change of variables $y \to yp^{v_\mathfrak{p}(z_0)b}$ gives a model whose reduction modulo $\mathfrak{p}$ equals

$$\mathcal{C}_2 : y^N = x^A(1-\tilde{z_0}x)^C\tilde{z_0}^D.$$

The second extra hypothesis implies that $\mathcal{C}_2$ is irreducible. Lemma 7.12 implies that the new part of the Jacobian of $\mathcal{C}_2$ has dimension $\frac{\phi(N)}{2}$. Then the Galois representation giving the action of the decomposition group at $\mathfrak{p}$ on the new part of $H^{1,\zeta_N}_{et}(\widehat{\mathcal{C}}, \mathbb{Q}_\ell)$ equals the sum of the contribution from $\mathcal{C}_1$ plus the contribution from $\mathcal{C}_2$. Theorem 7.22 applied to both $\mathcal{C}_1$ and $\mathcal{C}_2$ proves the statement. ☐

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