ON RANK 2 HYPERGEOMETRIC MOTIVES

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Abstract. Hypergeometric motives are family of motives associated to hypergeometric local systems. Their special features, in particular their rigidity, makes them more tractable than general motives. In the present article we prove most of the properties that they are expected to satisfy in the rank 2 case.

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1. Introduction

The theory of hypergeometric motives has its origins in the study of the complex hypergeometric series

$$ {}_2F_1(a,b;c|z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, $$

for z a complex number satisfying $|z| < 1$ (as done by Gauss [26], Kummer [37], Goursat [28] et al). Riemann (in [43], "Contribution a la théorie des fonctions représentables par la série de Gauss $F(a,b,c,x)$") studied the differential equation satisfied by this series. This differential equation

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yields a *rank two local system* on $\mathbb{P}^1\setminus\{0,1,\infty\}$ which in turn determines a 2-dimensional *monodromy representation*

$$\rho : \pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\}, z_0) \to \mathrm{GL}_2(\mathbb{C}),$$

describing how a chosen basis of local solutions at a regular point z_0 changes while looping around appropriately around the missing points $0, 1, \infty$. Riemann realized that the monodromy representation is a natural way to encode information of the differential equation and its solutions.

We denote by M_0, M_1 and M_∞ the local monodromy matrices around 0, 1 and ∞ respectively such that

$$M_0 M_1 M_\infty = I_2,$$

where I_2 is the identity matrix.

We are interested in algebraic incarnations of the hypergeometric series, so for the rest of the introduction we let a, b, c be rational numbers and let N be their common denominator. Under this assumption, a result of Levelt implies that we may choose a basis of local solutions such that the monodromy takes values in the number field $F := \mathbb{Q}(\zeta_N)$, with ζ_N a primitive N-th root of unity. Given a prime ideal $\mathfrak{p}$ of $\mathbb{Z}[\zeta_N]$ one can extend the monodromy representation to a *geometric* continuous representation

$$\rho_{\mathfrak{p}} : \mathrm{Gal}(\overline{\mathbb{Q}(z)}/\overline{\mathbb{Q}}(z)) \to \mathrm{GL}_2(F_{\mathfrak{p}}), \tag{1}$$

where $F_{\mathfrak{p}}$ denotes the $\mathfrak{p}$-adic completion of F at $\mathfrak{p}$ (as explained in §2) and z is a variable.

We expect the representation (1) to be *motivic*, namely to match the restriction of a representation of $\mathrm{Gal}(\overline{\mathbb{Q}(z)}/F(z))$ to $\mathrm{Gal}(\overline{\mathbb{Q}(z)}/\overline{\mathbb{Q}}(z))$ arising from a motive defined over $F(z)$. In particular and more precisely, for every specialization $z = \xi \in L$ of the parameter in a finite extension L/F there should exist a pure motive $\mathcal{H}(a, b; c \,|\, \xi)$ defined over L related to Gauss's hypergeometric series. Concretely, (see Conjecture 4.5) we should be able to determine from the defining hypergeometric data the characteristic polynomial of Frobenius, under the compatible system of Galois representations determined by $\mathcal{H}(a, b; c \,|\, \xi)$, for primes of good reduction, as well as obtain information on the action of inertia.

In more detail, let H denote the set of elements $i \in (\mathbb{Z}/N)^\times$ fixing c and the set $\{a, b\}$ modulo $\mathbb{Z}$; i.e., $ic \equiv c \bmod \mathbb{Z}$ and $i\{a, b\} \equiv \{a, b\} \bmod \mathbb{Z}$. Let $K = F^H$, after identifying H with a subgroup of $\mathrm{Gal}(F/\mathbb{Q})$ in a natural way. Then the following should hold (see Corollary 8.18):

(i) *The motive $\mathcal{H}(a, b; c \,|\, z)$ has a model defined over its field of moduli $K(z)$.*

(ii) *For generic values $\xi \in K$ the motive $\mathcal{H}(a, b; c \,|\, \xi)$ defined over K resulting from specializing (i) has coefficient field K.*
For $\xi \neq 0, 1 \in F$ we have the following.

(iii) *The prime ideals $\mathfrak{q}$ of F not diving N, ξ, ξ^{-1} and $\xi - 1$ are primes of good reduction for $\mathcal{H}(a, b; c \,|\, \xi)$.*

(iv) *Let $\mathfrak{q}$ be a prime ideal of F of good reduction for $\mathcal{H}(a, b; c \,|\, \xi)$. Then the trace of* $\mathrm{Frob}_{\mathfrak{q}}$ *on $\mathcal{H}(a, b; c \,|\, \xi)$ matches the finite field analogue of the function ${}_2F_1(a, b; c \,|\, \xi)$ (defined in §4).*

(v) *The primes $\mathfrak{q}$ of F not dividing N but dividing ξ, ξ^{-1} or $\xi - 1$ are (at worst)* tame *primes for $\rho_{\mathfrak{p}}$. Moreover, the conjugacy class of a generator of the image of inertia $\rho_{\mathfrak{p}}(I_{\mathfrak{q}})$ (for any prime ideal $\mathfrak{p}$ whose residual characteristic is prime to that of $\mathfrak{q}$) is that of $M_s^{k_s}$, where $s = 0, 1, \infty$ and*

$$k_0 := v_{\mathfrak{q}}(\xi), \qquad k_1 := v_{\mathfrak{q}}(\xi - 1), \qquad k_\infty := v_{\mathfrak{q}}(\xi^{-1}).$$

It is expected that all of the above properties actually hold for hypergeometric motives of any rank (see §3). In general if the rank is n the motive can be described as an isotypical component

of the middle cohomology of a corresponding Euler variety

$$V: \qquad y^N = \prod_{i=1}^{n-1} x_i^{A_i}(1-x_i)^{B_i}(1-zx_1\cdots x_{n-1})^{A_n} \tag{2}$$

under the action of the automorphism $y \mapsto \zeta_N y$ (see [36] for results on compactifications of these varieties). For $K = \mathbb{Q}$ the motive may also be seen as the top weight piece of the middle cohomology of a hypersurface in a torus (see [8], [44]). Partial results on properties (i)-(v) were obtained by Katz in [35] (see also [24]).

To define our hypergeometric motive we reason heuristically as follows. Assume first that $d = 1$. The hypergeometric series ${}_2F_1$ has the integral presentation

$$\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\,{}_2F_1(a,b;c\,|\,z) = \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}dx. \tag{3}$$

We may interpret this formula as an identity of periods. The right hand side (the integral) is a period of the holomorphic differential dx/y of the Euler curve $\mathcal{C}$ (see (64)); the first factor on the left hand side (a product of Gamma values) is a period of a Jacobi motive $J((a,b),(c,1))$ (defined in §5) and the second factor (the series ${}_2F_1$ itself) should correspond to a period of our hypergeometric motive $\mathcal{H}(\boldsymbol{\alpha},\boldsymbol{\beta}\,|\,\xi)$.

Suppose for the rest of the introduction that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ have rank 2. We prove (iii) and (iv) (see Theorem 8.15)

Theorem 1. *Let $\mathfrak{p}$ be a prime ideal of $\mathbb{Q}(\zeta_N)$ satisfying (iii). Then $\mathcal{H}(\boldsymbol{\alpha},\boldsymbol{\beta}\,|\,z)$ has good reduction at $\mathfrak{p}$ and the trace of the Frobenius automorphism $\mathrm{Frob}_{\mathfrak{p}}$ acting on $\mathcal{H}(\boldsymbol{\alpha},\boldsymbol{\beta}\,|\,z)$ equals $H_{\mathfrak{p}}(\boldsymbol{\alpha},\boldsymbol{\beta}\,|\,z)$.*

Definition 1.1. *Let H be the subgroup of elements in $(\mathbb{Z}/N)^\times$ that fix (under multiplication) the multisets $\{\alpha_1,\ldots,\alpha_n\}$ and $\{\beta_1,\ldots,\beta_n\}$ modulo $\mathbb{Z}$.*

The group H is naturally identified with a subgroup of $\mathrm{Gal}(F/\mathbb{Q})$. Since $-1 \in (\mathbb{Z}/N\mathbb{Z})^\times$ corresponds to complex conjugation, $K := F^H \subseteq \mathbb{Q}(\zeta_N)^+$ (or equivalently, it is totally real) if and only if $-1 \in H$.

Theorems 9.8 and 9.2 imply the following version of (i) (see also Corollary 9.12).

Theorem 2. *If $-1 \in H$ then for all specializations ξ of the parameter $\mathcal{H}(\boldsymbol{\alpha},\boldsymbol{\beta}\,|\,\xi)$ maybe defined over the totally real field K.*

Under some extra hypothesis (see Proposition 8.28) we can prove that for generic values of the parameter $\xi \in K$, the motive $\mathcal{H}(\boldsymbol{\alpha},\boldsymbol{\beta}\,|\,\xi)$ has coefficient field K. It seems to be true that always the field of definition of the motive is the same as the coefficient field (even when it is smaller than K, like in Example 9).

Let p be a rational prime. For a, b rational numbers, denote by $a \sim_p b$ the relation defined by the condition that the denominator of $a - b$ is a power of p. Extend this relation to vectors component-wise.

Theorem 3. *Let $\boldsymbol{\alpha},\boldsymbol{\beta}$ and $\boldsymbol{\alpha}',\boldsymbol{\beta}'$ be generic pairs of rational numbers (see 3.5) such that $\boldsymbol{\alpha} \sim_p \boldsymbol{\alpha}'$ and $\boldsymbol{\beta} \sim_p \boldsymbol{\beta}'$ for some p prime. Then $\mathcal{H}((a,b),(c,d)|\xi)$ is congruent to $\mathcal{H}((a',b'),(c',d')|\xi)$ modulo any prime ideal $\mathfrak{p}$ of a common field of definition dividing p.*

The previous results has two interesting applications: on the one hand, it provides a "lowering the level" result, given by removing a prime ideal p from the denominator of any hypergeometric motive's parameters. This is very useful for example in the study of Diophantine equations (as used in [27] to study solutions of the generalized Fermat equation).

On the other hand, it gives a "raising the level" result, since given a pair of rational parameters $\boldsymbol{\alpha},\boldsymbol{\beta}$ we can add a rational number r/p^s to each α_i and β_i, with r, s positive integers and p a prime

not dividing their denominator. Note that the resulting motive will a priory have wild ramification at primes dividing p.

Example 1. Consider the hypergeometric motive with parameters $(1/2, 1/2), (1, 1)$, corresponding (see Example 10) to a quadratic twist by $\left(\frac{-4}{\cdot}\right)$ of Legendre's family of elliptic curves

$$E_z : y^2 = x(1-x)(1-zx).$$

Adding/subtracting $1/3$ to the first parameters, we obtain the rational motive with parameters $(1/6, -1/6), (1, 1)$ corresponding (as proved in [32]) to the family of elliptic curves with equation

$$\widetilde{E_z} : y^2 = x^3 - \frac{x^2}{4} + \frac{z}{432}.$$

Then for any specialization ξ of the parameter z the elliptic curves are congruent modulo 3 as implied by Theorem 2.

We mostly restricted to rank two hypergeometric motives because of the applications to the study of solutions to the generalized Fermat's equation, following the program described by Darmon in [18], as explained in the article [27]. We expect, however, that our proofs extend to the arbitrary rank case by considering the general Euler variety (2).

The article is organized as follows: section §2 is devoted to rank 1 hypergeometric motives. Although the theory of rank 1 motives is quite elementary (solutions to the differential equation are algebraic), it gives a flavor of the general theory.

Section §3 starts recalling the definition of the rank n hypergeometric series, the differential equation it satisfies, and some of its well known properties. It includes a detailed description of the construction of the geometric representation (1) extending the monodromy one.

Section §4 contains a discussion of arbitrary rank n hypergeometric motives. Following computations of David Roberts and the third author, analogues of properties (i) to (v) are expected to hold. The section includes a proof of a weaker version of statement (ii) for general rank n hypergeometric motives (based on [6]). We end the section by recalling an algorithm to compute the Hodge numbers of the hypergeometric motive of parameters $\boldsymbol{\alpha}, \boldsymbol{\beta}$ and include different examples.

Section §5 is a short survey on Jacobi motives (following [54] and [56]). It includes results needed to interpret the product/quotient of Gamma values in (3) as a Jacobi motive. Section §6 gives the definition of the finite field analogue $H_q(\boldsymbol{\alpha}, \boldsymbol{\beta}|z)$ of the hypergeometric series ${}_nF_{n-1}$ (as studied in [29] and [34]). Proposition 6.2 summarizes the main properties satisfied by finite hypergeometric sums needed in the present article. There is an alternate p-adic definition of finite hypergeometric sums in terms of the p-adic Gamma function (as defined by Morita in [38]). Theorem 6.17 proves that both definitions coincide (using the Gross-Koblitz formula). The section ends (Theorem 6.22) with a relation between hypergeometric character sums and finite hypergeometric series. The result plays a crucial role while computing the zeta function of the Euler curve.

In Section §7 we compute the zeta function of a general superelliptic curve, relating the number of points of its isotypical components to finite character sums (see Theorem 7.4). In Section §8 we focus on the study of rank 2 hypergeometric motives. Previous results on rank 2 motives (like [4] and [24]) always assumed that $d = 1$. This a priory "innocent" assumption not only excludes many interesting motives, but also has deep consequences since it implies that Euler's curve is absolutely irreducible. For general values of d this is no longer true (as proved in Example 9); Euler's curve can have any number of irreducible components. To circumvent this problem, we start studying the case of irreducible curves, and use them to define a general rank 2 hypergeometric motive in Definition 8.14. In §8.3 we study the possible Hodge numbers of a rank 2 hypergeometric motive, a very reach theory. In Example 13 we compute the Hodge numbers of some curves studied by Shimura (in [50]). The last part of the section is devoted to computing the zeta function of the Euler curve $\mathcal{C}$ (and its factorization according to the action of $\boldsymbol{\mu}_N$). The main result (Theorem 8.15)

gives a precise relation between $H_{\mathfrak{p}}(\boldsymbol{\alpha},\boldsymbol{\beta}|z)$ and the trace of a Frobenius element $\mathrm{Frob}_{\mathfrak{p}}$ acting on $H^1_{\text{ét},\chi_0}(\widehat{\mathcal{C}},F_\lambda)$ (for $\mathfrak{p}$ a prime ideal of F of good reduction for $\mathcal{C}$), where $\widehat{\mathcal{C}}$ denotes the desingularization of the Euler curve. The result is an arithmetic counterpart to (3).

Section §9 is devoted to prove results on the field of definition of the motive (property (i) of the introduction). Our main result reads.

Theorem 9.2. *Let $(a,b),(c,d)$ be generic parameters such that $a+b$ and $c+d$ are integers and* (**Irr**) *holds. Then $\mathcal{H}((a,b),(c,d)|z)$ is defined over $K=F^H$. For $\xi\in K$, the coefficient field of $\mathcal{H}((a,b),(c,d)|\xi)$ is contained in K.*

Its proof is interesting on its own, since it consists on constructing involutions of Euler's curve. The section includes a study of the Galois representation attached to specializations of the hypergeometric motive. We prove irreducibility results (Theorem 9.9) and extension results (Theorem 9.8 and Corollary 9.12).

Section §10 is devoted to the study of hypergeometric motives defined over totally real fields (meaning that $-1\in H$). In this case we can prove a stronger irreducibility result (Theorem 10.1). The last section §11 contains the explicit statement and proof of congruences between hypergeometric motives. Although the proof is elementary, the result has deep implications in Darmon's program (as exploited in [27]). The present article ends with two appendices. The first one studies reduction types of the Euler curve and is used to relate the monodromy of the differential equation with the action of inertia on the hypergeometric motive $\mathcal{H}(\boldsymbol{\alpha},\boldsymbol{\beta}\,|\,z)$. It also includes a proof of (v) (see Theorem A.2 and A.4).

The second appendix addresses the following problem: keeping the previous notation, let $\mathfrak{p}$ be a prime ideal of F not dividing N but (say) dividing the numerator of ξ. Then it is a priory a tame prime for the motive, but it may actually be of good reduction (as follows from (4), since the matrix M_0 might have finite order). In this case the stated formula (proven in Theorem 8.15) does not apply. One can still compute the trace of the Frobenius element geometrically, by computing the stable model of Euler's equation (as explained in the appendix). It would be interesting to have an explicit description for wild primes.

The third author wrote a script to compute $H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|\xi)$ p-adically in terms of the p-adic Gamma function (a different algorithm to compute the L-series of a hypergeometric motive is given in [14] and [15]). The script can be downloaded from the github repository

https://github.com/frvillegas/frvmath.

Example 2. Here is an example of how the code works. Consider the hypergeometric motive $\mathcal{H}_z$ with parameters $\boldsymbol{\alpha}=(1/8,7/8),\boldsymbol{\beta}=(3/8,5/8)$, Tate twisted so that it has weight one (or equivalently so that it is an effective motive as explained in §4.1). A priory $\mathcal{H}_z$ is defined over $\mathbb{Q}(\sqrt{2})$ as explained in Example 14.

First we take $z=9$ and compute the action of Frobenius for the prime $p=7$.

```
? hgm(9,[1/8,7/8],[3/8,5/8],7)
3*7^-1 + 6 + 6*7 + 6*7^2 + 6*7^3 + 6*7^4 + 6*7^5 + 6*7^6 + 6*7^7 + 6*7^8 +
6*7^9 + 6*7^10 + 6*7^11 + 6*7^12 + 6*7^13 + 6*7^14 + 6*7^15 + 6*7^16 + 6*7^17 +
6*7^18 + O(7^19)
? recognizep(hgm(9,[1/8,7/8],[3/8,5/8],7))
-4/7
```

We conclude that the trace of Frobenius on $\mathcal{H}_9$ is -4. To get the full characteristic polynomial $L_7(T)$ of Frobenius we can proceed as follows

```
? hgmfrob(9,[1/8,7/8],[3/8,5/8],7)
(7^-1 + O(7^18))*x^2 + (4*7^-1 + O(7^19))*x + 1
? polrecognizep(hgmfrob(9,[1/8,7/8],[3/8,5/8],7))
```

```
1/7*x^2 + 4/7*x + 1
```

This case is special however since the specialization parameter is a square, which results in the field of definition being $\mathbb{Q}$. The characteristic polynomial of Frobenius at $p = 7$ is then

$$L_7(T) = 7T^2 + 4T + 1.$$

For a further discussion of this example see Example 9.

Example 3. For $z = 3$ on the other hand, we have

```
? L1=hgmfrob(3,[1/8,7/8],[3/8,5/8],7);
? L2=hgmfrob(3,[3/8,5/8],[1/8,7/8],7);
? polrecognizep(L1*L2)
1/49*x^4 + 6/49*x^2 + 1
```

In fact, looking at the polynomials `L1,L2` individually we find that the characteristic polynomial $L_{\mathcal{P}}(T)$ of Frobenius for primes $\mathcal{P}$ of $\mathbb{Q}(\sqrt{2})$ dividing 7 acting on $\mathcal{H}_3$ is

$$L_{\mathcal{P}}(T) = 7T^2 \pm 2\sqrt{2}T + 1.$$

Let χ be the quadratic character of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{2}))$ corresponding to the extension $\mathbb{Q}(\sqrt{\sqrt{2}+2})$ over $\mathbb{Q}(\sqrt{2})$. Computing for more primes, it is not hard to verify that the twist by χ of the Frobenius polynomials of the hypergeometric motive $\mathcal{H}_3$ match those of a Hilbert newform over the real quadratic field $\mathbb{Q}(\sqrt{2})$ of parallel weight 2 and level $3^2 \cdot \sqrt{2}^5$ for all prime ideals of $\mathbb{Q}(\sqrt{2})$ less than 200. This form is not in the LMFDB, but we can nevertheless uniquely identify it using its standardized labeling/ordering of forms. To this end, we compute the trace of the Hecke eigenvalue of each (Galois orbit) eigenform for primes up to norm 31 (sorted as explained in [17]). The form corresponds to the 11-th one in the resulting ordering. Here is the code in magma to compute the space of all relevant Hilbert modular forms.

```
> R<x> := PolynomialRing(IntegerRing());
> F := NumberField(x^2-2);  OF := Integers(F);
> P2:=Factorisation(2*OF)[1][1];
> M := HilbertCuspForms(F, 9*P2^5);
> decomp := NewformDecomposition(NewSubspace(M));
```

It seems plausible that the Faltings-Serre's method can be used to prove modularity of the corresponding motive and hence the equality of Frobenius polynomials for all p.

Remark. One important advantage of working p-adically is that there is typically no need to pass to an algebraic extension to compute the corresponding Gauss sums. For the above example, we would a priory need to compute in the field $\mathbb{Q}(\zeta_8)$ or possibly $\mathbb{Q}(\sqrt{2})$ if set up appropriately. As it stands we simply compute the trace of Frobenius in $\mathbb{Z}_p$ and then recognize it as an (algebraic) integer.

We finish this introduction with a few challenges that we expect to address in the near future:

- Can the inertial information be used to prove large image results for hypergeometric motives? (We do know (see [7]) that the Zariski closure of the geometric monodromy is big.)
- Extend the analogue of property (iii) to primes of the field of definition of $\mathcal{H}((a,b),(c,d)|z)$.
- Prove properties (i) to (v) for rank n hypergeometric motives.

Acknowledgments: We would like to thank David Roberts for many useful conversations.

2. Rank one hypergeometric motives

We start by considering a baby example of the theory, that of rank 1 motives. The definitions used in the present section will be explained in detail in subsequent sections for the general case of rank n hypergeometric motives.

Let α be a rational number of denominator $N > 1$. Consider the differential equation (see (8) for its general definition)

$$D(\alpha, 1) := z\frac{d}{dz} - z(z\frac{d}{dz} + \alpha) = z(1-z)\frac{d}{dz} - z\alpha = 0. \tag{4}$$

Its one-dimensional space of holomorphic local solutions on a disk centered at a base point $z_0 \neq 0, 1$ in the unit disk is spanned by the algebraic function $H(z) = (1-z)^{-\alpha}$ (for any fixed branch determination). Analytic continuation along closed loops based at z_0 determines the *monodromy representation*

$$\rho : \pi_1(\mathbb{C} \setminus \{1\}, z_0) \to \mathbb{C}^\times.$$

It is easy to verify that if γ is a simple loop around 1 (in counterclockwise direction) then $\rho(\gamma) = \exp\left(\frac{2\pi i}{N}\right)$. Let $F = \mathbb{Q}(\zeta_N)$. Algebraically, ρ corresponds to the Galois representation

$$\widetilde{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}(z)}/F(z)) \to \mathbb{C}^\times,$$

given by

$$\widetilde{\rho}(\sigma) = \frac{\sigma(\sqrt[N]{1-z})}{\sqrt[N]{1-z}}.$$

The representation is ramified only at 1 and ∞. For $\xi \in \mathbb{Q}$, $\xi \neq 1$, we can consider the specialization of $\widetilde{\rho}$ at ξ, corresponding to the extension of number fields $F(\sqrt[N]{1-\xi})/F$.

Let $\mathcal{O}_F$ denote the ring of integers of F, let $\mathfrak{q}$ a prime ideal of $\mathcal{O}_F$ unramified in the extension $F(\sqrt[N]{1-\xi})/F$ and let $\mathbb{F}_q$ denote the finite field $\mathcal{O}_F/\mathfrak{q}$ of $q = \mathcal{N}\mathfrak{q}$ elements. Since $N \mid q-1$, $\mathbb{F}_q$ contains the N-th roots of unity. Fix ϖ a generator of the character group of $\mathbb{F}_q^\times$ and ψ a non-trivial additive character of $\mathbb{F}_q$. For $\xi \in \mathbb{F}_q$, define (see §6)

$$H_q(\alpha, 1|\xi) := \frac{1}{1-q}\sum_{\varphi} \frac{\mathcal{J}(\alpha\varphi, \varphi)}{\mathcal{J}(\alpha, 1)}\varphi(\xi), \tag{5}$$

where the sum runs over characters φ of $\mathbb{F}_q^\times$,

$$\mathcal{J}(\alpha\varphi, \varphi) := g(\psi, \varpi^{(q-1)\alpha}\varphi)g(\psi^{-1}, \overline{\varphi}),$$

and $g(\psi, \varphi)$ denotes the standard Gauss sum (as recalled in (30)). Denote by ε the character $\varpi^{(q-1)\alpha}$ (a character of order N). Since $g(\psi^a, \varphi) = \overline{\varphi}(a)g(\psi, \varphi)$, we have

$$\frac{\mathcal{J}(\varepsilon\varphi, \varphi)}{\mathcal{J}(\varepsilon, 1)} = -\varphi(-1)\frac{g(\psi, \varepsilon\varphi)g(\psi, \overline{\varphi})}{g(\psi, \varepsilon)} = -\varphi(-1)J(\varepsilon\varphi, \overline{\varphi}),$$

where for φ, η characters, $J(\varphi, \eta)$ denotes the standard Jacobi sum (whose definition is recalled in Definition 6.20). Then (5) becomes

$$\frac{1}{q-1}\sum_{\varphi}\sum_{t\neq 1}\varphi(-\xi t)\varepsilon(t)\overline{\varphi}(1-t) = \frac{1}{q-1}\sum_{t\neq 1}\varepsilon(t)\sum_{\varphi}\varphi\left(\frac{-\xi t}{1-t}\right).$$

The last sum equals zero unless $-\xi t = 1 - t$ and $q-1$ in that case. Therefore

$$H_q(\alpha, 1|\xi) = \varepsilon(1-\xi)^{-1},$$

showing that our finite hypergeometric series $H_q(\alpha, 1|\xi)$ at a prime ideal $\mathfrak{q}$ matches the Dirichlet character of the extension $F(\sqrt[N]{1-\xi})/F$.

Consider now the same problem for parameters $1, -\alpha$. The differential equation becomes

$$D(1,-\alpha) = z(1-z)\frac{d}{dz} - \alpha = 0.$$

Its space of solutions is spanned by the function $G(z) = \left(\frac{z}{1-z}\right)^{\alpha}$. Adjoining it yields the abelian Galois extension $F\left(\sqrt[N]{\frac{z}{1-z}}\right)/F(z)$, unramified outside $\{0,1\}$. An elementary computation proves that both the differential equation and the field extension are related to the previous case by the change of variables $z \to 1/z$. This is consistent with the finite hypergeometric series behavior

$$H_q(1,-\alpha|\xi) = \frac{1}{1-q}\sum_{\varphi}\frac{J(\varphi,\varepsilon^{-1}\varphi)}{J(1,\varepsilon^{-1})}\varphi(\xi) = \frac{1}{1-q}\sum_{\varphi}\frac{J(\varepsilon\varphi^{-1},\varphi^{-1})}{J(\varepsilon,1)}\varphi(\xi) = H_q(\alpha,1|\xi^{-1}).$$

3. Hypergeometric Series and Monodromy Representation

For $\alpha \in \mathbb{C}$ and k a positive integer, let $(\alpha)_k$ denote the Pochammer symbol, defined by

$$(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1) = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}.$$

Definition 3.1. *Let $\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_{n-1}$ be complex numbers. The hypergeometric series with parameters $\boldsymbol{\alpha} := (\alpha_1,\ldots,\alpha_n), \boldsymbol{\beta} := (\beta_1,\ldots,\beta_{n-1},1)$ is defined as the convergent power series*

$$ {}_nF_{n-1}(\boldsymbol{\alpha},\boldsymbol{\beta}|z) = \sum_{k=0}^{\infty}\frac{(\alpha_1)_k\cdots(\alpha_n)_k z^k}{(\beta_1)_k\cdots(\beta_{n-1})_k k!}, \tag{6}$$

for $z \in \mathbb{C}$ with $|z|<1$.

We will briefly recall some the properties of this classical function and refer the reader to [7] and the references therein for details. For $\mathrm{Re}(\beta_i) > \mathrm{Re}(\alpha_i) > 0$, for $i = 1,\ldots n-1$, we have the following integral presentation (see formula (2.2.2) of [3])

$$ {}_nF_{n-1}(\boldsymbol{\alpha};\boldsymbol{\beta}|z) = \prod_{i=1}^{n-1}\frac{\Gamma(\beta_i)}{\Gamma(\alpha_i)\Gamma(\beta_i-\alpha_i)}\int_0^1\cdots\int_0^1\frac{\prod_{i=1}^{n-1}(x_i^{\alpha_i-1}(1-x_i)^{\beta_i-\alpha_i-1}dx_i)}{(1-zx_1\cdots x_{n-1})^{\alpha_n}}, \quad |z|<1. \tag{7}$$

As it is well-known the hypergeometric series ${}_nF_{n-1}$ satisfies the differential equation

$$D(\boldsymbol{\alpha};\boldsymbol{\beta})u = 0, \qquad D(\boldsymbol{\alpha};\boldsymbol{\beta}) := (\theta+\beta_1-1)\cdots(\theta+\beta_n-1) - z(\theta+\alpha_1)\cdots(\theta+\alpha_n), \tag{8}$$

where $\theta = z\frac{d}{dz}$. This equation has regular singularities at the points $z = 0,1,\infty$.

Remark 3.2. Note that in the definition of the series we assume $\beta_n = 1$. Since the order of the β_i's (or, for that matter, the order of the α_i's) is irrelevant we could have assumed any given $\beta_i = 1$ (adjusting the range for the parameters in (7)). We will now, in any case, relax this condition and let all β_i be arbitrary, in particular when considering the differential equation $D(\boldsymbol{\alpha};\boldsymbol{\beta})$.

We have ([7, §2]).

Proposition 3.3. *If the number $\beta_1,\ldots,\beta_n$ are distinct modulo $\mathbb{Z}$, n independent solutions of (8) are given by*

$$u_i(z) := z^{1-\beta_i}\,{}_nF_{n-1}(\boldsymbol{\alpha}_i,\boldsymbol{\beta}_i|z), \qquad i = 1,\ldots,n, \tag{9}$$

where

$$\boldsymbol{\alpha}_i := (1+\alpha_1-\beta_i,\ldots,1+\alpha_n-\beta_i), \qquad \boldsymbol{\beta}_i := (1+\beta_1-\beta_i,\ldots,1+\beta_n-\beta_i).$$

Lemma 3.4. *Let $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \in \mathbb{C}$. Then the change of variables $z \to u = 1/z$ sends solutions of the equation $D(\boldsymbol{\alpha}; \boldsymbol{\beta})$ to solutions of the equation $D(\boldsymbol{\beta}', \boldsymbol{\alpha}')$, where*

$$\boldsymbol{\alpha}' := (-\alpha_1 + 1, \dots, -\alpha_n + 1), \qquad \boldsymbol{\beta}' := (-\beta_1 + 1, \dots, -\beta_n + 1).$$

Proof. Set $u = 1/z$, so $z\frac{d}{dz} = -u\frac{d}{du}$. In the variable u, equation (8) becomes

$$\frac{(-1)^{n+1}}{u}\left((\theta - \alpha_1)\cdots(\theta - \alpha_n) - u(\theta - \beta_1 + 1)\cdots(\theta - \beta_n + 1)\right).$$

□

We are interested in the case when the parameters α_i, β_i are rational numbers, where we can expect a geometric origin for the differential equation. We restrict to this case from now on. Set N to be their least common denominator and define the quantities:

$$A_i := N(\beta_n - \alpha_i), \quad B_i := N(\alpha_i - \beta_i), \text{ for } i = 1, \dots, n-1, \quad A_n := N(\alpha_n - \beta_n), \quad B_n := N\beta_n. \tag{10}$$

Then u_n is a period of the middle cohomology of the twisted Euler's variety

$$V_n : \quad y^N = z^{B_n} \prod_{i=1}^{n-1} x_i^{A_i}(1 - x_i)^{B_i}(1 - zx_1 \cdots x_{n-1})^{A_n}. \tag{11}$$

Definition 3.5. *Let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$ and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_n)$ be n-tuples of complex numbers. We say that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are* generic *if the sets $\{\exp(2\pi i\alpha_j)\}, \{\exp(2\pi i\beta_k)\}$ are disjoint; equivalently, $\alpha_i \neq \beta_j \bmod \mathbb{Z}$ for $i, j = 1, \dots, n$. We will also say that such a pair determines a* hypergeometric data*.*

3.1. **The monodromy representation of the differential equation.** Keep the notation of the previous section. For X a topological space, and $x_0 \in X$, let $\pi_1(X, x_0)$ denote the fundamental group of X based at the point x_0. Recall the following definition.

Definition 3.6. *Let G be a group and let $\rho_i : G \to \mathrm{GL}_n(\mathbb{C})$, $i = 1, 2$, be two representations of the group G. The representations are isomorphic if there exists $M \in \mathrm{GL}_n(\mathbb{C})$ such that $\rho_1(g) = M\rho_2(g)M^{-1}$ for all $g \in G$.*

Definition 3.7. *Let z_0 be a complex number different from $0, 1$. The differential equation $D(\boldsymbol{\alpha}; \boldsymbol{\beta})$ has attached, up to isomorphism, a natural* monodromy representation

$$\rho : \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}, z_0) \to \mathrm{GL}_n(\mathbb{C}) \tag{12}$$

obtained as follows: pick a basis $\{f_1, \dots, f_n\}$ of n independent solutions to $D(\boldsymbol{\alpha}; \boldsymbol{\beta})$ around the point z_0. Given a loop $\gamma \in \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}, z_0)$ extend analytically each solution f_i along γ, to obtain a new basis $\{\tilde{f}_1, \dots, \tilde{f}_n\}$ of linearly independent solutions at z_0. The matrix $\rho(\gamma)$ is defined as the change of basis matrix.

Remark 3.8. The generic case is *rigid*: the representations ρ_1 and ρ_2 are isomorphic if and only if each monodromy matrix $\rho_1(\gamma_i)$ is individually conjugate to $\rho_2(\gamma_i)$ (for $i = 0, 1, \infty$) (see [7, Thm 3.5]).

Proposition 3.9. *The monodromy representation attached to the parameters $\boldsymbol{\alpha}, \boldsymbol{\beta}$ is irreducible if and only if the parameters are generic.*

Proof. See Propositions 2.7 and 3.3 of [7]. □

Proposition 3.10. *If the parameters $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ are generic then the monodromy representation for the parameters $\boldsymbol{\alpha}, \boldsymbol{\beta}$ is isomorphic to the one corresponding to the parameters $\boldsymbol{\alpha} + \boldsymbol{n}$, $\boldsymbol{\beta} + \boldsymbol{m}$ for any $\boldsymbol{n}, \boldsymbol{m} \in \mathbb{Z}^n$.*

Proof. See Corollary 2.6 of [7]. □

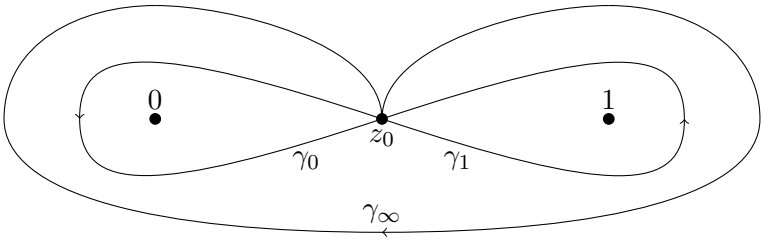


FIGURE 1. Fundamental group of $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}, z_0)$

Let $\gamma_0, \gamma_1, \gamma_\infty$ denote paths going through $0, 1, \infty$ respectively in counterclockwise direction as in Figure 1. It follows from Van Kampen's theorem that

$$\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}, z_0) = \langle \gamma_0, \gamma_1, \gamma_\infty \ : \ \gamma_0\gamma_1\gamma_\infty = 1 \rangle.$$

In particular, the monodromy representation is determined by the matrices

$$M_0 := \rho(\gamma_0), \qquad M_1 := \rho(\gamma_1), \qquad M_\infty := \rho(\gamma_\infty), \tag{13}$$

corresponding to the local monodromy matrices around $0, 1$ and ∞ respectively.

Definition 3.11. *A* pseudo-reflection *is a matrix M satisfying that $M - 1$ has rank one. Its determinant is called the special eigenvalue of M.*

Proposition 3.12. *Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ be generic. Let $a_j = \exp(2\pi i \alpha_j)$ and $b_j = \exp(2\pi i \beta_j)$ for $j = 1, \ldots, n$. Define*

$$q_\infty = \prod_{j=1}^{n} (t - a_j), \qquad q_0 = \prod_{j=1}^{n} (t - b_j), \tag{14}$$

Then M_∞ and M_0 are conjugate to the companion matrix of q_∞ and q_0 and M_1 is a pseudo-reflection with special eigenvalue

$$\mu := \exp\left(2\pi i \sum_{j=1}^{n} (\beta_j - \alpha_j)\right)$$

Proof. See Proposition 2.10 and Proposition 3.2 of [7]. □

Let $\boldsymbol{\alpha}, \boldsymbol{\beta}$ be generic rational parameters and let N be their least common denominator. Then a construction due to Levelt (see Theorem 3.5 of [7]) gives explicit matrices M_0, M_1, M_∞ for the monodromy representation, with image in $\mathrm{GL}_N(F)$, where $F = \mathbb{Q}(\zeta_N)$.

3.2. **Geometric representations.** A good reference for the statements of the present section is §6.3 of [48]. Let $\mathfrak{R}$ be an algebraic closed field, and X a projective smooth curve of genus g over $\mathfrak{R}$ (in our case $X = \mathbb{P}^1$, so $g = 0$). Let $P_1, \ldots, P_k$ be distinct points in $X(\mathfrak{R})$. Let $\overline{\mathfrak{R}(X)}$ be an algebraic closure of the function field $\mathfrak{R}(X)$ of X and let $\Omega \subset \overline{\mathfrak{R}(X)}$ be the maximal extension of $\mathfrak{R}(X)$ unramified outside the points $P_1, \ldots, P_k$. The algebraic fundamental group is defined by

$$\pi_1^{\mathrm{alg}}(X \setminus \{P_1, \ldots, P_k\}) := \mathrm{Gal}(\Omega/\mathfrak{R}(X)). \tag{15}$$

Definition 3.13. *Let G be a discrete group. The* profinite completion *of G is the topological group*

$$\widehat{G} := \varprojlim G/H,$$

where the inverse limit is with respect to finite index normal subgroups H of G.

Given a discrete group G, we denote by $\widehat{G}$ its profinite completion.

Definition 3.14. *Let g, k be positive integers. The group $\pi_1(g,k)$ is defined as the group with generators and relations*

$$\pi_1(g,k) = \langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_k \ : \ a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} c_1 \cdots c_k = 1 \rangle. \tag{16}$$

Theorem 3.15. *The algebraic fundamental group $\pi_1^{alg}(X \setminus \{P_1, \ldots, P_k\})$ is isomorphic to $\widehat{\pi_1(g,k)}$, the profinite completion of the group $\pi_1(g,k)$.*

Proof. See [48, Theorem 6.3.1]. This is a particular case of the well known general result: the algebraic fundamental group of a complex variety is isomorphic to the profinite completion of the topological fundamental group. □

Remark 3.16. As noted in [48], the canonical map $\pi_1(g,k) \to \widehat{\pi_1(g,k)}$ is in fact injective.

Let F be a number field, let $\mathcal{O}$ be its ring of integers and let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}$. We denote by $\mathcal{O}_\mathfrak{p}$ the completion of $\mathcal{O}$ at $\mathfrak{p}$ (and by $F_\mathfrak{p}$ its field of fractions).

Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be generic rational parameters. By Levelt's theorem the hypergeometric monodromy representation can be chosen to be defined over $\mathcal{O}$. Fix such a choice (unique up to isomorphism over F)

$$\rho : \quad \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, z_0) \to \mathrm{GL}_n(\mathcal{O}).$$

Corollary 3.17. *Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be generic rational parameters. The representation*

$$\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, z_0) \xrightarrow{\rho} \mathrm{GL}_n(\mathcal{O}) \xrightarrow{\iota} \prod_\mathfrak{p} \mathrm{GL}_n(\mathcal{O}_\mathfrak{p}),$$

Extends to a continuous representation (unique up to isomorphism)

$$\widehat{\rho} : \mathrm{Gal}(\overline{\mathfrak{R}(z)}/\mathfrak{R}(z)) \to \prod_\mathfrak{p} \mathrm{GL}_n(\mathcal{O}_\mathfrak{p}), \tag{17}$$

unramified outside $\{0, 1, \infty\}$.

Proof. By Van Kampen's theorem $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, z_0) \simeq \pi_1(0,3)$, so we can think of ρ as a representation $\rho : \pi_1(0,3) \to \mathrm{GL}_n(\mathcal{O})$. Consider the following natural morphisms

$$\pi_1(0,3) \xrightarrow{\rho} \mathrm{GL}_n(\mathcal{O}) \xrightarrow{\iota} \prod_\mathfrak{p} \mathrm{GL}_n(\mathcal{O}_\mathfrak{p}).$$

Since $\mathrm{GL}_n(\mathcal{O}_\mathfrak{p})$ is profinite, by the universal property of the profinite completion (see [42, Lemma 3.2.1]) we get a morphism

$$\widehat{\rho} : \widehat{\pi_1(0,3)} \to \prod_\mathfrak{p} \mathrm{GL}_n(\mathcal{O}_\mathfrak{p}).$$

Theorem 3.15 (with $X = \mathbb{P}^1$ and $\{P_1, P_2, P_3\} = \{0, 1, \infty\}$) gives an isomorphism $\widehat{\pi_1(0,3)} \simeq \mathrm{Gal}(\Omega/\mathfrak{R}(z))$, the later being a quotient of $\mathrm{Gal}(\overline{\mathfrak{R}(z)}/\mathfrak{R}(z))$ (corresponding to extensions unramified outside $\{0, 1, \infty\}$). This proves the existence statement. Uniqueness follows from the fact that any discrete group G is dense in its profinite completion $\widehat{G}$. □

Let $\rho_\mathfrak{p}$ be the map obtained by composing $\widehat{\rho}$ with the projection to $\mathrm{GL}_n(\mathcal{O}_\mathfrak{p})$. It follows from Corollary 3.17 that for rational generic parameters, the differential equation satisfied by the hypergeometric series induces a compatible family of Galois representations

$$\rho_\mathfrak{p} : \mathrm{Gal}(\overline{\mathbb{Q}(z)}/\overline{\mathbb{Q}}(z)) \to \mathrm{GL}_N(F_\mathfrak{p}). \tag{18}$$

4. Generalities on hypergeometric motives

We refer the reader to the survey [44] for an introduction to the subject (see also Katz's article [35]). Let $\boldsymbol{\alpha}, \boldsymbol{\beta}$ be vectors of generic rational numbers (as in Definition 3.5), and let N be their least common denominator and let r denote the number integral coordinates of $\boldsymbol{\alpha}, \boldsymbol{\beta}$. Keeping the previous notation, let $F = \mathbb{Q}(\zeta_N)$. Then for $\xi \in F$, $\xi \neq 0, 1$, there should exist a motive $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}|\xi)$, a *hypergeometric motive or HGM*, attached to the parameters $\boldsymbol{\alpha}, \boldsymbol{\beta}$ satisfying the following properties

(i) It is a pure motive of degree n and weight $r-1$ defined over F.
(ii) Its ℓ-ádic étale realization is related to the monodromy representation.
(iii) The primes of F of bad reduction belong to $S_{\mathrm{pt}} \cup S_{\mathrm{pw}}$ (*potentially tame* and *potentially wild* primes respectively), where

$$S_{\mathrm{pw}} := \{\mathfrak{p} \mid v_{\mathfrak{p}}(N) > 0\}, \qquad S_{\mathrm{pt}} := (S_0 \cup S_1 \cup S_\infty) \setminus S_{\mathrm{pw}},$$

and

$$S_0 := \{\mathfrak{p} \mid v_{\mathfrak{p}}(\xi) > 0\}, \qquad S_1 := \{\mathfrak{p} \mid v_{\mathfrak{p}}(\xi - 1) > 0\}, \qquad S_\infty := \{\mathfrak{p} \mid v_{\mathfrak{p}}(\xi) < 0\}.$$

We let S_g be the set of primes not in $S_{\mathrm{pt}} \cup S_{\mathrm{pw}}$.
(iv) For a prime $\mathfrak{p} \in S_g$ the trace of powers of the Frobenius automorphism at $\mathfrak{p}$ acting on $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}|\xi)$ are given by an explicit finite hypergeometric sum.

Remark 4.1. As is well-known (iv) yields a way to compute the characteristic polynomial of the Frobenius automorphism at $\mathfrak{p}$ acting on $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}|\xi)$ by means of Newton's formulas.

We present a more precise statement in Conjecture 4.5. We use the term motive to refer to a pure motive (as in §4 of [2]), i.e. a pair (X, p) where X is a projective variety and p is a correspondence on X which is an idempotent. We say that the motive has a *realization* over K if both the variety X and the correspondence p are defined over K.

Remark 4.2. The existence of a motive over a number field for general rigid local systems was proved by Katz in [35]. See §6 of [33] for a short exposition of the motivic interpretation.

Among the extensive literature on topics related to HGMs we may highlight the following as most relevant for our purposes: for *rational* HGMs (see Definition 4.8) [8] and [44] and for the specific case of rank $n = 2$ with $\beta_2 = 1$ [4] and [24].

Let H be the subgroup of $\mathrm{Gal}(F/\mathbb{Q})$ that fixes the set $\exp(2\pi i\boldsymbol{\alpha})$ and $\exp(2\pi i\boldsymbol{\beta})$. Via the natural identification of $\mathrm{Gal}(F/\mathbb{Q})$ with $(\mathbb{Z}/N)^\times$, the group H can be defined as the set of elements $j \in (\mathbb{Z}/N)^\times$ satisfying

$$j \cdot \{\alpha_1, \ldots, \alpha_n\} \equiv \{\alpha_1, \ldots, \alpha_n\} \bmod \mathbb{Z}, \quad \text{and} \quad j \cdot \{\beta_1, \ldots, \beta_n\} \equiv \{\beta_1, \ldots, \beta_n\} \bmod \mathbb{Z}. \tag{19}$$

Definition 4.3. *We define the* base field $K := \mathbb{Q}(\zeta_N)^H \subseteq \mathbb{Q}(\zeta_N)$ *of the hypergeometric data* $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ *as the field fixed by* H.

Definition 4.4. *Given a motive we will call the field generated by the traces of all Frobenius elements the* coefficient field *of the motive.*

We have (see Proposition 6.2)

$$\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}|\xi)^\sigma = \mathcal{H}(j\boldsymbol{\alpha}, j\boldsymbol{\beta}|\xi^\sigma), \tag{20}$$

where $1 \leq j \leq N-1$ coprime to N corresponds to σ. Hence it is natural to expect (and all numerical evidence supports) the following

Conjecture 4.5. *Let K be the base field of the hypergeometric data $\boldsymbol{\alpha}, \boldsymbol{\beta}$. Then*

(1) The motive $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}|z)$ has a realization $\mathcal{H}(z)$ over $K(z)$.

(2) *(Compatibility) The representation of the motive restricted to* $\mathrm{Gal}(\overline{K(z)}/\overline{K}(z))$ *is isomorphic to the geometric Galois representation* (18).
(3) *Let* L *be a finite extension of* K*. The specialization* $\mathcal{H}(\xi)$ *for generic* $\xi \in L$ *has coefficient field* K*.*
(4) *The primes* $\mathfrak{p} \in S_g$ *of* L *are of good reduction and the trace of Frobenius on* $\mathcal{H}(\xi)$ *is given by the finite hypergeometric sum* $H_{\mathfrak{p}}(\boldsymbol{\alpha}, \boldsymbol{\beta} \,|\, \xi)$*.*

Remark 4.6. For particular values of the parameter z, we might by able to descend to a motive over a proper subfield $L \subseteq K$. For example the motive of Example 14 has base field $\mathbb{Q}(\sqrt{2})$, but evaluated at squares is defined over $\mathbb{Q}$ and has coefficient field $\mathbb{Q}$.

Definition 4.7. *Let* $\rho \in \mathbb{Q}$*. The* hypergeometric twist by ρ *of the hypergeometric motive* $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ *is the motive with hypergeometric data* $(\rho\boldsymbol{\alpha}, \rho\boldsymbol{\beta})$*, where* $\rho(\alpha_1, \ldots, \alpha_n) = (\rho + \alpha_1, \ldots, \rho + \alpha_n)$*.*

It follows from Proposition 3.12 that the hypergeometric twist by ρ multiplies the local monodromies at 0 and ∞ of $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}|z)$ by the scalars λ and λ^{-1} respectively, where $\lambda = \exp(-2\pi i\rho)$. In particular, hypergeometric twisting does not change the projective monodromy representation. As a consequence, for example, the corresponding hypergeometric representations are simultaneously finite or infinite.

Example 4. A generic hypergeometric twist of a HGM typically increases the degree of its base field. But in some special cases it does not. For example, the following parameter set

$$\boldsymbol{\alpha} = (1/24, 11/24, 17/24, 19/24), \qquad \boldsymbol{\beta} = (1/4, 1/2, 3/4, 1)$$

has base field $\mathbb{Q}(\sqrt{-8})$. However, its twist by $-1/8$

$$\boldsymbol{\alpha}' = (1/3, 7/12, 2/3, 11/12) \qquad \boldsymbol{\beta}' = (1/8, 3/8, 5/8, 7/8)$$

has base field $\mathbb{Q}(i)$. The HGM of parameters $(\boldsymbol{\alpha}', \boldsymbol{\beta}')$ is number 38 in the Beukers-Heckman [7] list of algebraic hypergeometric functions; it has weight zero and corresponds to an Artin motive. The geometric monodromy group for $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}|z)$ is of order 2304 and is a central extension by C_2 of $S_4 \wr S_2$. However, the motive of parameters $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ itself does not actually appear in the Beukers-Heckman list since they mod out by twisting. There are only three cases of this kind in the Beukers-Heckman list for rank $n > 2$, all with base field $\mathbb{Q}(\sqrt{-8})$. Namely,

$$\begin{array}{cc} (1/24, 11/24, 17/24, 19/24) & (1/4, 1/2, 3/4, 1) \\ (1/24, 11/24, 17/24, 19/24) & (1/4, 5/8, 3/4, 7/8) \\ (1/8, 1/4, 3/8, 3/4) & (1/6, 1/3, 2/3, 5/6) \end{array}$$

Definition 4.8. *A pair* $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ *of vectors is* rational *if its base field is* $\mathbb{Q}$ *(equivalently,* $H = \mathrm{Gal}(F/\mathbb{Q})$*).*

Suppose for the rest of the section that the Conjecture 4.5 holds for the motive $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}|z)$. Then there exists a number field L with ring of integers $\mathcal{O}_L$ and a compatible family of Galois representations

$$\rho_{\mathfrak{p}} : \mathrm{Gal}(\overline{K(z)}/K(z)) \to \mathrm{GL}_n(L_{\mathfrak{p}}),$$

for primes $\mathfrak{p}$ of L extending the geometric representation (18). The potentially wild primes should be the ones dividing the denominator of $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$. We can prove a weaker version. Define

$$S(n, L) := \gcd\{\#\mathrm{GL}_n(\mathcal{O}_L/\mathfrak{p}) : \mathfrak{p} \text{ is a prime ideal of } \mathcal{O}_L\}. \tag{21}$$

The following result follows the lines of [18, Lemma 1.2], and its proof resembles that of [49, Corollary 2].

Theorem 4.9. *Keep the previous notation and assumptions and assume Conjecture 4.5 holds. Let $\xi \in \mathbb{P}^1(K) \setminus \{0, 1, \infty\}$ and let $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}|\xi)$ be the specialization at ξ. Let p, q, r denote the order of the monodromy representation at the points $0, 1, \infty$ respectively (possibly equal to ∞). Let $\mathfrak{n}$ be a prime ideal in $\mathcal{O}_K$ not dividing $S(n, L)$ and satisfying one of the following properties:*

- *$\mathfrak{n} \mid \xi$, p is finite and $p \mid v_{\mathfrak{n}}(\xi)$,*
- *$\mathfrak{n} \mid \xi - 1$, q is finite and $q \mid v_{\mathfrak{n}}(\xi - 1)$,*
- *$v_{\mathfrak{n}}(\xi) < 0$, r is finite and $r \mid v_{\mathfrak{n}}(\xi)$.*
- *$v_{\mathfrak{n}}(\xi) = v_{\mathfrak{n}}(\xi - 1) = 0$.*

Then the compatible family of Galois representations attached to $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}|\xi)$ is unramified at $\mathfrak{n}$.

Proof. Let $\mathfrak{n}$ be a prime ideal of K not dividing $S(n, L)$ and satisfying one of the required hypothesis. Then there exists a prime ideal $\mathfrak{p}$ of L such that $\mathfrak{n} \nmid \#\mathrm{GL}_n(\mathcal{O}_L/\mathfrak{p})$ (in particular, $\mathfrak{n}$ does not divide the norm of $\mathfrak{p}$). Let

$$\rho_{\mathfrak{p}} : \mathrm{Gal}(\overline{K}(z)/K(z)) \to \mathrm{GL}_n(L_{\mathfrak{p}})$$

be the $\mathfrak{p}$-adic representation attached to the motive $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}|z)$. Consider first the *geometric* part of the representation, namely its restriction to $G^{\mathrm{geom}} := \mathrm{Gal}(\overline{\mathbb{Q}(z)}/\overline{\mathbb{Q}}(z))$.

Part (2) of Conjecture 4.5 implies that the image of inertia at $z = 0$, $z = 1$ and $z = \infty$ under (the restriction to G^{geom} of) $\rho_{\mathfrak{p}}$ has order p, q and r respectively. Since the extension $\overline{\mathbb{Q}}(z)/K(z)$ is unramified at $z = 0$, $z = 1$ and $z = \infty$, the image of the inertia subgroup at these three points under $\rho_{\mathfrak{p}}$ also has order p, q and r respectively.

After choosing a lattice fixed by $\rho_{\mathfrak{p}}$, we can assume that our representation actually takes values in $\mathrm{GL}_n(\mathcal{O}_{\mathfrak{p}})$ (the completion of $\mathcal{O}_K$ at $\mathfrak{p}$). Let m be a positive integer, and let $\overline{\rho_{\mathfrak{p},m}}$ be the reduction of $\rho_{\mathfrak{p}}$ modulo $\mathfrak{p}^m$. The kernel of $\overline{\rho_{\mathfrak{p},m}}$ corresponds to a curve $X_{\mathfrak{p},m}$ which is a finite cover of $\mathbb{P}^1$ unramified outside $\{0, 1, \infty\}$. Then Theorem 1.2 of [6] implies that if one of our hypothesis is satisfied and if $\mathfrak{n}$ does not divide the order of the image of $\overline{\rho_{\mathfrak{p},m}}$, then the image of $I_{\mathfrak{n}}$ under the specialization map is trivial. But the order of the image of $\overline{\rho_{\mathfrak{p},m}}$ divides the order of $\mathrm{GL}_n(\mathcal{O}_L/\mathfrak{p}^m)$, whose order is supported at the same primes as $\mathrm{GL}_n(\mathcal{O}_L/\mathfrak{p})$. Then, if $\mathfrak{n} \nmid \#\mathrm{GL}_n(\mathcal{O}_L/\mathfrak{p})$, $\overline{\rho_{\mathfrak{p},m}}(I_{\mathfrak{n}}) = 1$ for all m, hence $\rho_{\mathfrak{p}}(I_{\mathfrak{n}}) = 1$, and the family is unramified at $\mathfrak{n}$. □

Suppose that the motive $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta} \mid z)$ has rank n and is rational. Then any potentially wild prime p satisfies $p - 1 \leq n$. The reason is that if a prime p divides the denominator of $\boldsymbol{\alpha}$ (respectively the denominator of $\boldsymbol{\beta}$), all values $\frac{i}{p}$ for $i = 1, \ldots, p - 1$ are also coordinates of $\boldsymbol{\alpha}$ (respectively $\boldsymbol{\beta}$).

Lemma 4.10. *If $L = K = \mathbb{Q}$ and $n > 1$, then $p \mid S(n, \mathbb{Q})$ if and only if $p - 1 \leq n$.*

Proof. Let p be a prime number such that $p - 1 = \phi(p) \leq n$. Then there is an injective map $\psi : \mathbb{Z}[\zeta_p] \to \mathrm{GL}_n(\mathbb{Z})$; for example, let M be the $n \times n$ matrix made up of two blocks on the diagonal (and zero elsewhere). The first block (of size $(p - 1) \times (p - 1)$) being the companion matrix of $(x^p - 1)/(x - 1)$ and the second one being the identity. Then $\mathbb{Z}[\zeta_p] \simeq \mathbb{Z}[M] \subset \mathrm{GL}_n(\mathbb{Z})$.

Since $n \geq 2$, it is always the case that $p \mid |\mathrm{GL}_n(\mathbb{Z}/p)|$. Let q be a rational prime number different from p. Since q does not ramify in $\mathbb{Z}[\zeta_p]/\mathbb{Z}$, the group $\mathrm{GL}_n(\mathbb{Z}/q)$ contains an element of order p (the image under the reduction map of $\psi(\zeta_p)$), so $p \mid \#\mathrm{GL}_n(\mathbb{Z}/q)$ hence $p \mid S(n, \mathbb{Q})$.

Reciprocally, let $1 \leq x \leq p-1$ be any integer and let r be its order in $\mathbb{F}_p^\times$. By Dirichlet's theorem on arithmetic progressions, there exists a rational prime q congruent to x modulo p. By hypothesis, $p \mid \#\mathrm{GL}_n(\mathbb{Z}/q) = q^*(q - 1)(q^2 - 1) \cdots (q^n - 1)$, so $x^i \equiv q^i \equiv 1 \pmod{p}$ for some $i \leq n$, i.e. any element modulo p has multiplicative order at most n, hence $p - 1 \leq n$. □

Corollary 4.11. *If $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta} \mid z)$ is a rational motive of rank n and $p \in S_{\mathrm{pw}}$ then $p \mid S(n, \mathbb{Q})$. Conversely, if $p \mid S(n, \mathbb{Q})$ then there exist a rational $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta} \mid z)$ with $p \in S_{\mathrm{pw}}$.*

Proof. The first statement follows from the lemma, since if $p \in S_{\mathrm{pw}}$, $p-1 \leq n$. To prove the second statement, suppose that p is a prime number with $p-1 \leq n$. Define $\boldsymbol{\alpha} = (\frac{1}{p}, \ldots, \frac{p-1}{p}, \frac{1}{2}, \ldots, \frac{1}{2})$ and $\boldsymbol{\beta} = (1, \ldots, 1)$. Then $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ are generic, the motive $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta} \,|\, z)$ is rational and $p \in S_{\mathrm{pw}}$. □

Remark 4.12. As described in (iii), the set of potentially wild primes consists of those primes of K dividing the least common denominator of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Theorem 4.9 should hold for all prime ideals that are not potentially wild, but we do not know how to prove this stronger statement when the rank is larger than 2. The rank 2 case is proved in Appendix A (see Theorems A.2 and A.4).

4.1. **Hodge numbers and normalization.** The Hodge numbers of a rational HGM can be computed using a formula conjectured by Corti and Golyshev (proved in general in [13, 23, 53]). The formula can be applied to arbitrary hypergeometric parameters by means of the *zig-zag diagram* (see §5 of [44]), giving the dimensions of the associated grading of the Hodge filtration (see [44, Figure 5.1] for a nice rank 5 example). Let us recall the procedure. Assume $\boldsymbol{\alpha}, \boldsymbol{\beta}$ are generic.

Algorithm 4.13. The zig-zag procedure

Set S the sequence of ordered parameters of $\boldsymbol{\alpha}$ as elements in $(0,1]$ and $\boldsymbol{\beta}$ as elements in $[0,1)$.
To each $s \in S$ associate the color red if $s \in \boldsymbol{\alpha}$ and blue if $s \in \boldsymbol{\beta}$.
$P \leftarrow 0$
for $i = 1 \ldots 2n$ ***do***
  Draw a point with the color $S[i]$ at (i, P).
  if *$S[i]$ is blue* ***then***
    $P \leftarrow P - 1$
  else
    $P \leftarrow P + 1$
  end if
end for

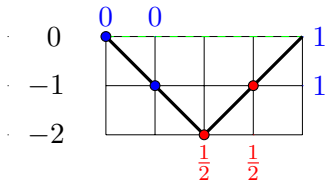
Let r be the number of parameters of $\boldsymbol{\alpha}, \boldsymbol{\beta}$ in $\mathbb{Z}$. Then the Hodge polynomial of the motive $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta} \,|\, z)$ equals

$$\sum_P x^{-p_2} y^{p_2+r-1},$$

where the sum runs over the blue points $P = (p_1, p_2)$ of the zig-zag procedure. In §8.3 we will prove the relation between the Hodge numbers of Euler's curve and the output of the zig-zag procedure for rank two hypergeometric motives.

Remark 4.14. In fact, this procedure can be extended to give the full mixed Hodge numbers of the middle cohomology of the corresponding toric model in the rational case, see [53].

Example 5. As a first example, consider the rational HGM with parameters $\boldsymbol{\alpha} = (1/2, 1/2)$, $\boldsymbol{\beta} = (0, 0)$. The zig-zag procedure gives the following picture



There are two integral entries in $\boldsymbol{\alpha}, \boldsymbol{\beta}$ so $r = 2$. The Hodge polynomial then equals $y + x$ (as expected). We will see below (see Remark 10) that $\mathcal{H}((1/2, 1/2), (0, 0) | z)$ is (a quadratic twist by

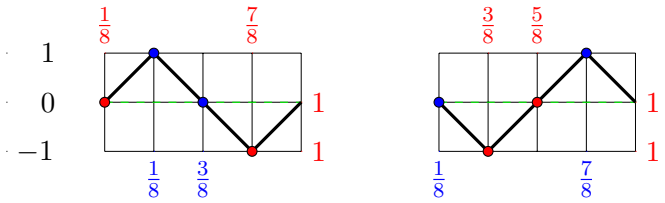


FIGURE 2. The Hodge numbers of $(\frac{1}{8},\frac{7}{8}),(\frac{3}{8},\frac{5}{8})$ and $(\frac{3}{8},\frac{5}{8}),(\frac{1}{8},\frac{7}{8})$

$\left(\frac{-4}{\cdot}\right)$ of) the first cohomology of Legendre's family of elliptic curves

$$E_z : y^2 = x(1-x)(1-zx).$$

Example 2.(continued) Consider the hypergeometric motive $\mathcal{H}_z$ of the introduction, with parameters $\boldsymbol{\alpha} = (1/8, 7/8), \boldsymbol{\beta} = (3/8, 5/8)$ defined over its base field, the real quadratic field $K = \mathbb{Q}(\sqrt{2}) \subset \mathbb{Q}(\zeta_8)$. For $j \in (\mathbb{Z}/8)^\times$ the Hodge numbers of $\mathcal{H}_z^{\sigma_j}$ are obtained by applying the zig-zag procedure to the parameters $\boldsymbol{\alpha} = (j/8, 7j/8), \boldsymbol{\beta} = (3j/8, 5j/8)$. The result is given in Figure 2. The Hodge polynomial in both cases is $x^{-1} + y^{-1}$.

Example 6. As a further example of a motive not defined over $\mathbb{Q}$, let $\boldsymbol{\alpha} = (\frac{1}{2},\frac{1}{2})$, $\boldsymbol{\beta} = (0,\frac{1}{4})$. Here $N = 4$, $r = 1$ and $K = \mathbb{Q}(i)$. We find the Hodge polynomials $1 + xy^{-1}$ and $1 + x^{-1}y$ for the respective complex embeddings K, as illustrated in Figure 3. The Hodge polynomial of the restriction of scalars $\mathcal{H}(z)$ to $\mathbb{Q}$ (the sum of $\mathcal{H}(\boldsymbol{\alpha},\boldsymbol{\beta}|z)^\sigma$ over the two complex embeddings σ of K) equals $x^{-1}y + 2 + xy^{-1}$.

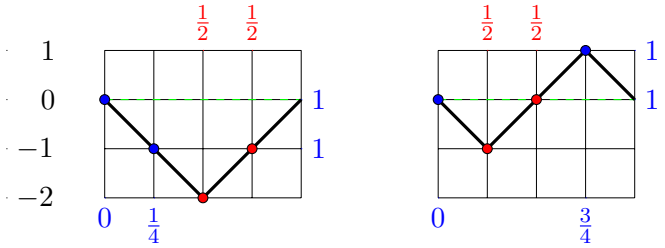


FIGURE 3. The Hodge numbers of $(\frac{1}{2},\frac{1}{2}),(0,\frac{1}{4})$ and $(\frac{1}{2},\frac{1}{2}),(0,\frac{3}{4})$

We can check this calculation numerically in Pari/GP using the third's author package. Taking for example $z = 3$ and $p = 17$, we get:

```
? polrecognizep(hgmfrob(3,[1/2,1/2],[3/4,1],17)*hgmfrob(3,[1/2,1/2],[1/4,1],17))
x^4 - 10/17*x^3 + 18/17*x^2 - 10/17*x + 1
```

This is the Euler factor $L_{17}(T)$ of $\mathcal{H}(3)$. By a well-known fact, its Newton polygon at $p = 17$ should lie above the Hodge polynomial. In fact they are actually equal in this case. Concretely, L_{17} has roots of absolute value $\{-1, 0, 0, 1\}$ with multiplicities matching the Hodge vector $(1, 2, 1)$ of $\mathcal{H}(z)$.

Remark 4.15. Taking a Tate twist has the effect of increasing or decreasing the exponents of the Hodge polynomial by an even integer. When working with Galois representations, it is customary to take the minimal Tate twist that makes the Hodge numbers of all Galois conjugates non-negative (making them the Hodge numbers of an *effective* motive) because this is the standard normalization for automorphic forms. In particular, the resulting Euler factors have integral coefficients. We will call this process the *effective normalization*.

In the present article (and also in the Pari/GP code written by the third author) we do not make any additional Tate twist when considering HGM's unless explicitly mentioned. The reader should bear this in mind when matching a HGM to an automorphic form.

The effective normalization is the Tate twist that shifts the minimum value of the zig-zag diagrams over all embeddings to zero. As an illustration, we have the following.

Example 6.(continued) The value $r-1=0$ so the effective normalization corresponds to a Tate twist by the minimum value of the zig-zag function over the two diagrams in Fig. 3 which equals -1. Therefore the effective normalization is the Tate twist

$$\mathcal{H}((\tfrac{1}{2},\tfrac{1}{2}),(\tfrac{1}{4},1)|z)(1)$$

In terms of Euler factors (like $L_{17}(T)$ above) it amounts to replacing T by qT.

Remark 4.16. Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be generic rational parameters and let N be their least common denominator. Let $F=\mathbb{Q}(\zeta_N)$ and let H be the subgroup of $(\mathbb{Z}/N)^\times$ defined in 1.1. Then for $\xi\in\mathbb{Q}$, the motive $\mathcal{H}(\boldsymbol{\alpha},\boldsymbol{\beta}|\xi)$ is expected to be defined over F^H and hence has $\phi(N)/|H|$ Hodge vectors (indexed by the embeddings of F^H into $\mathbb{C}$) instead of $\phi(N)$. This is consistent with the fact that if $j\in H$ then the Hodge vector of $\boldsymbol{\alpha},\boldsymbol{\beta}$ is the same as the one of $j\boldsymbol{\alpha},j\boldsymbol{\beta}$.

5. Jacobi motives

In this section we review some basic facts about Jacobi motives and set up the notation we will use. The main reference are the articles [55, 56], [1], [45] and [54] (the software package Magma [10] contains an implementation of Jacobi motives due to Mark Watkins).

Fix an integer $N>1$. For a prime ideal $\mathfrak{p}$ in $F:=\mathbb{Q}(\zeta_N)\subseteq\mathbb{C}$ of norm q, where ζ_N is a primitive N-st root of unity, let $\chi_\mathfrak{p}$ be the character

$$\chi_\mathfrak{p}:(\mathcal{O}_F/\mathfrak{p})^\times\to\mathcal{O}_F^\times. \tag{22}$$

of order N satisfying

$$\chi_\mathfrak{p}(x)\equiv x^{(q-1)/N}\bmod\mathfrak{p}.$$

Extend the definition by setting $\chi_\mathfrak{p}(x)=0$ if $\mathfrak{p}\mid x$. Here $\mathcal{O}_F$ is the ring of integers of F.

Remark 5.1. If L is a finite extension of F and $\widehat{\mathfrak{p}}$ is a prime ideal of L dividing $\mathfrak{p}$ we can similarly define a character $\widehat{\chi_\mathfrak{p}}:(\mathcal{O}_L/\widehat{\mathfrak{p}})^\times\to\mathbb{C}^\times$ by composing $\chi_\mathfrak{p}$ with the norm map. To avoid heavy notation, we will also denote by $\chi_\mathfrak{p}$ the character $\widehat{\chi_\mathfrak{p}}$.

We define Gauss sums as functions on $\frac{1}{N}\mathbb{Z}/\mathbb{Z}$ as follows. Fix a non-trivial additive character ψ on $\mathbb{F}_q$ and let

$$g(\psi,a,\mathfrak{p}):=\sum_{x\in\mathbb{F}_q^\times}\psi(x)\chi_\mathfrak{p}^{Na}(x),\qquad\qquad a\in\frac{1}{N}\mathbb{Z}/\mathbb{Z}. \tag{23}$$

The dependence on the choice of additive character is straightforward. For any non-zero $y\in\mathcal{O}_F$ coprime to $\mathfrak{p}$ if $\psi'(x):=\psi(y^{-1}x)$ then

$$g(\psi',a,\mathfrak{p})=\chi_\mathfrak{p}^{Na}(y)g(\psi,a,\mathfrak{p}). \tag{24}$$

Definition 5.2. *Let* $\boldsymbol{\theta}=\sum_i n_i\langle\theta_i\rangle\in\mathbb{Z}[\frac{1}{N}\mathbb{Z}/\mathbb{Z}]$ *satisfying*

$$\sum_i n_i\theta_i\equiv 0\bmod\mathbb{Z}. \tag{25}$$

The Jacobi sum *attached to* $\boldsymbol{\theta}$ *for the prime ideal* $\mathfrak{p}$ *is defined as*

$$\mathbf{J}(\boldsymbol{\theta})(\mathfrak{p}):=(-1)^{\sum_i n_i+1}\prod_i g(\psi,\theta_i,\mathfrak{p})^{n_i} \tag{26}$$

It is easy to verify, thanks to (25) and (24), that the definition is independent of the choice of the additive character ψ and therefore there is no need to include it in the notation.

By the main result of [56] the map

$$\mathfrak{p} \mapsto \mathbf{J}(\boldsymbol{\theta})(\mathfrak{p})$$

determines a Grössencharacter $\mathbf{J}(\boldsymbol{\theta})$ of F. In particular, it has an associated compatible system of l-adic Galois representations where $\mathbf{J}(\boldsymbol{\theta})(\mathfrak{p})$ equals the trace of $\mathrm{Frob}_{\mathfrak{p}}$. This is an incarnation of the *Jacobi motive* associated to $\boldsymbol{\theta}$ (see [1, §5]). We will denote this pure motive also by $\mathbf{J}(\boldsymbol{\theta})$ if there is no risk of confusion.

By [1, Theorem 1] (see also Remark 2.3.2 of loc. cit) the infinity type of the Grössencharacter associated to $\mathbf{J}(\boldsymbol{\theta})$ is given by

$$\sigma_j \mapsto \sum_i n_i\{j\theta_i\}, \qquad \gcd(j,N)=1, \tag{27}$$

where $\{\cdot\}$ denotes the fractional part of real numbers and $\sigma_j \in \mathrm{Gal}(F/\mathbb{Q})$ is the automorphism satisfying $\sigma_j(\zeta_N)=\zeta_N^j$. The (motivic) weight of $\mathbf{J}(\boldsymbol{\theta})$ equals

$$w := \sum_i n_i,$$

where the sum only includes indexes i for which $\theta_i \notin \mathbb{Z}$.

The motive $\mathbf{J}(\boldsymbol{\theta})$ appears (up to a Tate twist) in the middle cohomology $H^\bullet(X,F)$ of a smooth Fermat hypersurface X (see for example §10 of [1]). Moreover, if we define

$$p := \sum_i n_i\{\theta_i\}, \qquad q := \sum_i n_i\{-\theta_i\} \tag{28}$$

then (up to a Tate twist) $\mathbf{J}(\boldsymbol{\theta}) \subseteq H^{p,q}(X,F)$.

It will sometimes be convenient to use an alternative notation for Jacobi motives as a pair of tuples of rational numbers $\boldsymbol{a}=(a_1,\dots,a_r)$ and $\boldsymbol{b}=(b_1,\dots,b_s)$ with common denominator N such that

$$\sum_{i=1}^r a_i \equiv \sum_{i=1}^s b_i \bmod \mathbb{Z},$$

corresponding to

$$\boldsymbol{\theta} := \sum_{i=1}^r \langle a_i\rangle - \sum_{i=1}^s \langle b_i\rangle.$$

We will then simply write $\mathbf{J}(\boldsymbol{a},\boldsymbol{b})$ instead of $\mathbf{J}(\boldsymbol{\theta})$.

Example 7. Consider the Jacobi motive $\mathbf{J}(\boldsymbol{a},\boldsymbol{b})$ where

$$\begin{aligned}\boldsymbol{a} &:= (1/10,1/10,1/10,3/10,13/30,7/10,23/30,9/10)\\ \boldsymbol{b} &:= (1/5,1/3,2/5,2/3,4/5,1/5,3/10,1/2).\end{aligned}$$

It is perhaps not immediately obvious but this motive is an Artin motive. But we can quickly check that we have

$$\sum_i \{ja_i\} - \sum_i \{jb_i\} = 0, \qquad \gcd(j,30)=1,$$

and in particular $p=q=0$ for all j. Computations with MAGMA show that the associated Artin motive is given by a Dirichlet character of $\mathbb{Q}(\zeta_5)$ of order 10 and conductor $2^2\cdot 3\cdot(1-\zeta_5)^2$.

The Jacobi sums $\mathbf{J}(\boldsymbol{\theta})$ belong to F and for $\sigma_j \in \mathrm{Gal}(F/\mathbb{Q})$ as before we have

$$\sigma_j(\mathbf{J}(\boldsymbol{\theta}))(\mathfrak{p}) = \mathbf{J}(j\boldsymbol{\theta})(\mathfrak{p}),$$

where

$$j\boldsymbol{\theta} := \sum_i n_i \langle j\theta_i \rangle.$$

Let $T = \{\sigma \in \mathrm{Gal}(F/\mathbb{Q}) \ : \ \sigma(\mathbf{J}(\boldsymbol{\theta})) = \mathbf{J}(\boldsymbol{\theta})\}$. Then the base field of the motive $\mathbf{J}(\boldsymbol{\theta})$ is F^T (see §2.3 of [54]). Note its resemblance with Conjecture 4.5. In some very particular instances, the Grössencharacter $\mathbf{J}(\boldsymbol{\theta})$ can be defined over a proper subextension of F^T.

Example 8. Consider the Jacobi motive $\mathbf{J}(\boldsymbol{\theta})$ where

$$\boldsymbol{\theta} = \left\langle \frac{1}{3} \right\rangle + \left\langle \frac{2}{3} \right\rangle + \left\langle \frac{1}{5} \right\rangle + \left\langle \frac{4}{5} \right\rangle + \left\langle \frac{7}{15} \right\rangle + \left\langle \frac{8}{15} \right\rangle.$$

The set T of elements in $(\mathbb{Z}/15)^\times$ fixing θ equals $\{\pm 1\}$, so the base field of $\mathbf{J}(\boldsymbol{\theta})$ is $\mathbb{Q}(\zeta_{15})^+$ (the maximal totally real subfield of $\mathbb{Q}(\zeta_{15})$). However the values of the Jacobi sum attached to our choice of parameters at a prime ideal $\mathfrak{p}$ of $\mathbb{Q}(\zeta_{15})$ equals $\mathcal{N}\mathfrak{p}^3$ (see Lemma 9.3). Then there is a second extension of the Grössencharacter to $\mathrm{Gal}_{\mathbb{Q}}$ given by the cubic power of the cyclotomic character.

6. Finite Hypergeometric Sums

Let us recall Katz's definition (given in [34, p.258]) of the finite version of a hypergeometric series. Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n), \boldsymbol{\beta} = (\beta_1, \ldots, \beta_n) \in \mathbb{Q}^n$ be arbitrary vectors of rational numbers and let N be the common denominator of the α's and β's. For the remaining of the section we fix $q \equiv 1 \bmod N$ a prime power and a generator ϖ of $\widehat{\mathbb{F}_q^\times}$. Following [34, p. 258] (but note that our parameter z is the inverse of Katz's though it matches the normalization in [8]), define for $z \in \mathbb{F}_q$ the exponential sum (here $x_i, y_j \in \mathbb{F}_q^\times$):

$$\mathrm{Hyp}_q(\boldsymbol{\alpha}, \boldsymbol{\beta} | z) = \sum_{z x_1 \cdots x_n = y_1 \cdots y_n} \psi(x_1 + \cdots + x_n - y_1 - \cdots - y_n) \varpi(\boldsymbol{x})^{\boldsymbol{\alpha}(q-1)} \overline{\varpi}(\boldsymbol{y})^{\boldsymbol{\beta}(q-1)}, \tag{29}$$

where $\varpi(\boldsymbol{x})^{\boldsymbol{\alpha}(q-1)} = \varpi(x_1)^{\alpha_1(q-1)} \cdots \varpi(x_n)^{\alpha_n(q-1)}$ and similarly with β. Katz relates (using the Lefschetz trace formula) such finite hypergeometric sum to the trace of Frobenius acting on some concrete hypergeometric $\mathcal{D}$-module [34, Chap. 8.2].

It will be convenient to expand Hyp_q in terms of characters of $\mathbb{F}_q^\times$. The calculation is straightforward (see [34, §8.2.8]); we sketch it here for the reader's convenience. The coefficient c_φ of $\varphi \in \widehat{\mathbb{F}_q^\times}$ equals

$$c_\varphi = \frac{1}{q-1} \sum_{z \in \mathbb{F}_q^\times} \overline{\varphi}(z) \mathrm{Hyp}_q(\boldsymbol{\alpha}, \boldsymbol{\beta} | z).$$

Interchanging order of summation we get

$$c_\varphi = \frac{1}{q-1} \sum_{(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{T}_z} \psi(x_1 + \cdots + x_n - y_1 - \cdots - y_n) \varpi(\boldsymbol{x})^{\boldsymbol{\alpha}(q-1)} \overline{\varpi}(\boldsymbol{y})^{\boldsymbol{\beta}(q-1)} \varphi(x_1 \cdots x_n y_1^{-1} \cdots y_n^{-1})$$

and hence

$$c_\varphi = \frac{1}{q-1} \prod_{i=1}^n \sum_{x_i \in \mathbb{F}_q^\times} \psi(x_i) \varpi(x_i)^{(q-1)\alpha_i} \varphi(x_i) \prod_{i=1}^n \sum_{y_i \in \mathbb{F}_q^\times} \psi(-y_i) \overline{\varpi}(y_i)^{(q-1)\beta_i} \overline{\varphi}(y_i).$$

In terms of Gauss sums

$$g(\psi, \varphi) := \sum_{x \in \mathbb{F}_q^\times} \varphi(x) \psi(x), \tag{30}$$

we finally have

$$c_\varphi = \frac{1}{q-1}\prod_{i=1}^n g(\psi, \varpi^{(q-1)\alpha_i}\varphi)\prod_{i=1}^n g(\psi^{-1}, \overline{\varpi}^{(q-1)\beta_i}\overline{\varphi})$$

and therefore

$$\mathrm{Hyp}_q(\boldsymbol{\alpha},\boldsymbol{\beta}|z) = \frac{1}{q-1}\sum_\varphi\prod_{i=1}^n g(\psi, \varpi^{(q-1)\alpha_i}\varphi)\prod_{i=1}^n g(\psi^{-1}, \overline{\varpi}^{(q-1)\beta_i}\overline{\varphi})\varphi(z).$$

It is instructive to check the simplest non-trivial case where $n=1$, $\alpha_1 = 1/2, \beta_1 = 1$ and $q=p$ is an odd prime. In this case, a quick calculation shows that

$$\mathrm{Hyp}(1/2, 1\,|\,z) = g(\psi_{z-1}, \epsilon),$$

where $\psi_u(x) := \psi(ux)$ and ϵ is the quadratic character of $\mathbb{F}_p^\times$. It follows that

$$\mathrm{Hyp}(1/2, 1\,|\,z) = \epsilon(1-z)\sqrt{p^*}, \qquad\qquad p^* := (-1)^{(p-1)/2}p,$$

as is well known. In particular, the values $\mathrm{Hyp}(1/2, 1\,|\,\xi)$ as p varies, for fixed $\xi \in \mathbb{Q}$ say, do not lie in any given number field. We therefore cannot expect $\mathrm{Hyp}(1/2, 1\,|\,\xi)$ to be the trace of Frobenius of a motive. To achieve this we normalize the hypergeometric sum $\mathrm{Hyp}_q(\boldsymbol{\alpha},\boldsymbol{\beta}|z)$ by dividing by the appropriate constant so that the sum of its values over all $z \in \mathbb{F}_q^\times$ equals -1. Concretely, we consider the following.

Definition 6.1. *For $z \in \mathbb{F}_q$, define the finite hypergeometric sum $H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|z)$ by*

$$H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|z) := \frac{1}{1-q}\sum_\varphi \frac{\mathcal{J}(\boldsymbol{\alpha}\varphi,\boldsymbol{\beta}\varphi)}{\mathcal{J}(\boldsymbol{\alpha},\boldsymbol{\beta})}\varphi(z), \tag{31}$$

where

$$\mathcal{J}(\boldsymbol{\alpha}\varphi,\boldsymbol{\beta}\varphi) := \prod_{i=1}^n g(\psi, \varpi^{(q-1)\alpha_i}\varphi)\prod_{i=1}^n g(\psi^{-1}, \overline{\varpi}^{(q-1)\beta_i}\overline{\varphi}).$$

Note that for any integer a coprime with p we have

$$g(\psi^a, \varphi) = \overline{\varphi}(a)g(\psi,\varphi). \tag{32}$$

Therefore, we may alternatively define

$$\mathcal{J}^*(\boldsymbol{\alpha}\varphi,\boldsymbol{\beta}\varphi) = \varphi(-1)^n\prod_{i=1}^n g(\psi, \varpi^{(q-1)\alpha_i}\varphi)\prod_{i=1}^n g(\psi, \overline{\varpi}^{(q-1)\beta_i}\overline{\varphi}). \tag{33}$$

and then

$$H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|z) = \frac{1}{1-q}\sum_\varphi \frac{\mathcal{J}^*(\boldsymbol{\alpha}\varphi,\boldsymbol{\beta}\varphi)}{\mathcal{J}^*(\boldsymbol{\alpha},\boldsymbol{\beta})}\varphi(z).$$

To simplify the notation set $g(m) := g(\psi, \varpi^m)$ for $m \in \mathbb{Z}$. Then, explicitly,

$$H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|z) = \frac{1}{1-q}\sum_{m=0}^{q-2}\prod_{i=1}^n\left(\frac{g(m+\alpha_i(q-1))g(-m-\beta_i(q-1))}{g(\alpha_i(q-1))g(-\beta_i(q-1))}\right)\varpi((-1)^n z)^m. \tag{34}$$

The finite hypergeometric sum $H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|z)$ does not depend on the choice of the additive character ψ, but it does (in general) depend on the choice of the character ϖ. For our previous rank one example we have (see §2)

$$H_p(1/2, 1\,|\xi) = \epsilon(1-\xi),$$

where ϵ is the quadratic character of $\mathbb{F}_q$, matching

$$\sum_{n\geq 0}\frac{(1/2)_n}{(1)_n}\xi^n=\frac{1}{\sqrt{1-\xi}}.$$

Note that $\sum_{p\in\mathbb{F}_p^\times}H_p(1/2,1\,|\xi)=-\epsilon(1)=-1$ as desired.

6.1. Properties of $H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|z)$.

Proposition 6.2. *The hypergeometric sum $H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|z)$ satisfies the following basic properties*

(1) (Ordering) $H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|z)$ is independent of the ordering of $\{\alpha_1,\dots,\alpha_n\}$ and $\{\beta_1,\dots,\beta_n\}$.

(2) (Inversion) For $z\in\mathbb{F}_q^\times$ we have

$$H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|z)=H_q(-\boldsymbol{\beta},-\boldsymbol{\alpha}|z^{-1}).$$

(3) (Galois action) Let $z\in\mathbb{F}_q^\times$ and $\sigma\in\mathrm{Gal}_{\mathbb{Q}}$ with $\sigma(\zeta_N)=\zeta_N^j$ for some j coprime with N then

$$H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|z)^\sigma=H_q(j\boldsymbol{\alpha},j\boldsymbol{\beta}|z)$$

(4) (Coefficient Field) Let $z\in\mathbb{F}_q^\times$ and let H be the subgroup of $(\mathbb{Z}/N)^\times$ defined in 1.1. We have

$$H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|z)\in\mathbb{Q}(\zeta_N)^H.$$

(5) (Twists) Let $\rho\in\mathbb{Q}$ and $\rho\boldsymbol{\alpha}:=(\alpha_1+\rho,\dots,\alpha_n+\rho)$ and $\rho\boldsymbol{\beta}:=(\beta_1+\rho,\dots,\beta_n+\rho)$ then

$$H_q(\rho\boldsymbol{\alpha},\rho\boldsymbol{\beta}|z)=(-1)^{n(q-1)\rho}\overline{\varpi}(z)^{(q-1)\rho}\frac{\mathcal{J}(\boldsymbol{\alpha},\boldsymbol{\beta})}{\mathcal{J}(\rho\boldsymbol{\alpha},\rho\boldsymbol{\beta})}H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|z),$$

whenever both sides are well defined.

(6) (Non-generic) If $\alpha_1\equiv\beta_1 \bmod \mathbb{Z}$ let $\boldsymbol{\gamma}=(\alpha_2,\dots,\alpha_n)$ and $\boldsymbol{\delta}=(\beta_2,\dots,\beta_n)$. Then

$$H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|z)=q^\delta\left(\frac{\mathcal{J}((-\alpha_1)\boldsymbol{\gamma},(-\alpha_1)\boldsymbol{\delta})}{\mathcal{J}(\boldsymbol{\gamma},\boldsymbol{\delta})}+qH_q(\boldsymbol{\gamma},\boldsymbol{\delta})\right),$$

where

$$\delta=\begin{cases}0 & \text{if } \alpha_1\in\mathbb{Z}\\ -1 & \text{otherwise}\end{cases}$$

Proof. The first property is clear from its definition. To prove (2), it follows from (33) that

$$\mathcal{J}(\boldsymbol{\alpha}\varphi,\boldsymbol{\beta}\varphi)=\mathcal{J}(-\boldsymbol{\beta}\,\overline{\varphi},-\boldsymbol{\alpha}\,\overline{\varphi}).$$

The result follows from the change of variables $\varphi\to\overline{\varphi}$ in (31).
(3) Since σ is a ring morphism,

$$\sigma(H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|z))=\frac{1}{1-q}\sum_{\varphi}\prod_{i=1}^{n}\left(\frac{\sigma(g(\psi,\varpi^{(q-1)\alpha_i}\varphi))\sigma(g(\psi^{-1},\varpi^{-(q-1)\beta_i}\varphi^{-1})}{\sigma(g(\psi,\varpi^{(q-1)\alpha_i}))\sigma(g(\psi^{-1},\varpi^{-(q-1)\beta_i}))}\right)\sigma(\varphi(z)). \tag{35}$$

If $\sigma\in\mathrm{Gal}_{\mathbb{Q}}$, φ is a multiplicative character of $\mathbb{F}_q^\times$ and ψ is an additive character, then

$$\sigma(g(\psi,\varphi))=g(\psi^\sigma,\varphi^\sigma),$$

where $\varphi^\sigma(x)=\sigma(\varphi(x))$ (similarly for ψ). Let b be an integer (prime to $q-1$) satisfying $\sigma(\zeta_{q-1})=\zeta_{q-1}^b$, for ζ_{q-1} a primitive $(q-1)$-th root of unity. Then

$$\sigma(\varpi^{m+a(q-1)})=\varpi^{bm+ba(q-1)}=\varpi^{bm}\varpi^{ba(q-1)}.$$

The rational number ba equals ja up to translation by an integer. Then changing the summation order (replacing φ by $\sigma(\varphi)$), the left hand side of (35) becomes

$$\frac{1}{1-q}\sum_{\varphi}\prod_{i=1}^{n}\left(\frac{g(\psi,\varpi^{(q-1)j\alpha_i}\varphi))g(\psi^{-1},\varpi^{-(q-1)j\beta_i}\varphi^{-1})}{g(\psi,\varpi^{(q-1)j\alpha_i})g(\psi^{-1},\varpi^{-(q-1)j\beta_i})}\right)\varphi(z).$$

(4) For $z\in\mathbb{F}_q$, the standard properties of Gauss sums imply that the value $H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|z)$ is an element of $\mathbb{Q}(\zeta_{q-1})$ (see also [8, Proposition 3.2]). Note that all elements of $(\mathbb{Z}/(q-1))^\times$ that are congruent to 1 modulo N leave the sets $\{\alpha_1,\dots,\alpha_n\}$ and $\{\beta_1,\dots,\beta_n\}$ stable under multiplication (up to translation by integers), hence the second statement (and Galois theory) imply that actually $H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|z)\in F=\mathbb{Q}(\zeta_N)$. Since multiplication by elements of H also fix the sets $\{\alpha_1,\dots,\alpha_n\}$ and $\{\beta_1,\dots,\beta_n\}$, the result follows.
(5) It follows from (33) that

$$\mathcal{J}(\rho\boldsymbol{\alpha}\varphi,\rho\boldsymbol{\beta}\varphi)=\mathcal{J}(\boldsymbol{\alpha}\varphi\varpi^{(q-1)\rho},\boldsymbol{\beta}\varphi\varpi^{(q-1)\rho})(-1)^{n(q-1)\rho}.$$

Then

$$H_q(\rho\boldsymbol{\alpha},\rho\boldsymbol{\beta}|z)=\frac{1}{1-q}\sum_{\varphi}\frac{\mathcal{J}(\rho\boldsymbol{\alpha}\varphi,\rho\boldsymbol{\beta}\varphi)}{\mathcal{J}(\rho\boldsymbol{\alpha},\rho\boldsymbol{\beta})}\varphi(z)=(-1)^{n(q-1)\rho}\frac{\mathcal{J}(\boldsymbol{\alpha},\boldsymbol{\beta})}{\mathcal{J}(\rho\boldsymbol{\alpha},\rho\boldsymbol{\beta})}\overline{\varpi}(z)^{(q-1)\rho}H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|z).$$

(6) If ψ is an additive character of $\mathbb{F}_q$ and χ is a multiplicative character, then

$$g(\psi,\chi)g(\psi^{-1},\chi^{-1})=\begin{cases}q & \text{if } \chi=1,\\ 1 & \text{otherwise.}\end{cases}$$

If α_1 and β_1 are integers, then

$$H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|z)=\frac{1}{1-q}\left(\sum_{\varphi\neq 1}\frac{q\mathcal{J}(\boldsymbol{\gamma}\varphi,\boldsymbol{\delta}\varphi)}{\mathcal{J}(\boldsymbol{\gamma},\boldsymbol{\delta})}+1\right)=qH_q(\boldsymbol{\gamma},\boldsymbol{\delta})+1.$$

Similarly, if $\alpha_1\equiv\beta_1 \bmod \mathbb{Z}$ but they are not integers, let $\varphi_0=\varpi^{-(q-1)\alpha_1}$. Then

$$H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|z)=\frac{1}{1-q}\left(\sum_{\varphi\neq\varphi_0}\frac{\mathcal{J}(\boldsymbol{\gamma}\varphi,\boldsymbol{\delta}\varphi)}{\mathcal{J}(\boldsymbol{\gamma},\boldsymbol{\delta})}+\frac{\mathcal{J}(\boldsymbol{\gamma}\varphi_0,\boldsymbol{\delta}\varphi_0)}{q\mathcal{J}(\boldsymbol{\gamma},\boldsymbol{\delta})}\right)=H_q(\boldsymbol{\gamma},\boldsymbol{\delta})+\frac{\mathcal{J}((-\alpha_1)\boldsymbol{\gamma},(-\alpha_1)\boldsymbol{\delta})}{q\mathcal{J}(\boldsymbol{\gamma},\boldsymbol{\delta})}.$$

□

Recall that for L a finite extension of $F=\mathbb{Q}(\zeta_N)$ and $\hat{\mathfrak{p}}$ a prime ideal of L, we defined in Remark 5.1 an order N character $\widehat{\chi_{\mathfrak{p}}}$ of $(\mathcal{O}_L/\hat{\mathfrak{p}})^\times$.

Definition 6.3. *Let $\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathbb{Q}^n$ be generic parameters and let N be their least common denominator. Let L be a finite extension of F and let $\hat{\mathfrak{p}}$ be a prime ideal of L of norm q not dividing N. Let $\xi\in L$ satisfy $v_{\hat{\mathfrak{p}}}(\xi)=v_{\hat{\mathfrak{p}}}(\xi-1)=0$. The* $\hat{\mathfrak{p}}$-hypergeometric sum, *denoted $H_{\hat{\mathfrak{p}}}(\boldsymbol{\alpha},\boldsymbol{\beta}|\xi)$ is the value $H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|\xi)$ for ϖ a generator of $\widehat{\mathbb{F}_q^\times}$ satisfying $\varpi^{\frac{q-1}{N}}=\widehat{\chi_{\mathfrak{p}}}^{-1}$.*

The independence of ϖ follows from the following result.

Lemma 6.4. *Let $\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathbb{Q}^n$ be generic parameters and let N be their least common denominator. Let ϖ_1 and ϖ_2 be two generators of $\widehat{\mathbb{F}_q^\times}$ such that $\varpi_1^{\frac{q-1}{N}}=\varpi_2^{\frac{q-1}{N}}$. Let $\xi\in\mathbb{F}_q^\times$, $\xi\neq 1$. Then the values of $H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|z)$ for ϖ_1 and ϖ_2 are the same.*

Proof. Let $j \in (\mathbb{Z}/(q-1))^\times$ be such that $\varpi_2 = \varpi_1^j$. The hypothesis $\varpi_1^{\frac{q-1}{N}} = \varpi_2^{\frac{q-1}{N}}$ implies that $j \equiv 1 \pmod N$. The value $H_q(\boldsymbol{\alpha}, \boldsymbol{\beta}|z)$ taking ϖ_2 as a generator equals

$$\frac{1}{1-q}\sum_{\varphi}\frac{\prod_{i=1}^n g(\psi, \varpi_1^{(q-1)j\alpha_i}\varphi)g(\psi^{-1}, \overline{\varpi_1}^{(q-1)j\beta_i}\overline{\varphi})}{\prod_{i=1}^n g(\psi, \varpi_1^{(q-1)j\alpha_i})g(\psi^{-1}, \overline{\varpi_1}^{(q-1)j\beta_i})}\varphi(z),$$

which matches the value of the finite hypergeometric series for the generator ϖ_1 with parameters $j\boldsymbol{\alpha}$, $j\boldsymbol{\beta}$. Since $j \equiv 1 \pmod N$, $\boldsymbol{\alpha} = j\boldsymbol{\alpha} \pmod{\mathbb{Z}^n}$ and the same is true for $\boldsymbol{\beta}$ hence the result. □

6.2. **Relation with the p-adic Gamma function.** For the reader's convenience, we will use bold letters for p-adic functions. Let p be a prime number. Recall the following definition (see [38]).

Definition 6.5. *The* p-adic Gamma function *is the continuous function* $\boldsymbol{\Gamma} : \mathbb{Z}_p \to \mathbb{Q}_p$ *that at a positive integer* n *takes the value*

$$\boldsymbol{\Gamma}(n) = (-1)^n \prod_{\substack{i=1 \\ p\nmid i}}^{n-1} i. \tag{36}$$

As proved in [38, Lemma 1], the p-adic Gamma function satisfies the relation

$$\boldsymbol{\Gamma}(n + p^r m) \equiv \boldsymbol{\Gamma}(n) \pmod{p^r},$$

hence condition (36) determines it uniquely. An important property of the p-adic Gamma function is that it determines an analytic function.

Theorem 6.6 (Morita)**.** *Set* $Q = 8$ *if* $p = 2$ *and* $Q = 1$ *otherwise. Then the* p*-adic* $\boldsymbol{\Gamma}$ *function is an analytic function from* $Q\mathbb{Z}_p \to \mathbb{Q}_p$.

Proof. See [38, Theorem 3]. □

Let $q = p^f$ for a positive integer f. The following functions play an important role.

Definition 6.7. *Let* $\star \in \{0, \infty\}$. *The function* $\{\cdot\}^\star : \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}$ *is defined by*

$$\{x\}^\infty := x - [x], \qquad \text{and} \qquad \{x\}^0 := 1 - \{-x\}^\infty, \tag{37}$$

where $[x]$ *denotes the floor of* x *(so* $\{x\}^\infty \in [0,1)$ *while* $\{x\}^0 \in (0,1]$*). Consider the following functions*

$$\boldsymbol{\Gamma}_q^\star : \mathbb{Z}_{(p)}/\mathbb{Z} \to \mathbb{Q}_p, \qquad \boldsymbol{\Gamma}_q^\star(x) := \prod_{i=0}^{f-1} \boldsymbol{\Gamma}(\{p^i x\}^\star). \tag{38}$$

$$\eta_q^\star : \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}, \qquad \eta_q^\star(x) := \sum_{i=0}^{f-1} \{p^i x\}^\star. \tag{39}$$

$$\eta_{q,m}^\star : \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}, \qquad \eta_{q,m}^\star(x) := \eta_q^\star\left(x + \frac{m}{1-q}\right) - \eta_q^\star(x). \tag{40}$$

If $\boldsymbol{x} \in (\mathbb{Q}/\mathbb{Z})^n$, *we extend the last map linearly component-wise, namely*

$$\eta_{q,m}^\star(\boldsymbol{x}) = \sum_{i=1}^n \eta_{q,m}^\star(x_i).$$

Definition 6.8. *Let $x \in \mathbb{Q}/\mathbb{Z}$ be such that its denominator is not divisible by p. The (finite) q-*orbit *of x is the set*

$$\mathcal{O}_q(x) := \{x, qx, q^2x, \ldots\} \subset \mathbb{Q}/\mathbb{Z}.$$

We denote by $\mathrm{len}_q(x)$ its size.

Remark 6.9. If $x \in \mathbb{Q}/\mathbb{Z}$ and its q-orbit has r elements, then $(q^r - 1)x \in \mathbb{Z}$. In particular, the q-orbit of x has a unique element precisely when $\mathrm{den}(x) \mid q - 1$.

Lemma 6.10. *Let $x \in \mathbb{Q}/\mathbb{Z}$ be such that its denominator is not divisible by p. Then*

$$\prod_{u \in \mathcal{O}_q(x)} \mathbf{\Gamma}_q^\star(u) = \mathbf{\Gamma}_{q^{\mathrm{len}_q(x)}}^\star(x). \tag{41}$$

Similarly,

$$\sum_{u \in \mathcal{O}_q(x)} \eta_{q,m}^\star(u) = \eta_{q^{\mathrm{len}_q(x)},m}^\star(x). \tag{42}$$

Proof. By definition of the q-Gamma function

$$\prod_{u \in \mathcal{O}_q(x)} \mathbf{\Gamma}_q^\star(u) = \prod_{i=0}^{\mathrm{len}_q(x)-1} \prod_{j=0}^{f-1} \mathbf{\Gamma}^\star(\{p^j q^i x\}) = \prod_{i=0}^{f\,\mathrm{len}_q(x)-1} \mathbf{\Gamma}^\star(\{p^i x\}) = \mathbf{\Gamma}_{q^{\mathrm{len}_q(x)}}^\star(x).$$

The second statement is an additive version of the same proof. □

Definition 6.11. *Let p be a prime number, q a p-th power and n a positive integer. Let $x \in \mathbb{Z}_{(p)}/\mathbb{Z}$. For $\star \in \{0, \infty\}$, define the* p-adic Pochammer symbol

$$(x)_{q,n}^\star := \frac{\mathbf{\Gamma}_q^\star(x + \frac{n}{1-q})}{\mathbf{\Gamma}_q^\star(x)}. \tag{43}$$

Let $\mathcal{O}$ denote the ring of integers of $\mathbb{Q}_p(\zeta_{q-1})$ and let $\mathfrak{p}$ be its maximal ideal. For the rest of the section we let $\mathbb{F}_q := \mathcal{O}/\mathfrak{p}$.

Let Teich be the *Teichmuller character*

$$\mathrm{Teich} : \mathbb{F}_q^\times \to \mathcal{O}^\times \tag{44}$$

characterized by $\pi \circ \mathrm{Teich} = \mathrm{Id}$, where $\pi : \mathcal{O} \to \mathbb{F}_q$ is the reduction map. (This character is the p-adic version of the character $\chi_{\mathfrak{p}}$ of (22) for $N = q - 1$).

Definition 6.12. *Let $\boldsymbol{\alpha}, \boldsymbol{\beta} \in (\mathbb{Q}/\mathbb{Z})^n$. Let p be a prime number not dividing the denominator of $\boldsymbol{\alpha}$ nor $\boldsymbol{\beta}$ and let $q = p^f$. The* q-adic finite hypergeometric sum *is the function defined for $z \in \mathbb{F}_q$ by*

$$\boldsymbol{H}_q(\boldsymbol{\alpha}, \boldsymbol{\beta} | z) := \frac{1}{1-q} \sum_{m=0}^{q-2} \frac{(\alpha_1)_{q,m}^\infty \cdots (\alpha_n)_{q,m}^\infty}{(\beta_1)_{q,m}^0 \cdots (\beta_n)_{q,m}^0} (-p)^{\eta_{q,m}^\infty(\boldsymbol{\alpha}) - \eta_{q,m}^0(\boldsymbol{\beta})} \mathrm{Teich}(z)^m. \tag{45}$$

Remark 6.13. The routine "hgm" of the Pari/GP package described in Example 2 computes, given as input the specialization ξ (a rational number), the parameters $\boldsymbol{\alpha}, \boldsymbol{\beta}$ and a prime number p, the p-adic number $\boldsymbol{H}_q(\boldsymbol{\alpha}, \boldsymbol{\beta} | \xi)$ of (45). By default, the output is an element of the p-adic field $\mathbb{Q}_p$ computed with a default precision of 20 digits.

Definition 6.14. *Let $\boldsymbol{\alpha} \in (\mathbb{Q}/\mathbb{Z})^n$ and let $q = p^f$, with p not dividing the denominator of the coordinates of $\boldsymbol{\alpha}$. The vector $\boldsymbol{\alpha}$ is called* q-stable *if $\{q\alpha_1, \ldots, q\alpha_n\} = \{\alpha_1, \ldots, \alpha_n\}$.*

We will prove in Theorem 6.17 that $\boldsymbol{H}_q(\boldsymbol{\alpha},\boldsymbol{\beta}|z)$ is algebraic for q-stable parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. For that purpose, we start with some preliminaries.

Let $\boldsymbol{\psi} : \mathbb{F}_p \to \mathbb{Z}_p[\zeta_p]^\times$ be a non-trivial additive character. For $a \in \frac{1}{q-1}\mathbb{Z}/\mathbb{Z}$, define the *p-adic Gauss sum*

$$\boldsymbol{g}(\boldsymbol{\psi}, a, q) := \sum_{u\in\mathbb{F}_q^\times} \mathrm{Teich}(u)^{-a(q-1)}\boldsymbol{\psi}(\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}\, u). \tag{46}$$

Let $M = \mathbb{Q}(\zeta_{q-1})$ and let $L = M(\zeta_p)$. Let $\mathfrak{p}$ be a prime ideal of M dividing p and let $\mathfrak{q}$ be a prime ideal of L dividing $\mathfrak{p}$. Let $\iota : L \hookrightarrow L_{\mathfrak{q}}$ be the natural inclusion, where $L_{\mathfrak{q}}$ denotes the completion of L at $\mathfrak{q}$. Let $\psi : \mathbb{F}_q \to \mathbb{Z}[\zeta_p]^\times$ be the additive character satisfying $\iota\circ\psi = \boldsymbol{\psi}\circ\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}$. Let ϖ be a generator of $\widehat{\mathbb{F}_q^\times}$ such that $\iota\circ\varpi = \mathrm{Teich}^{-1}$. Then for $a \in \frac{1}{q-1}\mathbb{Z}/\mathbb{Z}$

$$\boldsymbol{g}(\boldsymbol{\psi}, a, q) = \iota(g(\psi, a, \mathfrak{p})), \tag{47}$$

where $g(\psi, a, \mathfrak{p}) \in L$ is the complex Gauss sum defined in (23).

Set $\zeta_p := \psi(1)$ and let $\pi \in \overline{\mathbb{Q}}$ satisfy

$$\pi^{p-1} = -p, \qquad \pi \equiv (\zeta_p - 1) \pmod{(\zeta_p-1)^2}.$$

Theorem 6.15 (Gross-Koblitz)**.** *Let $a \in \frac{1}{q-1}\mathbb{Z}$. Then*

$$\boldsymbol{g}(\boldsymbol{\psi}, a, q) = -\pi^{(p-1)\eta_q^\infty(a)}\boldsymbol{\Gamma}_q^\infty(a), \tag{48}$$

and

$$\frac{q}{\boldsymbol{g}(\boldsymbol{\psi}^{-1}, -a, q)} = -\pi^{(p-1)\eta_q^0(a)}\boldsymbol{\Gamma}_q^0(a). \tag{49}$$

Proof. The first result follows from [30, Theorem 1.7] and (47), taking (in Gross-Koblitz's notation) $N = q-1$. If we replace $\boldsymbol{\Gamma}_q^\infty$ and η_q^∞ by their $\boldsymbol{\Gamma}_q^0$ and η_q^0 counterparts, we find that

$$\pi^{(p-1)\eta_q^0(a)}\boldsymbol{\Gamma}_q^0(a) = \begin{cases} q & \text{if } a \in \mathbb{Z}, \\ \boldsymbol{g}(\boldsymbol{\psi}, a, q) & \text{if } a \notin \mathbb{Z}. \end{cases}$$

The formula follows from the well known relation

$$\boldsymbol{g}(\boldsymbol{\psi}, a, q)\cdot\boldsymbol{g}(\boldsymbol{\psi}^{-1}, -a, q) = \begin{cases} 1 & \text{if } a \in \mathbb{Z}, \\ q & \text{if } a \notin \mathbb{Z}. \end{cases}$$

□

Remark 6.16. The reason for the minus sign in the formulas in the Theorem (compared to [30, Theorem 1.7]) is due to a different choice of normalization for Gauss sums.

Theorem 6.17. *Let $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{Q}^n$ be q-stable vectors. Consider the decomposition of each set as a disjoint union of orbits (after relabeling)*

$$\{\alpha_1, \dots, \alpha_n\} = \mathcal{O}_q(\alpha_1) \cup \cdots \cup \mathcal{O}_q(\alpha_u), \qquad \{\beta_1, \dots, \beta_n\} = \mathcal{O}_q(\beta_1) \cup \cdots \cup \mathcal{O}_q(\beta_v).$$

To ease notation, let $l_i^\infty = \mathrm{len}_q(\alpha_i)$ and $l_i^0 = \mathrm{len}_q(\beta_i)$ be the size of the respective orbits. Then we have for $z \in \mathbb{F}_q$:

(1) The following formula holds

$$\boldsymbol{H}_q(\boldsymbol{\alpha},\boldsymbol{\beta}|z) = \frac{1}{1-q}\sum_{m=0}^{q-2}\prod_{i=1}^{u}\frac{\boldsymbol{g}(\boldsymbol{\psi}, \alpha_i + \frac{m}{1-q}, q^{l_i^\infty})}{\boldsymbol{g}(\boldsymbol{\psi}, \alpha_i, q^{l_i^\infty})}\prod_{i=1}^{v}\frac{\boldsymbol{g}(\boldsymbol{\psi}^{-1}, -\beta_i - \frac{m}{1-q}, q^{l_i^0})}{\boldsymbol{g}(\boldsymbol{\psi}^{-1}, -\beta_i, q^{l_i^0})}\,\mathrm{Teich}(z)^m. \tag{50}$$

(2) *Let L be a finite extension of $F = \mathbb{Q}(\zeta_N)$ and let $\mathfrak{p}$ be a prime ideal of L of norm q. Let $\iota : L \hookrightarrow L_{\mathfrak{p}}$ be the natural map to its completion at $\mathfrak{p}$. The global and local hypergeometric sums are related as follows. For $\xi \in L$ coprime with $\mathfrak{p}$ we have*

$$\iota(H_{\mathfrak{p}}(\boldsymbol{\alpha},\boldsymbol{\beta}|\xi)) = \boldsymbol{H}_q(\boldsymbol{\alpha},\boldsymbol{\beta}|\overline{\xi}), \tag{51}$$

where $\overline{\xi} \in (\mathcal{O}/\mathfrak{p})^\times$ is the reduction of ξ modulo $\mathfrak{p}$. In particular, $\boldsymbol{H}_q(\boldsymbol{\alpha},\boldsymbol{\beta}|\overline{\xi})$ is algebraic over $\mathbb{Q}$; more precisely, $\boldsymbol{H}_q(\boldsymbol{\alpha},\boldsymbol{\beta}|\overline{\xi}) \in \iota(F^H)$.

Proof. 1) Since $q(\alpha_i + \frac{m}{1-q}) = q\alpha_i + \frac{m}{1-q}$ in $\mathbb{Q}/\mathbb{Z}$, the decomposition in q-orbits of $\{\alpha_1 + \frac{m}{1-q}, \dots, \alpha_n + \frac{m}{1-q}\}$ mimics that of $\{\alpha_1, \dots, \alpha_n\}$. Then using Lemma 6.10

$$(\alpha_1)_{q,m}^\infty \cdots (\alpha_n)_{q,m}^\infty = \prod_{i=1}^n \frac{\boldsymbol{\Gamma}_q^\infty(\alpha_i + \frac{m}{1-q})}{\boldsymbol{\Gamma}_q^\infty(\alpha_i)} = \prod_{i=1}^u \frac{\boldsymbol{\Gamma}_{q^{l_i^\infty}}^\infty(\alpha_i + \frac{m}{1-q})}{\boldsymbol{\Gamma}_{q^{l_i^\infty}}^\infty(\alpha_i)}.$$

By Remark 6.9, both $(q^{l_i^\infty} - 1)\alpha_i$ and $(q^{l_i^\infty} - 1)(\alpha_i + \frac{m}{1-q})$ are integers, so Theorem 6.15 implies that the product equals

$$\prod_{i=1}^u \frac{\boldsymbol{g}(\boldsymbol{\psi}, \alpha_i + \frac{m}{1-q}, q^{l_i^\infty})}{\boldsymbol{g}(\boldsymbol{\psi}, \alpha_i, q^{l_i^\infty})} \pi^{(1-p)\left(\eta_{q^{l_i^\infty}}^\infty(\alpha_i + \frac{m}{1-q}) - \eta_{q^{l_i^\infty}}^\infty(\alpha_i)\right)}.$$

The same argument applies to the values $(\beta_i)_{q,m}^0$ using (49). Then the result follows from Lemma 6.10 and the fact that $\pi^{1-p} = (-p)$.

2) Since by assumption $\zeta_N \in L$ we have $N \mid q-1$. Hence, the sets $\{\alpha_1, \dots, \alpha_n\}$ and $\{\beta_1, \dots, \beta_n\}$ decompose into n-disjoint q-orbits of length 1 (so $l_i^\infty = l_i^0 = 1$). As $\iota(\chi_{\mathfrak{p}}) = \mathrm{Teich}^{(q-1)/N}$, $\iota(\chi_{\mathfrak{p}}^{N\alpha_i}(x)) = \mathrm{Teich}^{(q-1)\alpha_i}(x)$ the right hand side of (50) matches the $\iota(H_{\mathfrak{p}}(\boldsymbol{\alpha},\boldsymbol{\beta}|z))$. The algebricity statement follows from (47) and Proposition 6.2 (4). □

Remark 6.18. If $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$ be are not q-stable then most likely $H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|z)$ (as defined in (45)) is *not* algebraic.

Remark 6.19. If L is a finite extension of K and $\mathfrak{p}$ is a prime ideal of L of norm q, then the parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are q-stable. Hence by Theorem 6.17 we may extend the definition of $\mathfrak{p}$-hypergeometric sums to $\xi \in L$ as

$$H_{\mathfrak{p}}(\boldsymbol{\alpha},\boldsymbol{\beta}|\xi) := \frac{1}{1-q} \sum_{\varphi} \prod_{i=1}^u \frac{g(\psi, \chi_{\mathfrak{q}}^{-N\alpha_i}\varphi, \mathfrak{q})}{g(\psi, \chi_{\mathfrak{q}}^{-N\alpha_i}, \mathfrak{q})} \prod_{i=1}^v \frac{g(\psi^{-1}, \chi_{\mathfrak{q}}^{N\beta_i}\overline{\varphi}, \mathfrak{q})}{g(\psi^{-1}, \chi_{\mathfrak{q}}^{N\beta_i}, \mathfrak{q})} \varphi(\xi),$$

where the sum runs over characters of $\mathbb{F}_q^\times$. We expect this sum to be the trace of $\mathrm{Frob}_{\mathfrak{p}}$ of a motive defined over L (see Conjecture 4.5).

6.3. **Hypergeometric character sums.** The result of the present section will be used to relate the number of points of a variety to a finite hypergeometric sum. Although the definition of Gauss sums depend on the choice of an additive character, the products/quotients considered in the present section will not, so we omit writing the dependence to easy notation. Recall the following well known definition.

Definition 6.20. *Let φ, η be two multiplicative characters on $\mathbb{F}_q^\times$. The Jacobi sum attached to them is defined by*

$$J(\varphi, \eta) = \sum_{x \in \mathbb{F}_q} \varphi(x)\eta(1-x),$$

where $\varphi(0) = \eta(0) = 0$.

Lemma 6.21. *Let φ, η be characters of $\mathbb{F}_q$, with η non-trivial. Then*

$$J(\varphi, \eta) = \varphi(-1)\frac{g(\varphi^{-1}\eta^{-1})g(\varphi)}{g(\eta^{-1})}.$$

Proof. If φ and $\varphi\eta$ are non-trivial, then the left hand side is the usual Jacobi sum, and its value equals

$$J(\varphi, \eta) = \frac{g(\varphi)g(\eta)}{g(\varphi\eta)}.$$

For φ a non-trivial character, the equality $g(\varphi) = \varphi(-1)\frac{q}{g(\varphi^{-1})}$ applied to $\varphi = \varphi\eta$ and η implies that

$$J(\varphi, \eta) = \varphi(-1)\frac{g(\varphi)g(\varphi^{-1}\eta^{-1})}{g(\eta^{-1})}.$$

If $\varphi = 1$, the left hand side equals -1 (because $\varphi(0) = 0$), which equals the right hand side as well. At last, if $\varphi\eta = 1$, then the left hand side equals $J(\varphi, \varphi^{-1}) = -\varphi(-1)$, which also equals the value of the right hand side, since $g(1) = -1$. □

Let $\varepsilon_1, \eta_1, \ldots, \varepsilon_{n-1}, \eta_{n-1}, \chi$ be characters of $\mathbb{F}_q^\times$ (extended to $\mathbb{F}_q$ by setting their value at 0 to be 0). For $z \in \mathbb{F}_q$ let

$$H(z) = \sum_{x_1,\ldots,x_{n-1}} \prod_{i=1}^{n-1} \varepsilon_i(x_i)\eta_i(1-x_i)\chi^{-1}(1 - zx_1\cdots x_{n-1}). \tag{52}$$

Theorem 6.22. *Keep the previous notation, and set $\varepsilon_n = \chi$ and $\varepsilon_n\eta_n = 1$. Let $\boldsymbol{\alpha}, \boldsymbol{\beta}$ be rational numbers such that $\varpi^{(q-1)\alpha_i} = \varepsilon_i$ and $\varpi^{(q-1)\beta_i} = \varepsilon_i\eta_i$ for $i = 1, \ldots, n$. Then $H(z)$ equals*

$$H(z) = (-1)^{n-1}\left(\prod_{i=1}^{n-1}\varepsilon_i(-1)\right)\mathbf{J}((\boldsymbol{\alpha}, -\boldsymbol{\beta}), (\boldsymbol{\alpha} - \boldsymbol{\beta}))H_q(\boldsymbol{\alpha}, \boldsymbol{\beta}|z).$$

Proof. Add one more variable x_n to the definition of $H(z)$ defined by

$$x_n := zx_1\cdots x_{n-1}.$$

Then

$$H(z) = \frac{1}{q-1}\sum_{\varphi}\sum_{x_1,\ldots,x_n}\prod_{i=1}^{n-1}\varepsilon_i(x_i)\eta_i(1-x_i)\chi^{-1}(1-x_n)\varphi(zx_n^{-1}x_1\cdots x_{n-1}),$$

where the first sum ranges over all characters φ of $\mathbb{F}_q^\times$ (we are abusing notation, declaring x_n^{-1} to be 0 if x_n equals 0). Interchanging sum and product, we get

$$H(z) = \frac{1}{q-1}\sum_{\varphi}\left(\prod_{i=1}^{n-1}\left(\sum_{x_i}\varepsilon_i\varphi(x_i)\eta_i(1-x_i)\right)\cdot\sum_{x_n}\varphi^{-1}(x_n)\chi^{-1}(1-x_n)\right)\varphi(z). \tag{53}$$

Applying Lemma 6.21 we get that

$$\begin{aligned} H(z) &= \frac{1}{q-1}\sum_{\varphi}\left(\prod_{i=1}^{n-1}\varepsilon_i(-1)\varphi(-1)\frac{g(\varepsilon_i\varphi)g(\varepsilon_i^{-1}\eta_i^{-1}\varphi^{-1})}{g(\eta_i^{-1})}\right)\cdot\frac{g(\varphi\chi)g(\varphi^{-1})}{g(\chi)}\varphi(-1)\varphi(z) \\ &= \prod_{i=1}^{n-1}\frac{\varepsilon_i(-1)g(\varepsilon_i)g(\varepsilon_i^{-1}\eta_i^{-1})}{g(\eta_i^{-1})}\left(\frac{(-1)}{(q-1)}\sum_{\varphi}\left(\prod_{i=1}^{n}\frac{g(\varepsilon_i\varphi)g(\varepsilon_i^{-1}\eta_i^{-1}\varphi^{-1})}{g(\varepsilon_i)g(\varepsilon_i^{-1}\eta_i^{-1})}\right)\varphi(-1)^n\varphi(z)\right), \end{aligned}$$

as claimed (with the convention $\varepsilon_n = \chi$ and $\varepsilon_n\eta_n = 1$), the extra -1 coming from the fact that $g(1) = -1$. □

7. Zeta function of Superelliptic curves

For this section let N be a positive integer and L be a number field or a local field containing the N-th roots of unity. Let $f(x) \in L[x]$. Let $\mathcal{C}$ be the superelliptic curve with equation

$$\mathcal{C} : y^N = f(x). \tag{54}$$

Assume that the curve $\mathcal{C}$ is irreducible and the model is *minimal* in the sense that all irreducible factors of $f(x)$ have degree less than N. Let $\widehat{\mathcal{C}}$ be a smooth projective model. The group μ_N of N-th roots of unity in L acts on the L-rational points $\mathcal{C}(L)$ of $\mathcal{C}$ by

$$\zeta \cdot (x, y) = (x, \zeta y), \qquad \zeta \in \mu_N.$$

7.1. **The new part.** Let ℓ be a prime number. There is a natural decomposition

$$H^1_{\text{ét}}(\widehat{\mathcal{C}}, \mathbb{Q}_\ell) = H^1_{\text{ét}}(\widehat{\mathcal{C}}, \mathbb{Q}_\ell)^{\text{new}} \oplus H,$$

where H is the sum of the contributions of the curves

$$\mathcal{C}_d : \quad y^d = f(x),$$

for d a proper divisor of N.

Let $F := \mathbb{Q}(\zeta_N)$ and let λ be a prime ideal of F. Define

$$H^1_{\text{ét}}(\widehat{\mathcal{C}}, F_\lambda)^{\text{new}} := H^1_{\text{ét}}(\widehat{\mathcal{C}}, \mathbb{Q}_\ell)^{\text{new}} \otimes_{\mathbb{Q}_\ell} F_\lambda.$$

The action of the group μ_N on the curve $\mathcal{C}$ induces a decomposition

$$H^1_{\text{ét}}(\widehat{\mathcal{C}}, F_\lambda)^{\text{new}} := \bigoplus_{\chi} H^1_{\text{ét},\chi}(\widehat{\mathcal{C}}, F_\lambda), \tag{55}$$

where $H^1_{\text{ét},\chi}(\widehat{\mathcal{C}}, F_\lambda)$ is the χ-eigenspace for $\chi : \mu_N \to F_\lambda^\times$ a character of order N.

Let $\mathcal{O}$ be the discrete valuation ring consisting of the ring of integers of L (when L is a local field) or a completion of its ring of integers at a prime ideal (when L is a global field). Let $\mathfrak{p}$ be its maximal ideal and k its residue field, a finite field of characteristic p with q elements (so $N \mid q-1$). Note that $\text{Frob}_\mathfrak{p}$ preserves the decomposition (55) as L contains μ_N.

For the rest of the section we make the following assumptions.

Assumption 1. *Keeping the previous notation, the polynomial f, the integer N and the residual characteristic p satisfy the following properties:*

(1) $p \nmid N$,
(2) $p \nmid \text{disc}\left(\frac{f}{\gcd(f,f')}\right)$,
(3) The leading coefficient c of f is a unit in $\mathcal{O}$,
(4) The curve $\mathcal{C}$ is irreducible over $\overline{L}$.

These assumptions assure that the reduction of $\widehat{\mathcal{C}}$ modulo $\mathfrak{p}$ is smooth.

Remark 7.1. For $\mathcal{C}$ the Euler curve (64) specialized at $z = \xi \in \mathbb{P}^1(L) \setminus \{0, 1, \infty\}$, a prime ideal $\mathfrak{p}$ satisfies Assumption 1 precisely when $\mathfrak{p} \in S_g$ (see 4).

Definition 7.2. *Let ω be a character of $(\mathcal{O}/\mathfrak{p})^\times$. The* counting function $\mathcal{N}(\omega)$ *is defined by*

$$\mathcal{N}(\omega) := \sum_{x \in \mathbb{F}_q} \omega(f(x)) + \begin{cases} \omega(c) & \text{if } N \mid \deg(f), \\ 0 & \text{otherwise.} \end{cases} \tag{56}$$

Fix a generator ζ of $\mu_N \subset F \subset L$. Let $k \in \mathbb{F}_q$ be the reduction of ζ modulo $\mathfrak{p}$ and let $\alpha \in \overline{\mathbb{F}_q}$ be such that $\alpha^{q-1} = k$. Then $\varepsilon := \alpha^N \in \mathbb{F}_q$ (since it is a root of $x^{q-1} - 1$) and generates $\mathbb{F}_q^\times$.

Lemma 7.3. *The character $\chi_{\mathfrak{p}}$ (defined in (22)) satisfies*

$$\chi_{\mathfrak{p}}(\varepsilon) = \zeta. \tag{57}$$

Proof. By definition $\chi_{\mathfrak{p}}(\varepsilon) = \chi_{\mathfrak{p}}(\alpha^N) \equiv (\alpha^N)^{(q-1)/N} = \alpha^{q-1} = \kappa \equiv \zeta \pmod{\mathfrak{p}}$. □

For λ a prime ideal of F define the group morphism $\chi_{0,\lambda} : \boldsymbol{\mu}_N \to F_\lambda^\times$ to be the restriction of the map $\iota : F \hookrightarrow F_\lambda$ to $\boldsymbol{\mu}_N$.

Theorem 7.4. *Let λ be a prime ideal of F. Let $\mathcal{C}$ be the superelliptic curve given by (54) for $N > 1$ and suppose Assumption 1 holds. Then the trace of $\mathrm{Frob}_{\mathfrak{q}}$ on $H^1_{et,\chi_{0,\lambda}}(\widehat{\mathcal{C}}, F_\lambda)$ equals $-\iota_\lambda(\mathcal{N}(\chi_{\mathfrak{p}}))$.*

Proof. It is not hard to verify that

$$\mathrm{Tr}(\mathrm{Frob})|_{H^1_{\text{ét}}(\widehat{\mathcal{C}},F_\lambda)^{\text{new}}} = -\sum_{\omega}\sum_{x\in k}\omega(f(x)) - \begin{cases} \sum_\omega \omega(c) & \text{if } N \mid \deg(f), \\ 0 & \text{otherwise,} \end{cases} \tag{58}$$

where ω runs over characters of $k^\times$ of order N and c denotes the leading coefficient of f. For $\chi : \boldsymbol{\mu}_N \to F_\lambda^\times$ a character of order N, let A_χ be the matrix of $\mathrm{Frob}_{\mathfrak{p}}$ acting on $\mathrm{H}^1_{\text{ét},\chi}(\widehat{\mathcal{C}}, \mathrm{F}_\lambda)$. By (58)

$$\sum_{\chi}\mathrm{Tr}(A_\chi) = -\sum_{\omega}\iota_\lambda(\mathcal{N}(\omega)).$$

Let $X/\mathbb{F}_q$ be a non-singular variety and let $\sigma \in \mathrm{Aut}_{\mathbb{F}_q}(X)$. By Lefschetz's trace formula

$$\#\{x \in X(\overline{\mathbb{F}_q}) : \mathrm{Frob}_q(\sigma(x)) = x\} = \sum_{i=1}^{2\dim(X)} (-1)^i \mathrm{Tr}(\mathrm{Frob}_q \circ \sigma)|_{H^i_{\text{ét}}(X,F_\lambda)}. \tag{59}$$

Let σ_i be the automorphism of $\widehat{\mathcal{C}}$ defined by $(x, y) \mapsto (x, \zeta^i y)$. Then (59) with $\sigma = \sigma_i$ gives

$$\#\{(x,y) \in \widehat{\mathcal{C}}(\mathbb{F}_q) : x \in \mathbb{F}_q, y^{q-1} = k^{-iq}\} = 1 + q - \sum_{\chi}\chi(\zeta)^i\,\mathrm{Tr}(A_\chi) + \text{“old contribution”}, \tag{60}$$

where *“old contribution”* comes from the curves $\mathcal{C}_d$ for a proper divisors $d \mid N$.

Since $k \in \mathbb{F}_q$, $k^q = k$ and

$$\{(x,y) \in \widehat{\mathcal{C}}(\mathbb{F}_q) : x \in \mathbb{F}_q, y^{q-1} = k^{-iq}\} = \{(x,y) \in \widehat{\mathcal{C}}(\mathbb{F}_q) : x \in \mathbb{F}_q, y^{q-1} = k^{-i}\}.$$

Recall that $\alpha \in \overline{\mathbb{F}_q}$ satisfies $\alpha^{q-1} = k$ and $\varepsilon := \alpha^N$. Then the map $y \leftrightarrow \tilde{y} := y\alpha^i$ gives a bijection between the sets

$$\{(x,y) \in \widehat{\mathcal{C}}(\mathbb{F}_q) \text{ fixed by } \mathrm{Frob}_q \circ \sigma_i\} \leftrightarrow \{(x,\tilde{y}) \in \mathbb{F}_q^2 : \varepsilon^{-i}\tilde{y}^N = f(x)\}, \tag{61}$$

Indeed,

$$y^{q-1} = k^{-i} \Leftrightarrow \frac{\tilde{y}^{q-1}}{(\alpha^{q-1})^i} = k^{-i} \Leftrightarrow \tilde{y}^q = \tilde{y}.$$

The elements of the right hand side of (61) are the $\mathbb{F}_q$-points of the twisted curve

$$\mathcal{C}_i : y^N = \varepsilon^i f(x).$$

Then the left hand side of (60) equals

$$\#\widehat{\mathcal{C}_i}(\mathbb{F}_q) = 1 + q + \sum_{\omega}\omega(\varepsilon^i)\mathcal{N}(\omega) + \text{“old contribution”}$$

so for all i

$$\sum_{\chi}\chi(\zeta)^i\,\mathrm{Tr}(A_\chi) = -\sum_{\omega}\iota_\lambda(\omega(\varepsilon))^i\iota_\lambda(\mathcal{N}(\omega)).$$

It follows that $\mathrm{Tr}(A_{\chi_{0,\lambda}}) = -\iota_\lambda(\mathcal{N}(\omega))$ for ω the character of $\mathbb{F}_q$ satisfying $\iota_\lambda(\omega(\varepsilon)) = \chi_{0,\lambda}(\zeta) = \iota_\lambda(\zeta)$, which by Lemma 7.3 corresponds to $\omega = \chi_{\mathfrak{p}}$. □

8. Rank two Hypergeometric Motives

The main result of [8] provides an explicit formula relating the function $H_p(\vec{\alpha},\vec{\beta}|\xi)$ to the number of points of a non-singular projective variety V when the parameters are rational. Such relations allow to realize the motive $\mathcal{H}(\vec{\alpha},\vec{\beta}|\xi)$ in V. A nice instance of their result is the following result (see for example [39]).

Theorem 8.1. *Let p be an odd prime power and $\xi \in \mathbb{F}_p$ and $\xi \neq 0,1$. Let E_ξ be the Legendre elliptic curve with affine equation*

$$E_\xi : y^2 = x(x-1)(x-\xi)$$

Then the set of $\mathbb{F}_p$-rational points (including the one at infinity) equals

$$|E_\xi(\mathbb{F}_p)| = p + 1 - (-1)^{(p-1)/2} H_p((1/2,1/2),(1,1)|\xi).$$

Equivalently,

$$H_p((1/2,1/2),(1,1)|\xi) = (-1)^{(p-1)/2} a_p(E_\xi), \tag{62}$$

where $a_p(E_\xi)$ is the trace of the Frobenius endomorphism acting on E_ξ.

Our definition of the geometric realization of the hypergeometric motive (Definition 8.11) yields an analogue of (62). The hypergeometric motive appears in H^1 of the associated Euler curve (Definition 8.2) tensored with a Jacobi motive and the explicit relation between the corresponding trace of Frobenius and the finite hypergeometric sum is a particular instance of Theorem 8.15 below.

Definition 8.2. *Let (a,b),(c,d) be rational numbers, and let N be their least common denominator. Define the quantities*

$$A = (d-b)N, \qquad B = (b+1-c)N, \qquad C = (1+a-d)N, \qquad D = (d-1)N. \tag{63}$$

Assume (A,B,C,D) are non-negative. Define the Euler curve attached to $(a,b),(c,d)$ as the affine curve over $\mathbb{Q}(z)$ with equation

$$\mathcal{C} : y^N = x^A(1-x)^B(1-zx)^C z^D. \tag{64}$$

Remark 8.3. The Euler curve depends only on (A,B,C,D) mod N so it maybe defined for any rationals $(a,b),(c,d)$ with common denominator N.

Definition 8.4. *Let $(a,b),(c,d)$ be a pair of generic rationals parameters. The parameters satisfy condition* (**Irr**) *if the Euler curve (64) is irreducible over $\overline{\mathbb{Q}(z)}$.*

Lemma 8.5. *Let (a,b),(c,d) be generic rational numbers and let N be their least common denominator. Then condition* (**Irr**) *holds if and only if*

$$N = \mathrm{lcm}\{\mathrm{den}(d-b), \mathrm{den}(b-c), \mathrm{den}(a-d)\}. \tag{65}$$

Proof. By making the change of variables (over $\overline{\mathbb{Q}(z)}$) $y = \sqrt[N]{z^D}y$, it is enough to study the curve

$$y^N = x^A(1-x)^B(1-zx)^C,$$

which is irreducible if and only of $\gcd(A,B,C,N) = 1 = \gcd((d-b)N,(b-c)N,(a-d)N,N)$. The later equality holds if and only if

$$\mathrm{lcm}\{\mathrm{den}(d-b),\mathrm{den}(b-c),\mathrm{den}(a-d)\} = N = \mathrm{lcm}\{\mathrm{den}(a),\mathrm{den}(b),\mathrm{den}(c),\mathrm{den}(d)\}.$$

□

Remark 8.6. If r denotes the quotient of N by $\mathrm{lcm}\{\mathrm{den}(d-b),\mathrm{den}(b-c),\mathrm{den}(a-d)\}$, then the curve $\mathcal{C}$ decomposes as the union of r irreducible components defined over $\mathbb{Q}(\zeta_r)[\sqrt[r]{z}]$. This phenomena does not appear when $d=1$ since then condition (**Irr**) always holds.

Example 9. Continuing with Example 2, let $a = 1/8, b = 7/8, c = 3/8, d = 5/8$ so $N = 8$. The Euler curve is defined by

$$\mathcal{C}: \quad y^8 = x^6(1-x)^4(1-zx)^4z^5.$$

Over $\mathbb{Q}(w)$, where $w^2 = z$, it becomes the union of the curves

$$\mathcal{C}_\pm: \quad y^4 = \pm wx^3(1-x)^2(1-w^2x)^2.$$

These curves have genus two with hyperelliptic model

$$y^2 = wx(x^2 \mp w)(x^2 \mp w^{-1})$$

The map

$$\iota: \quad (x,y) \mapsto (x^{-1}, yx^{-3})$$

yields an involution of these curves and it is not to hard to see that the quotient of $\mathcal{C}_\pm$ by ι equals

$$E_\pm : y^2 = x^3 - 4wx^2 \mp w(w-1)^2x.$$

For $d \in \mathbb{Z}$, let θ_d be the quadratic character of the extension $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$. It can be verified (using the involution $(x,y) \to (x^{-1}, -yx^{-3})$) that actually

$$\mathrm{Jac}(\mathcal{C}_\pm) \simeq E_\pm(w) \oplus E_\pm(w) \otimes \theta_{-1}.$$

(See Examples 12 and 14 for more on this curve.)

Remark 8.7. The point $P := (0,0)$ is a 2-torsion point on $E_\pm(w)$ and

$$E_\pm(w)/\langle P \rangle \simeq E_\mp(w) \otimes \theta_{-2}.$$

Over the cyclotomic extension $\mathbb{Q}(\zeta_8)$

$$\mathrm{Jac}(\mathcal{C}_\pm) \simeq E_+(w) \oplus E_+(w).$$

If we now take $w = 3$, so $z = 9$ as in Example 2, the curve $E_+(w)$ specializes to the elliptic curve

$$E: \quad y^2 = x^3 - 60x + 176,$$

of conductor 576 and CM by $\mathbb{Q}(\sqrt{-12})$. We can quickly verify for small primes p split in $\mathbb{Q}(\sqrt{2})$ that the trace of Frobenius Frob_p on $H^1(E)$ and on $\mathcal{H}_9$ indeed agree (the latter may be computed in $\mathbb{Z}_p$ as

```
? e=ellinit([0,0,0,-60,176]);
? forprime(p=5,100,if(kronecker(2,p)==1,
    print(p,"  ",ellap(e,p),"  ",recognizep(hgm(9,[1/8,-1/8],[3/8,-3/8],p)*p))))

7  -4  -4
17  0  0
23  0  0
31  -4  -4
41  0  0
47  0  0
71  0  0
73  -10  -10
79  -4  -4
89  0  0
97  14  14
```

with the GP code already mentioned).

8.1. **Hypergeometric motive definition.** In this section we give a definition of a hypergeometric motive for general parameters in the case of rank two. For this we will use the Euler curve. Many issues arise, however, if the Euler curve is reducible.

We proceed as follows. First, we consider the case where the Euler curve is irreducible, see Definition 8.11, and $(a,b),(c,d)$ are generic and rational but otherwise arbitrary.

Second, we give an alternative definition, see Definition 8.14, which includes the case where the Euler curve is reducible, as a twist of the motive attached to the (irreducible) Euler curve for parameters $(a-d,b-d),(c-d,1)$ (in analogy with formula (9)).

The fact that both definitions essentially agree when the Euler curve is irreducible is not entirely obvious; we prove it in Theorem 8.23. Neither is it clear why the definitions are independent of the ordering of a,b,c,d since this is not true for either the corresponding Euler curve or Jacobi motive. However, we will prove that this is the case in Proposition 2.

We start by studying the analogue of the character $(-1)^{(p-1)/2}$ in (62).

Definition 8.8. *Let N be a positive integer, and let $\varkappa_N$ be the character of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_N))$ (of order dividing 2) corresponding to the abelian extension $\mathbb{Q}(\zeta_{2N})/\mathbb{Q}(\zeta_N)$.*

Lemma 8.9. *Let $\mathfrak{p}$ be a prime ideal of $F=\mathbb{Q}(\zeta_N)$ not dividing $2N$. Then*

$$\varkappa_N(\mathfrak{p})=\chi_{\mathfrak{p}}(-1)=(-1)^{(\mathcal{N}\mathfrak{p}-1)/N}, \tag{66}$$

where $\chi_{\mathfrak{p}}$ is defined in (22).

Proof. The second equality follows from the fact that $\chi_{\mathfrak{p}}(-1)\equiv(-1)^{(\mathcal{N}\mathfrak{p}-1)/N} \bmod \mathfrak{p}$ by definition. Let $t=v_2(N)\geq 0$. Let $\mathcal{N}\mathfrak{p}=p^r$ for p a rational prime number, so r is the order of p modulo N. Then

$$(-1)^{(\mathcal{N}\mathfrak{p}-1)/N}=(-1)^{(p^r-1)/2^t}.$$

Then the right hand side equals 1 if $p^r\equiv 1 \pmod{2^{t+1}}$ and -1 otherwise. By the Chinese reminder theorem it equals 1 if $p^r\equiv 1 \pmod{2N}$ and -1 otherwise; note that $p^r\equiv 1 \pmod{2N}$ precisely when $\mathfrak{p}$ splits in $\mathbb{Q}(\zeta_{2N})/F$. □

Corollary 8.10. *Suppose that $t=v_2(N)\geq 1$. Let κ be a primitive Dirichlet character of conductor 2^{t+1} thought as a character of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Then the quadratic character $\varkappa$ equals the restriction of κ to $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$.*

Proof. The fixed field of the character κ corresponds to the Galois extension $\mathbb{Q}(\zeta_{2^{t+1}})$. Then the restriction of κ to $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ has as fixed field $\mathbb{Q}(\zeta_{2N})=\mathbb{Q}(\zeta_N)\cap\mathbb{Q}(\zeta_{2^{t+1}})$. □

Let $H^1_{\chi_0}(\widehat{\mathcal{C}},F)^{\mathrm{new}}$ denote the χ_0 isotypical component of the de Rham cohomology group $H^1(\widehat{\mathcal{C}},F)=H^1(\widehat{\mathcal{C}},\mathbb{Q})\otimes_{\mathbb{Q}}F$, where $\chi_0:\boldsymbol{\mu}\to F^\times$ is the identity character. It is known to be of rank two (see [4] formula (16)).

Definition 8.11. *Let $(a,b),(c,d)$ be a pair of generic rational numbers satisfying condition* (**Irr**) *and let N be their least common denominator. The hypergeometric motive with parameters $(a,b),(c,d)$ is defined as*

$$\mathcal{H}((a,b),(c,d)|z):=H^1_{\chi_0}(\widehat{\mathcal{C}},F)^{new}\otimes\mathbf{J}((-a,-b,c,d),(c-b,d-a))^{-1}\varkappa_N^{(d-b)N}. \tag{67}$$

Example 10. Let $\boldsymbol{\alpha}=(1/2,1/2)$ and $\boldsymbol{\beta}=(1,1)$ as in Example 1. Formula (63) gives the values $A=1$, $B=1$, $C=1$, $D=0$ and $N=2$. Then Euler's curve corresponds to Legendre's elliptic curve

$$E_z: y^2=x(1-x)(1-zx).$$

The Jacobi motive $\mathbf{J}((-1/2,-1/2,1,1)(1/2,1/2))$ is trivial. Then the hypergeometric motive $\mathcal{H}((1/2,1/2),(1,1)|z)$ corresponds to the quadratic twist by the character $\left(\frac{-4}{\cdot}\right)$ of Legendre's elliptic curve (as expected from (62)).

Formula (67) involves three different motives, with a priory different fields of definition (see Example 11). This is the reason why we cannot prove a full version of Conjecture 4.5 (but we can under some extra hypotheses, for example, conditions that make the Jacobi motive to be defined over $\mathbb{Q}$). To prove cases not covered by our assumptions, one probably needs to study other geometric varieties where the HGM appears. Let us illustrate the situation with some examples.

Example 11. Let $a=\frac{1}{15}, b=\frac{4}{15}, c=\frac{1}{8}$ and $d=\frac{3}{8}$. The Jacobi motive for these parameters is $\mathbf{J}(\boldsymbol{\theta})$, where

$$\boldsymbol{\theta}=\left\langle\frac{14}{15}\right\rangle+\left\langle\frac{11}{15}\right\rangle+\left\langle\frac{1}{8}\right\rangle+\left\langle\frac{3}{8}\right\rangle-\left\langle\frac{103}{120}\right\rangle-\left\langle\frac{37}{120}\right\rangle.$$

Since 19 is the only element in $(\mathbb{Z}/120)^\times$ that fixes (under multiplication) the sets $\{\frac{14}{15},\frac{11}{15},\frac{1}{8},\frac{3}{8}\}$ and $\{\frac{103}{120},\frac{37}{120}\}$ (in $\mathbb{Q}/\mathbb{Z}$), the Jacobi motive is defined over $F_0:=\mathbb{Q}(\zeta_{120})^{\sigma_{19}}$. The base field K of $\mathcal{H}((\frac{1}{15},\frac{4}{15}),(\frac{1}{8},\frac{3}{8})|z)$, however, is that fixed by $H=\langle 49,91\rangle$. In particular, $K\subsetneq F_0$.

On the other hand, if we take $a=\frac{1}{8}, b=\frac{7}{8}, c=\frac{3}{8}$ and $d=\frac{5}{8}$ then the Jacobi motive is defined over $\mathbb{Q}$ (since multiplication by any odd integer fixes the sets $\{\frac{1}{8},\frac{3}{8},\frac{5}{8},\frac{7}{8}\}$ and $\{\frac{1}{2}\}$), while the base field of the corresponding hypergeometric motive is $\mathbb{Q}(\sqrt{2})$.

Definition 8.12. *Let $\alpha=\frac{a}{N}$ be a rational number, with a,N coprime integers. Define*

$$\eta_\alpha:\mathrm{Gal}(\overline{\mathbb{Q}(z)}/\mathbb{Q}(z,\zeta_N))\to\overline{\mathbb{Q}}^\times,$$

to be the character that factors through $\mathrm{Gal}(\mathbb{Q}(\sqrt[N]{z},\zeta_N)/\mathbb{Q}(z,\zeta_N))$ *and whose value at σ equals*

$$\eta_\alpha(\sigma)=\left(\frac{\sigma(\sqrt[N]{z})}{\sqrt[N]{z}}\right)^a.$$

Remark 8.13. The character η_α is a realization of the hypergeometric motive $\mathcal{H}(\alpha,1|1-z)$ studied in §2.

Let $\xi\in F=\mathbb{Q}(\zeta_N)$, $\xi\neq 0$ and consider the specialization of η_α at $z=\xi$. Let $\mathfrak{p}$ be a prime ideal of F of norm q, not dividing N and such that $v_\mathfrak{p}(\xi)=0$. Let $\widetilde{\mathfrak{p}}$ be a prime ideal of $\mathbb{Q}(\sqrt[N]{\xi},\zeta_N)$ dividing $\mathfrak{p}$. It follows from its definition that

$$\mathrm{Frob}_{\widetilde{\mathfrak{p}}}(\sqrt[N]{\xi})\equiv(\sqrt[N]{\xi})^q\pmod{\widetilde{\mathfrak{p}}},$$

hence

$$\frac{\mathrm{Frob}_{\widetilde{\mathfrak{p}}}(\sqrt[N]{\xi})}{\sqrt[N]{\xi}}\equiv\xi^{\frac{q-1}{N}}\pmod{\mathfrak{p}}.$$

This implies that the specialization of η_α at ξ satisfies

$$\eta_\alpha(\mathrm{Frob}_\mathfrak{p})=\chi_\mathfrak{p}(\xi)^a, \tag{68}$$

where $\chi_\mathfrak{p}$ is the character defined in (22).

Definition 8.14. *Let $(a,b),(c,d)$ be generic rational numbers, and let N be their least common denominator. Set $F=\mathbb{Q}(\zeta_N)$. The hypergeometric motive $\mathcal{H}((a,b),(c,d)|z)$ is defined by*

$$\mathcal{H}((a-d,b-d),(c-d,1)|z)\otimes\mathbf{J}((-a,-b,c,d),(d-a,d-b,c-d))^{-1}\eta_d(z), \tag{69}$$

where the hypergeometric motive $\mathcal{H}((a-d,b-d),(c-d,1);z)$ (base changed to $F(z)$ if necessary) is defined in 8.11.

This definition of $\mathcal{H}((a,b),(c,d)|z)$ has the disadvantage of making more difficult to understand its descent to smaller fields. In this regard, it is not clear whether there is a more suitable definition of the motive when condition (**Irr**) is not satisfied.

Example 12. Continuing with Example 2, take $a = 1/8, b = 7/8, c = 3/8, d = 5/8$, so $N = 8$. The values $\tilde{a} = a - d = 1/2$, $\tilde{b} = b - d = 1/4$ and $\tilde{c} = c - d = 3/4$ have denominator 4 so $\mathcal{H}((a-d, b-d), (c-d, 1); z)$ has base field $\mathbb{Q}(i) \subsetneq F := \mathbb{Q}(\zeta_8)$. Also, the curve $\mathcal{C}$ is reducible (as described in detail in Example 9). The Euler curve of parameters $(\tilde{a}, \tilde{b}), (\tilde{c}, 1)$ has the affine equation

$$\widetilde{\mathcal{C}} : y^4 = x^3(1-x)^2(1-zx)^2.$$

It is an irreducible curve isomorphic to a twist of an irreducible component of $\mathcal{C}$. Note that $\mathcal{H}((1/2, 1/4), (3/4, 1)|z)$ is defined over its base field $\mathbb{Q}(i)$ whereas $\mathcal{H}((1/8, 7/8), (3/8, 5/8)|z)$ has base field $\mathbb{Q}(\sqrt{2})$ (see Example 4 and Corollary 9.12).

Theorem 8.15. *Let $(a, b), (c, d)$ be generic rational parameters and let N be their least common denominator. Let L be a finite extension of F. Let $\xi \in L$ with $\xi \neq 0, 1$. Let $\mathfrak{p}$ be a prime ideal of L satisfying that $\mathfrak{p} \nmid N$ and $v_{\mathfrak{p}}(\xi(\xi-1)) = 0$. Then the trace of the Frobenius automorphism $\mathrm{Frob}_{\mathfrak{p}}$ acting on $\mathcal{H}((a, b), (c, d)|\xi)$ equals $H_{\mathfrak{p}}((a, b), (c, d)|\xi)$.*

Before proving the statement, recall that we gave two different definitions of $\mathcal{H}((a, b), (c, d)|\xi)$ depending on whether (**Irr**) holds or not. The theorem holds for both definitions. We need the following auxiliary result.

Lemma 8.16. *Let $(a, b), (c, d)$ be rational generic parameters and let N be their least common denominator. Let $\mathfrak{p}$ be a prime ideal of L not dividing N and such that $v_{\mathfrak{p}}(\xi(\xi-1)) = 0$. Then*

$$H_{\mathfrak{p}}((a-d, b-d), (c-d, 1)|\xi) = \chi_{\mathfrak{p}}(\xi)^{-dN}\, \mathbf{J}((-a, -b, c, d), (d-a, d-b, c-d))H_{\mathfrak{p}}((a, b), (c, d)|\xi).$$

Proof. Let $q = \mathcal{N}\mathfrak{p}$ and let ϖ be a generator of the character group $\widehat{\mathbb{F}_q^\times}$ so that $\varpi^{(q-1)/N} = \chi_{\mathfrak{p}}^{-1}$. Then [8, Theorem 3.4] implies that

$$H_{\mathfrak{p}}((a-d, b-d), (c-d, 1)|\xi) = \varpi(\xi)^{d(q-1)} \frac{g(\chi_{\mathfrak{p}}^{-aN})g(\chi_{\mathfrak{p}}^{cN})}{g(\chi_{\mathfrak{p}}^{(d-a)N})g(\chi_{\mathfrak{p}}^{(c-d)N})} \frac{g(\chi_{\mathfrak{p}}^{-bN})g(\chi_{\mathfrak{p}}^{dN})}{g(\chi_{\mathfrak{p}}^{(d-b)N})g(1)} H_{\mathfrak{p}}((a, b), (c, d)|\xi).$$

From its definition (Definition 5.2) the middle factor equals the stated Jacobi motive. As a side remark, the first statement of [8, Theorem 3.4] is correct, but the second one is not. The numerator should be the denominator and vice-versa (since the function S_q is the product of H_q with a Gauss sum involving the same parameters, as follows from (3.1) of loc. cit.) □

Proof of Theorem 8.15. Suppose that (**Irr**) holds and let $\mathcal{C}$ denote the Euler curve as defined in (64). The hypothesis on $\mathfrak{p}$ implies (by Remark 7.1) that Assumption 1 holds. The genericity condition on the parameters imply that $a-c, b-c, a-d, b-d \notin \mathbb{Z}$, so $N \nmid A$, $N \nmid B$, $N \nmid C$ and $N \nmid \deg(f(x)) = A + B + C = (a-c)N$. By Theorem 7.4 the trace of Frobenius on $\mathrm{H}^1_{\mathrm{et},\chi}(\widehat{\mathcal{C}}, \mathbb{Q}_\ell)$ equals

$$-\mathcal{N}(\chi_{\mathfrak{p}}; \xi) = -\chi_{\mathfrak{p}}(\xi)^D \sum_{x \in \mathbb{F}_q} \chi_{\mathfrak{p}}(x^A(1-x)^B(1-\xi x)^C),$$

where $A = (d-b)N$, $B = (b-c)N$, $C = (a-d)N$ and $D = dN$. Set $\alpha_1 = (d-b)N$, $\alpha_2 = (d-a)N$, $\beta_1 = (d-c)N$ and $\beta_2 = 0$. Then Theorem 6.22 implies that

$$-\mathcal{N}(\chi_{\mathfrak{p}}; \xi) = \chi_{\mathfrak{p}}(\xi)^D \chi_{\mathfrak{p}}(-1)^A\, \mathbf{J}((\alpha_1, -\beta_1), (\alpha_1 - \beta_1))H_q(\boldsymbol{\alpha}, \boldsymbol{\beta}|\xi). \tag{70}$$

By definition (6.3) the value $H_{\mathfrak{p}}(\boldsymbol{\alpha}, \boldsymbol{\beta}|\xi)$ equals the value $H_q(\boldsymbol{\alpha}, \boldsymbol{\beta}|\xi)$ choosing a generator ϖ satisfying $\varpi^{(q-1)/N} = \chi_{\mathfrak{p}}^{-1}$, hence

$$H_q(\boldsymbol{\alpha}, \boldsymbol{\beta}|\xi) = H_{\mathfrak{p}}(((a-d)N, (b-d)N), ((c-d)N, 1)|\xi).$$

Then (70) equals

$$\chi_{\mathfrak{p}}(\xi)^{dN} \chi_{\mathfrak{p}}^{(d-b)N}(-1)\, \mathbf{J}((d-b, c-d), (c-b))H_{\mathfrak{p}}\left((a-d, b-d), (c-d, 1)|\xi\right).$$

Using Lemma 8.16 this value equals

$$\chi_{\mathfrak{p}}^{(d-b)N}(-1)\,\mathbf{J}((-a,-b,c,d),(c-b,d-a))H_{\mathfrak{p}}\left((a,b),(c,d)|\xi\right),$$

which matches the motive of Definition 8.11 (recall that $\varkappa_N(\mathfrak{p}) = \chi_{\mathfrak{p}}(-1)$ by Lemma 9.4).

To prove that $\mathcal{H}((a,b),(c,d)|\xi)$ also matches (69), we apply the proven result to the parameters $(a-d,b-d),(c-d,0)$ and are led to prove the equality

$$H_q((a-d,b-d),(c-d,1)|\xi)\,\mathbf{J}((-a,-b,c,d),(d-a,d-b,c-d))^{-1}\eta_d(\xi) = H_q((a,b),(c,d)|\xi),$$

which follows from Lemma 8.16 since $\eta_d(\mathrm{Frob}_{\mathfrak{p}})(\xi) = \chi_{\mathfrak{p}}(\xi)^{dN}$. □

Lemma 8.17. *Let L be a finite extension of F and let $\xi \in L$, with $\xi \neq 0,1$. The coefficient field of the motive $\mathcal{H}((a,b),(c,d)|\xi)$ (i.e. the field extension of $\mathbb{Q}$ generated by the trace of Frobenius' elements) is contained in K.*

Proof. Follows from Theorem 8.15 and Proposition 6.2, part (3). □

As a corollary we get the following slightly softer version of Conjecture 4.5 (basically where the field K is replaced by F).

Corollary 8.18. *Let $\boldsymbol{\alpha} = (a,b), \boldsymbol{\beta} = (c,d)$ be rational generic parameters with denominator N. Let $F = \mathbb{Q}(\zeta_N)$ and $K = F^H$, with H as in Definition 1.1. Then*

1. *The motive $\mathcal{H}(\boldsymbol{\alpha},\boldsymbol{\beta}|z)$ has a realization $\mathcal{H}(z)$ over $F(z)$.*
2. *The representation of the motive restricted to $\mathrm{Gal}(\overline{K(z)}/\overline{F}(z))$ is isomorphic to the geometric Galois representation (18).*
3. *Let L be a finite extension of F. The specialization $\mathcal{H}(\xi)$ for generic $\xi \in L$ has coefficient field K.*
4. *For primes $\mathfrak{p}$ of L of good reduction the trace of Frobenius on $\mathcal{H}(\xi)$ is given by the finite hypergeometric sum $H_{\mathfrak{p}}(\boldsymbol{\alpha},\boldsymbol{\beta}\,|\,\xi)$.*

Proof. The first two statements follow from Definitions 8.11 and 8.14 and Theorem 8.20; the third statement follows from the last lemma, while the last one follows from Theorem 8.15. □

Remark 8.19. Let $(a,b),(c,d)$ be rational generic parameters with denominator N. Let $\xi \in \mathbb{Q}$ and let $\mathfrak{p}$ be a prime ideal of F not dividing N, but dividing ξ. Suppose that the monodromy at 0 is finite of order m and that $m \mid v_{\mathfrak{p}}(\xi)$. Then Theorem 4.9 implies that the motive $\mathcal{H}((a,b),(c,d)|\xi)$ is unramified at $\mathfrak{p}$, so the trace of the action of $\mathrm{Frob}_{\mathfrak{p}}$ on our motive is well defined but is not given by $H_{\mathfrak{p}}((a,b),(c,d)|\xi)$. Nevertheless, there is an explicit formula, see Appendix B.

8.2. **The Galois representation.** The motive $H^1_{\chi_0}(\widehat{\mathcal{C}},F)^{\mathrm{new}}$ has attached a family of 2-dimensional Galois representations indexed by prime ideals $\mathfrak{p}$ of F (as defined by Serre in [47]). Denote by

$$\rho_{\mathcal{H}((a,b),(c,d)|z),\mathfrak{p}} : \mathrm{Gal}(\overline{\mathbb{Q}(z)}/F(z)) \to \mathrm{GL}_2(F_{\mathfrak{p}}), \tag{71}$$

the representation of the hypergeometric motive $\mathcal{H}((a,b),(c,d)|z)$.

Theorem 8.20. *The Galois representation $\rho_{\mathcal{H}((a,b),(c,d)|z),\mathfrak{p}}$ restricted to $\mathrm{Gal}(\overline{\mathbb{Q}(z)}/\overline{\mathbb{Q}}(z))$ matches the representation given in Corollary 3.17.*

Proof. The Galois representation attached to the Jacobi motive does not depend on z and its restriction to $\mathrm{Gal}(\overline{\mathbb{Q}(z)}/\overline{\mathbb{Q}}(z))$ is trivial. Then to prove the result it is enough to prove that the image under $\rho_{\mathcal{H}((a,b),(c,d)|z),\mathfrak{p}}$ of the inertia group at $z=0$ (respectively $z=1$ and $z=\infty$) of $\overline{\mathbb{Q}}(z)$ is conjugate to the matrix M_0 (respectively M_1 and M_∞), which is proved in Theorems A.2 and A.4. □

Definition 8.21. *The hypergeometric motives $\mathcal{H}((a,b),(c,d)|z)$ and $\mathcal{H}((a',b'),(c',d')|z)$ are called* isogenous, *denoted by $\mathcal{H}((a,b),(c,d)|z) \simeq \mathcal{H}((a',b'),(c',d')|z)$, if their Galois representations are isomorphic.*

Lemma 8.22. *Let $\rho_i : \mathrm{Gal}(\overline{\mathbb{Q}(z)}/F(z)) \to \mathrm{GL}_2(\overline{\mathbb{Q}_p})$ for $i = 1, 2$ be two irreducible continuous representations satisfying that their restrictions to the subgroup $\mathrm{Gal}(\overline{\mathbb{Q}(z)}/\overline{\mathbb{Q}}(z))$ are irreducible and isomorphic. Then there exists $\chi : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \overline{\mathbb{Q}_p}^{\times}$ such that $\rho_1 \simeq \rho_2 \otimes \chi$.*

Proof. See [40, Corollary 2.5]. □

Theorem 8.23. *Let $(a,b),(c,d)$ be a pair of generic rational numbers. Then the following properties hold:*

(1) (Twist) With the above notation

$$\mathcal{H}((a,b),(c,d)|z) \simeq \mathcal{H}((a-d,b-d),(c-d,1)|z) \otimes \mathbf{J}((-a,-b,c,d),(d-a,d-b,c-d))^{-1}\eta_d(z)$$

(2) (Permutation) Let $(a,b),(c,d)$ be generic parameters. Then

$$\mathcal{H}((a,b),(c,d)|z) \simeq \mathcal{H}((b,a),(c,d)|z) \simeq \mathcal{H}((a,b),(d,c)|z) \simeq \mathcal{H}((b,a),(d,c)|z).$$

(3) (Inverse) The motives $\mathcal{H}((a,b),(c,d)|z)$ and $\mathcal{H}((-c,-d),(-a,-b)|z^{-1})$ are isomorphic.

Proof. (1) If the Euler curve is reducible (i.e., (**Irr**) does not hold) the claim follows from definition. Otherwise, by Theorem 8.20 the restriction of both Galois representations to $\mathrm{Gal}(\overline{\mathbb{Q}(z)}/\overline{\mathbb{Q}}(z))$ is isomorphic to their respectively monodromy representations. Using the description of the monodromy matrices given in 3.12 it is easy to verify that the two monodromy representations are indeed isomorphic. The genericity hypothesis on the parameters implies that the monodromy representations are irreducible (by Proposition 3.9) so the lemma implies that the two representations are isomorphic up to a twist by a character χ independent of the variable z. If $\xi \in F$, $\xi \neq 0, 1$, then the two specialized representations are isomorphic up to a twist by χ. By Theorem 8.15 for any prime ideal $\mathfrak{p}$ of F not dividing N nor $\xi(\xi-1)$, the trace of the Frobenius element $\mathrm{Frob}_{\mathfrak{p}}$ is the same for both representations, so the character χ is trivial.

(2) All four motives have isomorphic monodromy representations (i.e. the restriction of their Galois representations to $\mathrm{Gal}(\overline{\mathbb{Q}(z)}/\overline{\mathbb{Q}}(z))$ are isomorphic). By Lemma 8.22 they are isomorphic up to the twist by a finite order character χ (independent of the parameter z). Let $\xi \in \mathbb{Q}$ be any number satisfying $\xi \neq 0, 1$, and let $\mathfrak{p}$ be a prime ideal of F not dividing $N\xi(\xi-1)$. Then Theorem 8.15 implies that the trace of $\mathrm{Frob}_{\mathfrak{p}}$ on $\mathcal{H}(\boldsymbol{\alpha},\boldsymbol{\beta}|\xi)$ equals $\mathcal{H}_{\mathfrak{p}}(\boldsymbol{\alpha},\boldsymbol{\beta}|\xi)$, but it is clear from its definition that

$$\mathcal{H}_{\mathfrak{p}}((a,b),(c,d)|\xi) = \mathcal{H}_{\mathfrak{p}}((b,a),(c,d)|\xi) = \mathcal{H}_{\mathfrak{p}}((a,b),(d,c)|\xi) = \mathcal{H}_{\mathfrak{p}}((b,a),(d,c)|\xi),$$

so χ must be trivial.

(3) Let N be the least common multiple of the denominators of a, b, c, d and assume (for simplicity) that condition (**Irr**) holds. Keeping the previous notation, the Euler curve for the parameters $(-c,-d),(-a,-b)$ at the variable z^{-1} has equation

$$y^N = x^A(1-x)^C(1-x/z)^B z^{D-A}.$$

The change of variables $x = zx'$, $y = y'$ provide an isomorphism between this curve and Euler's curve with parameters $(a,b),(c,d)$. Clearly the quadratic character $(-1)^{d-b}$ is invariant under the substitution $(a,b),(c,d) \to (-c,-d),(-a,-b)$ and the same holds for the Jacobi motive $\mathbf{J}((-a,-b,c,d),(c-b,d-a))$ appearing in the motive definition (67). □

Remark 8.24. It is likely that the isogenies in the theorem are actually isomorphisms but we have not attempted to do this except for (3).

8.3. **Hodge numbers.** For rank two motives, we can give an explicit relation between the Hodge vector of $\mathcal{H}((a,b),(c,d)|z)$ and the output of the zig-zag procedure, since the Hodge numbers of the Euler curve (and of its eigenspaces) as well as the ones of the Jacobi motive are well known.

Theorem 8.25. *Let $(a,b),(c,d)$ be generic rational parameters. Let r be the number of parameters among a,b,c,d that are in $\mathbb{Z}$. Then:*

(1) The motive $\mathcal{H}((a,b),(c,d)|z)$ is pure of weight $r-1$.

(2) For j coprime to N let $\sigma_j \in \mathrm{Gal}(F/\mathbb{Q})$ be the automorphism sending $\zeta_N \to \zeta_N^j$. Then the Hodge polynomial of $\mathcal{H}((a,b),(c,d)|z)$ for the embedding σ_j equals

$$\sum_P x^{-p_2} y^{p_2+r-1},$$

where the sum runs over the blue points $P=(p_1,p_2)$ of the zig-zag procedure applied to the parameters $(aj,bj),(cj,dj)$.

Proof. By definition, the hypergeometric motive $\mathcal{H}((a,b),(c,d)|z)$ satisfies

$$\mathcal{H}((a,b),(c,d)|z) \otimes \mathbf{J}((-a,-b,c,d),(c-b,d-a)) = H^1_{\chi_0}(\widehat{\mathcal{C}},F)^{\mathrm{new}} \varkappa_N^{(d-b)N}.$$

The space of differentials of the first kind on $\mathcal{C}$ has an action of μ_N (the N-th roots of unity); the dimension of the ζ_N^j eigenspace (for $\gcd(j,N)=1$) is given by the formula

$$\left\{\frac{jA}{N}\right\} + \left\{\frac{jB}{N}\right\} + \left\{\frac{jC}{N}\right\} - \left\{\frac{j(A+B+C)}{N}\right\}, \tag{72}$$

where $\{a\}$ denotes the fractional part of a (see for example §4, formula (21) of [59]).

The motivic weight of the Jacobi motive equals $n=2-r$. The Hodge number (corresponding to the embedding σ_j) of the Jacobi motive is given (see (28)) by

$$p := \{-aj\}+\{-bj\}+\{cj\}+\{dj\}-\{(c-b)j\}-\{(d-a)j\}, \quad p+q=n.$$

Table 8.2 contains, for $a \le b \in (0,1]$ and $c \le d \in [0,1)$, all possible outcomes of the zig-zag procedure together with the hodge polynomial of the Jacobi motive and of Euler's curve for the embedding $\sigma(\zeta_N)=\zeta_N$ in the case $r=0$. The statement follows by comparing the last two columns.

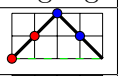
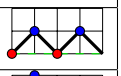
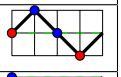
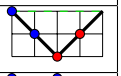
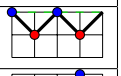
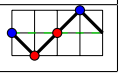
Case	Condition	**J**	Euler	$\mathcal{H}((a,b),(c,d)\|z)$	ZigZag
I	$a \le b < c \le d$	x^2	$x+y$	$x^{-1}+x^{-2}y$	
II	$a < c < b < d$	xy	$2y$	$2x^{-1}$	
III	$a < c \le d < b$	xy	$x+y$	$y^{-1}+x^{-1}$	
IV	$c \le d < a \le b$	y^2	$x+y$	$xy^{-2}+y^{-1}$	
V	$c < a < d < b$	xy	$2x$	$2y^{-1}$	
VI	$c < a \le b < d$	xy	$x+y$	$x^{-1}+y^{-1}$	

TABLE 8.1. Possible Hodge diagrams in the rank 2 case

It is easy to verify that the values of the columns (namely **J**, Euler and $\mathcal{H}((a,b),(c,d)|z)$) remain the same while permuting (a,b) and (c,d) in cases I, III, IV and VI. In case II, when $b<c<a<d$

or when $a < d < b < c$, Euler's curve has Hodge polynomial $2x$ while $\mathbf{J}$ has Hodge polynomial x^2, so the value for $\mathcal{H}$ stays the same. A similar phenomena occurs in case V.

When r is non-zero, the p-value of the Jacobi motive remains the same, but the motivic weight decreases by r hence the same occurs to q. □

Example 13. Let $\mathcal{C}$ be the Euler curve with equation

$$\mathcal{C} : y^5 = x(1-x)(1-tx),$$

as studied by Shimura in [50] (case (4) of the table in the first page of his paper). It corresponds to the parameters $\left(\frac{1}{5}, \frac{4}{5}\right), \left(\frac{3}{5}, 1\right)$ (using formula (63)). The field $\mathbb{Q}(\zeta_5)$ has four embeddings into $\mathbb{C}$, parametrized (once we choose a fifth root of unity in $\mathbb{C}$) by the elements $i \in (\mathbb{Z}/5)^\times$. It is easy to verify that the values $i = 1, 4$ give type V, the value $i = 2$ gives type IV, while the value $i = 3$ gives type VI (see Figure 8.1). We summarize the individual Hodge numbers for each motive (hypergeometric, Euler and Jacobi respectively) in the Table 8.2 (note that here $r = 1$).

i	Case	$\mathcal{H}^{j,-j}$					Euler$^{j,1-j}$					Jacobi$^{j,1-j}$				
j		-2	-1	0	1	2	-2	-1	0	1	2	-2	-1	0	1	2
1	V			2						2					1	
2	IV			1	1				1	1				1		
3	VI		1	1					1	1					1	
4	V			2					2					1		
Total		0	1	6	1	0	0	0	4	4	0			2	2	

TABLE 8.2. Hodge values

Remark 8.26. We see that the $\mathcal{H}((\frac{1}{5}, 45), (\frac{3}{5}, 1)|z)$ is the *half-twist* of H^1 of the Euler curve $\mathcal{C}$, in the notation of van Geemen [52], with respect to the CM type $\Sigma := \{\sigma_1, \sigma_2\}$ of $\mathbb{Q}(\zeta_5)$.

Similarly, we can study the case (6) of Shimura's table (in [50]). It is given by Euler's curve with equation

$$\mathcal{C} : y^7 = x(1-x)(1-tx),$$

corresponding to the parameters $\left(\frac{1}{7}, \frac{6}{7}\right), \left(\frac{5}{7}, 1\right)$. Now the embeddings are parametrized by the elements $i \in (\mathbb{Z}/7)^\times$. The values $i = 1, 2, 5, 6$ yield zig-zag diagrams of type V, with Hodge polynomial 2 while for $i = 3$ we get type IV with Hodge polynomial $1 + xy^{-1}$ and for $i = 4$ type VI with polynomial $1 + x^{-1}y$. Summarizing, the Hodge polynomials are

$$2, 2, 1 + xy^{-1}, 1 + x^{-1}y, 2, 2.$$

8.4. On the coefficient field of the hypergeometric motive.

Lemma 8.27. *Let a, b, c, d be generic rational parameters such that $c \not\equiv d \bmod \mathbb{Z}$. Let N be their least common denominator. Let L be a finite extension of F. Let $\xi \in L$, $\xi \neq 0, 1$, be such that there exists a prime ideal $\mathfrak{p}$ of L satisfying*

- $\mathfrak{p} \nmid N$,
- $v_{\mathfrak{p}}(\xi) > 0$ *and prime to* N.

Then both ζ_{c+d} and $\zeta_c + \zeta_d$ belong to the coefficient field of $\mathcal{H}((a, b), (c, d)|\xi)$. Similarly, if $a \not\equiv b$ and there exists a prime ideal $\mathfrak{p}$ of L not dividing N such that $v_{\mathfrak{p}}(\xi) < 0$ and is prime to N then both ζ_{a+b} and $\zeta_a + \zeta_b$ belong to the coefficient field of $\mathcal{H}((a, b), (c, d)|\xi)$.

Proof. Recall that the geometric representation attached to $\mathcal{H}((a,b),(c,d)|z)$ has monodromy matrices M_∞ conjugate to $\begin{pmatrix}\zeta_a & 0\\ 0 & \zeta_b\end{pmatrix}$ at ∞ if $a \not\equiv b \bmod \mathbb{Z}$ and M_0 conjugate to $\begin{pmatrix}\zeta_{-c} & 0\\ 0 & \zeta_{-d}\end{pmatrix}$ at 0 if $c \not\equiv d$.

Let $\mathfrak{p}$ be a prime ideal of L satisfying the hypothesis. Then by Corollary A.3 (see also [6, Theorem 1.2]) the image of inertia $I_\mathfrak{p}$ is generated by M_0^r (up to conjugation), so [20, Lemma 15] implies that the coefficient field contains both $\zeta_{r(c+d)}$ and $\zeta_{-rc}+\zeta_{-rd}$, which is equivalent to say that it contains ζ_{c+d} and $\zeta_{-c}+\zeta_{-d}$ (because the extension $\mathbb{Q}(\zeta_c,\zeta_d)/\mathbb{Q}$ is abelian). The second statement follows from a similar argument. □

Proposition 8.28. *With the hypothesis of the previous lemma, let $\xi \in L$ be such that there exists prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ of L prime to N satisfying:*

- *$v_\mathfrak{p}(\xi) > 0$ and prime to N,*
- *$v_\mathfrak{q}(\xi) < 0$ and prime to N.*

Then if (**Irr**) *holds and $|H| = 4$, the coefficient field of $\mathcal{H}((a,b),(c,d)|\xi)$ equals F^H.*

Proof. Let M denote the coefficient field. By Theorem 9.2, M is contained in F^H hence $[F:M] \geq |H|$. The hypothesis on ξ (and the last lemma) imply that $\mathbb{Q}(\zeta_{a+b}, \zeta_a+\zeta_b, \zeta_{c+d}, \zeta_c+\zeta_d) \subset M \subset F^H \subset F$, so $[F:M] \leq 4$ (because $F = \mathbb{Q}(\zeta_a,\zeta_b,\zeta_c,\zeta_d)$). Since $|H| = 4$, $M = F^H$. □

Example 14. The last proposition suggests that some genericity condition on the specialization parameter ξ is needed for the coefficient field to match F^H. Consider the case of parameters $(1/8,7/8),(3/8,5/8)$ (studied in Examples 2 and 9). The group H equals $\langle -1\rangle \subseteq (\mathbb{Z}/8)^\times$, so for any specialization ξ of the parameter, the motive is defined over $\mathbb{Q}(\sqrt{2})$ (by Theorem 9.2). But in Example 9 we showed that when ξ is a square, the field of definition is $\mathbb{Q}$ (the motive actually corresponds to a elliptic curve over $\mathbb{Q}$).

9. Extension to K

Let H be the group (19) for generic parameters $(a,b),(c,d)$.

Lemma 9.1. *The group H is a subgroup of $\mathbb{Z}/2 \times \mathbb{Z}/2$.*

Proof. For S a finite set, let $\mathrm{Perm}(S)$ denote the group of bijective maps on S. Set $S = \{a,b\}\times\{c,d\}$ (we are not assuming $a \neq b$ nor $c \neq d$). Then there is a group morphism

$$\psi : H \to \mathrm{Perm}(\{a,b\}) \times \mathrm{Perm}(\{c,d\}),$$

sending an element h to the bijective map given by multiplication by h on the sets $\{a,b\}$ and $\{c,d\}$. Since $\mathrm{Perm}(\{a,b\}) \times \mathrm{Perm}(\{c,d\}) \hookrightarrow \mathbb{Z}/2 \times \mathbb{Z}/2$, it is enough to prove that ψ is injective. Let N be the least common denominator of a,b,c,d. Let $i \in (\mathbb{Z}/N)^\times$ be an element in the kernel of ψ, so i satisfies

$$ia = a, \qquad ib = b, \qquad ic = c, \qquad id = d,$$

as elements of $\mathbb{Q}/\mathbb{Z}$. Write $a = \frac{\alpha}{N}$, $b = \frac{\beta}{N}$, $c = \frac{\gamma}{N}$, $d = \frac{\delta}{N}$ with $\gcd(\alpha,\beta,\gamma,\delta,N) = 1$. Then there are integers x_1,x_2,x_3,x_4 such that the following congruence holds

$$\alpha x_1 + \beta x_2 + \gamma x_3 + \delta x_4 \equiv 1 \pmod{N}.$$

Multiplying by i we get that

$$i \equiv i\alpha x_1 + i\beta x_2 + i\gamma x_3 + i\delta x_4 \equiv \alpha x_1 + \beta x_2 + \gamma x_3 + \delta x_4 \equiv 1 \pmod{N}.$$

□

As stated in Conjecture 4.5, it is expected that $K = F^H$ is both the field of definition and the coefficient field of the motive $\mathcal{H}((a,b),(c,d)|z)$.

Theorem 9.2. *Let $(a,b),(c,d)$ be generic parameters such that $a+b$ and $c+d$ are integers and* (**Irr**) *holds. Then $\mathcal{H}((a,b),(c,d)|z)$ descends to F^H and for any specialization its coefficient field is contained in F^H.*

We start proving some preliminary results.

Lemma 9.3. *Let $(a,b),(c,d)$ be generic rational parameters satisfying that $a+b$ and $c+d$ are integers. Let N be their least common denominator. Set*

$$\delta = \begin{cases} 1 & \text{if } a,c \notin \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathfrak{p}$ be a prime ideal of $\mathbb{Q}(\zeta_N)$ of norm q, not dividing N. Then

$$\mathbf{J}((-a,-b,c,d),(c-b,d-a))(\mathfrak{p}) = q^{\delta}.$$

Proof. The hypothesis $a+b \in \mathbb{Z}$ (respectively $c+d \in \mathbb{Z}$) imply that we can replace b by $-a$ (respectively d by $-c$) in the Jacobi motive

$$\mathbf{J}((-a,-b,c,d),(c-b,d-a))(\mathfrak{p}) = \mathbf{J}((-a,a,c,-c),(a+c,-a-c))(\mathfrak{p}).$$

If χ is a character of $\mathbb{F}_q^\times$,

$$g(\chi)g(\overline{\chi}) = \chi(-1)\begin{cases} q & \text{if } \chi \neq 1, \\ 1 & \text{if } \chi = 1. \end{cases}$$

The genericity condition on the coefficients imply that $a+c = a-d \notin \mathbb{Z}$ and at most one of a,c is an integer. □

It follows that if $a+b$ and $c+d$ are integers then the hypergeometric motive $\mathcal{H}((a,b),(c,d)|z)$ is (up to a Tate twist) the twist by $\varkappa_N$ (see Definition 8.8) of the motive coming from Euler's curve. Assume that $t = v_2(N) > 0$ (as otherwise $\varkappa_N$ is trivial). Then Corollary 8.10 implies that $\varkappa_N$ matches the restriction to $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ of a character κ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of conductor 2^{t+1}.

Lemma 9.4. *If $(a,b),(c,d)$ are generic parameters such that $a+b$ and $c+d$ are integers, then the character κ is at most quadratic when restricted to $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$.*

Proof. We can assume that $t > 0$ (as otherwise $\varkappa_N$ and κ are trivial). Since $-1 \in H$ (complex conjugation), a simple Galois theory exercise shows that $\mathbb{Q}(\zeta_{2^{t+1}}) \cap F^H = \mathbb{Q}(\zeta_{2^t})^+$. Then κ restricted to $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ factors through $\mathrm{Gal}(\mathbb{Q}(\zeta_{2^{t+1}})/\mathbb{Q}(\zeta_{2^t})^+)$, a group isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$ if $t > 1$ and $\mathbb{Z}/2$ if $t = 1$, hence κ is at most quadratic. □

Our goal is to construct a motive $\mathfrak{M}$ defined over K whose base change to F is isomorphic to Euler's motive and define $\mathcal{H}((a,b),(c,d)|z)$ as in (67). Since κ is (at most) quadratic and the Jacobi motive corresponds to a Tate twist, for any specialization ξ of z, the coefficient field of $\mathcal{H}((a,b),(c,d)|\xi)$ is the same as that of $\mathfrak{M}$. The motive $\mathfrak{M}$ appears in the quotient of Euler's curve $\mathcal{C}$ by involutions that we now describe.

Proposition 9.5. *Let a,b,c,d be generic parameters such that $a+b=c+d=0$. For $j \in H$ define a morphism $\iota_j : \mathcal{C} \to \mathcal{C}$ by*

- *If $j=-1$,*

$$\iota_{-1}(x,y) = \left(\frac{1}{zx}, \frac{1}{y}\right). \tag{73}$$

- *If $ja = a+r$ and $jc = -c+s$, for some $r,s \in \mathbb{Z}$,*

$$\iota_j(x,y) = \left(\frac{x-1}{zx-1}, \frac{x^{r-s}(1-x)^{-r-s}(1-xz)^{r+s}z^{-s}}{y^j}\right). \tag{74}$$

- *If $ja = -a + r$ and $jc = c + s$, for $r, s \in \mathbb{Z}$,*

$$\iota_j(x,y) = \left(\frac{zx-1}{z(x-1)}, \frac{x^{r-s}(1-x)^{-r-s}(1-xz)^{r+s}z^{-s}}{y^j} \right). \tag{75}$$

The map ι_j is an involution of $\mathcal{C}$ and the new part of $\mathcal{C}/\iota_j$ has genus $\phi(N)/2$.

Proof. Follows from a standard computation. □

Proof of Theorem 9.2. Let N be the least common denominator of the parameters a, b, c, d. Let $j \in H$ and let ι_j be the involution defined in Proposition 9.5. The new part of $\mathcal{C}/\iota_j$ has genus $\phi(N)/2$. Abusing notation, denote by ζ_N the map on $\mathcal{C}$ given by $\zeta_N(x,y) = (x, \zeta_N y)$. It follows easily from its definition that

$$\iota_j \circ \zeta_N = \begin{cases} \zeta_N^{-j} \circ \iota_j & \text{if } j \neq -1, \\ \zeta_N^{-1} \circ \iota_j & \text{if } j = -1. \end{cases} \tag{76}$$

Then over the field $F^{\langle j \rangle}$ there is an action of the ring $\mathbb{Z}[\zeta_N + \zeta_N^{-j}]$ (respectively $\mathbb{Z}[\zeta_N + \zeta_N^{-1}]$) on $H^1(C/\iota, F)$. Looking at the χ_0 isotypical component, we get a 2-dimensional representation with coefficient field $F^{\langle j \rangle}$. If H is cyclic, this concludes the proof; when H is not cyclic, take the quotient by two different involutions attached to elements on H (getting a representation over F^H). □

The irreducibility assumption (**Irr**) in the theorem is not a strong one.

Lemma 9.6. *Let $(a,b),(c,d)$ be generic parameters such that $a+b$ and $c+d$ are integers. If* (**Irr**) *does not hold then $H = \{\pm 1\}$.*

Proof. By Lemma 8.5 the values $N := \operatorname{lcm}\{\operatorname{den}(a), \operatorname{den}(c)\}$ and $N' := \operatorname{lcm}\{\operatorname{den}(a+c), \operatorname{den}(a-c)\}$ are different so $N = 2N'$. This implies that $v_2(\operatorname{den}(a)) = v_2(\operatorname{den}(c))$. If $j \in H$ and $j \neq \pm 1$ then $ja = a \pmod{\mathbb{Z}}$ and $jc = -c \pmod{\mathbb{Z}}$ or vice-versa. But then $v_2(j-1) = v_2(j+1) = v_2(\operatorname{den}(a))$, a contradiction. □

A result similar to Theorem 9.2 can be proved under the following hypothesis.

Theorem 9.7. *Let $(a,b),(c,d)$ be generic parameters such that* (**Irr**) *holds. Suppose furthermore that $H = \langle j \rangle$ where the action of multiplication by j is not trivial on both sets $\{a,b\}$ and $\{c,d\}$. Then $\mathcal{H}((a,b),(c,d)|z)$ descends to F^H and its coefficient field is contained in F^H.*

Proof. Consider the *twisted* Euler curve

$$\widetilde{\mathcal{C}} : y^N = (-1)^{(b-d)N} x^A (1-x)^B (1-xz)^C z^D, \tag{77}$$

where the exponents are as in (63). Then

$$\mathcal{H}((a,b),(c,d)|z) = H^1_{\chi_0}(\widetilde{\mathcal{C}}, F)^{\text{new}} \otimes \mathbf{J}((-a,-b,c,d),(c-b,d-a))^{-1}.$$

The action of multiplication by j fixes both the Jacobi motive and the twisted Euler curve. Then, as in the previous case, we can define an involution ι_j and the proof follows mutatis mutandis that of Theorem 9.2. □

When the group H is cyclic, we can prove a weaker general version of the result.

Theorem 9.8. *Let $(a,b),(c,d)$ be generic parameters such that $|H| \leq 2$. Then for all $\xi \in K$, $\xi \neq 0, 1$, the Galois representation of $\mathcal{H}((a,b),(c,d)|\xi)$ extends to Gal_K and its coefficient field is contained in a quadratic extension of K.*

Let us start with some auxiliary needed results.

Theorem 9.9. *Let $(a,b),(c,d)$ be generic rational parameters. For ξ outside a thin set of K, the Galois representation $\rho_{\mathcal{H}((a,b),(c,d)|z),\mathfrak{p}}$ of Gal_F is absolutely irreducible.*

Proof. By Theorem 8.20 the Galois representation $\rho_{\mathcal{H}((a,b),(c,d)|z),\mathfrak{p}}$ extends the monodromy representation ρ, an absolutely irreducible representation (by [7, Proposition 3.3]). After the choice of a stable lattice we can assume that our representation $\rho_{\mathcal{H}((a,b),(c,d)|z),\mathfrak{p}}$ takes values in $\mathrm{GL}_2(\mathcal{O}_{\mathfrak{p}})$. Then there exists a positive integer n such that the reduction

$$\rho_{\mathcal{H}((a,b),(c,d)|z),\mathfrak{p},n} : \mathrm{Gal}(\overline{\mathbb{Q}(z)}/F(z)) \to \mathrm{GL}_2(\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^n)$$

is irreducible. The field fixed by its kernel is a finite extension $L(z)/F(z)$. Then Hilbert's irreducibility theorem (see for example §3.4 of [48]) implies that for ξ outside a thin set, $\mathrm{Gal}(L(\xi)/F) = \mathrm{Gal}(L(z)/F(z))$, so the representation $\rho_{\mathcal{H}((a,b),(c,d)|\xi),\mathfrak{p}}$ is irreducible. □

Example 15. Let $F = \mathbb{Q}(\zeta_3)$, where ζ_3 is a third root of unity. The motive $\mathcal{H}((1/3, 2/3), (1, 1)|z)$ matches (as discovered in [32]) the motive attached to the rational elliptic curve with equation

$$E_z : y^2 + xy + \frac{z}{27}y = x^3.$$

For $z = 9/8$ the elliptic curve has complex multiplication by $\mathbb{Z}[\zeta_3]$ over $F = \mathbb{Q}(\sqrt{-3})$ (as it is easy to verify). The map: $x = u^2x' + r$, $y = u^3y' + u^2sx' + t$, with

$$u = \zeta_3, \quad r = -\frac{\zeta_3 + 2}{12}, \quad s = \frac{\zeta_3 - 1}{2}, \quad t = \frac{\zeta_3 + 2}{24}.$$

is an endomorphism of the curve defined over F. It follows that the Galois representation attached to

$$\mathcal{H}((1/3, 2/3), (1, 1)|9/8)$$

is reducible.

Example 16 (Shimura's Example 13 continued)**.** Take $\xi = 1/2$, corresponding to the curve

$$\mathcal{C} : y^5 = x(1 - x)(1 - x/2).$$

The map $\iota(x, y) = (2-x, -y)$ is an involution of $\mathcal{C}$. The quotient curve can be computed using [10].

```
A<x,y>:=AffineSpace(Rationals(),2);
C:=Curve(A,y^5-x*(1-x)*(1-x/2));
G:=AutomorphismGroup(C);
CG,prj := CurveQuotient(G);
```

It is given by the hyperelliptic model

$$\mathcal{C}_1 : y^2 = x(x^5 - 8).$$

This implies that the corresponding hypergeometric motive at $\xi = 1/2$ is reducible. To obtain an isogeny complement to the Jacobian of $\mathcal{C}_1$ in the Jacobian of $\mathcal{C}$ we use the code developed in [16] and we find the hyperelliptic curve

$$\mathcal{C}_2 : y^2 = x(x^5 - 16).$$

The two curves $\mathcal{C}_1$ and $\mathcal{C}_2$ are actually isomorphic over the field $\mathbb{Q}(\sqrt[5]{2})$, an isomorphism being $(x, y) \to (\sqrt[5]{2}x, \sqrt[5]{8}y)$. Their Jacobians are not isogenous over $F = \mathbb{Q}(\zeta_5)$; this can be verified by computing their Frobenius polynomial at 11

```
P<x> := PolynomialRing(RationalField());
C := HyperellipticCurve(x*(x^5-8));
D := HyperellipticCurve(x*(x^5-16));
LPolynomial(ChangeRing(C,GF(11)));
> 121*x^4 + 121*x^3 + 51*x^2 + 11*x + 1
LPolynomial(ChangeRing(D,GF(11)));
> 121*x^4 - 99*x^3 + 41*x^2 - 9*x + 1
```

Both curves have an order 5 automorphism sending $(x,y) \to (\zeta_5^2 x, \zeta_5 y)$ so correspond to abelian surfaces with complex multiplication by $\mathbb{Z}[\zeta_5]$. Hence, the motive $\mathcal{H}((\frac{1}{5},\frac{4}{5}),(\frac{3}{5},1)|\frac{1}{2})$ corresponds to the sum of two (distinct) Hecke characters.

A result of Bailey (see [5]) gives the following identity

$$ {}_2F_1\left(a, 1-a; c|\frac{1}{2}\right) = \frac{\Gamma(\frac{c}{2})\Gamma(\frac{c+1}{2})}{\Gamma(\frac{c+a}{2})\Gamma(\frac{1+c-a}{2})}.$$

A similar formula is expected to hold motivically: the left hand side corresponding to a hypergeometric motive and the right hand side to a Jacobi motive. Concretely, taking

$$a_1 = 1/5, c_1 = 2/5, \qquad a_2 = 6/5, c_2 = 2/5$$

we could expect

$$\mathcal{H}\left(\left(\frac{1}{5},\frac{4}{5}\right),\left(\frac{3}{5},1\right)\middle|\frac{1}{2}\right) = \mathbf{J}\left(\left(\frac{1}{5},\frac{7}{10}\right),\left(\frac{4}{5},\frac{1}{10}\right)\right) \oplus \mathbf{J}\left(\left(\frac{1}{5},\frac{7}{10}\right),\left(\frac{3}{10},\frac{3}{5}\right)\right). \tag{78}$$

We verified that the Euler factors of both sides coincide for primes up to 500 (it seems like a nice exercise to provide a proof).

Question: Let S denote the set of values where the specialization of the Galois representation is reducible. Besides the proven statement on S being thin, can more be said? Is it finite?

Lemma 9.10. *Let L/K be a cyclic extension of number fields, and let $\rho : \mathrm{Gal}_L \to \mathrm{GL}_d(\overline{\mathbb{Q}_p})$ be an irreducible continuous Galois representation. Then the representation ρ extends to a representation $\tilde{\rho} : \mathrm{Gal}_K \to \mathrm{GL}_d(\overline{\mathbb{Q}_p})$ if and only if for all $\tau \in \mathrm{Gal}(L/K)$, the representation $\rho^\tau : \mathrm{Gal}_L \to \mathrm{GL}_d(\overline{\mathbb{Q}_p})$ defined by $\rho^\tau(\sigma) := \rho(\tau\sigma\tau^{-1})$ is isomorphic to ρ.*

Proof. See [19, Lemma 3.4] (even when the result is stated for L/K quadratic, the proof works the same for cyclic extensions). □

Remark 9.11. The result is only valid for cyclic extensions; it is not hard to construct examples of abelian extensions L/K with Galois group $\mathbb{Z}/2 \times \mathbb{Z}/2$ where the last result does not hold. It is however true for abelian extensions that a twist of our representation does extend.

Proof Theorem 9.8. Start assuming that $\rho_{\mathcal{H}((a,b),(c,d)|\xi),\mathfrak{p}}$ is absolutely irreducible (by Theorem 9.9 and Remark 9.11 this occurs for ξ outside a thin set). Since H is cyclic, Lemma 9.10 implies that it is enough to prove that for all $\sigma \in H$, the representation $\rho^\sigma_{\mathcal{H}((a,b),(c,d)|\xi),\mathfrak{p}}$ is isomorphic to $\rho_{\mathcal{H}((a,b),(c,d)|\xi),\mathfrak{p}}$, or equivalently (since $\rho_{\mathcal{H}((a,b),(c,d)|\xi),\mathfrak{p}}$ is irreducible) that for all prime ideals $\mathfrak{q}$ of F on a density one set both representations have the same trace at $\mathrm{Frob}_\mathfrak{q}$. By Theorem 8.15 and Proposition 6.2, if $\sigma(\zeta_N) = \zeta_N^j$, it amounts to verify that

$$H_\mathfrak{q}((a,b),(c,d)|\xi) = H_\mathfrak{q}((ja,jb),(jc,jd)|\xi),$$

which follows from the definition of H.

If the representation is reducible, then there exists Hecke characters χ_1, χ_2 of Gal_F such that (up to semisimplification)

$$\rho^{\mathrm{ss}}_{\mathcal{H}((a,b),(c,d)|\xi),\mathfrak{p}} \simeq \chi_{1,\mathfrak{p}} \oplus \chi_{2,\mathfrak{p}}.$$

Let σ be the non-trivial element of $\mathrm{Gal}(F/K)$. Either ${}^\sigma\chi_1 = \chi_1$ or ${}^\sigma\chi_1 = \chi_2$. In the first case, both characters χ_i, $i = 1, 2$, extend to Gal_K, so the extension is defined as the sum of the two extended characters. In the second case, the extension is defined as the induced representation $\mathrm{Ind}_{\mathrm{Gal}_F}^{\mathrm{Gal}_K} \chi_1$ (which is isomorphic to the induction of χ_2). □

Corollary 9.12. *If $a+b$ and $c+d$ are integers, then for $\xi \in K$, $\xi \neq 0, 1$, the Galois representation attached to $\mathcal{H}((a,b),(c,d)|\xi)$ extends to Gal_K.*

Proof. If (**Irr**) holds, the result follow from Theorem 9.2. Otherwise, Lemma 9.6 implies that H is cyclic, so the result follows from Theorem 9.8. □

10. Hypergeometric Motives over totally real fields

Our understanding of L-series coming from Galois representations is still very limited. There are a few instances where some general results are known, including odd 2-dimensional representations of weight two over totally real fields. In this case, it is known (see for example [51, Theorem 1.1.1]) that the L-series extends meromorphically to the whole complex plane and satisfies a functional equation (it is furthermore expected that the extension is holomorphic). In this section we study hypergeometric motives defined over totally real fields.

According to Conjecture 4.5 the motive $\mathcal{H}((a,b),(c,d)|z)$ is defined over a totally real field precisely when $-1 \in H$; i.e. when

$$\{a,b\} = \{-a,-b\}, \qquad \{c,d\} = \{-c,-d\}.$$

Equivalently, the motive is defined over a totally real field when one of the following (non-disjoint) cases occurs:

- $a = -b$ and $c = -d$, or equivalently, $a+b \in \mathbb{Z}$ and $c+d \in \mathbb{Z}$,
- $a = -b$ (respectively $c = -d$), $c = -c$ and $d = -d$ (respectively $a = -a$ and $b = -b$).
- $a = -a$, $b = -b$, $c = -c$ and $d = -d$.

The last case (for generic parameters) corresponds to the motive with parameters $(1/2,1/2),(1,1)$ or $(1,1)(1/2,1/2)$. It is a rational motive corresponding to Legendre's family of elliptic curves

$$E_z : y^2 = x(1-x)(1-zx).$$

In this case modularity for specializations $z \neq 0,1$ in $\mathbb{Q}$ is known by the work of Wiles and others (see [58] and [12]). We consider the remaining cases. For a rational number a, the relation $a \equiv -a \pmod{\mathbb{Z}}$ implies that either $a \in \mathbb{Z}$ or $a \in \frac{1}{2} + \mathbb{Z}$, so we can separate (up to a quadratic twist obtained by adding $1/2$ to all parameters) totally real motives into two disjoint cases (that we study separately):

(1) Motives with parameters $(a,b),(c,d)$ with $a+b \in \mathbb{Z}$ and $c+d \in \mathbb{Z}$,
(2) Motives with parameters $(a,-a),(1/2,1)$ or $(1/2,1),(c,-c)$.

10.1. **Case (1).** We start by computing the Hodge polynomials as explained in §4.1 (using Theorem 8.25). If two of the parameters are integers, then for all embeddings $\sigma : K \hookrightarrow \mathbb{C}$, the zigzag

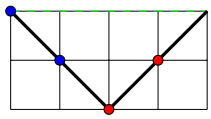


Figure 4. Zigzag outcome when $a,b \in \mathbb{Z}$ or $c,d \in \mathbb{Z}$

procedure gives the diagram in Figure 4 and the Hodge polynomial equals $x+y$. Then for any $\xi \in K \setminus \{0,1\}$, the Galois representation of Gal_K attached to $\mathcal{H}((a,b),(c,d)|\xi)$ has Hodge-Tate weights $\{0,1\}$.

When no parameter is an integer, the zig-zag procedure gives the diagram in Figure 5. In this case, the Hodge polynomial equals $x^{-1}+y^{-1}$. Then for any $\xi \in K \setminus \{0,1\}$ the Galois representation of Gal_K attached to $\mathcal{H}((a,b),(c,d)|\xi)$ has Hodge-Tate weights $\{-1,0\}$.

In both cases, the L-series attached to $\mathcal{H}((a,b),(c,d)|\xi)$ extends meromorphically to the whole complex plane and satisfies a functional equation (by [51, Theorem 1.1.1]). As already mentioned, it is expected that the L-series is actually holomorphic, and comes (up to a Tate twist) from

FIGURE 5. The zigzag procedure in case (1) for $a < c$ and $a > c$

a parallel weight 2 Hilbert modular form defined over K. In some cases one can use modularity lifting theorems and congruences to prove modularity of all specializations of the motive (as done in [27] for the parameters $(\frac{1}{2p}, -\frac{1}{2p}), (1,1)$ for example). In other cases (like the motive corresponding to the parameters $(3/8, -3/8), (1/8, -1/8)$ described in the introduction) one can prove modularity of the motive for some particular specializations (like $\xi = 3$) using the Faltings-Serre method.

10.2. **Case (2).** The zig-zag procedure gives (for all embeddings $\sigma : K \hookrightarrow \mathbb{C}$) the diagram shown in Figure 6. Its Hodge polynomial is 2, so it should correspond to a parallel weight 1 modular form. This is indeed the case, since such motives have finite monodromy as studied by Schwarz (in [46]),

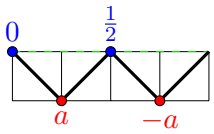


FIGURE 6. The zigzag procedure in case (2)

corresponding to dihedral projective image. A detailed description will be given in a subsequent article by the last two authors.

In both cases, we can prove a stronger version of Theorem 9.9.

Theorem 10.1. *Let $(a,b),(c,d)$ be rational generic parameters, and let $\xi \in K$, $\xi \neq 0, 1$. Suppose that $a+b$ and $c+d$ are integers. Then the Galois representation of the motive $\mathcal{H}((a,b),(c,d)|\xi)$ extended to Gal_K is irreducible.*

Proof. Recall that any Hecke character of the idèle group of a totally real field is (up to a finite order character) a multiple of the norm character (the reason is that if $d = [K : \mathbb{Q}]$, the compatibility relation at the units of K impose $d-1$ relations among the d possible characters at the archimedean places). Suppose then that the representation is reducible, namely

$$\rho_\lambda = \begin{pmatrix} \chi_\ell^a \varepsilon_1 & * \\ 0 & \chi_\ell^b \varepsilon_2 \end{pmatrix}, \tag{79}$$

for some $a, b \in \mathbb{Z}$ and $\varepsilon_1, \varepsilon_2$ finite order characters of the idèle group of K, where χ_ℓ denotes the cyclotomic character (the precise values of a and b can be given using the zig-zag procedure as previously explained). The representation ρ_λ is also reducible while restricted to Gal_F and the same holds for any twist, so the 2-dimensional representation coming from $H^{1,\zeta_N^i}_{\text{ét}}(\widehat{\mathcal{C}}, \mathbb{Q}_\ell)$ also looks like (79) (since by Lemma 9.3, the Jacobi motive is a power of the cyclotomic character).

The Hodge-Tate weights of the Galois representation attached to the new part of Euler's curve are $\{0, \ldots, 0, 1, \ldots, 1\}$, each one with multiplicity $\frac{\phi(N)}{2}$. From the decomposition

$$H^1_{\text{ét}}(\widehat{\mathcal{C}}, F_\lambda)^{\text{new}} := \bigoplus_\chi H^1_{\text{ét},\chi}(\widehat{\mathcal{C}}, F_\lambda),$$

it follows that for each i prime to N the representation of $H^1_{\text{ét},\chi}(\widehat{\mathcal{C}}, F_\lambda)$ has a decomposition like (79) for some characters $\varepsilon_1^{(\chi)}, \varepsilon_2^{(\chi)}$ (but the exponents a, b, are independent of χ, since they are Galois conjugate of each other). In particular, $a = 0$ and $b = 1$ or vice-versa.

Let $\mathfrak{q}$ be a prime ideal of F not dividing $N\ell$ such that $v_\mathfrak{q}(\xi(\xi-1)) = 0$ and also $\varepsilon_1^{(\chi)}(\mathfrak{q}) = \varepsilon_2^{(\chi)}(\mathfrak{q}) = 1$ for all order N character χ (there are infinitely many such prime ideals by Chebotarev density theorem). Then the trace of $\text{Frob}_\mathfrak{q}$ acting on the new part of the Jacobian equals $(\mathcal{N}(q)+1)\phi(N)$.

On the other hand, Weil's conjectures imply that such trace cannot be larger than $2\phi(N)\sqrt{\mathcal{N}(q)}$, which is a contradiction, since $x + 1 > 2\sqrt{x}$ if $x > 1$. □

11. Congruences

Congruences between Galois representation have many applications in number theory. In Langlands' program they play a crucial role in the proof of results like base change, Serre's conjectures, potential modularity, the Shimura-Taniyama conjecture, etc. In the study of Diophantine equations, they also play a crucial role, like in Wiles' proof of Fermat's last theorem. Hypergeometric motives have the property that they satisfy many congruences.

Definition 11.1. *Let ℓ be a rational prime. Define on $\mathbb{Q}/\mathbb{Z}$ the relation defined by the condition that the denominator of $a - b$ is an ℓ-th power. Denote it by $a \sim_\ell b$.*

The following result is easy to verify.

Lemma 11.2. *The relation $\sim_\ell$ is an equivalence relation.*

Extend the definition to $(\mathbb{Q}/\mathbb{Z})^n$ component-wise.

Theorem 11.3. *Let ℓ be a prime number, and let $(a,b),(c,d)$ and $(a',b'),(c',d')$ be two pairs of generic rational parameters. Let F be the composite of the fields of definition of the motives attached to both parameters. Then if $(a,b) \sim_\ell (a',b')$ and $(c,d) \sim_\ell (c',d')$, the base change of both motives to F specialized at $\xi \in F$, $\xi \neq 0, 1$, are congruent modulo $\mathfrak{l}$, for $\mathfrak{l}$ any prime ideal of their coefficient field dividing ℓ.*

Proof. By Theorem 8.15 the trace of the hypergeometric motive at a prime $\mathfrak{p}$ of good reduction matches a finite hypergeometric sum. From its very definition (equation (34)) the finite hypergeometric sum is a sum and product/quotients of Gauss sums of the form $g(m+\alpha(q-1),\mathfrak{p})$, where α is one of the parameters (and $q = p^r$ is the order of the residue field $\mathcal{O}_F/\mathfrak{p}$). To prove the statement it is enough to verify that if $\alpha \sim_\ell \beta$ then $g(m+\alpha(q-1),\mathfrak{p}) \equiv g(m+\beta(q-1),\mathfrak{p}) \pmod{\mathfrak{l}}$. Recall that if ω denotes a generator of the multiplicative group $\mathbb{F}_q^\times$, then

$$g(m+\alpha(q-1)) = \sum_{x \in \mathcal{O}_F/\mathfrak{p}} \omega^m(x)\omega^{\alpha(q-1)}(x)\psi(x), \tag{80}$$

where ψ is an additive character of $\mathbb{F}_q^\times$. The last sum is an element of $\mathbb{Z}[\zeta_p, \zeta_{q-1}]$. Let $\mathfrak{l}$ be a prime ideal in this ring dividing ℓ. Then

$$\frac{\omega^{\alpha(q-1)}(x)}{\omega^{\beta(q-1)}(x)} = \omega^{(\alpha-\beta)(q-1)}(x) = \omega^{(q-1)/\ell^r}(x),$$

for some non-negative integer r. In particular, it is a root of unity whose order divides ℓ^r, so congruent to 1 modulo $\mathfrak{l}$. This implies that $g(m+\alpha(q-1)) \equiv g(m+\beta(q-1)) \pmod{\mathfrak{l}}$ as claimed. □

Some explicit examples of lowering/raising the level using congruences between hypergeometric motives were given at the introduction.

Appendix A. Tame inertia of an Hypergeometric Motive

by Elisa Lorenzo García and Ariel Pacetti

Let $\mathcal{O}$ be a Henselian discrete valuation ring, let π be a local uniformizer, let K be its field of fractions and let k its residue field. For the present applications, have in mind the case in which $\mathcal{O} = k(z)$ is the coordinate ring of $\mathbb{P}^1_k$. Keeping the article's notation, let (a,b), (c,d) be a pair of generic parameters and let N be their least common denominator. Set

$$A = (d-b)N, \qquad B = (b-c)N, \qquad C = (a-d)N, \qquad D = dN. \tag{81}$$

Define Euler's curve $\mathcal{C}$ over $\mathcal{O}$ by

$$\mathcal{C} : y^N = x^A(1-x)^B(1-\pi x)^C\pi^D. \tag{82}$$

Assume for the rest of the appendix that the following two conditions hold:

- The curve $\mathcal{C}$ is irreducible (i.e. (**Irr**) holds).
- The residual characteristic of K ($\mathrm{char}(k)$) does not divide N.
- K contains the N-th roots of unity.

Fix μ a root of unity in K (a generator of $\boldsymbol{\mu}_N$) and ζ a primitive N-th root of unity in $\mathbb{C}$. Let $\mathcal{Y}^{\mathrm{st}}$ denote the semistable model of $\mathcal{C}$ and let $\mathcal{A}$ be the abelian variety $\mathrm{Pic}^0(\mathcal{Y}^{\mathrm{st}})$. Let $\mathcal{A}_0$ denote its special fiber. It contains a toric part T_0 and an abelian part $\mathcal{B}_0$ so that

$$\mathcal{A}_0^0/T_0 \simeq \mathcal{B}_0$$

(see [9], Example 8, page 246 and also [31] Expose IX, equation (2.1.2)). Let $n = \dim \mathcal{A}$, $\mu = \dim T_0$ and $a = \dim \mathcal{B}_0$ so that $n = \mu + a$.

Let $\ell \equiv 1 \pmod N$ be a prime number different from the characteristic of K (so $\mathbb{Q}_\ell$ contains the N-th roots of unity). Let $T_\ell(\mathcal{A})$ denote the Tate module of the generic fiber of $\mathcal{A}$. Since $\mathcal{Y}^{\mathrm{st}}$ is semistable, the action of the inertia group I_K on $T_\ell(\mathcal{A})$ is unipotent with echelon rank 2, i.e. there exists a subspace U such that I_K acts trivially on U and on $T_\ell(\mathcal{A})/U$ (see [31, Proposition 3.5], page 350).

Let Sp_2 denote the $\mathbb{Q}_\ell[I_K]$-module of dimension 2 where the action of the inertia group I_K factors through the maximal pro-ℓ-group $I_K(\ell)$, and a generator g acts like the matrix $\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$. Let $\Upsilon = (V,E)$ denote the dual graph of of $\overline{\mathcal{Y}^{\mathrm{st}}}$ (the special fiber of $\mathcal{Y}^{\mathrm{st}}$). Then μ (the rank of the toric part T_0) equals the rank of $\mathrm{H}^1(\Upsilon,\mathbb{Z})$ and there is an isomorphism of $\mathbb{Q}_\ell[I_K]$-modules

$$T_\ell(\mathcal{A}) \simeq \left(\left(\mathrm{H}^1(\Upsilon,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell\right) \otimes \mathrm{Sp}_2\right) \oplus \bigoplus_{\tilde{Y}\in V} T_\ell(\mathrm{Pic}^0(\tilde{Y})), \tag{83}$$

where the sum runs over the irreducible components of $\overline{\mathcal{Y}^{\mathrm{st}}}$. To describe the action of I_K on $T_\ell(\mathcal{A})$ it suffices to compute the semistable model of $\mathcal{C}$ and its dual graph.

The (**Irr**) assumption and the generic condition on the parameters imply that

(i) $\gcd(N,A,B,C) = 1$,
(ii) $N \nmid A$, $N \nmid B$, $N \nmid C$ and $N \nmid A+B+C$.

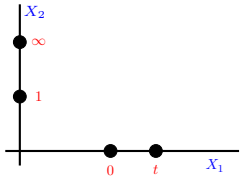
A simple change of variables transforms equation (82) into

$$\pi^{A+B-D}y^N = x^A(\pi-x)^B(1-x)^C. \tag{84}$$

Up to a twist (coming from the change of variables $y \to \pi^{(A+B-D)/N}y$), it is enough to study the curve

$$\widetilde{\mathcal{C}} : y^N = x^A(\pi-x)^B(1-x)^C. \tag{85}$$

The defining polynomial has now roots $0, 1, \pi$, so its cluster picture looks like (see [21]). Then the decorated graph of $(\mathcal{X},\mathcal{D})$ equals



Define the following quantities:

- $d_1 := \gcd(N, C, A+B)$, $d_2 := \gcd(N, A, B)$.
- $e := \gcd(N, A+B)$, $t := \frac{e}{d_1 d_2}$.

It is clear that $d_1 \mid e$ and $d_2 \mid e$. Condition (i) implies that d_1 and d_2 are prime to each other, hence $t \in \mathbb{Z}$. If $d \mid N$, we denote by ζ_d the root d-th root of unity $\zeta^{N/d}$.

The special fiber $\overline{\mathcal{Y}^{\mathrm{st}}}$ has d_1 components over $\overline{X_1}$ (setting $x_1 = x$, $y_1 = y$) given by

$$\mathcal{C}_1^{(i)} : y_1^{\frac{N}{d_1}} = \zeta_{2d_1}^{B+2i} x_1^{\frac{A+B}{d_1}} (1-x_1)^{\frac{C}{d_1}}, \qquad i = 0, \dots, d_1 - 1, \tag{86}$$

corresponding to the irreducible components of the curve

$$\mathcal{C}_1 : y_1^N = (-1)^B x_1^{A+B} (1-x_1)^C.$$

Over $\overline{X_2}$ the special fiber has d_2 components, setting $x_2 = \pi x$ and $y = \pi^{\frac{A+B}{N}} y_2$, they are given by the equation

$$\mathcal{C}_2^{(j)} : y_2^{\frac{N}{d_2}} = \zeta_{d_2}^{i} x_2^{\frac{A}{d_2}} (1-x_2)^{\frac{B}{d_2}}, \qquad j = 0, \dots, d_2 - 1, \tag{87}$$

corresponding to the irreducible components of the curve

$$\mathcal{C}_2 : y_2^N = x_2^A (1-x_2)^B.$$

The components $\mathcal{C}_1^{(i)}$ and $\mathcal{C}_2^{(j)}$ intersect at e different points as in Figure 7.

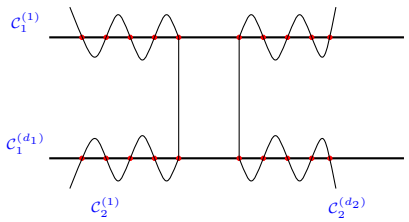


FIGURE 7. Intersection of Special Fiber of $\mathcal{Y}^{\mathrm{st}}$

Denote by $\widetilde{\mathcal{A}}$ the Jacobian of $\widetilde{\mathcal{C}}$ and by $\widetilde{\mathcal{A}}^{\mathrm{new}}$ its new part. Let η denote the character of Gal_K corresponding to the extension $K(\sqrt[N]{t})/K$, and let θ be the character corresponding to the (at most quadratic) extension $K(\sqrt[N]{-1})/K$.

Theorem A.1. *Let ℓ be a prime number different from the characteristic of K.*

- *If $N \nmid A+B$, set $r_i = \frac{\varphi(N)}{\varphi(N/d_i)}$ for $i = 1, 2$. Then*

$$T_\ell(\widetilde{\mathcal{A}}^{new}) \simeq T_\ell(\mathcal{A}_1)^{r_1} \oplus T_\ell(\mathcal{A}_2)^{r_2} \otimes \eta^{-(A+B)},$$

  where $\mathcal{A}_1$ and $\mathcal{A}_2$ are abelian varieties with complex multiplication over K.
- *Otherwise, as $\mathbb{Q}_\ell[I_K]$-modules*

$$T_\ell(\widetilde{\mathcal{A}}^{new}) \simeq \bigoplus_{i=1}^{\varphi(N)} \mathrm{Sp}_2 \otimes \theta^B.$$

Proof. Start assuming that $N \nmid A+B$, so the curves $\mathcal{C}_1^{(i)}$ and $\mathcal{C}_2^{(j)}$ have positive genus. Let μ be a generator of $\boldsymbol{\mu}_N$. The action of μ gives an isomorphism between points of $\mathcal{C}_1^{(i)}$ and of $\mathcal{C}_1^{(i+1)}$ (for i an element of $\mathbb{Z}/d_1$). Then

$$T_\ell(\mathrm{Jac}(\mathcal{C}_1^1)) \simeq T_\ell(\mathrm{Jac}(\mathcal{C}_1^i)) \qquad \text{for all } i.$$

The same holds for $\mathcal{C}_2^{(j)}$. Let $\mathcal{A}_1$ (respectively $\mathcal{A}_2$) be the new part of $\mathrm{Jac}(\mathcal{C}_1^{(1)})$ (resp. $\mathrm{Jac}(\mathcal{C}_2^{(1)})$); $\mathcal{A}_1$ consists of the eigenspaces for the action of $\boldsymbol{\mu}_{\frac{N}{d_1}}$ whose eigenvalues are primitive N/d_1-th roots of unity.

Let v be an element of $\mathrm{Jac}(\mathcal{C}_1^{(1)})$ on the ζ^{d_1}-eigenspace for the action of μ^{d_1}. Let $i \in (\mathbb{Z}/N)^\times$ be an element congruent to 1 modulo N/d_1 and consider the element in $\mathrm{Jac}(\mathcal{C}_1) \simeq \bigoplus_j \mathrm{Jac}(\mathcal{C}_1^{(j)})$ defined by

$$w_i = \zeta^{i(d_1-1)}v + \mu\cdot\zeta^{i(d_1-2)}v + \cdots + \mu^{d_1-2}\cdot\zeta^i v + \mu^{d_1-1}\cdot v$$

It is clear from its definition that $\mu\cdot w_i = \zeta^i w_i$ (since $\mu^{d_1}\cdot v = \zeta^{d_1}v$ and $i \equiv 1 \pmod{\frac{N}{d_1}}$), so w_i belongs to the ζ^i-eigenspace for the action of μ on $\mathrm{Jac}(\mathcal{C}_1)$. Furthermore, the map $v \to w_i$ gives an isomorphism between the ζ^{d_1}-eigenspace for the action of μ^{d_1} on $\mathcal{A}_1$ and the ζ^i-eigenspace for the action of μ on $\mathrm{Jac}(\mathcal{C}_1)$, so as Galois modules

$$T_\ell(\mathrm{Jac}(\mathcal{C}_1)^{\mathrm{new}}) \simeq T_\ell(\mathcal{A}_1)^{r_1},$$

where $r_1 = \frac{\varphi(N)}{\varphi(N/d_1)}$ (the number of possible values of i).

A similar result holds for the curve $\mathcal{C}_2$ and its components, but note that $\mathcal{C}_2^{(j)}$ is not a special fiber of $\widetilde{\mathcal{C}}$ but a twist of it by $\eta^{-(A+B)}$ is (due to the change of variables $y = \pi^{\frac{A+B}{N}}y_2$).

Condition (ii) (and the hypothesis $N \nmid A+B$) implies that the curve $\mathcal{C}_1^{(i)}$ (respectively $\mathcal{C}_2^{(j)}$) has genus $\varphi(N/d_1)/2$ (respectively $\varphi(N/d_2)/2$) and has an action of $\boldsymbol{\mu}_{\frac{N}{d_1}}$ (respectively $\boldsymbol{\mu}_{\frac{N}{d_2}}$), so its Jacobian has complex multiplication (see [57]).

Suppose then that $N \mid A+B$. Then $e = N$ and the components $\mathcal{C}_1^{(i)}$ and $\mathcal{C}_2^{(j)}$ have all genus zero. By (83) the Galois representation is a twist of the representation Sp_2 by a character coming from the action of the inertia group I_K on the graph cohomology. The edges of the graph $\mathrm{H}^1(\Upsilon,\mathbb{Z})$ correspond to the components $\mathcal{C}_1^{(i)}$ and $\mathcal{C}_2^{(j)}$, which are defined over K, so the inertia group I_K acts trivially on them. The intersection points of $\mathcal{C}_1^{(i)}$ and $\mathcal{C}_2^{(j)}$ correspond to the desingularizations points at $(0,0)$ of the first curve and at ∞ of the second. By §3.1.1 and §3.1.2 of [4], the intersection points of the desingularized curves are defined over the extension $K(\sqrt[N]{(-1)^B})$, giving the extra twist by θ^B. □

A.1. **The hypergeometric motive.** Keeping the previous notation, let $\mathcal{A}^\zeta$ denote the submotive of $\mathcal{A}$ given by the ζ-eigenspace for the action of μ. Then (as done in 8.11)

$$\mathcal{H}((a,b),(c,d)|z) = \mathcal{A}^\zeta(-1)\otimes\mathbf{J}((-a,-b,c,d),(c-b,d-a))^{-1}\varkappa_K^{(d-b)N}.$$

The motive $\mathbf{J}((-a,-b,c,d),(c-b,d-a))^{-1}$ is a Hecke character (independent of the variable z), so it is unramified while restricted to $\mathrm{Gal}(\overline{\mathbb{Q}(z)}/\overline{\mathbb{Q}}(z))$ (the same holds for $\varkappa_K$).

Let $\ell \equiv 1 \pmod N$ be a prime and let

$$\rho_\ell : \mathrm{Gal}(\overline{\mathbb{Q}(z)}/\overline{\mathbb{Q}}(z)) \to \mathrm{GL}_2(\mathbb{Q}_\ell)$$

denote the 2-dimensional Galois representation attached to $\mathcal{H}((a,b),(c,d)|z)$. Recall the definition (as given in Proposition 3.12) of the monodromy matrix M_0 around 0 for the parameters $(a,b),(c,d)$

$$M_0 = \begin{cases} \begin{pmatrix} e^{-c} & 0 \\ 0 & e^{-d} \end{pmatrix} & \text{if } c-d \notin \mathbb{Z}, \\ \begin{pmatrix} e^{-c} & 1 \\ 0 & e^{-c} \end{pmatrix} & \text{if } c-d \in \mathbb{Z}, \end{cases}$$

where $e^a := \exp(2\pi i a)$.

Theorem A.2. *The image of the inertia group at $z=0$ under ρ_ℓ is spanned (up to GL_2-conjugation) by the matrix M_0. Furthermore, if $s \in \mathbb{Q}[z]$ and we consider Euler's curve with parameter s, then the same holds, but replacing M_0 by $M_0^{v(s)}$, where $v(s)$ denote the valuation of s at 0.*

Proof. Let $\mathcal{O} = \mathbb{Q}[[z]]$, the completion of $\mathbb{Q}[z]$ at the prime ideal (z). It is a Henselian complete valuation ring with local uniformizer z, so we can apply the results of the previous section to the curve $\widetilde{\mathcal{C}}$. To relate the curve $\mathcal{C}$ to $\widetilde{\mathcal{C}}$ we need to twist by $\eta^{A+B-D} = \eta^{-cN}$. Since $A+B=(c-d)N$, if $c-d \notin \mathbb{Z}$ we are in the first case of Theorem A.1, so inertia acts on the ζ-eigenspace like the sum of the two characters

$$\eta^{-c} \oplus \eta^{-d}.$$

When $c-d \in \mathbb{Z}$, we are in the second case of Theorem A.1, but note that the character θ is trivial on $\mathrm{Gal}(\overline{\mathbb{Q}(z)}/\overline{\mathbb{Q}}(z))$. □

Corollary A.3. *Let L be a number field containing F and let $\xi \in L$, $\xi \neq 0,1$. Let $\mathfrak{p}$ be a prime ideal of L not dividing N. Then the image of the inertial group $I_\mathfrak{p}$ on $\mathcal{H}((a,b),(c,d)|\xi)$ is spanned (up to conjugation) by $M_0^{v_\mathfrak{p}(\xi)}$.*

Proof. Mimics the previous theorem, noting that the Hecke character $\mathbf{J}((-a,-b,c,d),(c-b,d-a))^{-1}$ and the quadratic character $\varkappa_N$ are only ramified at primes dividing N. □

An analogous result holds for the inertia group at 1 and ∞. Define the matrices

$$M_\infty = \begin{cases} \begin{pmatrix} e^a & 0 \\ 0 & e^b \end{pmatrix} & \text{if } a-b \notin \mathbb{Z}, \\ \begin{pmatrix} e^a & 1 \\ 0 & e^a \end{pmatrix} & \text{if } a-b \in \mathbb{Z}. \end{cases} \qquad M_1 = \begin{cases} \begin{pmatrix} e^\delta & 0 \\ 0 & 1 \end{pmatrix} & \text{if } \delta := a+b-c-d \notin \mathbb{Z}, \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \text{otherwise.} \end{cases}$$

Theorem A.4. *The image of the inertia group at $z=\infty$ (respectively at $z=1$) under ρ_ℓ is spanned (up to GL_2-conjugation) by the matrix M_∞ (respectively M_1).*

Proof. Mimics the last theorem. For the first statement, the local parameter is $1/z$. The change of variables $x' = zx$, $y' = (1/z)^b y$ (on the equation for $\mathcal{C}$) gives the component

$$\mathcal{C}_1 : y'^N = x'^A (1-x')^C.$$

Similarly, the second component is obtained by setting $y' = (1/z)^a y$, with equation

$$\mathcal{C}_2 : y'^N = (-1)^C x^{A+B} (1-x)^B.$$

The result then follows from the same arguments as in the last theorem. For the second statement, consider the two components of $\widetilde{\mathcal{C}}$ given by

$$\mathcal{C}_1 : y^N = x^A (1-x)^{B+C}, \qquad \mathcal{C}_2 : y'^N = x'^B (1-x')^C,$$

where the second curve is obtained by looking at the special fiber of the curve obtained via the change of variables $x' = \frac{(x-1)}{(z-1)}$, $y' = (1-z)^{-\frac{B+C}{N}} y$. The result follows from the fact that $B+C = \delta$ and that the twist needed to relate $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ is unramified at $z-1$. □

Appendix B. Stable models and the trace of Frobenius

The goal of the present appendix is to give a formula for the trace of a Frobenius element $\mathrm{Frob}_{\mathfrak{p}}$ when the motive has good reduction at $\mathfrak{p}$ but the equation used to define Euler's curve is singular at $\mathfrak{p}$. There are three different cases, depending on whether the specialization ξ reduces to 0, 1 or ∞ modulo $\mathfrak{p}$. More concretely, let $(a,b),(c,d)$ be generic rational parameters and let N be their least common denominator. Let $F = \mathbb{Q}(\zeta_N)$ and let $\mathfrak{p}$ be a prime ideal of F not dividing N. Consider the following three cases:

(1) The monodromy matrix M_0 has finite order r_0, $v_{\mathfrak{p}}(\xi)$ is positive and divisible by r_0.
(2) The monodromy matrix M_1 has finite order r_1, $v_{\mathfrak{p}}(\xi-1)$ is positive and divisible by r_1.
(3) The monodromy matrix M_∞ has finite order r_∞, $v_{\mathfrak{p}}(\xi)$ is negative and divisible by r_∞.

The following three theorems describe the value of the trace of $\mathrm{Frob}_{\mathfrak{p}}$ in each case (assuming an extra technical condition that holds in most instances).

Theorem B.1. *Let $(a,b),(c,d)$ be rational generic parameters and let N be their least common denominator. Let $\xi \in \mathbb{Q}$. Suppose that condition (1) is satisfied. Write $\xi = p^{v_{\mathfrak{p}}(\xi)}\tilde{\xi}$. Suppose that the following two extra hypotheses hold:*

$$\gcd(N,(d-b)N,(b-c)N) = 1, \quad \text{and} \quad \gcd(N,(d-c)N,(a-d)N) = 1.$$

Then the trace of $\mathrm{Frob}_{\mathfrak{p}}$ acting on $\mathcal{H}((a,b),(c,d)|\xi)$ equals

$$-\chi_{\mathfrak{p}}(-1)^{(d-b)(\mathcal{N}\mathfrak{p}-1)}\,\mathbf{J}((-a,-b,c,d),(c-b,d-a))(\mathfrak{p})^{-1}\cdot \left(\chi_{\mathfrak{p}}(\tilde{\xi})^{dN} J(\chi_{\mathfrak{p}}^{(d-b)N},\chi_{\mathfrak{p}}^{(b-c)N}) + \chi_{\mathfrak{p}}(-1)^{(b-c)N}\chi_{\mathfrak{p}}(\tilde{\xi})^{cN} J(\chi_{\mathfrak{p}}^{(d-c)N},\chi_{\mathfrak{p}}^{(a-d)N})\right), \tag{88}$$

where $J(\chi_1,\chi_2)$ denotes the usual Jacobi sum.

Proof. The proof follows standard arguments (see [11], [22] and [41]). Let $A = (d-b)N$, $B = (b-c)N$, $C = (a-d)N$ and $D = dN$. Then Euler's curve is defined by the equation

$$\mathcal{C} : y^N = x^A(1-x)^B(1-\xi x)^C p^{v_{\mathfrak{p}}(\xi)D}\tilde{\xi}^D. \tag{89}$$

The order of M_0 is $\mathrm{lcm}\{\mathrm{den}(c),\mathrm{den}(d)\}$, hence the hypothesis of being in case (1) implies that both $v_{\mathfrak{p}}(\xi)c$ and $v_{\mathfrak{p}}(\xi)d$ are integers. The change of variables $y \to yp^{-v_{\mathfrak{p}}(\xi)d}$ is defined over F. The new model reduces modulo $\mathfrak{p}$ to the curve

$$\mathcal{C}_1 : y^N = x^A(1-x)^B\tilde{\xi}^D.$$

The first extra hypothesis implies that $\mathcal{C}_1$ is irreducible. Since the parameters are generic, $N \nmid A$ and $N \nmid B$. Furthermore, $N \nmid A+B = N(d-c)$ because the matrix M_0 has finite order. Then $\dim(\mathrm{Jac}(\mathcal{C}_1)^{\mathrm{new}}) = \frac{\phi(N)}{2} > 0$, so $\mathcal{C}_1$ is a component of the stable model of $\mathcal{C}$ (see [11] for more details).

To find the other component, consider the change of variables $x \to xp^{-v_{\mathfrak{p}}(\xi)}$, which transforms (89) into

$$y^N p^{v_{\mathfrak{p}}(\xi)(A+B-D)} = x^A(p^{v_{\mathfrak{p}}(\xi)} - x)^B(1-\tilde{\xi}x)^C\tilde{\xi}^D.$$

From its definition, $A+B-D = -cN$ so after the change of variables $y \to yp^{v_{\mathfrak{p}}(\xi)c}$ we get an equation whose reduction modulo $\mathfrak{p}$ equals

$$\mathcal{C}_2 : y^N = (-1)^B x^{A+B}(1-\tilde{\xi}x)^C\tilde{\xi}^D.$$

The second extra hypothesis implies that $\mathcal{C}_2$ is irreducible. The genericity condition on the parameters implies that $N \nmid C$, $N \nmid A+B$ and $N \nmid A+B+C = N(a-c)$ so $\dim(\mathrm{Jac}(\mathcal{C}_2)^{\mathrm{new}}) = \frac{\phi(N)}{2}$ and $\mathcal{C}_2$ is another component of the stable reduction of $\mathcal{C}$. Then the Galois representation of the decomposition group at $\mathfrak{p}$ acting on $H^1_{\text{ét},\chi}(\widehat{\mathcal{C}}, \mathbb{Q}_\ell)$ is the direct sum of the contribution from $\mathcal{C}_1$ and that from $\mathcal{C}_2$ (both being abelian varieties with complex multiplication by $\mathbb{Z}[\zeta_N]$).

To compute the trace of $\mathrm{Frob}_\mathfrak{p}$ acting on $\mathcal{C}_1$ (respectively $\mathcal{C}_2$), let $\chi_\mathfrak{p}$ be as in (22). Then we can apply Theorem 7.4 to $\mathcal{C}_1$ to get that the trace of $\mathrm{Frob}_\mathfrak{p}$ acting on $H^{1,\zeta_N}_{et}(\widehat{\mathcal{C}_1}, \mathbb{Q}_\ell)$ equals

$$-\sum_{x\in\mathbb{F}_q} \chi_\mathfrak{p}(x^A(1-x)^B\tilde{\xi}^D) = -\chi_\mathfrak{p}(\tilde{\xi})^D J(\chi_\mathfrak{p}^A, \chi_\mathfrak{p}^B).$$

The same computation for $\mathcal{C}_2$ gives that the trace of $\mathrm{Frob}_\mathfrak{p}$ on the (new part of its) étale cohomology equals

$$-\chi_\mathfrak{p}(-1)^B \chi_\mathfrak{p}(\tilde{\xi})^{D-A-B} J(\chi_\mathfrak{p}^{A+B}, \chi_\mathfrak{p}^C).$$

The result follows from the definition of A, B, C and D together with the relation between the hypergeometric motive and Euler's curve given in (67). □

With the same technique one can prove the following results for the other two cases.

Theorem B.2. *Let $(a,b),(c,d)$ be rational generic parameters and let N be their least common denominator. Let $\xi \in \mathbb{Q}$. Suppose that condition (2) is satisfied. Let $v = v_\mathfrak{p}(\xi - 1)$ and write $\xi = 1 + p^v\tilde{\xi}$. Suppose that the following two extra hypotheses hold:*

$$\gcd(N, (d-b)N, (a+b-c-d)N) = 1, \quad \textit{and} \quad \gcd(N, (b-c)N, (a-d)N) = 1.$$

Then the trace of $\mathrm{Frob}_\mathfrak{p}$ *acting on* $\mathcal{H}((a,b),(c,d)|\xi)$ *equals*

$$-\chi_\mathfrak{p}(-1)^{(d-b)(\mathcal{N}\mathfrak{p}-1)}\,\mathbf{J}((-a,-b,c,d),(c-b,d-a))(\mathfrak{p})^{-1}\cdot \left(J(\chi_\mathfrak{p}^{(d-b)N}, \chi_\mathfrak{p}^{(a+b-c-d)N}) + \chi_\mathfrak{p}(-1)^{(a-d)N}\chi_\mathfrak{p}(\tilde{\xi})^{(a-b)N} J(\chi_\mathfrak{p}^{(d-b)N}, \chi_\mathfrak{p}^{(a-d)N})\right). \tag{90}$$

Theorem B.3. *Let $(a,b),(c,d)$ be rational generic parameters and let N be their least common denominator. Let $\xi \in \mathbb{Q}$. Suppose that condition (3) is satisfied. Write $\xi = p^{v_\mathfrak{p}(\xi)}\tilde{\xi}$. Suppose that the following two extra hypotheses hold:*

$$\gcd(N, (a-b)N, (b-c)N) = 1, \quad \textit{and} \quad \gcd(N, (d-b)N, (a-d)N) = 1.$$

Then the trace of $\mathrm{Frob}_\mathfrak{p}$ *acting on* $\mathcal{H}((a,b),(c,d)|\xi)$ *equals*

$$-\chi_\mathfrak{p}(-1)^{(d-b)(\mathcal{N}\mathfrak{p}-1)}\,\mathbf{J}((-a,-b,c,d),(c-b,d-a))(\mathfrak{p})^{-1}\cdot \left(\omega(\tilde{\xi})^{bN} J(\omega^{(d-b)N}, \omega^{(a-d)N}) + \omega(-1)^{(a-d)N}\omega(\tilde{\xi})^{aN} J(\omega^{(a-b)N}, \omega^{(b-c)N})\right). \tag{91}$$

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