ON TRANSFORMATION PROPERTIES OF HYPERGEOMETRIC MOTIVES AND DIOPHANTINE EQUATIONS

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Abstract. Over the last two hundred years different transformation formulas for Gauss' hypergeometric function ${}_2F_1$ were discovered. The goal of the present article is to study their arithmetic analogue for the underlying hypergeometric motive. As an application, we show how these transformation properties can be used in the study of some Diophantine equations.

1. Introduction

Let a, b, c be complex numbers, and let $z \in \mathbb{C}$ with $|z| < 1$. Recall the definition of Gauss' hypergeometric function

$$ {}_2F_1(a, b; c|z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \tag{1} $$

where for n a non-negative integer, $(a)_n$ is the Pocchammer symbol defined by

$$ (a)_n = \begin{cases} 1 & \text{if } n = 0, \\ a(a+1)\cdots(a+n-1) & \text{if } n > 0. \end{cases} $$

The hypergeometric function ${}_2F_1$ satisfies the differential equation

$$ z(1-z)\frac{d^2}{dz^2} + [c - (a+b+1)z]\frac{d}{dz} - ab = 0. \tag{2} $$

The differential equation has three singular points in $\mathbb{P}^1$, namely at the points $\{0, 1, \infty\}$. There are many transformation formulas satisfied by Gauss' hypergeometric function (discovered mostly during the XIX century by Gauss, Kummer, Goursat et al).

Example 1. Formula 49 in [11] (page 77), which matches formula (38) in [10] (page 119), states the following relation:

$$ {}_2F_1\left(a, b; \frac{a+b+1}{2}|z\right) = {}_2F_1\left(\frac{a}{2}, \frac{b}{2}; \frac{a+b+1}{2}|4z(1-z)\right). \tag{3} $$

With some manipulation, it is not hard to verify that both functions satisfy the same differential equation. Riemann realized that instead of studying the differential equation itself, one should focus on the so called *monodromy representation*. Let $z_0 \in \mathbb{C} \setminus \{0, 1\}$ be a complex "base point". Let $\{f_1, f_2\}$ be a basis of solutions of the differential equation. If η is a loop based at z_0, one can holomorphically extend the solutions f_1 and f_2 along η and obtain a new basis of solutions. This gives a representation

$$ \rho : \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, z_0) \to \mathrm{GL}_2(\mathbb{C}), \tag{4} $$

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that sends a loop η to the change of basis matrix.

With a little more generality, given a, b, c, d complex numbers, consider the differential equation

$$z^2(1-z)\frac{d^2}{dz^2} + z[c+d-1-(a+b+1)z]\frac{d}{dz} + (c-1)(d-1) - abz = 0. \tag{5}$$

This differential equation has a similar monodromy representation, that we denote by ρ. The case $d = 1$ corresponds to Gauss' hypergeometric function, while the general case is a "twist" of it (as explained in [9]).

For ease of notation, we denote by $\exp(a)$ the complex number $e^{2\pi i a}$.

Definition 1.1. *We say that the parameters* $(a, b), (c, d)$ *are* generic *if the sets* $\{\exp(a), \exp(b)\}$ *and* $\{\exp(c), \exp(d)\}$ *are disjoint.*

In the present article we will only study parameters that are generic, so from now we assume this is always the case. Under this assumption the following properties hold:

(1) The monodromy representation depends only on the values a, b, c, d up to integral translation (as proven in [2, Proposition 2.5])
(2) The monodromy representation is irreducible (as proven in [2, Proposition 3.3]).
(3) The monodromy representation is *rigid* (as proven in [2, Theorem 3.5]).

Let us explain in more detail the notion of being rigid. Two representations ρ_1, ρ_2 are *isomorphic* if they are the same up to a choice of a basis for the vector space. Equivalently, once a basis for each vector space is chosen, they are isomorphic if there exists a matrix $C \in \mathrm{GL}_2(\mathbb{C})$ such that $\rho_1(\eta) = C\rho_2(\eta)C^{-1}$ for any loop η.

A direct application of Van Kampen's theorem proves that the fundamental group $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, z_0)$ is generated by three loops $\{\eta_0, \eta_1, \eta_\infty\}$, where η_i is a simple loop in counterclockwise direction around the point i, for $i = 0, 1, \infty$, satisfying the condition $\eta_0\eta_1\eta_\infty$ being homotopically trivial. The monodromy representation is then uniquely determined by the values

$$M_0 := \rho(\eta_0), \qquad M_1 := \rho(\eta_1), \qquad M_\infty := \rho(\eta_\infty).$$

Two representations ρ, ρ' are isomorphic if the triple of matrices (M_0, M_1, M_∞) are conjugated to (M_0', M_1', M_∞') (by an invertible matrix). The rigidity property means that it is enough for each M_i being conjugated to M_i' for $i = 0, 1, \infty$ for the two representations to be isomorphic (so the conjugacy class of each monodromy matrix determines the representation).

A result of Level (Theorem 1.1 of his Ph.D. thesis, or [2, Theorem 3.5]) states that given generic parameters $(a, b), (c, d)$, there exists a (unique up to isomorphism) monodromy representation whose representatives for the conjugacy monodromy matrices M_i around the point i, for $i = 0, 1, \infty$ are

$$M_0 = \begin{cases} \begin{pmatrix} \exp(-c) & 0 \\ 0 & \exp(-d) \end{pmatrix} & \text{if } c-d \notin \mathbb{Z}, \\ \begin{pmatrix} \exp(-c) & 1 \\ 0 & \exp(-c) \end{pmatrix} & \text{otherwise,} \end{cases} \qquad M_1 = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & \exp(c+d-a-b) \end{pmatrix} & \text{if } a+b-c-d \notin \mathbb{Z}, \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \text{otherwise,} \end{cases}$$

and

$$M_\infty = \begin{cases} \begin{pmatrix} \exp(a) & 0 \\ 0 & \exp(b) \end{pmatrix} & \text{if } a-b \notin \mathbb{Z}, \\ \begin{pmatrix} \exp(a) & 1 \\ 0 & \exp(a) \end{pmatrix} & \text{otherwise.} \end{cases}$$

The monodromy representation attached to (the differential equation of) an hypergeometric series is unramified outside $\{0, 1, \infty\}$ and the monodromy matrix at 1 is a pseudo-reflexion (a matrix whose eigenvalue 1 has an eigenspace of codimension 1). Any monodromy representation satisfying these two property comes from a differential equation as (5). We will use the term *hypergeometric* monodromy representation for them.

Definition 1.2. *Let $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ be a continuous map, and let ρ be an hypergeometric monodromy representation. The* pullback *of ρ by φ is the monodromy representation*

$$\varphi^*(\rho) : \pi_1\left(\mathbb{P}^1 \setminus \varphi^{-1}(\{0,1,\infty\})\right) \to \mathrm{GL}_2(\mathbb{C}), \tag{6}$$

defined by $\varphi^*(\rho)(\eta) = \rho(\varphi(\eta))$.

A priory the pullback of an hypergeometric representation needs not be hypergeometric (as for example it might ramify in more than three points).

Definition 1.3. *An* hypergeometric relation *is a pair (φ, ρ) consisting of an algebraic map $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ and an hypergeometric monodromy representation ρ satisfying that $\varphi^*(\rho)$ is also hypergeometric.*

It seems like a natural question to study the following problem.

Problem: determine/classify hypergeometric relations (φ, ρ).

This problem (presented in a little different way) was intensively studied by Kummer (in [11]) and Goursat (in [10]). Goursat obtained a complete list of algebraic transformations satisfied by Gauss' hypergeometric function for small degrees. For a modern approach (including historical references) see the article [14].

Example 1 (continued). Let us study Example 1 in more detail. Let $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ be the degree 2 cover sending $z \to 4z(1-z)$. To compute the ramification points, we just compute the discriminant of the polynomial $4z(1-z) - t$, which equals $16(t-1)$. Then the cover is ramified precisely at the points 1 and ∞.

The preimage of the ramified points are $1/2$ and ∞ respectively. The preimage of 0 consists of the two points $\{0, 1\}$. Let ρ denote the hypergeometric monodromy representation with parameters $\left(\frac{a}{2}, \frac{b}{2}\right)\left(\frac{a+b+1}{2}, 1\right)$. Then the pullback $\varphi^*(\rho)$ is unramified outside the set $\{0, 1, 1/2, \infty\}$.

A representative for the conjugacy class of ρ at η_1 is

$$M_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since $1/2$ is a ramified point, $\varphi^*(\rho)(\eta_{1/2}) = M_1^2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, so the pullback is unramified at $1/2$. On the other hand,

$$\varphi^*(\rho)(\eta_0) = \varphi^*(\rho)(\eta_1) = \rho(0) = \begin{pmatrix} \exp(\frac{a+b+1}{2}) & 0 \\ 0 & 1 \end{pmatrix}$$

The fact that $\varphi^*(\rho)$ at 1 is a pseudo-reflection implies that the pullback is an hypergeometric monodromy representation, so (φ, ρ) is an hypergeometric relation. We can also compute the missing monodromy matrix

$$\varphi^*(\rho)(\eta_\infty) = \rho(\eta_\infty)^2 = \begin{pmatrix} \exp(a) & 0 \\ 0 & \exp(b) \end{pmatrix}.$$

The rigidity property then implies that the monodromy representation of $\varphi^*(\rho)$ matches the monodromy representation with parameters $(a, b), (\frac{a+b+1}{2}, 1)$ as expected.

Knowing that the two monodromy representations are the same, does not necessarily imply a simple relation between the corresponding Gauss' hypergeometric functions, since the space of solutions is two dimensional! It does however imply that one hypergeometric series is a linear combination of other two ones, but an equality like (33) is not always true, and when it does hold, the proof requires some extra work.

Example 2. Consider the transformation formula obtained from [11] (formula 67 in page 81):

$$(7)\quad {}_2F_1\left(a,b;\frac{a+b+1}{2}|z\right)=\frac{\cos(a-b)\frac{\pi}{2}}{\cos(a+b)\frac{\pi}{2}}\,{}_2F_1\left(a,b;\frac{a+b+1}{2}|1-z\right)+$$
$$\frac{\Gamma(\frac{a+b-1}{2})\Gamma(\frac{a+b-3}{2})}{\Gamma(a-1)\Gamma(b-1)}(1-z)^{\frac{1-a-b}{2}}\,{}_2F_1\left(\frac{a-b+1}{2},\frac{b-a+1}{2};\frac{3-a-b}{2}|1-z\right).$$

The monodromy matrices of the hypergeometric series ${}_2F_1(a,b,\frac{a+b+1}{2})$ at $0,1$ and ∞ are

$$M_0=\begin{pmatrix}\exp(-\frac{a+b+1}{2}) & 0\\ 0 & 1\end{pmatrix},\qquad M_1=\begin{pmatrix}\exp(-\frac{a+b+1}{2}) & 0\\ 0 & 1\end{pmatrix},M_\infty=\begin{pmatrix}\exp(a) & 0\\ 0 & \exp(b)\end{pmatrix}.$$

Since the monodromy matrices at 0 and at 1 are the same, the map that swaps 0 and 1 (sending z to $1-z$) preserves the monodromy representation. Then the first hypergeometric summand on the right hand side satisfies the same differential equation as the left hand side function. The hypergeometric series appearing in the second term on the right has monodromy matrices

$$N_0=\begin{pmatrix}\exp(\frac{a+b+1}{2}) & 0\\ 0 & 1\end{pmatrix},\quad N_1=\begin{pmatrix}\exp(-\frac{a+b+1}{2}) & 0\\ 0 & 1\end{pmatrix},\quad N_\infty=\begin{pmatrix}\exp(\frac{a-b+1}{2}) & 0\\ 0 & \exp(\frac{b-a+1}{2})\end{pmatrix}.$$

We need to compose with the map that swaps the monodromy at 0 and 1 (so we get the right monodromy matrix at 0) and then twist by the character ramified at $\{1,\infty\}$ (corresponding to the term $(1-z)^{\frac{1-a-b}{2}}$). This multiplies the monodromy matrix at 1 (namely N_0 because of the swap) by $\exp(-\frac{a+b+1}{2})$ and the monodromy matrix at ∞ by $\exp(\frac{1+a+b}{2})$. Then the monodromy representation also coincides with the previous one. However, in this case, a linear combination of the two solutions is needed to get a match of hypergeometric series.

In the present article we are interested on the *arithmetic* side of the previous transformation formulas. Suppose that the parameters $(a,b),(c,d)$ besides being generic are also rational numbers. Let N be their least common denominator. Then (as proved in [9]) there exists an hypergeometric motive $\mathcal{H}((a,b),(c,d)|z)$ defined over $F:=\mathbb{Q}(\zeta_N)$ having the hypergeometric series as a period.

The profinite completion of the fundamental group $\pi_1\left(\mathbb{P}^1\setminus\varphi^{-1}(\{0,1,\infty\})\right)$ is isomorphic to $\mathrm{Gal}(\Omega/\overline{\mathbb{Q}}(z))$, where Ω denotes the maximal extension of $\overline{\mathbb{Q}}(z)$ unramified outside $\{0,1,\infty\}$ (see [13, Theorem 6.3.1]). Then for each prime ideal $\mathfrak{p}$ of F there is a continuous Galois representation

$$\rho_{\mathfrak{p}}:\mathrm{Gal}(\overline{\mathbb{Q}(z)}/\overline{\mathbb{Q}}(z))\to\mathrm{GL}_2(F_{\mathfrak{p}}),$$

"extending" the monodromy representation (after identifying the group $\pi_1(\mathbb{P}^1\setminus\{0,1,\infty\},z_0)$ with a subgroup of $\mathrm{Gal}(\Omega/\overline{\mathbb{Q}}(z))$). By [9, Remark 6.28] this "geometric" representation extends to an "arithmetic" one, namely for each prime ideal $\mathfrak{p}$ of F there exists a Galois representation

$$(8)\qquad \rho_{\mathfrak{p}}:\mathrm{Gal}(\overline{\mathbb{Q}(z)}/F(z))\to\mathrm{GL}_2(F_{\mathfrak{p}}),$$

extending the monodromy representation. The goal of the present article is to relate the classical hypergeometric series relations with relations between the associated hypergeometric motives.

2. Hypergeometric motives

The hypergeometric motive $\mathcal{H}((a,b),(c,d)|z)$ defined in [9] appears (up to a twist by a Hecke character) in the middle cohomology of Euler's curve. Consider the following problem:

Problem: Let $(a,b),(c,d)$ and $(\alpha,\beta),(\gamma,\delta)$ be two pairs of generic rational parameters. How to determine if the hypergeometric motives $\mathcal{H}((a,b),(c,d)|z)$ and $\mathcal{H}((\alpha,\beta),(\gamma,\delta)|z)$ are isomorphic?

Let us start studying a related problem.

Proposition 2.1. *Let H be a normal subgroup of G and let $\rho : H \to \mathrm{GL}_n(L)$ be an irreducible representation of H. If the representation extends to an n-dimensional representation of G, then the extension is unique up to a twist by a character of G/H.*

Proof. The results follows from an easy application of Schur's lemma. Suppose that $\rho_i : G \to \mathrm{GL}_n(K)$, $i = 1, 2$ are two extensions of ρ. Then for any $g \in G$ and any $h \in H$, the equality

$$\rho_1(g)\rho_1(h)\rho_1(g)^{-1} = \rho_1(ghg^{-1}) = \rho_2(ghg^{-1}) = \rho_2(g)\rho_2(h)\rho_2(g)^{-1},$$

implies that $\rho_2(g^{-1})\rho_1(g)$ must commute with ρ, hence by Schur's lemma it is a scalar matrix of the form $\chi(g) \cdot 1_n$, for some $\chi(g) \in L^\times$ (where 1_n denotes the $n \times n$ identity matrix). The fact that ρ_1 and ρ_2 are representations implies that χ is a character, and the fact that they are extensions of ρ implies that χ is trivial on H. □

Corollary 2.2. *Let K be a number field and let $\rho_i : \mathrm{Gal}(\overline{\mathbb{Q}(t)}/K(t)) \to \mathrm{GL}_2(\overline{\mathbb{Q}_p})$ for $i = 1, 2$ be two irreducible continuous representations. Suppose that the following two conditions hold:*

(1) $\rho_1|_{\mathrm{Gal}_{\overline{\mathbb{Q}}(t)}} \simeq \rho_2|_{\mathrm{Gal}_{\overline{\mathbb{Q}}(t)}}$,
(2) The restriction $\rho_1|_{\mathrm{Gal}_{\overline{\mathbb{Q}}(t)}}$ is irreducible.

Then there exists $\chi : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \to \overline{\mathbb{Q}_p}^\times$ such that $\rho_1 \simeq \rho_2 \otimes \chi$.

Proof. Just take $H := \mathrm{Gal}(\overline{\mathbb{Q}(t)}/\overline{\mathbb{Q}}(t))$ and $G := \mathrm{Gal}(\overline{\mathbb{Q}(t)}/K(t))$ in the previous proposition and recall that $\mathrm{Gal}(\overline{\mathbb{Q}}(t)/K(t)) \simeq \mathrm{Gal}(\overline{\mathbb{Q}}/K)$. □

The corollary gives an answer to our problem: the two hypergeometric motives are isomorphic if and only if the following two properties hold:

(1) The monodromy representations of both parameters are isomorphic.
(2) At one specialization of the parameter the motives are isomorphic.

One implication is clear: if the motives are isomorphic, (2) must hold, and also the restriction of their Galois representations to $\pi_1\left(\mathbb{P}^1 \setminus \varphi^{-1}(\{0, 1, \infty\})\right)$ must also be isomorphic, so (1) holds. For the other implication, the first property assures that both motives have the same geometric representation, so by Corollary 2.2 they differ by a twist which does not depend on the parameter. Then the twist is determined by any specialization.

Verifying whether condition (1) holds is a linear algebra problem in most cases. To check whether condition (2) holds is more challenging.

2.1. **Finite hypergeometric sums.** Let $N > 1$ be an integer and $F = \mathbb{Q}(\zeta_N)$. Let $\mathfrak{p}$ be a prime ideal in F prime to N and let $\mathbb{F}_q$ denote its residue field. Let ψ be an additive character of $\mathbb{F}_q^\times$. For ω a character of $\mathbb{F}_q^\times$, denote by $g(\psi, \omega)$ Gauss' sum

$$g(\psi, \omega) := \sum_{x \in \mathbb{F}_q^\times} \omega(x)\psi(x). \tag{9}$$

Following [15], for x an integer prime to $\mathfrak{p}$, let $\chi_\mathfrak{p}(x)$ denote the N-th root of unity congruent to $x^{(q-1)/N}$ modulo $\mathfrak{p}$. Extend the definition by setting $\chi_\mathfrak{p}(x) = 0$ if $\mathfrak{p} \mid x$. This determines a character

$$\chi_\mathfrak{p} : (\mathbb{Z}[\zeta_N]/\mathfrak{p})^\times \to \mathbb{C}^\times. \tag{10}$$

Definition 2.3. *Let $(a, b), (c, d)$ be generic rational parameters, and let N be their least common denominator. Let $\mathfrak{p}$ be a prime ideal of F. For $z \in \mathbb{F}_q$, define the* finite hypergeometric sum *$H_\mathfrak{p}((a, b), (c, d)|z)$ by*

$$\frac{1}{1-q}\sum_{\omega} \frac{g(\psi, \chi_\mathfrak{p}^{-aN}\omega)g(\psi, \chi_\mathfrak{p}^{-bN}\omega)g(\psi, \chi_\mathfrak{p}^{cN}\omega^{-1})g(\psi, \chi_\mathfrak{p}^{dN}\omega^{-1})}{g(\psi, \chi_\mathfrak{p}^{-aN})g(\psi, \chi_\mathfrak{p}^{-bN})g(\psi, \chi_\mathfrak{p}^{cN})g(\psi, \chi_\mathfrak{p}^{dN})}\omega(z), \tag{11}$$

where the sum runs over characters of $\mathbb{F}_q^\times$.

The definition of the finite hypergeometric sum does not depend on the choice of the additive character. Let $z \in \mathbb{Q}$, with $z \neq 0, 1$. Let $\mathfrak{p}$ a prime ideal of F not dividing N such that $v_\mathfrak{p}(z(z-1)) = 0$. Then the motive $\mathcal{H}((a,b),(c,d)|z)$ is unramified at $\mathfrak{p}$ and the trace of the Frobenius element $\mathrm{Frob}_\mathfrak{p}$ acting on $\mathcal{H}((a,b),(c,d)|z)$ equals $H_\mathfrak{q}((a,b),(c,d)|z)$ (as proven in [9, Theorem 6.29]).

Then we can give a more combinatorial criteria to determine whether two hypergeometric motives are isomorphic.

(1) Determine whether the monodromy representations of both parameters are isomorphic.
(2) Prove that there exists $z \in \mathbb{Q}$ satisfying that for all primes $\mathfrak{p}$ of F but finitely many (or outside a set of density zero) the following equality holds:

$$H_\mathfrak{p}((a,b),(c,d)|z) = H_\mathfrak{p}((\alpha,\beta),(\gamma,\delta)|z).$$

2.2. **Special values.** Although the specializations $z = 0, 1, \infty$ give singular motives, in some cases it still makes sense to compute the value of a Frobenius element at it. This flexibility allows in many concrete cases to prove condition (2). Recall the following definition (from [15] and §5 of [9]).

Definition 2.4. *Let* $\mathbf{a} = (a_1, \ldots, a_r)$ *and* $\mathbf{b} = (b_1, \ldots, b_s)$ *be two sets of rational numbers and let* N *be their least common denominator. The Jacobi motive attached to* $\mathbf{a}, \mathbf{b}$ *at a prime ideal* $\mathfrak{p}$ *of* $F = \mathbb{Q}(\zeta_N)$ *is defined as*

$$\mathbf{J}(\mathbf{a},\mathbf{b})(\mathfrak{p}) = (-1)^{r+s+1} \frac{g(\psi, \chi_\mathfrak{p}^{Na_1}) \cdots g(\psi, \chi_\mathfrak{p}^{Na_r}) g(\psi, \chi_\mathfrak{p}^{N\sum_j b_j - N\sum_i a_i})}{g(\psi, \chi_\mathfrak{p}^{Nb_1}) \cdots g(\psi, \chi_\mathfrak{p}^{Nb_s})}. \tag{12}$$

Lemma 2.5. *Let* $(a,b),(c,1)$ *be rational generic parameters. Let* $\varphi(z)$ *be a rational function vanishing at* $z = 0$ *with order* $v > 0$. *Then if* $vc \notin \mathbb{Z}$, *for any prime ideal* $\mathfrak{p}$ *of* F *not dividing* N, *the trace of* $\mathrm{Frob}_\mathfrak{p}$ *on* $\mathcal{H}((a,b),(c,1)|\varphi(z))$ *specialized at* $z = 0$ *equals* 1.

Proof. The proof follows the lines of [9, Appendix A] (see also [7]). Write $\varphi(z) = z^v \tilde{\varphi}(z)$, so $\tilde{\varphi}(0) \neq 0$. Our motive is part of the middle cohomology of Euler's curve

$$y^N = x^A(1-x)^B(1 - z^v\tilde{\varphi}(z)x)^C,$$

where $A = -bN$, $B = (b-c)N$ and $C = aN$. The semistable model has two components, namely

$$\mathcal{C}_1 : y^N = x^A(1-x)^B,$$

and

$$\mathcal{C}_2 : z^{v(A+B)}y^N = (-1)^B x^{A+B}(1 - \tilde{\varphi}(0)x)^C.$$

The second curve is a twist by $\sqrt[N]{z^{cvN}}$ of a non-singular curve (both curves have positive genus and complex multiplication). Since cv is not an integer, the action of inertia at $z = 0$ on $\mathcal{C}_2$ is non-trivial hence the only contribution comes from $\mathcal{C}_1$. Let $\mathfrak{p}$ be a prime ideal of F with norm q. Since $\mathcal{C}_1$ has complex multiplication, the 1-dimensional contribution at a prime ideal $\mathfrak{p}$ equals (by [9, Appendix A])

$$-\frac{g(\psi, \chi_\mathfrak{p}^{bN}) g(\psi, \chi_\mathfrak{p}^{(c-b)N})}{g(\psi, \chi_\mathfrak{p}^{cN})} = \mathbf{J}((b, c-b),(c))(\mathfrak{p}). \tag{13}$$

The motive $\mathcal{H}((a,b),(c,d)|z)$ (defined in [9, Definition 6.8]) is the twist of the contribution coming from Euler's curve times the character

$$\mathbf{J}((-a,-b,c,1),(c-b,-a))^{-1}(\mathfrak{p})\chi_\mathfrak{p}(-1)^{bN} = \mathbf{J}((-b,c)(c-b))^{-1}(\mathfrak{p})\chi_\mathfrak{p}(-1)^{bN}. \tag{14}$$

Recall that if χ is a non-trivial character in $\mathbb{F}_q^\times$, then

$$g(\psi,\chi) = \chi(-1)\frac{q}{g(\psi,\chi^{-1})}. \tag{15}$$

Using the genericity condition and that c is not an integer we get the following relation for (13)

$$\mathbf{J}((b,c-b)(c))(\mathfrak{p}) = -\frac{g(\psi,\chi_{\mathfrak{p}}^{bN})g(\psi,\chi_{\mathfrak{p}}^{(c-b)N})}{g(\psi,\chi_{\mathfrak{p}}^{cN})} = -\frac{\chi_{\mathfrak{p}}(-1)^{bN}g(\psi,\chi_{\mathfrak{p}}^{bN})g(\psi,\chi_{\mathfrak{p}}^{-cN})}{g(\psi,\chi_{\mathfrak{p}}^{(b-c)N})} = \mathbf{J}((-b,c),(c-b))(\mathfrak{p})\chi_{\mathfrak{p}}(-1)^{bN}.$$

Then the product of (13) and (14) equals 1 as claimed. $\square$

Lemma 2.6. *Let a,b be rational numbers which are not integers and let N be their least common denominator. Let $\varphi(z)$ be a rational function vanishing at 0. Then for any prime ideal $\mathfrak{p}$ of F not dividing N, the trace of $\mathrm{Frob}_{\mathfrak{p}}$ on $\mathcal{H}((a,b),(1,1)|\varphi(z))$ specialized to $z=0$ equals 1.*

Proof. The proof follows the idea of [7] (see §2 of loc. cit. for details). Let N be the least common multiple of the denominators of a,b and let $F=\mathbb{Q}(\zeta_N)$. Set $\varphi(z)=z^v\tilde{\varphi}(z)$, with $\tilde{\varphi}(0)\neq 0$. Euler's curve for this parameters has equation

$$\mathcal{C} : y^N = x^{-bN}(\varphi(z)-x)^{bN}(1-x)^{aN}.$$

Assume that $\gcd(N,bN)=\gcd(N,aN)=1$ (the other case is a little more technical, see [7]). The semistable model of $\mathcal{C}$ consists of the two (genus zero) irreducible curves

$$\mathcal{C}_1 : y^N = x^{-bN}(\tilde{\varphi}(0)-x)^{bN}.$$

and

$$\mathcal{C}_2 : y^N = (-1)^{bN}(1-x)^{aN}$$

The two curves intersect in N points defined over the extension $F(\sqrt[N]{(-1)^{bN}})$ (independent of the function φ). The Jacobian of Euler's curve matches the first cohomology group of the just described component graph tensored with the 2-dimensional Steinberg representation. Then if $\mathfrak{p}$ is a prime ideal of F not dividing N, the trace of Frobenius at the ζ_N-eigenpart of the Jacobian matches the value of the character giving the extension $F(\sqrt[N]{(-1)^{bN}})/F$ at $\mathfrak{p}$, whose value is $\chi_{\mathfrak{p}}(-1)^{bN}$. Since $c=d=1$, the normalization (14) also takes the value takes the same value $\chi_{\mathfrak{p}}(-1)^{bN}$ hence the quotient is trivial. $\square$

Recall the following transformation formula

Proposition 2.7. *The motives $\mathcal{H}((a,b),(c,d)|z)$ and $\mathcal{H}((-c,-d),(-a,-b)|z^{-1})$ are isomorphic.*

Proof. See [9, Proposition 6.33]. $\square$

The proposition gives results analogous to the previous lemmas but specializing at $z=\infty$.

Lemma 2.8. *Let a,b,c be rational numbers which are not integers.*

1. *Let N be the least common denominator of a,b,c. Let $\varphi(z)$ be a rational function, and z_0 a point on $\mathbb{P}^1$ where the function has a pole of order v. Then if $vc\notin\mathbb{Z}$, for any prime ideal $\mathfrak{p}$ of $\mathbb{Q}(\zeta_N)$ not dividing N, the trace of $\mathrm{Frob}_{\mathfrak{p}}$ acting on $\mathcal{H}((c,1),(a,b)|\varphi)$ specialized at z_0 equals 1.*
2. *Let N be the least common denominator of a,b. Let $\varphi(z)$ be a rational function having a pole at z_0. Then for any prime ideal $\mathfrak{p}$ of $\mathbb{Q}(\zeta_N)$ not dividing N, the trace of $\mathrm{Frob}_{\mathfrak{p}}$ acting on $\mathcal{H}((1,1),(a,b)|\varphi(z))$ specialized at z_0 equals 1.*

At last, we have the following result regarding specializations at $z=1$.

Lemma 2.9. *Let $(a,b),(c,d)$ be rational generic parameters such that $a+b-c-d\notin\mathbb{Z}$. Then for any prime ideal $\mathfrak{p}$ of F not dividing N, the trace of $\mathrm{Frob}_{\mathfrak{p}}$ acting on $\mathcal{H}((a,b),(c,d)|1)$ equals*

$$\mathbf{J}((a-c,a-d,b-c,b-d),(a,b,-c,-d,a+b-c-d))(\mathfrak{p}). \tag{16}$$

Proof. The proof follows the lines of [9, Theorem A.3]. We consider the equation

$$\mathcal{C}: y^N = x^A(1-x)^B(1-zx)^C z^D,$$

where $A=(d-b)N$, $B=(b-c)N$, $C=(a-d)N$ and $D=dN$. The semistable model has two components that intersect in a single point, with equations (setting $x=1-(z-1)x'$)

$$\mathcal{C}_1: y^N = x^A(1-x)^{B+C}, \qquad \mathcal{C}_2: y^N(z-1)^{-B-C} = (1-(z-1)x')^A x'^B(1+zx')^C z^D.$$

The hypothesis $a+b-c-d\notin\mathbb{Z}$ implies that $B+C=(a+b-c-d)N$ is not divisible by N, so the curve $\mathcal{C}_2$ has singular reduction (it attains good reduction over the extension $\sqrt[N]{z-1}$), so it does not contribute to the trace. The reduction of $\mathcal{C}_1$ is the curve

$$y^N = x^{(d-b)N}(1-x)^{(a+b-c-d)N},$$

whose new part contributes equals $\mathbf{J}((b-d,c+d-a-b),(c-a))(\mathfrak{p})$. Dividing this quantity by the normalization and using (15), we obtain the equality

$$\begin{aligned} H_{\mathfrak{p}}((a,b),(c,d)|1) &= \chi_{\mathfrak{p}}(-1)^{(d-b)N}\frac{J((b-d,c+d-a-b),(c-a))(\mathfrak{p})}{J((-a,-b,c,d),(c-b,d-a))(\mathfrak{p})} = \\ &= J((b-c,b-d,a-c,a-d),(a,b,-c,-d,a+b-c-d)). \end{aligned}$$

□

Lemma 2.10. *Let $(a,b),(c,d)$ be rational generic parameters such that $a+b$ and $c+d$ are integers. Then for any prime ideal $\mathfrak{p}$ of F not dividing N, the trace of $\mathrm{Frob}_{\mathfrak{p}}$ acting on $\mathcal{H}((a,b),(c,d)|1)$ equals $\mathcal{N}\mathfrak{p}^{\delta}$, where*

$$\delta = \begin{cases} 0 & \text{if } a\in\mathbb{Z} \text{ or } c\in\mathbb{Z},\\ 1 & \text{otherwise.}\end{cases}$$

Proof. The proof is similar to that of Lemma 2.8. We can assume without loss of generality that $a=-b$ and $c=-d$. Set $A=(a-c)N$, $B=-(a+c)N$, $C=(a+c)N=-B$ and $D=-cN$, so we need to study the reduction at $z=1$ of Euler's curve

$$\mathcal{C}: y^N = x^A(1-x)^B(1-zx)^{-B}z^{-cN}.$$

One component is given by its reduction modulo $z-1$, namely the curve

$$\mathcal{C}_1: y^N = x^A.$$

To get the other component, let $x' = \frac{(1-x)}{(z-1)}$, so the equation for $\mathcal{C}$ transforms into

$$y^N = (1-(z-1)x')^A x'^B (zx'-1)^{-B} z^{-cN}.$$

Its reduction modulo $z-1$ gives the curve

$$\mathcal{C}_2: y^N = x'^B(x'-1)^{-B}.$$

Both curves $\mathcal{C}_1$ and $\mathcal{C}_2$ have genus zero, and intersect at N points defined over F, giving just the classical Steinberg representation. To prove the formula we need to compute the contribution from the Jacobi motive.

$$\mathbf{J}((-a,a,-c,c),(a+c,-a-c)) = \frac{g(\psi,\chi_{\mathfrak{p}}^{-aN})g(\psi,\chi_{\mathfrak{p}}^{aN})g(\psi,\chi_{\mathfrak{p}}^{-cN})g(\psi,\chi_{\mathfrak{p}}^{cN})}{g(\psi,\chi_{\mathfrak{p}}^{(a+c)N})g(\psi,\chi_{\mathfrak{p}}^{-(a+c)N})}.$$

Recall from (15) that

$$g(\psi,\chi)g(\psi,\chi^{-1})=\chi(-1)\cdot\begin{cases}\mathcal{N}\mathfrak{p} & \text{if } \chi\neq 1,\\ 1 & \text{otherwise.}\end{cases}$$

Let δ as in the statement. Since the parameters are generic, $a+c\notin\mathbb{Z}$, so

$$\mathbf{J}((-a,a,-c,c),(a+c,-a-c))=\mathcal{N}\mathfrak{p}^{\delta}.$$

□

3. Kummer's S_4 transformations

In [11, §8], Kummer listed twenty four transformations $\{T_i\}_{i=1}^{24}$ that send a solution of the differential equation (2) to another solution. For each transformation he wrote down the explicit relations satisfied between the corresponding hypergeometric functions. In this section we compute the analogous relations between the hypergeometric motives.

Definition 3.1. *Let $\alpha=\frac{r}{N}$ be a rational number, with r,N coprime integers. Define the character*

$$\theta_\alpha:\mathrm{Gal}(\overline{\mathbb{Q}(z)}/\mathbb{Q}(z,\zeta_N))\to\overline{\mathbb{Q}}^{\times},$$

to be the character that factors through $\mathrm{Gal}(\mathbb{Q}(\sqrt[N]{z},\zeta_N)/\mathbb{Q}(z,\zeta_N))$ whose value at σ equals

$$\theta_\alpha(\sigma)=\left(\frac{\sigma(\sqrt[N]{z})}{\sqrt[N]{z}}\right)^{r}.$$

The character is unramified outside $\{0,\infty\}$. Let $z_0\in\mathbb{Q}\setminus\{0\}$ and let $\mathfrak{p}$ be a prime ideal of $\mathbb{Q}(\zeta_N)$ of norm q not dividing Nz_0. Then an easy computation (see §6.5 of [9]) shows that the specialization of θ_α at z_0 satisfies

$$\theta_\alpha(\mathrm{Frob}_{\mathfrak{p}})=\chi_{\mathfrak{p}}(z_0)^{r}. \tag{17}$$

Similarly, for each rational number α there is a character ramified only at $\{1,\infty\}$.

Definition 3.2. *Let $\alpha=\frac{r}{N}\in\mathbb{Q}$ as before. Let $\eta_\alpha:\mathrm{Gal}(\overline{\mathbb{Q}(z)}/\mathbb{Q}(z,\zeta_N))\to\overline{\mathbb{Q}}^{\times}$ be the character defined by*

$$\eta_\alpha(\sigma)=\left(\frac{\sigma(\sqrt[N]{1-z})}{\sqrt[N]{1-z}}\right)^{r}. \tag{18}$$

Theorem 3.3. *Let a,b,c be rational numbers satisfying that $(a,b),(c,1)$ is generic. Let N denote their least common denominator and let $F=\mathbb{Q}(\zeta_N)$. Kummer's transformations imply the equality of the following twenty four hypergeometric motives defined over F.*

i) $\mathcal{H}((a,b),(c,1)|z)$,
ii) $\mathcal{H}((c-a,c-b),(c,1)|z)\otimes\eta_{c-a-b}$,
iii) $\mathcal{H}((a-c,b-c),(-c,1)|z)\otimes\mathbf{J}((a-c,b-c,c),(a,-c,b))\theta_{-c}$,
iv) $(-1)^c\mathcal{H}((-a,-b),(-c,1)|z)\otimes\mathbf{J}((a-c,-b,c),(a,-c,c-b))\theta_{-c}\eta_{c-a-b}$,
v) $(-1)^a\mathcal{H}((a,b),(a+b-c,1)|1-z)\otimes\mathbf{J}((a-c,c-a-b),(-c,c-b))$,
vi) $(-1)^{b+c}\mathcal{H}((a-c,b-c),(a+b-c,1)|1-z)\otimes\mathbf{J}((a-c,b-c,c-a-b),(a,-c,-a))\theta_{-c}$,
vii) $(-1)^a\mathcal{H}((c-a,c-b),(c-a-b,1)|1-z)\otimes\mathbf{J}((a-c,c-a,a+b-c),(a,-c,b))\eta_{c-a-b}$,
viii) $(-1)^{b+c}\mathcal{H}((-a,-b),(c-a-b,1)|1-z)\otimes\mathbf{J}((-b,a+b-c),(a,-c))\theta_{-c}\eta_{c-a-b}$,
ix) $\mathcal{H}((a,a-c),(a-b,1)|\frac{1}{z})\otimes\mathbf{J}((a-c,b-a),(b,-c))\theta_{-a}$,
x) $\mathcal{H}((b,b-c),(b-a,1)|\frac{1}{z})\otimes\mathbf{J}((b-c,a-b),(a,-c))\theta_{-b}$,
xi) $(-1)^c\mathcal{H}((-a,c-a),(b-a,1)|\frac{1}{z})\otimes\mathbf{J}((a-c,c-a,a-b),(a,-c,c-b))\theta_{a-c}\eta_{c-a-b}$,
xii) $(-1)^c\mathcal{H}((-b,c-b),(a-b,1)|\frac{1}{z})\otimes\mathbf{J}((b-c,c-b,b-a),(b,-c,c-a))\theta_{b-c}\eta_{c-a-b}$,

1	y	7	$\frac{1}{1+y(z-1)}$	13	$\frac{y}{y+z-1}$	19	$\frac{1}{z(1-y)}$
2	$\frac{1-y}{1-zy}$	8	$\frac{1+y(z-1)}{z}$	14	$\frac{1}{1-y}$	20	$1+\frac{1-z}{zy}$
3	$\frac{1}{zy}$	9	$\frac{y}{z}$	15	$\frac{1-y}{z}$	21	$\frac{y}{z(y-1)}$
4	$\frac{y-z^{-1}}{y-1}$	10	$\frac{1}{y}$	16	$\frac{y+z-1}{yz}$	22	$\frac{y-1}{y}$
5	$\frac{y}{y-1}$	11	$\frac{z-y}{z(1-y)}$	17	$\frac{y}{1+zy-z}$	23	$1+\frac{(1-z)y}{z}$
6	$\frac{y-1}{zy}$	12	$\frac{y-1}{y-z}$	18	$1-y$	24	$\frac{1}{z+(1-z)y}$

TABLE 3.1. Change of variables

xiii) $(-1)^a\mathcal{H}((a,c-b),(a-b,1)|\frac{1}{1-z})\otimes\mathbf{J}((a-c,b-a),(b,-c))\eta_{-a}$,
xiv) $(-1)^b\mathcal{H}((b,c-a),(b-a,1)|\frac{1}{1-z})\otimes\mathbf{J}((b-c,a-b),(a,-c))\eta_{-b}$,
xv) $(-1)^{b+c}\mathcal{H}((a-c,-b),(a-b,1)|\frac{1}{1-z})\otimes\mathbf{J}((a-c,-b,b-a),(a,-c,-a))\theta_{-c}\eta_{c-a}$,
xvi) $(-1)^{a+c}\mathcal{H}((b-c,-a),(b-a,1)|\frac{1}{1-z})\otimes\mathbf{J}((b-c,-a,a-b),(b,-c,-b))\theta_{-c}\eta_{c-b}$,
xvii) $\mathcal{H}((a,c-b),(c,1)|\frac{z}{z-1})\otimes\eta_{-a}$,
xviii) $\mathcal{H}((b,c-a),(c,1)|\frac{z}{z-1})\otimes\eta_{-b}$,
xix) $(-1)^c\mathcal{H}((a-c,-b),(-c,1)|\frac{z}{z-1})\otimes\mathbf{J}((a-c,-b,c),(a,-c,c-b))\theta_{-c}\eta_{c-a}$,
xx) $(-1)^c\mathcal{H}((b-c,-a),(-c,1)|\frac{z}{z-1})\otimes\mathbf{J}((b-c,-a,c),(b,-c,c-a))\theta_{-c}\eta_{c-b}$,
xxi) $(-1)^a\mathcal{H}((a,a-c),(a+b-c,1)|\frac{z-1}{z})\otimes\mathbf{J}((a-c,c-a-b),(c-b,-c))\theta_{-a}$,
xxii) $(-1)^b\mathcal{H}((b,b-c),(a+b-c,1)|\frac{z-1}{z})\otimes\mathbf{J}((b-c,c-a-b),(c-a,-c))\theta_{-b}$,
xxiii) $(-1)^a\mathcal{H}((-a,c-a),(c-a-b,1)|\frac{z-1}{z})\otimes\mathbf{J}((a-c,c-a,a+b-c),(a,-c,b))\theta_{a-c}\eta_{c-a-b}$,
xxiv) $(-1)^b\mathcal{H}((-b,c-b),(c-a-b,1)|\frac{z-1}{z})\otimes\mathbf{J}((b-c,c-b,a+b-c),(b,-c,a))\theta_{b-c}\eta_{c-a-b}$,

where $(-1)^a$ is the quadratic character of F whose value at a prime ideal $\mathfrak{p}$ equals $\chi_{\mathfrak{p}}(-1)^{Na}$.

Proof. For ε,ω,χ characters of $\mathbb{F}_q$ and $z\in\mathbb{F}_q$, define the function

$$H(z)=\sum_{x\in\mathbb{F}_q}\varepsilon(x)\omega(1-x)\chi^{-1}(1-zx). \tag{19}$$

Setting $\varepsilon=\chi_{\mathfrak{p}}^{aN}$, $\omega=\chi_{\mathfrak{p}}^{(c-a)N}$ and $\chi=\chi_{\mathfrak{p}}^{bN}$, it follows from [9, Theorem 4.10] that

$$H(z)=\frac{\varepsilon(-1)g(\psi,\varepsilon)g(\psi,\varepsilon^{-1}\omega^{-1})}{g(\psi,\omega^{-1})}H_{\mathfrak{p}}((a,b),(c,1)|z). \tag{20}$$

The function $H(z)$ is a finite version of the integral defining the hypergeometric series. The stated transformation results for $H(z)$ follow from applying each change of variables suggested in [6] (Table 1). The order of the transformations listed in Dwork's table is not the same as Kummer's original article, so to avoid confusions, in Table 3.1 we reordered the entries of Dwork's table. The change of variables corresponds to set $x=f(y)$ for each entry. Although this transformation might not be well defined at one of the points $\{0,1,1/z\}$ the term in the definition of $H(z)$ takes the value 0 at all such points.

For example, the second formula corresponds to the change of variables $x=\frac{1-y}{1-zy}$. Then (19) becomes

$$\sum_{y\in\mathbb{F}_q}\varepsilon\left(\frac{1-y}{1-zy}\right)\omega\left(\frac{y(1-z)}{1-zy}\right)\chi^{-1}\left(\frac{1-z}{1-zy}\right)=\omega\chi^{-1}(1-z)\sum_{y\in\mathbb{F}_q}\omega(y)\varepsilon(1-y)(\varepsilon\omega\chi^{-1})^{-1}(1-zy),$$

which matches the series for the parameters $(c-a,c-b),(c,1)$ at z twisted by η_{c-a-b} (corresponding to the value outside the summation). The first term in (20) is the same for both parameters, proving the formula. Similar computations prove all other cases. $\square$

Remark 1. As a safety check, there is code available at the author's web page `https://sweet.ua.pt/apacetti` to verify each of these twenty four transformations for specializations of the variable.

Remark 2. As is well known, the twenty four transformations found by Kummer are not closed under composition, i,e. they do not form a group (see §2 [6] for a description of them). It is not hard to compute the group generated by these 24 transformations. If we forget the "twist" appearing in the previous formulas, we can represent each transformation by the way it acts on the parameters (a,b,c) (corresponding to a linear transformation in $\mathrm{GL}_3(\mathbb{Z})$) and its action on z (a Möbius transformation of $\mathbb{P}^1\setminus\{0,1,\infty\}$). The resulting group H has order 144, corresponding to the small group number 189, isomorphic to $C_2\times(C_3\rtimes S_3)$. Then the previous list of 24 transformations can be enlarged to a list containing 144 ones.

4. Hypergeometric relations and covers

As mentioned in the introduction, Goursat in [10] proved many quadratic transformation formulas satisfied by Gauss' hypergeometric function coming from hypergeometric pairs (φ,ρ) (as well as some larger degree ones). The strategy he used is not that different from the two steps mentioned before for hypergeometric motives (our strategy is largely influenced by his):

(1) Prove that both sides of the equality satisfy the same differential equation.
(2) Prove that the value of both functions at a well chosen point coincide.

Corollary 2.2 together with the results obtained in §2.2 can probably be used to prove arithmetic versions of each of the seventy six transformations discovered by Kummer in [11] as well as the one hundred and thirty seven ones discovered by Goursat in [10]. We content ourselves studying some of them.

Theorem 4.1. *Let (φ,ρ) be a hypergeometric pair, where $\rho=\mathcal{H}((a,b),(c,1)|z)$. Let a',b',c' be rational numbers such that the monodromy representation of $\mathcal{H}((a,b),(c,d)|z)$ is isomorphic to the monodromy representation of $\mathcal{H}((a',b'),(c',1)|\varphi(z))$. Suppose that φ vanishes at $z=0$ with order $v>0$ and one of the following two hypothesis is satisfied:*

(1) $c\notin\mathbb{Z}$ and $vc'\notin\mathbb{Z}$,
(2) $c,c'\in\mathbb{Z}$.

Then there is an isomorphism of hypergeometric motives

$$\mathcal{H}((a,b),(c,1)|z)\simeq\mathcal{H}((a',b'),(c',1)|\varphi(z)). \tag{21}$$

Proof. Since both motives have the same monodromy representation, Corollary 2.2 implies that they are a twist of each other. Then (as explained in §2) it is enough to prove that they are isomorphic for a particular specialization of the parameter z (like $z=0$). Under the first hypothesis, Lemma 2.5 implies that the trace of the Frobenius automorphism $\mathrm{Frob}_{\mathfrak{p}}$ acting of the left hand side motive equals 1 for any prime ideal $\mathfrak{p}$ of F not dividing N.

When the first hypothesis is satisfied, Lemma 2.5 also implies that the right hand side also evaluates to 1 at $z=0$. When the second hypothesis holds the result follows from Lemma 2.6. ☐

As an application to Example 1 we have the following.

Corollary 4.2. *Let a,b be rational numbers satisfying:*

- *They are not integers,*
- *$\frac{a-b+1}{2}$, $\frac{a+1}{2}$ and $\frac{b+1}{2}$ are not integers.*

Then the arithmetic version of (33) is the following isomorphism between hypergeometric motives

$$\mathcal{H}\left((a,b),\left(\frac{a+b+1}{2},1\right)|z\right)=\mathcal{H}\left(\left(\frac{a}{2},\frac{b}{2}\right),\left(\frac{a+b+1}{2},1\right)|4z(1-z)\right). \tag{22}$$

Proof. The hypothesis imply that both parameters are generic. We already proved in the introduction that the monodromies of both motives are isomorphic. The result then follows from the last theorem (as the trace at a Frobenius element $\mathrm{Frob}_{\mathfrak{p}}$, for $\mathfrak{p}$ a prime ideal not dividing N, of both sides at $z = 0$ equals 1). $\square$

Remark 3. The isomorphism (22) has deep implications. Let N be the least common multiple of the denominators of $\frac{a}{2}$, $\frac{b}{2}$ and $\frac{a+b+1}{2}$, and let N' be the least common multiple of the denominators of a, b and $\frac{a+b+1}{2}$. Clearly $N' \mid N$, but they might be different. The field of definition of the motive $M_1 := \mathcal{H}\left(\left(\frac{a}{2}, \frac{b}{2}\right), \left(\frac{a+b+1}{2}, 1\right) | z\right)$ is generically $K(\zeta_N)$, while the field of definition of $M_2 := \mathcal{H}\left((a, b), \left(\frac{a+b+1}{2}, 1\right) | z\right)$ is $\mathbb{Q}(\zeta_{N'})$. The isomorphism then implies that if $N \neq N'$, the motive M_1 can be defined over an extension smaller than the expected one. However, this is only true for specializations lying in the image of φ (a thin set). This goes in accordance with the expectation that for most specializations of the parameter (outside a thin set), the coefficient field of the motive is the one conjectured by David Roberts and Fernando Rodriguez Villegas (see [9, Conjecture 3.2]). See also Example 7.21 of [9] for a more naive example of this phenomena.

The arithmetic version of Example 2 is the following.

Proposition 4.3. *Let a, b be rational numbers satisfying:*

- *They are not integers,*
- *$\frac{a-b+1}{2}$ is not an integer.*

Then the arithmetic version of (7) are the isomorphisms

$$\mathcal{H}\left((a, b), \left(\frac{a+b+1}{2}, 1\right) | z\right) = \mathcal{H}\left((a, b), \left(\frac{a+b+1}{2}, 1\right) | 1 - z\right), \tag{23}$$

and

$$\mathcal{H}\left((a, b), \left(\frac{a+b+1}{2}, 1\right) | z\right) = \eta_{(1-a-b)/2}\, \mathrm{J}\left(\left(\frac{a+b+1}{2}, \frac{a+b+1}{2}\right), (a, b)\right) \mathcal{H}\left(\left(\frac{a-b+1}{2}, \frac{b-a+1}{2}\right), \left(\frac{1-a-b}{2}, 1\right) | 1 - z\right). \tag{24}$$

Proof. In Example 2 we proved that the monodromy representation of the three motives involved are isomorphic. The result then follows by evaluating at $z = 0$ and using Lemma 2.5 (or Lemma 2.6) and Lemma 2.9. $\square$

Let us state two more examples of the method.

Proposition 4.4. *Let (a, b) be rational numbers which are not integers. Then the following hypergeometric motives are isomorphic:*

$$\mathcal{H}((a, b), (\frac{a+b+1}{2}, 1)|z) = \mathbf{J}\left(\left(\frac{a+1}{2}, \frac{b+1}{2}\right), \left(\frac{a+b+1}{2}, \frac{1}{2}\right)\right) \mathcal{H}\left(\left(\frac{a}{2}, \frac{b}{2}\right), \left(\frac{1}{2}, 1\right) |(1 - 2z)^2\right). \tag{25}$$

Proof. Consider the map $\varphi(z) = (2z - 1)^2$ ramified at $\{0, \infty\}$ (the preimage of 0 being $\frac{1}{2}$). The preimage of 1 being the two points 0 and 1. It then follows from a simple verification (using the monodromy matrices definition given in (1) and the fact that our parameters are generic) that the two monodromy representations are indeed isomorphic. Specialize both motives at $z = 0$, so the left hand side is evaluated at 0 while the right one at 1. Lemma 2.5 (if $\frac{a+b+1}{2}$ is not an integer) or Lemma 2.6 (in the other case) imply that

$$\mathcal{H}\left((a,b),\left(\frac{a+b+1}{2},1\right)|0\right)=1.$$

Lemma 2.9 implies that

$$\mathcal{H}\left(\left(\frac{a}{2},\frac{b}{2}\right),\left(\frac{1}{2},1\right)|1\right)=\mathrm{J}\left(\left(\frac{a+1}{2},\frac{b+1}{2}\right),\left(\frac{a+b+1}{2},\frac{1}{2}\right)\right)= \mathrm{J}\left(\left(\frac{a+b+1}{2},\frac{1}{2}\right),\left(\frac{a+1}{2},\frac{b+1}{2}\right)\right)^{-1}, \tag{26}$$

so the Jacobi motive relating them equals $\mathrm{J}\left(\left(\frac{a+1}{2},\frac{b+1}{2}\right),\left(\frac{a+b+1}{2},\frac{1}{2}\right)\right)$. ☐

Remark 4. Note that the cover $\varphi(z)=(2z-1)^2$ sends 1 to 1, so we can apply the same reasoning as before for the value $z=1$, in which case the same result follows from the easy equality

$$\mathrm{J}\left(\left(\frac{1+a+b}{2},\frac{1-a-b}{2}\right),\left(\frac{1-a+b}{2},\frac{1+a-b}{2}\right)\right)=1. \tag{27}$$

The following is the arithmetic version of formula 69 in [11].

Theorem 4.5. *Let (a,b) be rational integers satisfying:*

- *a,b are not integers,*
- *The numbers $\frac{-a+b+1}{2}$, $\frac{a-b+1}{2}$ and $\frac{b+1}{2}$ are not integers.*

Then the following hypergeometric motives are isomorphic:

$$\mathcal{H}\left((a,b),\left(\frac{a+b+1}{2},1\right)|z\right)=\mathcal{H}\left(\left(\frac{a}{2},\frac{a+1}{2}\right),\left(\frac{a+b+1}{2},1\right)\middle|\frac{4z^2-4z}{4z^2-4z+1}\right)\chi_{-a},$$

where $\chi_{a/N}$ is the Hecke character corresponding to the extension $\mathbb{Q}(\zeta_N,\sqrt[N]{(2-z)^a})/\mathbb{Q}(\zeta_N,z)$.

Proof. The map $\varphi(z)=\frac{4z^2-4z}{4z^2-4z+1}$ is ramified at 1 and ∞. It maps $\{0,1\}\to 0$, $\infty\to 1$ and $1/2\to\infty$. The monodromy matrices of the right hand side motive are:

$$M_0=M_1=\begin{pmatrix}e^{\frac{-a-b-1}{2}} & 0\\ 0 & 1\end{pmatrix},\qquad M_{1/2}=\begin{pmatrix}e^a & 0\\ 0 & e^a\end{pmatrix},\qquad M_\infty=\begin{pmatrix}e^{-a+b} & 0\\ 0 & 1\end{pmatrix}.$$

Twisting by η_{-a} kills the ramification at $1/2$ and transforms the ramification at ∞ into $\begin{pmatrix}e^a & 0\\ 0 & e^b\end{pmatrix}$, which matches the monodromy representation of the left hand side. The result then follows by evaluating both quantities at $z=0$ (using Theorem 4.1). ☐

At last, let us study an example of a degree 4 cover (formula 131 of [10]).

Theorem 4.6. *Let a be a rational integer, such that none of $4a$, $2a+1/6$, $a+1/2$ and $a-1/6$ is an integer. Then the following motives are isomorphic*

$$\mathcal{H}\left((4a,2a+1/6),(2/3,1)|z\right)=\eta_{-3a}\mathcal{H}\left((a,1/6-a),(2/3,1)\middle|-\frac{(z+8)^3z}{64(1-z)^3}\right)$$

Proof. The cover $\varphi=-\frac{(z+8)^3z}{64(1-z)^3}$ ramifies at the points $0,1$ and ∞. The monodromy matrices of the left hand side motive are

$$N_0=\begin{pmatrix}\exp(1/3) & 0\\ 0 & 1\end{pmatrix},\qquad N_1=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\qquad N_\infty=\begin{pmatrix}\exp(a) & 0\\ 0 & \exp(1/6-a)\end{pmatrix}.$$

The two points above 1 are $10 \pm 6\sqrt{2}$ and have ramification degree 2, so the monodromy matrix of the pullback is trivial for them both. The preimage of 0 consists of the points 0 (unramified) and -8 with ramification degree 3, so -8 is unramified for the pullback. At last, the preimage of ∞ consists of the points 1 (with ramification degree 3) and ∞, which is unramified. Then the monodromy matrices at $0, 1$ and ∞ of the pullback are (respectively)

$$\begin{pmatrix} \exp(1/3) & 0 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} \exp(3a) & 0 \\ 0 & \exp(-3a) \end{pmatrix}, \qquad \begin{pmatrix} \exp(a) & 0 \\ 0 & \exp(1/6-a) \end{pmatrix}.$$

The twist by η_{-3a} multiplies the monodromy matrix at 1 by $\exp(-3a)$ and the one at ∞ by $\exp(3a)$ so both monodromy representations are isomorphic. The result follows from Theorem 4.1. $\square$

Remark 5. Each stated isomorphism gives a non-trivial relation between sums and products of Gauss sums (corresponding to the trace of Frobenius of the right and the left hand side of the isomorphism). Is there a direct way to prove such type of formulas?

5. Applications to Diophantine equations

In [4] Darmon presented a general program to study solutions of the generalized Fermat equation

$$Ax^p + By^q = Cz^r. \tag{28}$$

It is expected that for fixed values of A, B, C, there are only finitely many *primitive* solutions for any triple of exponents (p, q, r) satisfying $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$. Recall the following definition.

Definition 5.1. *A solution* (α, β, γ) *of (28) is called primitive if* $\gcd(\alpha, \beta, \gamma) = 1$.

Darmon's program consists on attaching to the equation exponents (namely p, q, r) what he calls a *Frey representation*, a representation of $\mathrm{Gal}(\overline{\mathbb{Q}(z)}/F(z))$ (for some number field F) into $\mathrm{GL}_2(\mathbb{F})$ for some finite field $\mathbb{F}$. If (α, β, γ) is a solution to (28), then the specialization of the family at $z_0 := \frac{A\alpha^p}{C\gamma^q}$ corresponds to a finite extension of F with little ramification.

Following the ideas used in Wiles' proof of Fermat's last theorem, the representation should be the reduction of a representation attached to a modular form, and (assuming various conjectures) such modular form should not exist if one of the exponents (say p) is sufficiently large. Proving the conjectured missing results is nowadays a deep problem.

In [8] the authors gave a similar approach replacing the finite field $\mathbb{F}$ with a p-adic field, using the theory of hypergeometric motives. Hypergeometric motives give more flexibility to prove some of the expected properties, but the theory is not completely understood. For example, if $(a, b), (c, d)$ are generic rational parameters, and N is their least common denominator, we do not understand the action of inertia on $\mathcal{H}((a, b), (c, d)|z)$ at primes of $F = \mathbb{Q}(\zeta_N)$ dividing N.

While studying the family of exponents (p, p, q), the motive is part of the middle cohomology of an hyperelliptic curve, where the reduction type at odd primes is well understood (see for example [5] and [1]). For the family (q, q, p) this is not the case, the motive appears in Euler's curve which is superelliptic, a case where the description of wild inertia is still not fully understood.

However, in the remarkable article [3], the authors manage to relate a solution of the equation

$$x^q + y^q = z^p, \tag{29}$$

with an hyperelliptic curve $\mathcal{C}_t$. The constructed curve gives a *Frey representation* (as defined by Darmon in [4]) but ramifies at $\{\pm 2i, \infty\}$ (see [3, Proposition 2.32]). An extra transformation is needed to transform this set into $\{0, 1, \infty\}$. Due to the way the curve is constructed, if (α, β, γ) is a solution of (29), the curve $\mathcal{C}_t$ is not evaluated at the usual point $t_0 = \frac{\alpha^q}{\gamma^p}$, but at a different one satisfying a "mysterious" algebraic relation with t_0. The purpose of this section is to explain how to attach naturally an hyperelliptic curve to Fermat equation with exponents (q, q, p) from two

Motive	M_0	M_1	M_∞
$\mathcal{M}_{(p,p,q)}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \zeta_{2q} & 0 \\ 0 & \zeta_{2q}^{-1} \end{pmatrix}$
$\mathcal{M}^s_{(q,q,p)}$	$\begin{pmatrix} \zeta_q^s & 0 \\ 0 & \zeta_q^{-s} \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \zeta_{2q} & 0 \\ 0 & \zeta_{2q}^{-1} \end{pmatrix}$
$\mathcal{M}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -\zeta_{4q}^{-1} & 0 \\ 0 & \zeta_{4q} \end{pmatrix}$

TABLE 5.1. Monodromy matrices

different hypergeometric relations. This approach might prove useful while studying other families of exponents.

The three hypergeometric motives involved are:

(1) The hypergeometric motive $\mathcal{H}\left(\left(\frac{1}{2q}, -\frac{1}{2q}\right), (1,1)|z\right)$ (that we denote by $\mathcal{M}_{(p,p,q)}$ to ease notation) associated to the exponents (p,p,q) in [8] (see also [4]).

(2) The hypergeometric motive $\mathcal{H}\left(\left(\frac{1}{2q}, -\frac{1}{2q}\right), \left(\frac{s}{q}, -\frac{s}{q}\right)|z\right)$ (that we denote by $\mathcal{M}^s_{(q,q,p)}$) attached to the exponents (q,q,p) in [8], for any $s \in \{1,\dots,q-1\}$.

(3) The hypergeometric motive $\mathcal{M} = \mathcal{H}\left(\left(\frac{1}{2} - \frac{1}{4q}, \frac{1}{4q}\right)(1,1)|z\right)$.

For N a positive integer, denote by $\zeta_N = \exp(\frac{1}{N})$, a primitive N-th root of unity. The monodromy matrices around $\{0,1,\infty\}$ of the three motives are given in Table 5.1. Ideally, we seek for degree 2 covers π_1 and π_2 such that the relation between the three motives is explained by the following diagram

$$\begin{tikzcd} \mathcal{M}_{(p,p,q)} \arrow[dr, "\pi_1"] & & \mathcal{M}^s_{(q,q,p)} \arrow[dl, "\pi_2"'] \\ & \mathcal{M} & \end{tikzcd} \tag{30}$$

By looking at Table 5.1 it is clear that some adjustments are needed, since a degree 2 cover tends to have the effect of duplicating a monodromy in the pullback, but the monodromy matrices of $\mathcal{M}^s_{(q,q,p)}$ are all different.

Let $\theta_{-1/4}$ be the order 4 character of $\mathrm{Gal}_{\mathbb{Q}(i,z)}$ of Definition 3.1 (corresponding to the extension $\mathbb{Q}(i,\sqrt[4]{z})/\mathbb{Q}(i,z)$), whose monodromy matrix at 0 equals $\exp(-1/4)$ and at ∞ equals $\exp(1/4)$. Set $s_0 = \frac{q+1}{2}$. Then the monodromy matrices at 0, 1 and ∞ of the twist $\mathcal{M}^{s_0}_{(q,q,p)} \otimes \theta_{-1/4}$ are respectively

$$\begin{pmatrix} \zeta_{4q}^{q+2} & 0 \\ 0 & \zeta_{4q}^{q-2} \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} \zeta_{4q}^{q+2} & 0 \\ 0 & \zeta_{4q}^{q-2} \end{pmatrix}.$$

5.1. **The map π_1:** Let π_1 be the degree 2 cover of $\mathbb{P}^1$ given by the map $\pi_1(z) = 4z(1-z)$. For b a rational number, let η_b be the character of Definition 3.2.

Proposition 5.2. *Let a, b be a rational numbers such that a is not an integer and $b \pm a \notin \mathbb{Z}$. Then*

$$\eta_{-b}\mathcal{H}((a,-a),(b,1)|z) \simeq \mathcal{H}\left(\left(\frac{b-a}{2}, \frac{a+b+1}{2}\right)(b,1)|4z(1-z)\right).$$

Proof. The proof mimics the previous section ones. The monodromy matrices for the hypergeometric motive on the left hand side (if b is not an integer) are

$$M_0 = \begin{pmatrix} \exp(-b) & 0 \\ 0 & 1 \end{pmatrix}, \qquad M_1 = \begin{pmatrix} \exp(-b) & 0 \\ 0 & 1 \end{pmatrix}, \qquad M_\infty = \begin{pmatrix} \exp(b+a) & 0 \\ 0 & \exp(b-a) \end{pmatrix}. \tag{31}$$

The monodromy matrices for the right hand side hypergeometric motive are

$$N_0 = \begin{pmatrix} \exp(-b) & 0 \\ 0 & 1 \end{pmatrix}, \qquad N_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad N_\infty = \begin{pmatrix} \exp(\frac{b-a}{2}) & 0 \\ 0 & \exp(\frac{a+b+1}{2}) \end{pmatrix}. \tag{32}$$

The cover π_1 is ramified at the points $\{1, \infty\}$ (with preimages $1/2$ and ∞ respectively), and the preimage of 0 are the two points $\{0, 1\}$. Then the monodromy representations are isomorphic. The result then follows from Theorem 4.1. The case $b = 1$ follows similarly. ☐

The left part of (30) corresponds to $a = \frac{1}{2q}$ and $b = 1$ in the last proposition, getting the equality

$$\mathcal{H}\left(\left(\frac{1}{2q}, \frac{-1}{2q}\right), (1,1)|z\right) = \mathcal{H}\left(\left(\frac{1}{2} - \frac{1}{4q}, \frac{1}{4q}\right), (1,1)|4z(1-z)\right) \tag{33}$$

5.2. **The map π_2:** Let $(a,b), (c,d)$ be generic parameters and let N be their least common multiple. Since the motive is defined over $F = \mathbb{Q}(\zeta_N)$, the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ acts on it. If $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ satisfies that $\sigma(\zeta_N) = \zeta_N^j$, then the motive $\sigma(\mathcal{H}((a,b),(c,d)|z))$ equals the motive $\mathcal{H}((ja,jb),(jc,jd)|z)$ (as proved in [9, Proposition 4.6]). Instead of looking at the motive $\mathcal{M}$, consider its Galois conjugate $\widetilde{\mathcal{M}}$ by the element σ corresponding to $j = q+2 \in (\mathbb{Z}/2q)^\times$.

Let π_2 be the degree 2 cover of $\mathbb{P}^1$ given by $\pi_2(z) = \frac{-(z-1)^2}{4z}$.

Proposition 5.3. *For q an odd prime, and σ as before, the following motives are isomorphic*

$$\mathcal{H}\left(\left(\frac{1}{2q}, -\frac{1}{2q}\right), \left(\frac{q+1}{2q}, -\frac{q+1}{2q}\right)|z\right) \otimes \theta_{-\frac{1}{4}} \simeq \mathcal{H}\left(\left(\frac{1}{2} - \frac{1}{4q}, \frac{1}{4q}\right), (1,1)\left|\frac{-(z-1)^2}{4z}\right)^\sigma (-1),$$

where the (-1) denotes a Tate twist by the inverse of the cyclotomic character.

Proof. Since the character $\theta_{-\frac{1}{4}}$ has monodromy $\exp(-1/4)$ at 0 and $\exp(1/4)$ at ∞, the left hand side has monodromy matrices

$$M_0 = \begin{pmatrix} \exp(\frac{q+2}{4q}) & 0 \\ 0 & \exp(\frac{q-2}{4q}) \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad M_\infty = \begin{pmatrix} \exp(\frac{q+2}{4q}) & 0 \\ 0 & \exp(\frac{q-2}{4q}) \end{pmatrix}.$$

The motive $\mathcal{H}\left(\left(\frac{1}{2} - \frac{1}{4q}, \frac{1}{4q}\right), (1,1)\,|z\right)^\sigma$ has monodromy matrices

$$N_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad N_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad N_\infty = \begin{pmatrix} \exp(\frac{q+2}{4q}) & 0 \\ 0 & \exp(\frac{q-2}{4q}) \end{pmatrix}.$$

The map π_2 is ramified at 0 and 1, with preimages 1 and -1 respectively. The preimage of ∞ are the points 0 and ∞, so the two monodromy representations are isomorphic. To prove the statement we specialize at the value $z = 1$. The specialization of the character $\theta_{-\frac{1}{4}}$ is trivial, so the trace of the left hand side at the Frobenius element of a prime ideal $\mathfrak{p}$ equals $\mathcal{N}\mathfrak{p}$ by Lemma 2.10. The value of the right hand side equals 1 by Lemma 2.6. ☐

Set $z_0 = \frac{\alpha^q}{\gamma^p}$. Then up to a twist by $\theta_{-\frac{1}{4}}$ Proposition 5.3 implies that instead of considering the superelliptic curve studied by Darmon, we can consider the motive $\mathcal{M}(1)$ evaluated at the point

$\frac{-(z_0-1)^2}{4z_0} = -\frac{\beta^{2q}}{4\alpha^q\gamma^p}$. Then Proposition 5.2 implies that we can instead consider the hyperelliptic curve coming from the exponents (p,p,q) evaluated at u_0 satisfying

$$4u_0(u_0-1) = \frac{\beta^{2q}}{4\alpha^q\gamma^p}.$$

The tame primes of the hyperelliptic curve are the ones dividing $2\alpha\beta\gamma$ as expected, but we can get information at the ramification at the prime q (as exploited in [3] and [1]).

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