ON THE EQUATION $x^2 + dy^6 = z^p$ FOR SQUARE-FREE $1 \leq d \leq 20$

FRANCO GOLFIERI, ARIEL PACETTI, AND LUCAS VILLAGRA TORCOMIAN

ABSTRACT. The purpose of the present article is to show how the modular method together with different techniques can be used to prove non-existence of primitive non-trivial solutions of the equation $x^2+dy^6 = z^p$ for square-free values $1 \leq d \leq 20$ following the approach of [PT20]. The main innovation is to make use of the symplectic argument over ramified extensions to discard solutions, together with a multi-Frey approach to deduce large image of Galois representations.

INTRODUCTION

The study of Diophantine equations has been a major research area since ancient times, and it got a lot of attention during the last twenty years after Wiles' proof of Fermat's last theorem. Of particular interest is to understand the set of solutions to the generalized Fermat's equation

$$AX^p + BY^q = CZ^r,$$

for $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$. In the present article, we focus on the particular equation

$$\mathcal{C}_d : x^2 + dy^6 = z^n. \tag{1}$$

Specializing (1) at $y = 1$ gives the so called Lebesgue–Nagell equation (see for example [Coh93] and the references therein). Not so long ago, all solutions of the Lebesgue-Nagell equation where obtained for $1 \leq d \leq 100$ (in [BMS06b]). Going back to equation (1), we are mostly interested in the case $n > 3$ so that $\frac{1}{2} + \frac{1}{6} + \frac{1}{n} < 1$. In the article [BC12] the authors studied equation (1) for $d = 1$ (as part of their study of generalized Fermat's equations). Their main result is that for $n \geq 3$ there are no *non-trivial* and *primitive* solutions. Let us recall these two notions.

Definition. A solution (A, B, C) to (1) is called *primitive* if $\gcd(A, B, C) = 1$. A solution is called *non-trivial* if $ABC \neq 0$.

Contrary to Fermat's problem, equation (1) is non-homogeneous, hence it does not determine a projective variety, but an affine surface. In particular, the set of primitive and non-primitive solutions are very different. Moreover, there are many non-primitive solutions!

Proposition (Granville)**.** *Let $p > 3$ be a prime number. The equation*

$$\mathcal{C}_d : x^2 + dy^6 = z^p,$$

has infinitely many non-primitive solutions.

Proof. Suppose that $p \equiv 1 \pmod 6$. Let $u, v \in \mathbb{Z}$ arbitrary so that the value $r = u^2 + dv^6$ does not equal ± 1. Then the point $(ur^{(p-1)/2}, vr^{(p-1)/6}, r)$ lies in $\mathcal{C}_d$. If $p \equiv 5 \pmod 6$, then the point $(ur^{(5p-1)/2}, vr^{(5p-1)/6}, r^5)$ lies in $\mathcal{C}_d$. □

Remark 1. The ABC conjecture implies that besides the solutions with $z = \pm 1$ (which are finite by Faltings' theorem), all other primitive ones are finite. In particular, this is the case when $d > 0$.

Remark 2. Sometimes when $d < 0$ a solution to Pell's equation provides a point in $\mathcal{C}_d$ with $z = \pm 1$. Such a point provides a non-trivial solution for all values of p, making the method to fail in general. This is the case for example for $(u, v, d) = (31, 2, -15)$. For this reason, we restrict to positive values of d.

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In [PT20] a general strategy to prove non-existence of non-trivial solutions was presented, including a few examples (for both positive and negative values of d, see also [PT21]). Let us recall how the method works: to a putative solution (A, B, C) of (1) of prime exponent p, we attach the $\mathbb{Q}$-curve

$$E_{A,B} : y^2 + 6B\sqrt{-d}xy - 4d(A + B^3\sqrt{-d})y = x^3. \tag{2}$$

Its discriminant equals $-2^8 3^3 d^4 C^p (A + B^3\sqrt{-d})^2$. Note that the trivial solution $(\pm 1, 0, 1)$ corresponds to a rational curve with complex multiplication (CM from now on) by $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$.

The main result of the aforementioned article consists of constructing a character χ such that $\rho_E \otimes \chi$ descends to a rational representation, which is modular by Serre's conjectures (say attached to a newforms $f_{A,B}$). In particular, an explicit formula for the level and the Nebentypus of the modular form was presented.

Note that the discriminant of the curve $E_{A,B}$ is "almost" a perfect p-th power (except for the primes 2, 3 and the ones dividing d). Then the general strategy is to apply Ribet's lowering the level result (as in [Rib91]) to get a congruence between $f_{A,B}$ and another newform whose level is not divisible by primes dividing C, i.e. its level is only divisible by $2, 3$ and the primes dividing d. Ribet's result is only valid when the image of the residual representation (modulo p) is absolutely irreducible.

Problem 1: Give an explicit constant N_d such that if $p > N_d$ then the Galois representation $\rho_{E_{A,B},p}$ has absolutely irreducible residual image.

Then if $p > N_d$, there exists a newform g in $S_2(\Gamma_0(N), \varepsilon)$ where N is divisible only by $2, 3$ and the primes dividing d such that $f_{A,B} \equiv g \pmod{p}$. In [PT20, Theorem 6.3] an explicit formula for N is given.

The remaining obstruction for deciding whether equation (1) has a non-trivial solution or not, comes from the challenge to discard all newforms in the space $S_2(\Gamma_0(N), \varepsilon)$ (as not coming from a possible solution).

Problem 2: How can we discard the forms with complex multiplication appearing in such space?

Modular forms with CM are in general harder to discard, as for example the trivial solutions correspond to such a curves! However, trivial solutions are the unique ones corresponding to an elliptic curve with CM under our assumption that d is square-free when $d \neq 2$ (by Lemma 3.1), hence we can just discard them while searching for non-trivial solutions.

Problem 3: How can we discard the non-CM forms?

The purpose of the present article is to show how different approaches appearing in the literature (and some generalizations) can be combined to answer Problem 3 and in particular prove non-existence (under certain hypothesis) of solutions to equation (1) for most integral square-free values of d in the range $[1, 20]$.

Regarding Problem 1, let us recall the following result due to Ellenberg ([Ell04] and [Ell05]).

Theorem. *(Ellenberg) Let $E/\mathbb{Q}(\sqrt{-d})$ (with $d > 0$) be a $\mathbb{Q}$-curve satisfying that there exists a prime $p > 3$ of multiplicative reduction for E. Then there exist an integer N_d such that the projective image of the residual representation of $\rho_{E,p}$ is surjective for all primes $\mathfrak{p}$ of norm greater than N_d.*

In [Ell04] it is explained how to get an explicit bound for N_d (see also [DU09, Theorem 6]). Concretely, let N be a any positive integer, and χ the character corresponding to $K/\mathbb{Q}$ (of conductor $\mathfrak{f}$). Let $\mathcal{F}$ be a Petterson-orthogonal basis for $S_2(\Gamma_0(N))$. Define

$$(a_m, L_\chi)_N = \sum_{f \in \mathcal{F}} a_m(f) L(f \otimes \chi, 1).$$

If $M|N$, define $(a_m, L_\chi)_N^M$ as the contribution from the old forms of level M. Then Ellenberg's result [Ell04] states that for any prime p for which

$$(a_1, L_\chi)_{p^2}^{p-\text{new}} = (a_1, L_\chi)_{p^2} - p(p^2 - 1)^{-1}(a_1 - p^{-1}\chi(p)a_p, L_\chi)_p \tag{3}$$

is non-zero, the residual image is large (see Section 3.1 for more details). Note that Ellenberg's result is strong enough to solve Problem 1 and Problem 2, since modular forms with complex multiplication do not have surjective projective image. A new problem appears while trying to apply Ellenberg's result, namely.

Problem 4: How can be assure that the curve $E_{A,B}$ has a prime $p > 3$ of multiplicative reduction?

In Section 2 we present a solution to Problem 4, in particular giving a complete answer to Problem 2.

The article is organized as follows: the first section contains four different approaches to attack Problem 3, namely properties that newforms related to real solutions must satisfy! Each approach can be thought of as a "real solution test", so while trying to discard a particular newform, we run the four different tests on it. If the form fails to pass one test, we automatically discard it. The four different tests include the so called Mazur's trick, the study of the local inertial type, the application of the symplectic argument (as explained in [FK16], with some improvements for $\mathbb{Q}$-curves), and the property that all our curves contain a rational point of order 3 over $\mathbb{Q}(\sqrt{-d})$.

The second section contains one of the novelties of the present article, namely to use a multi-Frey approach to give a complete answer to Problem 4. As will be explained in Section 2, we succeed to run the multi-Frey approach thanks to Cremona's list of elliptic curves of conductor up to 500000 (available at [LMF21]).

The last section contains applications of all the previous ones to study solutions of (1) for $1 \leq d \leq 20$ square-free. Regarding such examples, there are three cases that cannot be handled for computational reasons (the level of the space of modular forms needed to be computed is too large). They correspond to the values $d = 10$, $d = 14$ and $d = 17$. The symplectic argument used to discard newforms only provides "partial results", namely, it only allows to discard primes p satisfying certain congruence conditions. In particular, in some cases we can only prove that equation (1) does not have any solution for an explicit positive density of primes. This is the case for $d = 7$, $d = 11$ and $d = 15$. Full new results are obtained for $d = 5$, $d = 13$ and $d = 19$.

The computations of the present article were done using Pari/GP ([Par21]), Magma ([BCP97]) and Sage ([Sag21]). The Pari script implemented to compute Ellenberg's bound, the one implemented (in Magma) to apply Mazur's trick and the computations made in Magma (as well as their outputs) to certify the present results can be found in the web page `http://sweet.ua.pt/apacetti/research.html`.

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1. Some tests to eliminate forms

To easy notation, during the whole article, we will denote by K the field $\mathbb{Q}(\sqrt{-d})$.

1.1. **Mazur's trick.** Let $f \in S_2(\Gamma_0(N), \varepsilon)$ be a newform, where N is the level obtained in [PT20] after applying the lowering the level result of Ribet to a putative solution (A, B, C). Let p and ℓ be different prime numbers such that ℓ does not ramifying in $K/\mathbb{Q}$. Let $\mathfrak{l}$ be a prime in K dividing ℓ. Let f^{BC} denote the base change of f to K. Recall that the q-expansion of f^{BC} is given by the following rule:

$$a_{\mathfrak{l}}(f^{\mathrm{BC}}) = \begin{cases} a_\ell(f) & \text{if } \mathcal{N}\mathfrak{l} = \ell, \\ a_\ell(f)^2 - 2\ell\varepsilon(\ell) & \text{if } \mathcal{N}\mathfrak{l} = \ell^2. \end{cases} \tag{4}$$

The idea behind Mazur's trick is to study equation (1) modulo ℓ, to get information on $a_{\mathfrak{l}}(E_{A,B})$. Let

$$S_\ell = \{(\tilde{A}, \tilde{B}, \tilde{C}) \in \mathbb{F}_{\mathcal{N}(\mathfrak{l})} \ : \ \tilde{A}^2 + d\tilde{B}^6 = \tilde{C}^\ell\},$$

discarding the trivial solution $(0, 0, 0)$, where $\mathcal{N}(\mathfrak{l})$ stands for the norm of the ideal $\mathfrak{l}$. For each point $(\tilde{A}, \tilde{B}, \tilde{C}) \in S_\ell$, consider the curve $E_{\tilde{A},\tilde{B}}$ over $\mathbb{F}_{\mathcal{N}(\mathfrak{l})}$. Then, either:

- The curve $E_{\tilde{A},\tilde{B}}$ is non-singular, in which case if (A, B, C) is an integral solution reducing to $(\tilde{A}, \tilde{B}, \tilde{C})$, we must have that $a_{\mathfrak{l}}(E_{A,B}) = a_{\mathfrak{l}}(E_{\tilde{A},\tilde{B}})$ and furthermore
$$\chi(\mathfrak{l})a_{\mathfrak{l}}(E_{\tilde{A},\tilde{B}}) \equiv a_{\mathfrak{l}}(f^{\mathrm{BC}}) \pmod{p},$$
or
- The curve $E_{A,B}$ has bad reduction at $\mathfrak{l}$ in which case we are in the lowering the level hypothesis, and
$$a_\ell(f)^2 \equiv \varepsilon^{-1}(\ell)(\ell + 1)^2 \pmod{p}.$$

In both cases, each side of the congruence can be computed, and if they happen to be different, we get a finite list of candidates for the prime p dividing their difference. This method is very powerful, and running it for a few different values of ℓ allows to discard all newforms whose coefficient field (of the base change to $\mathbb{Q}(\sqrt{-d})$) does not match $\mathbb{Q}(\chi)$. Sometimes, it does discard all possible forms (providing non-existence of solutions of (1)), however in many instances, there are newforms which systematically pass the test.

1.2. **The local type.** Let K_λ be a finite extension of $\mathbb{Q}_\ell$. The local-Langlands correspondence gives a bijection between the set of automorphic representations of $\mathrm{GL}_2(K_\lambda)$ and representations of the Weil-Deligne group of K_λ (we will follow Carayol's normalization of [Car86]). Recall that a Weil-Deligne representation consists of a two dimensional complex representation ρ of the Weil group $W(K_\lambda)$ together with a monodromy operator N. Let $\omega_1 : W(\mathbb{Q}_\ell) \to \mathbb{C}^\times$ be the unramified character sending Frobenius to $\|\ell\|_\ell$ (and use the same notation for its restriction to $W(K_\lambda)$). For $\ell \neq 2$, there are three different local types for a Weil-Deligne representation ρ with trivial Nebentypus:

(1) **Principal Series**: the endomorphism $N = 0$ and $\rho = \chi \oplus \chi^{-1}\omega_1^{1-k}$ for some quasi-character $\chi : W(\mathbb{Q}_p)^{\mathrm{ab}} \mapsto \mathbb{C}^\times$.
(2) **Steinberg or Special Representation**: The endomorphism N is given by the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and the representation ρ equals $\omega_1^r \left(\begin{smallmatrix} \chi\omega_1 & 0 \\ 0 & \chi \end{smallmatrix}\right)$ for some quasi-character $\chi : W(K_\lambda) \mapsto \mathbb{C}^\times$.
(3) **Supercuspidal Representation**: the endomorphism $N = 0$ and $\rho = \mathrm{Ind}_{W(E)}^{W(K_\lambda)} \varkappa$ where E is a quadratic extension of K_λ, and $\varkappa : W(E)^{\mathrm{ab}} \to \mathbb{C}^\times$ is a quasi-character which does not factor through the norm map with a quasi-character of $W(K_\lambda)^{\mathrm{ab}}$.

Remark 3. There is a big difference between a principal series representation and a supercuspidal one, as the first one is reducible (and decomposable) while the second one is not.

The local inertial type of an automorphic representation of $\mathrm{GL}_2(\mathbb{Q}_\ell)$ is the isomorphism class of its restriction to the inertia subgroup (which is related to the restriction of its Weil-Deligne counterpart).

Proposition 1.1. *Let $\rho : \mathrm{Gal}_K \to \mathrm{GL}_2(\overline{\mathbb{Q}_p})$ be a modular Galois representation, and let $\mathfrak{l}$ be a prime of K whose residual characteristic is prime to p. If the local type of ρ at $\mathfrak{l}$ is principal series (respectively supercuspidal) given by a character χ (respectively $\varkappa$) of order n prime to p then its reduction is also of principal series type (respectively supercuspidal) given by a character of order n.*

Proof. Let K_λ denote the completion of K at the ideal $\mathfrak{l}$. By local class field theory, the character χ appearing in the restriction $\rho|D_\mathfrak{l}$ to a decomposition group at $\mathfrak{l}$ comes from a Hecke character $\chi : W(K_\lambda) \to \overline{\mathbb{Q}_p}$. The kernel of the reduction map $\overline{\mathbb{Z}_p} \to \overline{\mathbb{F}_p}$ is a pro-p-group. Since the order of χ is prime to p, the reduced character $\tilde{\chi}$ has the same order as χ, proving the first statement. In the supercuspidal case, the residual character $\tilde{\varkappa}$ has the same order as $\varkappa$, hence we are only led to prove it does not factor through the norm map (so the residual representation is also irreducible). If $\tilde{\varkappa} = \mathcal{N} \circ \tilde{\theta}$, for some character $\tilde{\theta} : W(E) \to \overline{\mathbb{F}_p}$, let $\theta : W(E) \to \overline{\mathbb{Q}_p}$ be the Teichmuller lift of $\overline{\theta}$. Then $\kappa = \mathcal{N} \circ \theta$ (as both characters have order prime to p and their reductions coincide). □

Remark 4. The representations ρ appearing in the present article come from $\mathbb{Q}$-curves (whose modularity is known due to Serre's conjectures). Since the coefficient field of an elliptic curve is the rational one, any character (in both the principal series or the supercuspidal type) appearing at a prime of bad reduction has order n with $\varphi(n) \leq 2$. In particular, $n \in \{1, 2, 3, 4, 6\}$. Then for $p > 3$, the local type is preserved by congruences.

The ℓ-inertial type of the p-adic representation ρ is the isomorphism class of the restriction of ρ to the decomposition group of ℓ. There is an algorithm for given a rational newform and a prime dividing the level, compute its local type (based on [LW12]) which is implemented both in Sage and Magma. The problem is that in most instances, the space $S_2(\Gamma_0(N), \varepsilon)$ has very large dimension, making the computation unfeasible (from a computational point of view).

1.3. **The symplectic argument.** The symplectic argument is a powerful tool to study congruences between elliptic curves. The idea is to consider not only when two elliptic curves have isomorphic residual representations, but add information on how the isomorphism relates their Weil's pairing. Its first version

(and application) appeared in [KO92] (see [FK16] for the different historical applications and the latest results). Let us state two instances of the symplectic argument that will be used in the present article.

Theorem 1.2. *Let $\ell \neq p$ be primes with $p \geq 3$. Let E, E' be elliptic curves over $\mathbb{Q}_\ell$ with multiplicative reduction. Suppose that $E[p] \simeq E'[p]$ as $\mathrm{Gal}_{\mathbb{Q}_\ell}$-modules. Assume furthermore that $p \nmid v_\ell(\Delta(E))$. Then $p \nmid v_\ell(\Delta(E'))$ and furthermore, $E[p]$ and $E'[p]$ are symplectically isomorphic $\Leftrightarrow \left(\frac{v_\ell(\Delta(E))/v_\ell(\Delta(E'))}{p}\right) = 1$.*

Furthermore, both curves cannot be symplectically and anti-symplectically isomorphic.

Proof. See [KO92] (and [FK16, Theorem 13]). □

We also need a version for ramified extensions. Recall that if K is a local field and E/K is an elliptic curve with potentially good reduction, the *defect* of E is the degree of the minimal extension over K^{ur} where E attains good reduction.

Theorem 1.3. *Let $\ell \equiv 2 \pmod 3$ be a prime number, K be a quadratic ramified extension of $\mathbb{Q}_\ell$, and let $p \neq \ell$ be a prime with $p \geq 5$. Let E, E' be two elliptic curves over K with a point of order 3 on $K_{\mathfrak{l}}$ (where $\mathfrak{l}$ is a prime in K dividing ℓ), with potentially good reduction of defect 3. Set $r = 0$ if $v(\Delta(E)) \equiv v(\Delta(E')) \pmod 3$ and $r = 1$ otherwise. Suppose that $E[p]$ and $E'[p]$ are isomorphic as Gal_K-modules. Then $E[p]$ and $E'[p]$ are symplectically isomorphic $\Leftrightarrow \left(\frac{\ell}{p}\right)^r = 1$.*

Furthermore, both curves cannot be symplectically and anti-symplectically isomorphic.

Proof. The proof mimics Case 1 in page 79 of [FK16]. A key property of our hypothesis is that the fact that $\ell \equiv 2 \pmod 3$ and that $K/\mathbb{Q}_\ell$ is ramified, implies that there are no cubic roots of unity in K (besides the trivial one), hence the extension of K obtained by adding the p-torsion points is not abelian (by Corollary 5 of [FK16]).

The fact that there are no cubic roots of unity in K, implies that raising to the third power is a bijection on the residue fields, hence Lemma 15 of loc. cit. holds without any change. In particular, the same matrix representation of Frobenius and inertia given in Lemma 25 holds. With these ingredients at hand, it is clear that the proof of Case 1 in page 79 of [FK16] holds mutatis mutandis. □

1.4. **Using the 3-torsion information.** Consider the following general situation: Let E and $\tilde{E}$ be rational elliptic curves. Let N be a prime number, and suppose that E has an N-torsion point and that for a large prime p there is an isomorphism of Galois modules $E[p] \simeq \tilde{E}[p]$. How does the N-torsion point of E affect the curve $\tilde{E}$? We thank Professor Samir Siksek for explaining us the following result.

Theorem 1.4. *There exists an explicit constant M (depending only on the conductor of $\tilde{E}$) such that if $p \geq M$, then the elliptic curve $\tilde{E}$ has a N-rational point, or it is N-isogenous to a curve with an N-rational torsion point.*

Proof. Since the curve E has an N-rational point, the semisimplification of its residual representation $\overline{\rho_{E,N}}^{\mathrm{ss}}$ is isomorphic to $1 \oplus \chi_N$ (the cyclotomic character). In particular, $a_q(E) \equiv 1 + q \pmod N$ for all primes $q \neq N$ of good reduction. On the other hand, since $E[p] \simeq \tilde{E}[p]$, $a_q(E) \equiv a_q(\tilde{E}) \pmod p$. By Hasse's bound, the numbers $|a_q(E)| \leq 2\sqrt{q}$, so if $p > 4\sqrt{q}$, $a_q(E) = a_q(\tilde{E})$. Since $\tilde{E}$ is modular (by [BCDT01]), it is attached to a newform in $S_2(\Gamma_0(\mathfrak{N}))$ ($\mathfrak{N}$ being its conductor). Let M be the Sturm bound for modular forms in $S_2(\Gamma_0(\mathfrak{N}))$ (as in [Ste07, Theorem 9.21]), so two forms in $S_2(\Gamma_0(\mathfrak{N}))$ are congruent modulo p if and only if they eigenvalues are congruent modulo p for all primes up to M.

If $p \geq 4\sqrt{M}$, the newform attached to $\tilde{E}$ is congruent modulo N to an Eisenstein series. In particular, $a_q(\tilde{E}) \equiv q + 1 \pmod N$ for all primes $q \neq N$ of good reduction. This implies that $\overline{\rho_{\tilde{E},N}}^{\mathrm{ss}} \simeq 1 \oplus \chi_N$ (by the Brauer-Nesbitt theorem), hence either

$$\overline{\rho_{\tilde{E},N}} \simeq \left(\begin{smallmatrix} 1 & * \\ 0 & \chi_N \end{smallmatrix}\right) \qquad \text{or} \qquad \overline{\rho_{\tilde{E},N}} \simeq \left(\begin{smallmatrix} \chi_N & * \\ 0 & 1 \end{smallmatrix}\right).$$

In the first case, $\tilde{E}$ has a rational point of order N, while in the second case, it has a rational subgroup of order N, whose quotient is an (isogenous) curve with a point of order N. □

Remark 5. We will apply the previous result to elliptic curves over an imaginary quadratic field with a point of order 3 and try to discard possible curves $\tilde{E}$ which do not have a 3-torsion point, nor a 3-isogenous curve

with a point of order 3. Although effective Sturm bounds for Bianchi modular forms are hard to get, we could use the fact that our curves are $\mathbb{Q}$-curves, hence are related to rational newforms to get one. However, from a computational point of view, it is more effective to overcome this issue via computing (if it exists) the first prime $\mathfrak{q}$ such that $a_{\mathfrak{q}}(\tilde{E}) \not\equiv 1 + \mathcal{N}(\mathfrak{q}) \pmod 3$. If such a prime exists, then E and $\tilde{E}$ cannot have Galois representations which are congruent modulo p for $p > 4\sqrt{\mathcal{N}\mathfrak{q}}$.

2. The multi-Frey approach

The idea of the multi-Frey technique (as developed by S. Siksek in [BMS06a] and [BMS08]) is to attach not one elliptic curve to a putative solution, but many of them. In this way, even if Mazur's trick fails for a particular value in S_ℓ for one curve, it may work for another one. This is precisely the case while dealing with equation (1) for $d = 1$ (see Section 5 of [BC12]). Unfortunately, using the same method for other values of d, we did not succeed to discard any solution that passed Mazur's trick for our original $\mathbb{Q}$-curve. However, the existence of a new curve attached to a solution turns out to be very useful to deal with Problem 4. More concretely, to a putative solution (A, B, C) of (1), attach the rational elliptic curve

$$\widetilde{E_{A,B}} \quad : \quad Y^2 = X^3 + 3dB^2X + 2dA. \tag{5}$$

Its discriminant equals $\Delta(\widetilde{E_{A,B}}) = -1728 \cdot d^2 \cdot C^p$ and its j-invariant $\frac{1728 \cdot d \cdot B^6}{C^p}$. Since (A, B, C) is primitive, $\gcd(d, C) = 1$, hence it has multiplicative reduction at all primes dividing C and additive reduction at all primes $\ell > 3$ dividing d if $\ell^6 \nmid d$.

2.0.1. *Application to Problem 4.* Consider the particular case of equation (1) when n is prime and z is only divisible by the primes 2 or 3, i.e. the equation

$$C_p : x^2 + dy^6 = (2^a 3^b)^p. \tag{6}$$

To a putative solution (A, B, C) we attach the rational elliptic curve as in (5), with discriminant $-2^{ap-6}3^{bp+3}d^2$ and c-invariants as follows:

$$c_4 = -2^4 3^2 dy^2, \quad c_6 = -2^6 3^3 dx. \tag{7}$$

We thank Professor Mike Bennett for the following local description of the curve.

Proposition 2.1. *The model is minimal at all primes $\ell \geq 3$. Furthermore, suppose that d is 6-th power-free. Then*

- *The curve has additive reduction at each prime $\ell \mid d$, $\ell > 3$.*
- *If $3 \nmid d$, the model is minimal at 3 and $v_3(N_E) \in \{2, 3\}$, with the latter case possible only if $b = 0$ or $(b, p) = (1, 2)$.*
- *If $v_3(d) \in \{1, 2, 4, 5\}$, then E is minimal at 3 and $v_3(N_E) = 5$.*
- *If $v_3(d) = 3$, E is minimal at 3 and $v_3(N_E) \in \{2, 3\}$.*
- *If $2 \nmid d$, and our model is minimal at $\ell = 2$, we have $v_2(N_E) \in \{2, 3, 4, 5, 6\}$. If it is non-minimal (whereby necessarily $ap \geq 6$) a minimal model has c-invariants $c_4 = -3^2 dB^2$, $c_6 = -3^3 dA$, minimal discriminant $\Delta_E = -2^{ap-6}3^{bp+3}d^2$, and $v_2(N_E) \in \{0, 1\}$.*
- *If $2 \mid d$, E is minimal at 2 if $8 \nmid d$. Furthermore,*
  (1) If $v_2(d) = 1$, $v_2(N_E) \in \{2, 3, 4, 7\}$.
  (2) If $v_2(d) = 2$, $v_2(N_E) = 6$.
- *If $\nu_2(d) = 3$, then either E is minimal at 2 and $v_2(N_E) \in \{4, 5\}$, or E is non-minimal at 2, $2 \mid B$, and a minimal model has $2 \nmid N_E$.*
- *If $v_2(d) = 4$, then E is minimal at 2 and $v_2(N_E) = 6$.*
- *If $v_2(d) = 5$, then E is not minimal at 2 and a minimal model has $v_2(N_E) \in \{2, 3, 4\}$.*

Proof. Follows from a straightforward computation using [Pap93]. □

For a particular value of d, we can search Cremona's tables ([Cre97]) of elliptic curves, as available on the LMFDB [LMF21] and from their discriminant, we get a finite list of possibilities for a and b.

Theorem 2.2. *Let $2 \leq d \leq 19$ be a square-free positive integer. Then for each value of d, the value of p in (6) is bounded by the values of Table 2.1.*

d	*Bound*	d	*Bound*	d	*Bound*	d	*Bound*
2	3	3	2	5	2	6	–
7	7	10	–	11	5	13	–
14	–	15	3	17	2	19	–

TABLE 2.1. Bound for curves having a multiplicative reduction prime. The "–" symbol means that C cannot be supported in $\{2,3\}$.

Proof. Let us explain the algorithm in one example: let $d=2$. Then Proposition 2.1 gives that $N_E = 2^\alpha 3^\beta$ with $\alpha \in \{2,3,4,7\}$ and $\beta \in \{2,3\}$. Now we do a search for all the elliptic curves of conductor $2^\alpha 3^\beta$ which satisfy that its invariants (c_4, c_6, Δ) are compatible with the invariants of our Frey curve, for instance, that are negative numbers (see equation (7)) and that $288 \mid c_4$ and $3456 \mid c_6$. There is a single curve matching these three requirements labeled 1152-r-2 in the LMFDB. Formula (7) allows us to recover the value of (x,y), namely $(5,1)$. Then, factorizing x^2+2y^6 gives that p must be 3.

A similar computation was done to each other value of d, giving the finite list of possibilities for the value of p presented in the table. We want to stress that the described method fails for $d=19$ as it involves computing elliptic curves of conductor 623808. There are two possibilities to try to follow the same approach: one is to compute such set of elliptic curves using Cremona's algorithm (this was done for us by Professor John Cremona), while the other is to use the tables of elliptic curves with bad reduction at a small set of primes as described in [vKM16] (whose database is available at https://bmatschke.github.io/solving-classical-diophantine-equations/). However, in this particular case, there is no need to perform such a computation. The reason is that since 2 is inert, $2 \nmid C$ (by [PT20, Lemma 5.3]) and the same happens for the prime 3 (by [PT20, Lemma 5.4]). □

Remark 6. The last argument actually proves that there is no solution for all values of d which satisfy $d \not\equiv 7 \pmod 8$ and $d \not\equiv 2 \pmod 3$.

3. Application to (1) with $1 \le d \le 20$ and square-free

In this section, we apply the previous procedures to study solutions of (1) for $1 \le d \le 20$, square-free (and show what are its limitations). The case $d=1$ was considered in [BC12], the case $d=2$ in [PT20] and the case $d=3$ in [Kou20], so we restrict to values $d \ge 5$.

To discard all CM newforms, we need to understand which solutions provide elliptic curves with CM.

Lemma 3.1. *Let d be square-free and (A,B,C) be a solution of (1). Then the curve $E_{A,B}$ has CM if and only if the solution is trivial or $(A,B,C,d,p) = (\pm 5, \pm 1, 3, 2, 3)$, with complex multiplication by $\mathbb{Z}[\sqrt{-2}]$.*

Proof. The j-invariant $j(E_{A,B}) = 2^4 3^3 \sqrt{-d} B^3 \cdot \frac{(4A-5B^3\sqrt{-d})^3}{C^p (A+B^3\sqrt{-d})^2} \in \mathbb{Q}(\sqrt{-d})$. If E has CM by an order $\mathcal{O} \subset K$ its j-invariant lies in the Hilbert class field of $\mathcal{O}$ (an extension of K). The extension $\mathbb{Q}(j(\mathcal{O}))$ is quadratic imaginary if and only if it is the rational field. The imaginary part of $j(E_{A,B})$ factors as

$$864AB^3 \cdot \frac{(2A^2 - 25dB^6)(16A^2 - 11dB^6)}{C^{3p}}. \tag{8}$$

It vanishes if and only if $A=0$ (providing a non-primitive solutions), $B=0$ (trivial solutions) or one of the last two terms vanishes. Since d is square-free and our solution is primitive, this can only occur when $d=2$, $A=\pm 5$ and $B=\pm 1$, corresponding to the point $(\pm 5, \pm 1, 3)$ for $p=3$. □

3.1. Explicit Ellenberg's bounds.

Proposition 3.2. *The bound N_d in Ellenberg's result can be taken as: $N_5 = 1033$, $N_6 = 1289$, $N_7 = 337$, $N_{11} = 557$, $N_{13} = 3491$, $N_{15} = 743$ and $N_{19} = 1031$.*

Proof. Let χ the quadratic character attached to $K/\mathbb{Q}$ and q its conductor. Let $N = p^2$, with $N \ge 400$, and let $\sigma = q^2/2\pi$. Then Ellenberg [Ell05, Theorem 1] give us the following formula:

$$(a_1, L_\chi)_{p^2} = 4\pi e^{-2\pi/\sigma N \log(N)} - E^{(3)} + E_3 - E_2 - E_1 + (a_1, B(\sigma N \log(N))), \tag{9}$$

where bounds for $(a_1, B(\sigma N \log(N))), E_1, E_2, E_3$ and $E^{(3)}$ are specified in [Ell05, Theorem 1]. Denote the previous bounds as bound1(p,q), $E_1(p,q)$, $E_2(p,q)$, $E_3(p,q)$ and $E_5(p,q)$ respectively. In order to obtain a simpler formula for the bound of $E^{(3)}$ (following Ellenberg's notation) one splits the sum depending on whether $c \leq p^4$ or $c > p^4$. In this way (as in [DU09, Lemma 8]), we obtain:

$$|E^{(3)}| \leq 16\pi^3 \left(\frac{12\phi(q)\log^2(p)}{\pi p^2} + \frac{q^2 \log(p^2)}{4\pi p} \left(\zeta^2\left(\frac{3}{2}\right) - \sum_{k=1}^{p^2} \frac{\tau(k)}{k^{3/2}} \right) \right), \tag{10}$$

where ϕ is Euler's function and $\tau(k) = \sum_{d|k} 1$. Let $E_4(p,q)$ denote the right hand side. Let $F(p,q) := 4\pi e^{-2\pi^2/p^2 q \log(p)} - E_4(p,q) - E_3(p,q) - E_2(p,q) - E_1(p,q) - \text{bound1}(p,q)$. Then from Equations (3), (9), (10) we have:

$$(a_1, L_\chi)_{p^2}^{p-\text{new}} \geq F(p,q) - \frac{1}{p^2-1}(a_p, L_\chi)_p - \frac{p}{p^2-1}(a_1, L_\chi)_p. \tag{11}$$

In order to bound $(a_p, L_\chi)_p$ and $(a_1, L_\chi)_p$ we use Ellenberg's bound [Ell04, Theorem 3.13], obtaining a function $F_2(p,q,m)$ such that $(a_m, L_\chi)_p \leq F_2(p,q,m)$. Therefore, using this bound and the Inequality (11) we have that:

$$(a_1, L_\chi)_{p^2}^{p-\text{new}} \geq F(p,q) - \frac{1}{p^2-1}F_2(p,q,p) - \frac{p}{p^2-1}F_2(p,q,1). \tag{12}$$

Note that, since $4\pi e^{-2\pi^2/p^2 q \log(p)}$, $-E_4(p,q)$, $-E_3(p,q)$, $-E_2(p,q)$, $-E_1(p,q)$ and $-\text{bound1}(p,q)$ are increasing functions (in p), the function $F(p,q)$ is increasing. One can also deduce that $-\frac{1}{p^2-1}F_2(p,q,p)$ and $-\frac{p}{p^2-1}F_2(p,q,1)$ are also increasing functions, hence $(a_1, L_\chi)_{p^2}^{p-\text{new}}$ is increasing. Then, in order to assure that $(a_1, L_\chi)_{p^2}^{p-\text{new}}$ is non-zero, it is enough to get the first prime p where the right hand side of (12) is positive. We implemented this code, following the above notation, in Pari/GP, the resulting file being labeled "ellenberg.gp" (available at http://sweet.ua.pt/apacetti/research).

The values $d = 5, 6, 7, 11, 13, 15, 19$ correspond to characters of conductors $20, 24, 7, 44, 52, 15, 19$ respectively. Running our script (EllenbergBound(p,q)), we obtain that the first prime p for which the bound is positive for $d = 5, 6, 7, 11, 13, 15, 19$ equals 1033, 1289, 337, 557, 3491, 743 and 1031 respectively. □

Remark 7. In some instances the previous bound can be improved for each value of d by a finite computation (see [Ell04, Proposition 3.9]) via computing for each $p < N_d$ whether there is a newform f satisfying:

- $f \in S_2(\Gamma_0(dp^2))$ with $w_p f = f$ and $w_d f = -f$ or
- $f \in S_2(\Gamma_0(d'p^2))$ for d' a proper divisor of d with $w_p f = f$,

for which $L(f,\chi) \neq 0$. In [Kou20, Proposition 5.4] a script to perform such computation is given. The problem with this approach is that it is almost impossible to compute the space of newforms for large values of N. In particular, it will definitely not work for values of d different from 7 in the previous list (and we did not try to run it for $d = 7$ as it will probably not work either).

3.2. **The case $d = 5$:** Let $K = \mathbb{Q}(\sqrt{-5})$ and let $\mathfrak{p}_2 = \langle 2, 1+\sqrt{-5}\rangle$ so that $2 = \mathfrak{p}_2^2$. Let $\mathfrak{p}_3 = \langle 3, 1+\sqrt{-5}\rangle$ so that $3 = \mathfrak{p}_3\overline{\mathfrak{p}_3}$. By the results of [PT20], the following holds.

- The curve $E_{A,B}$ reduction type IV at $\mathfrak{p}_2$, conductor exponent 2 and $v_{\mathfrak{p}_2}(\Delta(E_{A,B})) = 4$.
- If $3 \mid AB$ then the curve $E_{A,B}$ has reduction type II or III at both $\mathfrak{p}_3$ and $\overline{\mathfrak{p}_3}$.
- If $3 \nmid AB$ and $p \geq 5$ then we can assume that $v_{\mathfrak{p}_3}(N) = 1$ while $v_{\overline{\mathfrak{p}_3}}(N) = 2$.
- At the prime $\mathfrak{p}_5 = \sqrt{-5}$ the curve has reduction type IV*, with $v_{\mathfrak{p}_5}(\Delta) = 8$. In particular, $e = 3$.

In all cases, the curve is not the quadratic twist of a curve with good reduction at primes dividing $2, 3, 5$. Following the notation of [PT20], $Q_{+-} = \{5\}$ while the other sets are empty. In particular, ε_5 is the character of order 4 (and conductor 5), while $\varepsilon_2 = \delta_{-1}$, hence ε is a character of order 4 and conductor 20. By Theorem 6.3 of loc. cit., we need to compute the spaces $S_2(\Gamma_0(2^4 \cdot 3^2 \cdot 5^2), \varepsilon)$ and $S_2(\Gamma_0(2^4 \cdot 3^3 \cdot 5^2), \varepsilon)$.

Remark 8. There are two possible choices for the character χ needed to twist the representation $\rho_{E_{A,B}}$ so that is descends to $G_{\mathbb{Q}}$, due to the fact that $\text{Cl}(\mathbb{Q}(\sqrt{-5})) = 2$. Both choices differ by the quadratic twist of the character attached to the class group (corresponding to the extension $K(\sqrt{-1})/K$).

• The space $S_2(\Gamma_0(2^4\cdot 3^2\cdot 5^2),\varepsilon)$ has 15 newform Galois conjugate classes, 7 of them with CM (which cannot correspond to solutions). Mazur's trick (as explained in Section 1.1), proves that all other forms cannot correspond to solutions if $p>5$ but for four forms, corresponding to the forms $8, 11, 12$ and 13 in Magma's order (their coefficient field is a degree 8 extension of $\mathbb{Q}$, containing the field of fourth roots of unity). By Eichler-Shimura we know that all such forms have attached a geometric object (providing a GL_2-type representation), but a priori the dimension of its building block could be 2 (see [Que09]). A first sanity check (when possible) is to run Quer's algorithm (implemented in Magma [BCP97]) to make sure that the building block has dimension one, hence there is an elliptic curve attached to our 8-dimensional abelian variety. Also, Quer's algorithm gives the field of definition of the building block. The four forms have inner twists, and have a building block of dimension one (this can be checked with the "BrauerClass" routine and with the "DegreeMap" one in Magma) define over the quadratic field $\mathbb{Q}(\sqrt{-5})$, hence we are led to compute such curves.

Before giving equations for the elliptic curves attached to our modular forms, let us make an important observation. We know that we cannot have a congruence between two elliptic curves with different local types (at least if $p\geq 5$), hence we could check whether the modular forms come as twist of forms of smaller level. The forms f_8, f_{12} and f_{13} are quadratic twists of forms whose level have 3 to the first power while the form f_{11} is a quadratic twist of a form with good reduction at 3, hence it cannot come from a solution!

We searched for elliptic curves over $\mathbb{Q}(\sqrt{-5})$ with good reduction outside $\{\mathfrak{p}_2,\mathfrak{p}_3,\overline{\mathfrak{p}_3},\mathfrak{p}_5\}$ using Magma's implementation (while running the routine it is useful to have a bound on the elliptic curve's conductor) with the command "EllipticCurveWithGoodReductionSearch", getting 81952 elliptic curves. Using a few a_p's of the newforms (using (4) and twisting by χ^{-1}) we can discard all curves but the ones related to the previous newforms. Note that since $\mathbb{Q}(\sqrt{-5})$ does not have class number one, there is no minimal equation. By Remark 8 there are (at least) two elliptic curves attached to the modular forms f_i (which are quadratic twists of each other by $K(\sqrt{-1})/K$), and their complex conjugate (which are isogenous to their twist by $\sqrt{-3}$) hence we will only give an equation for one of these four curves.

Remark 9. Each curve E that is a candidate to match our modular form has the property that it is isogenous to the quadratic twist by $\sqrt{-3}$ of its Galois conjugate. In particular, they are $\mathbb{Q}$-curves, and (after twisting) corresponds to a modular newform in $S_2(\Gamma_0(2^4\cdot 3^2\cdot 5^2),\varepsilon)$. Computing a few a_p's allows to certify that the curve is really the building block of the newform we started with.

The curves are the following:

$$E_8 : y^2 = x^3 + (\sqrt{-5}+1)x^2 + (294\sqrt{-5}+132)x + -1206\sqrt{-5} - 6282. \tag{13}$$

$$E_{11} : y^2 = x^3 + (\sqrt{-5}+1)x^2 + (-186\sqrt{-5}-168)x + 1334\sqrt{-5} - 382. \tag{14}$$

The set of curves attached to the form f_{12} is more interesting, as the curves have an extra isogeny of degree 2, hence we get eight different curves (where the isogeny graph of each curve has 4 elements).

$$E_{12} : y^2 = x^3 + (-\sqrt{-5}-1)x^2 + (102\sqrt{-5}-168)x + (810\sqrt{-5}-162), \tag{15}$$

and its two isogenous curve

$$y^2 = x^3 + (-\sqrt{-5}-1)x^2 + (22\sqrt{-5}-8)x + (-38\sqrt{-5}-50).$$

At last, the form f_{13} is related to the elliptic curve

$$E_{13} : y^2 = x^3 + (\sqrt{-5}+1)x^2 + (414\sqrt{-5}-2268)x + (-9666\sqrt{-5}+33318). \tag{16}$$

Here are some trivial observations: the curve E_8 is the quadratic twist of a curve with good reduction at 2, hence it cannot be congruent to a curve $E_{A,B}$ (for $p>2$). The curve E_{11} is the quadratic twist of a curve with good reduction at $\mathfrak{p}_3$ (after twisting by 3) hence can also be discarded (we already knew this fact). The curve E_{12} is the quadratic twist of a curve with good reduction at $\mathfrak{p}_5$ hence can also be discarded. So we only need to discard the curve E_{13}. An easy computation (using Sage) shows that the curve E_{13} does not have a 3-torsion point (neither does its 3-isogenous curve), hence we can use the strategy described in Theorem 1.4 and Remark 5. Concretely, $a_{\mathfrak{p}_{43}}(E_{13})=1$ for both primes in $\mathbb{Q}(\sqrt{-5})$ dividing 43. In particular, $a_{\mathfrak{p}_{43}}(E_{13})\not\equiv 43+1 \pmod 3$, hence E_{13} cannot come from a solution if $\ell\geq 4\sqrt{43}=26.22...$

• The space $S_2(\Gamma_0(2^4\cdot 3^3\cdot 5^2),\varepsilon)$ has 15 newform Galois conjugate classes, 6 of which have CM (hence we can discard them). For the remaining ones, Mazur's trick discard all of them when $p>5$ except four forms corresponding to the places $7,8,9$ and 10 in Magma's output. It is important to remark that twisting by $\sqrt{-3}$ preserves the space, hence we can actually group the forms in pairs. The twist of the seventh form is the ninth one, while the twist of the eighth form is the tenth. Proceeding as before, we compute equations for their building blocks and obtain

$$E_7 : y^2 = x^3 + (-24\sqrt{-5}-60)x + (112\sqrt{-5}+136), \tag{17}$$

and

$$E_8 : y^2 + (\sqrt{-5}+1)xy = x^3 + (\sqrt{-5}+1)x^2 + (-45\sqrt{-5}-60)x + (190\sqrt{-5}+100). \tag{18}$$

Note that the first curve has good reduction at the prime dividing 5, hence cannot be attached to a solution. The second one has conductor valuation 3 at both primes dividing 3, which only occurs for solutions satisfying $3 \mid AB$. Once again, $a_{\mathfrak{p}_7}(E_8)=1$ for both primes dividing 7, hence this cannot correspond to a solution if $\ell \geq 4\sqrt{7} = 10.58....$ Together with Remark 7, we get the following result.

Theorem 3.3. *The equation $x^2+5y^5=z^p$ has no non-trivial solution if $p\geq 1033$.*

3.3. **The case $d=6$:** this case was solved in [PT20, Theorem 7.2].

3.4. **The case $d=7$:** This case turns to be very interesting, since the Lebesgue-Nagell equation is hard to solve. In [Coh93] (Section 6) Cohn conjectured that all possible solutions of

$$x^2+7=z^n \tag{19}$$

with $n\geq 3$ have $|x|\in\{1,3,5,11,181\}$. The conjecture was studied in [CS03] and completely solved in [BMS06b]. Going back to the general equation $x^2+7y^6=z^p$, following the notation of [PT20], $Q_{-+}=\{7\}$ and all other ones are empty, so ε is the quadratic character of conductor 21. According to Theorem 6.3 in [PT20] we need to compute the spaces $S_2(\Gamma_0(2^a\cdot 3^b\cdot 7^2),\varepsilon)$, where $a\in\{1,3\}$ and $b\in\{1,3\}$.

• The space $S_2(\Gamma_0(2\cdot 3\cdot 7^2),\varepsilon)$ has two newforms Galois orbits. The second one can be discarded using Mazur's trick for $p>7$, but it does not discard the first one. It corresponds to the elliptic curve:

$$E_1 : y^2 + xy + \frac{1+\sqrt{-7}}{2}y = x^3 - x^2 + (950\sqrt{-7}-46)x + \frac{7285\sqrt{-7}+200449}{2}. \tag{20}$$

In fact, this curve corresponds to the solution $181^2+7\cdot 1^6 = 2^{15}$ (notice that this comes from a solution of (19)). The curve has discriminant $-1\cdot(\sqrt{-7})^8\cdot\mathfrak{p}_2^{13}\cdot\overline{\mathfrak{p}_2}^{39}\cdot 3^3$ and a point of order 3 over K, namely, the point $(-5,-2\sqrt{-7}+319)$. Let $\mathfrak{p}_2=\langle\frac{1+\sqrt{-7}}{2}\rangle$ and $\overline{\mathfrak{p}_2}$ is Galois conjugate. The curve has multiplicative reduction at both primes, hence we can apply Theorem 1.2. Recall that $v_{\mathfrak{p}_2}(E_{A,B})\equiv 8 \pmod p$ (and the same is true for $\overline{\mathfrak{p}_2}$), hence both curves are symplectically isomorphic for $\mathfrak{p}_2$ if and only of $\left(\frac{2\cdot 13}{p}\right)=1$ while for $\overline{\mathfrak{p}_2}$ if and only if $\left(\frac{2\cdot 13\cdot 3}{p}\right)=1$. Clearly both cases cannot occur if $\left(\frac{3}{p}\right)=-1$.

• The space $S_2(\Gamma_0(2^2\cdot 3\cdot 7^2),\varepsilon)$ has four Galois orbits of newforms, the first one having CM. The other three forms can be discarded using Mazur's trick for $p>7$.

• The space $S_2(\Gamma_0(2^2\cdot 3^3\cdot 7^2),\varepsilon)$ has seven Galois orbits of newforms. The first three of them have CM (hence can be discarded), while the other four ones, can be discarded using Mazur's trick for $p>7$.

• The space $S_2(\Gamma_0(2\cdot 3^3\cdot 7^2),\varepsilon)$ has six Galois orbits of newforms. The last three ones can be discarded using Mazur's trick for $p>7$, but not the first three ones. They correspond to the following elliptic curves:

$$E_1 : y^2 + xy + \frac{-1+\sqrt{-7}}{2}y = x^3 - x^2 + \frac{115\sqrt{-7}-91}{2}x + \frac{443\sqrt{-7}+529}{2}. \tag{21}$$

Its discriminant equals $-1\cdot(\sqrt{-7})^8\cdot\mathfrak{p}_2^9\cdot\overline{\mathfrak{p}_2}^3\cdot 3^3$. It has the 3 K-rational torsion point $(-5,-2\sqrt{-7}+22)$ (hence cannot be discarded using the method of Section 1.4).

$$E_2 : y^2 + xy + \frac{-1+\sqrt{-7}}{2}y = x^3 - x^2 + \frac{31\sqrt{-7}-91}{2}x + \frac{121\sqrt{-7}-157}{2}. \tag{22}$$

Its discriminant equals $-1\cdot(\sqrt{-7})^8\cdot\mathfrak{p}_2^6\cdot\overline{\mathfrak{p}_2}^2\cdot 3^3$. It has the 3 K-rational torsion point $(-5,-2\sqrt{-7}+8)$.

$$E_3: y^2+xy+\frac{-1+\sqrt{-7}}{2}y=x^3-x^2-\left(\frac{53\sqrt{-7}+91}{2}\right)x-\left(\frac{201\sqrt{-7}+59}{2}\right). \tag{23}$$

Its discriminant equals $-1\cdot(\sqrt{-7})^8\cdot\mathfrak{p}_2^9\cdot\overline{\mathfrak{p}_2}^3\cdot 3^3$. It has the 3 K-rational torsion point $(-5,-2\sqrt{-7}-6)$.

Note that in all three cases, we can use again the symplectic argument at both primes dividing 2, getting that the curve $E_{A,B}$ cannot be symplectic isomorphic to any of the curves if $\left(\frac{3}{p}\right)=-1$. In particular, we get the following result.

Theorem 3.4. *Let $p\geq 337$ be a prime number such that $p\equiv 5,7 \pmod{12}$. Then there are no non-trivial solutions of the equation*

$$x^2+7y^6=z^p.$$

3.5. **The case $d=10$:** Theorem 6.3 of [PT20] implies that to apply the current approach we need to compute the space $S_2(2^8\cdot 3^3\cdot 5^2,\varepsilon)$, where ε is a character of order 4 and conductor 5. Such a computation is nowadays unfeasible.

3.6. **The case $d=11$:** The prime 2 is inert in $\mathbb{Q}(\sqrt{-11})$ while the prime 3 splits (as in the case $d=5$). The only non-empty set (following the notation of [PT20]) is $Q_{--}=\{11\}$, hence the Nebentypus is trivial. In particular, we need to compute the spaces $S_2(\Gamma_0(2^2\cdot 3^2\cdot 11^2))$ and $S_2(\Gamma_0(2^2\cdot 3^3\cdot 11^2))$.

• The space $S_2(\Gamma_0(2^2\cdot 3^3\cdot 11^2))$ has 43 Galois conjugate classes of newforms, 11 of them having CM. All other forms in this space can be discarded using Mazur's trick for $p>13$. In particular it follows that there are no non-trivial primitive solutions if $3\mid AB$.

• The space $S_2(\Gamma_0(2^2\cdot 3^2\cdot 11^2))$ has 26 Galois orbits of newforms, 6 of them corresponding to forms with CM. For the other ones, Mazur's trick discards all of them for $p>11$ but two of them whose coefficient field equals $\mathbb{Q}(\sqrt{3})$ (corresponding to the newforms 15 and 20). The second one is a quadratic twist of a form of level $2^2\cdot 11^2$, hence cannot be congruent to our curve $E_{A,B}$. To discard the form f_{15}, we need to find an equation for the curve (as explained before, we made a sanity check using Quer's algorithm and computed the dimension of the building block and its field of definition). Since the Nebentypus is trivial, there are two different ways to search for such a curve. One option is to compute all possible curves with a given set of ramification using magma; we postpone the second construction. Discarding the ones that do not match our form, we end up with the curve

$$E_{15}: y^2=x^3+\frac{1+\sqrt{-11}}{2}x^2+\frac{1907\sqrt{-11}-1615}{2}x--19479\sqrt{-11}-31012. \tag{24}$$

Note that it has a point of order 3, hence we cannot discard it using that our solutions are related to elliptic curves with a 3-torsion point. In particular, we need to use the symplectic argument as explained in Section 1.3. The discriminant of E_{15} equals $3^9\cdot(\sqrt{-11})^8\cdot 2^{20}$. Applying Theorem 1.3 at the prime $\sqrt{-11}$ to both E_{15} and a curve $E_{A,B}$, we get that they are symplectically isomorphic because the discriminant valuation of both curves is the same.

Consider the prime $\mathfrak{p}_3=\langle\frac{1-\sqrt{-11}}{2}\rangle$ of multiplicative reduction of both curves. In particular, we are in the classical case (i.e. the completion equals $\mathbb{Q}_3$) so we can use Theorem 1.2. The curve $\tilde{E}$ has discriminant valuation 9 at $\mathfrak{p}_3$. Since the curve $E_{A,B}$ has discriminant valuation $3p-9$ at $\mathfrak{p}_3$, they are symplectically isomorphic if and only if $(-1/p)=1$. In particular, when $(-1/p)=-1$ we get a contradiction.

Theorem 3.5. *The equation $x^2+11y^6=z^p$ has no non-trivial solution if $p\geq 557$ and either one of the following conditions is satisfied:*

- *Either x or y is divisible by 3,*
- *The prime $p\equiv 3 \pmod 4$.*

A different approach to compute the curve. We take the opportunity to explain in detail a method to compute the elliptic curve as explained in [Cre92]. We thank Professor John Cremona for kindly explaining some computational aspects of it. The Jacobi map (and Eichler-Shimura's relation) allows to given a weight

two newform f, construct a lattice Λ_f attached to an abelian variety of GL_2-type (whose complex points correspond to $\mathbb{C}^d/\Lambda_f$, where $d = [\mathbb{Q}_f : \mathbb{Q}]$).

In our particular case, let $f = f_{15}$ (to avoid heavy notation). Recall that $\mathbb{Q}_f = \mathbb{Q}(\sqrt{3})$, a quadratic extension, hence we can construct a rank 4 lattice obtained by integrating the homology against the basis $\{f, f^\sigma\}$, where σ is the non-trivial endomorphism of $\mathrm{Gal}(\mathbb{Q}(\sqrt{3})/\mathbb{Q})$. We want to "split" the lattice as a sum of two rank 2 ones, providing the building block E_f we are searching for. Following Cremona's notation, it is easy to verify that for all $\sigma \in \mathrm{Gal}_{\mathbb{Q}}$,

$$\sigma(a_p(f)) = a_p(f)\chi(p),$$

where χ is the quadratic character corresponding to the quadratic extension $\mathbb{Q}(\sqrt{-11})$. By [Cre92, Theorem 2], the endomorphism algebra of A_f equals the quaternion algebra $B = \left(\frac{3,-11}{2}\right) \simeq M_2(\mathbb{Q})$ (since 3 is the norm of $\frac{1+\sqrt{-11}}{2}$). This confirms that A_f is isogenous to the product of two elliptic curves over $\mathbb{Q}(\sqrt{-11})$. To split the surface, we need to find a zero divisor inside the endomorphism ring, and clearly in B, the element $1 + 2i + j$ is such an element (where $i^2 = 3$, $j^2 = -11$ and $ij = -ji$). In terms of the lattice, recall that the twisting operator η_χ (as an elements of the endomorphism ring) in the chosen basis has matrix $\left(\begin{smallmatrix} 0 & \sqrt{-11} \\ \sqrt{-11} & 0 \end{smallmatrix}\right)$ (see [Shi73] Section 2). In particular, $\eta_\chi^2 = -11$ (the endomorphism given by multiplication by -11), and $f^\sigma = \frac{\eta_\chi f}{\sqrt{-11}}$. In particular, $(1+2i+j)f = (1+2\sqrt{3})f + \sqrt{-11}f^\sigma$, so we can multiply the lattice (on the left) by the vector $[1 + 2\sqrt{3}, \sqrt{-11}]$ to get the complex lattice we are searching for. Note that the zero divisor is not unique, a rational multiple of it will give another isomorphic (over $\mathbb{C}$) elliptic curve, hence we just compute the j-invariant of the elliptic curve obtained via this process, and find the twist corresponding to the curve that matches f.

Here is how to do it on magma. Just a remark to explain the computation: internally Magma works with rational basis, so instead of computing the period matrix relative to the pair $\{f, f^\sigma\}$, it does so using the basis $\{\frac{f+f^\sigma}{2}, \frac{f-f^\sigma}{2\sqrt{3}}\}$, hence we need to multiply by the inverse of such a matrix to get the right lattice.

```
SetDefaultRealFieldPrecision(100);
M:=ModularSymbols(2^2*3^2*11^2,2);
S:=NewSubspace(CuspidalSubspace(M));
new:=NewformDecomposition(S);
f:=new[15];
PP:=Periods(f,2000);
```

This gives the matrix of periods

$$\begin{pmatrix} -0.45416987304689526385002426684 9 - 6.3108872417680944432938285222 6 \cdot 10^{-30}\mathrm{I} & -0.04843566510741159901130962320 58 + 3.6287601640166543048939514003 0 \cdot 10^{-29}\mathrm{I} \\ 0.22708493652344763192501213331 1 - 0.02465697700837897583907132794 19\mathrm{I} & 0.02421783255370579950565481161 33 - 0.12963541783305462672084489812 2\mathrm{I} \\ -1.51694105800301602495812050226 - 0.46287718452430080767974866114 0\mathrm{I} & 0.05756010864750703538542845273 90 - 0.41356323050754285600160603088 8\mathrm{I} \\ -0.15443143886233023340804770183 8 + 0.41356323050754285600160600516 7\mathrm{I} & 0.20286710396974183241935732242 1 + 0.15429239484143360255991623453 8\mathrm{I} \end{pmatrix}$$

To compute the j-invariant of our curve (and recognize it as an algebraic integer), it is better to work in Pari/GP. Here is how the computation finishes

```
\p 30
Periods=[-0.45416987304689526385002426684 9 - 6.3108872417680944432938285222 6E-30*I,
-0.04843566510741159901130962320 58 + 3.6287601640166543048939514003 0E-29*I;
0.22708493652344763192501213331 1 - 0.02465697700837897583907132794 19*I,
0.02421783255370579950565481161 33 - 0.12963541783305462672084489812 2*I;
-1.51694105800301602495812050226 - 0.46287718452430080767974866114 0*I,
0.05756010864750703538542845273 90 - 0.41356323050754285600160603088 8*I;
-0.15443143886233023340804770183 8 + 0.41356323050754285600160600516 7*I,
0.20286710396974183241935732242 1 + 0.15429239484143360255991623453 8*I];
A=[1/2,1/2;1/2/sqrt(3),-1/2/sqrt(3)];
Candidate=[1+2*sqrt(3),sqrt(-11)]*1/A*Periods~;
lindep(Candidate)
%4 = [-7, -15, 1, -4]~
```

This proves that the third element (in $\mathbb{C}$) is an integral combination of the other three ones.

```
lindep([Candidate[1],Candidate[2],Candidate[4]]);
% 5 = [-1, -3, -2]~
```

Then our lattice is the spanned by the second and fourth elements! We compute the elliptic curve and its j-invariant over the complex numbers.

```
W=ellperiods([Candidate[2],Candidate[4]]);
E=ellinit([0,0,0,-elleisnum(W,4,1)/4,-elleisnum(W,6,1)/4]);
algdep(E.j,2)
%8 = 531441*x^2 - 37711872000*x + 1441792000000000
```

It is easy to verify that the j-invariant of the curve (24) is a root of the given polynomial.

3.7. **The case $d = 13$:** This case can be completely studied following the method explained in [PT20]. Following the same notation, we have that the only non-empty set is $Q_{++} = \{13\}$, the Nebentypus ε has order 2 and conductor $4 \cdot 3 \cdot 19$, while χ is a character of order 4. We have to compute the spaces $S_2(\Gamma_0(2^4 \cdot 3 \cdot 13^2), \varepsilon)$ and $S_2(\Gamma_0(2^4 \cdot 3^3 \cdot 13^2), \varepsilon)$.

• The space $S_2(\Gamma_0(2^4 \cdot 3 \cdot 13^2)$ has 29 conjugacy classes, 9 of them with CM hence can be discarded. The remaining forms can all be eliminated using Mazur's trick for $p > 13$.

• The space $S_2(\Gamma_0(2^4 \cdot 3^3 \cdot 13^2), \varepsilon)$ has 68 conjugacy classes, 8 of them with CM. The remaining forms can all be discarded once again with Mazur's trick for $p > 29$.

Theorem 3.6. *Let $p \geq 3491$ be a prime number. Then there are no non-trivial solutions of the equation*

$$x^2 + 13y^6 = z^p.$$

3.8. **The case $d = 14$:** Following the notation of [PT20], $Q_{-+} = \{7\}$, hence the possible solutions are related to forms in the space $S_2(\Gamma_0(2^8 \cdot 3^2 \cdot 7^2), \varepsilon)$ and $S_2(\Gamma_0(2^8 \cdot 3^3 \cdot 7^2), \varepsilon)$, where ε is the quadratic character of conductor 28. Nowadays such computation is unfeasible.

3.9. **The case $d = 15$:** Let $K = \mathbb{Q}(\sqrt{-15})$. The prime 2 splits as $2 = \mathfrak{p}_2\overline{\mathfrak{p}_2}$, where $\mathfrak{p}_2 = \langle 2, \frac{1+\sqrt{-15}}{2} \rangle$ while the prime 3 is inert in K. By Lemmas 5.3 and 5.5 of [PT20] the following holds:

- $2 \nmid AB$ if and only if the curve $E_{A,B}$ has multiplicative reduction at $\mathfrak{p}_2$ and $\overline{\mathfrak{p}_2}$.
- At the prime $\mathfrak{p}_5 = \langle 5, \sqrt{-15} \rangle$, the curve $E_{A,B}$ has reduction type IV* and $v_{\mathfrak{p}_5}(\Delta) = 8$. In particular, $e = 3$.

As in the case $d = 5$, the unique non-empty set is $Q_{+-} = \{5\}$. By [PT20, Theorem 6.3] we need to compute the spaces $S_2(\Gamma_0(2 \cdot 3^5 \cdot 5^2), \varepsilon)$ and $S_2(\Gamma_0(2^2 \cdot 3^5 \cdot 5^2), \varepsilon)$ where the Nebentypus ε has order 4 and conductor $3 \cdot 5$.

• The space $S_2(\Gamma_0(2 \cdot 3^5 \cdot 5^2), \varepsilon)$ has 21 conjugacy classes. Mazur's trick allows to eliminate, for $p > 7$, all the newforms except for the first six ones (in Magma's order) which do not have CM and have building blocks of dimension 1. Since the prime 2 splits in K, the character χ needed to descend our Galois representation is unramified at 2, hence the building block has multiplicative reduction at the primes dividing 2, and the same holds for $E_{A,B}$. In particular, we can assume $2 \nmid AB$. To discard the remaining forms we searched for elliptic curves over $\mathbb{Q}(\sqrt{-15})$ with good reduction outside $\{\mathfrak{p}_2, \overline{\mathfrak{p}_2}, \mathfrak{p}_3, \mathfrak{p}_5\}$ and got 111264 elliptic curves. Using a few a_p's of the newforms we discarded most of the forms, and end with the following equations for our curves (up to conjugation):

$$E_1 : y^2 + xy - \frac{1-\sqrt{-15}}{2}y = x^3 - x^2 - \frac{421 + 23\sqrt{-15}}{2}x - \frac{2185 + 191\sqrt{-15}}{2}, \tag{25}$$

$$E_2 : y^2 + xy - \frac{1-\sqrt{-15}}{2}y = x^3 - x^2 - (2\sqrt{-15} - 8)x - \frac{7 + 7\sqrt{-15}}{2}, \tag{26}$$

$$E_3 : y^2 + xy - \frac{1-\sqrt{-15}}{2}y = x^3 + \frac{1+\sqrt{-15}}{2}x^2 + \frac{(-111 + 87\sqrt{-15})}{2}x + \frac{65 + 375\sqrt{-15}}{2}, \tag{27}$$

$$E_4 : y^2 + xy + \frac{1+\sqrt{-15}}{2}y = x^3 - x^2 + -211 + 137\sqrt{-15}x + \frac{1973 + 2333\sqrt{-15}}{2}, \tag{28}$$

$$E_5 : y^2 + xy + \frac{1+\sqrt{-15}}{2}y = x^3 + \frac{1-\sqrt{-15}}{2}x^2 - \frac{111 + 375\sqrt{-15}}{2}x + \frac{-9823 + 2793\sqrt{-15}}{2}, \tag{29}$$

and

$$E_6 : y^2 + xy + \frac{1+\sqrt{-15}}{2}y = x^3 - x^2 - (79\sqrt{-15}+211)x - \left(\frac{1339\sqrt{-15}+835}{2}\right). \tag{30}$$

The curves E_2, E_3, E_4 and E_5 have good reduction at $\mathfrak{p}_5$ so can be discarded. The curves E_1 and E_6 do have a 3-torsion point (for example $(-11, 2-4\sqrt{-15})$ and $(-11, -4\sqrt{-15}-21)$ respectively) hence we need to use the symplectic argument to discard them. Their discriminant valuations are the following:

- The curve E_1 has minimal discriminat with valuation 6 at $\mathfrak{p}_2$, 2 at $\overline{\mathfrak{p}_2}$, 14 at $\mathfrak{p}_3$ and 8 at $\mathfrak{p}_5$.
- The curve E_6 has minimal discriminat with valuation 12 at $\mathfrak{p}_2$, 4 at $\overline{\mathfrak{p}_2}$, 14 at $\mathfrak{p}_3$ and 8 at $\mathfrak{p}_5$.

Recall that $v_{\mathfrak{p}_2}(E_{A,B}) \equiv v_{\overline{\mathfrak{p}_2}}(E_{A,B}) \equiv 8 - 12 \pmod{p}$ (since $2 \nmid AB$ the model is not minimal). If we apply Theorem 1.2 at both primes dividing 2, we get:

- the method at the prime $\mathfrak{p}_2$ gives that E_1 (respectively for E_6) and $E_{A,B}$ are symplectically isomorphic if and only if $\left(\frac{-6}{p}\right) = 1$ (respectively $\left(\frac{-3}{p}\right) = 1$).
- the method at the prime $\overline{\mathfrak{p}_2}$ gives that E_1 (respectively E_6) and $E_{A,B}$ are symplectically isomorphic if and only if $\left(\frac{-2}{p}\right) = 1$ (respectively $\left(\frac{-1}{p}\right) = 1$).

In particular, if $\left(\frac{3}{p}\right) = -1$ we get a contradiction.

We can do a little better; note that $v_{\mathfrak{p}_5}(E_{A,B}) = 8 = v_{\mathfrak{p}_5}(E_1)$ (respectively E_6) hence we can also apply the symplectic argument (Theorem 1.3) at $\mathfrak{p}_5$ getting that the curves are always symplectically isomorphic. Then we can use this information to add more primes to the final result. To discard both curves when $\left(\frac{3}{p}\right) = 1$, we need the extra hypothesis $\left(\frac{-2}{p}\right) = 1$ and $\left(\frac{-1}{p}\right) = -1$. In particular, if $p \equiv 5, 7, 15, 17, 19 \pmod{24}$ then we get a contradiction (we increased the percentage of primes from 1/2 to 5/8).

• The space $S_2(\Gamma_0(2^2 \cdot 3^5 \cdot 5^2), \varepsilon)$ has 33 conjugacy classes. In this case Mazur's trick is enough to discard all the newforms for $p > 11$, except for the first 12 newforms, which have CM.

Theorem 3.7. *The equation $x^2 + 15y^6 = z^p$ has no non-trivial primitive solution if $p \geq 743$ and either one of the following conditions is satisfied:*

- $2 \mid xy$.
- $p \equiv 5, 7, 15, 17, 19 \pmod{24}$.

3.10. **The case $d = 17$:** Following [PT20] notation, the unique non-empty set equals $Q_{+-} = \{17\}$. Then the Nebentypus ε has order 16 and conductor $4 \cdot 17$. We need to discard forms in the spaces $S_2(\Gamma_0(2^4 \cdot 3^2 \cdot 17^2), \varepsilon)$ and $S_2(\Gamma_0(2^4 \cdot 3^3 \cdot 17^2), \varepsilon)$. Although the dimension of the first space lies in the computable range (close to the limit), Magma gives an internal error while computing it. The second space is too large for computations.

3.11. **The case $d = 19$:** The unique non-empty set in [PT20] notation is the set $Q_{-+} = \{19\}$. The Nebentypus has order 2, conductor $3 \cdot 19$ and we have to discard newforms in $S_2(2^2 \cdot 3 \cdot 19^2, \varepsilon)$ and $S_2(2^2 \cdot 3^3 \cdot 19^2, \varepsilon)$.

• The space $S_2(\Gamma_0(2^2 \cdot 3 \cdot 19^2), \varepsilon)$ has 10 conjugacy classes of newforms, three of them with CM. Mazur's trick allow to discard all non-CM forms for $p > 19$.

• The space $S_2(\Gamma_0(2^2 \cdot 3^3 \cdot 19^2), \varepsilon)$ has 18 conjugacy classes of newforms. Mazur's trick allows once again to discard all the newforms for $p > 19$, except for three of them that have CM. From the above analysis, the following holds:

Theorem 3.8. *Let $p \geq 1031$ be a prime number. Then there are no non-trivial solutions of the equation*

$$x^2 + 19y^6 = z^p.$$

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FAMAF-CIEM, Universidad Nacional de Córdoba. C.P:5000, Córdoba, Argentina.
Email address: franco.golfieri@mi.unc.edu.ar

Center for Research and Development in Mathematics and Applications (CIDMA), Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal
Email address: apacetti@ua.pt

FAMAF-CIEM, Universidad Nacional de Córdoba. C.P:5000, Córdoba, Argentina.
Email address: lucas.villagra@unc.edu.ar