

ON THE GENERALIZED FERMAT EQUATION OF SIGNATURE $(5, p, 3)$

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ABSTRACT. In this article we study solutions to the generalized Fermat equation $x^q + y^p + z^r = 0$ using hypergeometric motives within the framework of the modular method. In doing so, we give an explicit description of the ramification behavior at primes dividing $2qr$ and analyze the contribution of trivial solutions. We identify a general obstruction to the modular method that accounts for its failure in many instances. As an application, assuming a standard large image conjecture, we prove that the previous equation admits no nontrivial primitive solutions (a, b, c) with $3 \nmid c$, when $q = 5$, $r = 3$ and p is a prime sufficiently large.

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1. INTRODUCTION

Fermat's Last Theorem inspired a broader investigation of equations of similar form, most notably the *generalized Fermat equation*

$$(1) \quad x^q + y^p + z^r = 0.$$

A solution $(a, b, c) \in \mathbb{Z}^3$ to (1) is called *primitive* if $\gcd(a, b, c) = 1$, and is said to be *trivial* if $abc = 0$. We refer to the triple of exponents (q, p, r) as the *signature* of (1). It is expected (see Beal's conjecture in [Mau97]) that if $\max\{q, p, r\} > 2$, then there are no non-trivial primitive solutions to (1). We recommend the reader to look at the very nice survey [BMS16].

The strategy followed by Wiles in [Wil95], based on Galois representations and modularity, is commonly referred to as the *modular method*. Its main steps are the following.

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- (I) Attach to a putative solution to (1) a geometric object $\mathcal{C}$ defined over a small degree number field K so that the p -th member of the family of Galois representations attached to $\mathcal{C}$ has small residual ramification set (independent of the solution).
- (II) Prove that the geometric object is *modular*, i.e. its L -series matches one coming from an automorphic form.
- (III) Prove that the p -th residual representation in Step (I) is absolutely irreducible, and compute its conductor $\mathfrak{n}$.
- (IV) Compute the space of newforms of level $\mathfrak{n}$ and prove that none of them is related to a solution to (1).

The modular method is a remarkably powerful tool for proving non-existence of solutions while varying along a line in the space of signatures. For instance, the line $q = p = r$ corresponds to Fermat's Last Theorem. Our choice of notation for the exponents in (1) reflects the fact that we allow p to be the varying parameter, in accordance with the notation used in [GP24].

In [Dar00], Darmon described a program based on the modular method to prove that the generalized Fermat equation (1) has no non-trivial solutions when one of the exponents is a varying prime $p \geq 3$. Darmon's idea consists on constructing a family of residual Galois representations (indexed by a parameter t), called *Frey representations*, with the property that if (a, b, c) is a solution to (1), then the specialization at $t_0 = -a^q/c^r$ is unramified outside a small and explicitly controlled set of primes, namely those dividing $2qr$. The specialized representation coincides with the one obtained in Step (I) of the original modular method approach. Darmon's program has been successfully applied to the cases of signatures $(5, p, 5)$ [BCDF25] and $(p, p, 5)$ [CK22]. Until now, these applications have focused on situations where the construction of the Frey representation is explicit, via Frey abelian varieties arising as the Jacobians of Frey hyperelliptic curves.

1.1. Hypergeometric motives: a new approach. In the recent article [GP24], Golfieri and the first author show how the geometric objects traditionally used as sources of Galois representations in Step (I) can be replaced by a more combinatorial one, namely a hypergeometric motive. Within this new context, a first goal of the present article is to give an explicit formula for the conductor at the so-called *potentially wild primes*, as required in Step (III).

Theorem 1. *Attached to a hypothetical solution (a, b, c) to*

$$x^q + y^p + z^r = 0, \quad \text{with } q > r > 2, \ p > 2,$$

there exist two Galois representations

$$\rho_{\mathfrak{p}}^{\pm} : \text{Gal}(\bar{K}/K) \rightarrow \text{GL}_2(K_{\mathfrak{p}}),$$

where $K = \mathbb{Q}(\zeta_q)^+ \cdot \mathbb{Q}(\zeta_r)^+$ and $\mathfrak{p}$ is a prime in K above p , satisfying the following. If $\bar{\rho}_{\mathfrak{p}}^{\pm}$ is irreducible, then:

- (i) $\bar{\rho}_{\mathfrak{p}}^+$ *is unramified outside pqr and is finite at p*
- (ii) $\bar{\rho}_{\mathfrak{p}}^-$ *is unramified outside $2pqr$ and is finite at p .*

The conductor exponent at primes dividing qr is described in Corollary 3.11, Proposition 3.14, and the conductor exponent at 2 for $\bar{\rho}_{\mathfrak{p}}^-$ is given in Proposition 3.15.

1.2. **A general obstruction.** The existence of the trivial solutions

$$\{(\pm 1, \mp 1, 0), (\pm 1, 0, \mp 1), (0, \pm 1, \mp 1)\}$$

to equation (1) constitutes an obstruction to the modular method. Following the approach of [GP24], we prove that only the first and last pair of solutions correspond to abelian varieties with complex multiplication, and we provide explicit formulas for their conductors (see Lemmas 4.4 and 4.5).

It is well known that sometimes the modular method fails to prove non-existence of non-trivial primitive solutions. In some cases (like for signature $(3, p, 2)$) the failure can be explained by the existence of particular solutions to (3), but in others the reason for the failure remains mysterious.

If $(a, c, s, \ell, m, n, u, v) \in \mathbb{Z}^2 \times \mathbb{Z}_{\geq 0}^6$, with $ac \neq 0$, satisfies the relation

$$(2) \quad q^s r^\ell a^q \pm q^m r^n + q^u r^v c^r = 0,$$

then the specialization of Frey's representation (and also of the associated hypergeometric motive) at $t_0 := -a^q q^{s-u} r^{\ell-v} / c^r$ is also unramified outside qr . We call solutions to (2) *ghost solutions*; they are studied in §5. If the motive attached to a ghost solution is modular and its conductor lies among the possible conductors of Hilbert modular forms that must be eliminated in the modular method, then the method fails. This phenomenon for Frey representations appears not to have been previously observed.

In the particular case $q = 5, r = 3$ we found several solutions to (2), listed in Table 7.3. All of them are modular, and two of them lie in the space of Hilbert modular forms $S_2(\Gamma_0(3^3 \cdot (\sqrt{5})^3))$ over $\mathbb{Q}(\sqrt{5})$, one of the spaces that must be analyzed in the elimination step. In the present article we prove that all other forms without complex multiplication can be discarded using classical elimination arguments. It is unclear to us whether ghost solutions explain in general the failure of the modular method, namely that any non-CM form that cannot be discarded arises from a specialization coming from a solution to (2).

1.3. **Signature $(5, p, 3)$.** The main focus of the present article is the study of solutions to the generalized Fermat equation

$$(3) \quad x^5 + y^p + z^3 = 0.$$

According to Beal's conjecture, equation (3) is expected to admit only trivial primitive solutions when $p \geq 3$. For $p = 2$, it is known that there are infinitely many solutions, and they are parametrized in [Edw04]. The cases $p = 3, 4, 5$ were completely solved in [Bru00, SS12, Poo98], respectively. Therefore, we may assume that $p \geq 7$.

Equation (3) is a case of particular interest for several reasons. First, modularity of the hypergeometric motive attached to any solution has been established in [GP24, Theorem 6.3]. Secondly, the associated spaces of Hilbert modular forms are defined over $K = \mathbb{Q}(\sqrt{5})$, making the required computations feasible in practice.

Following the approach of [GP24], we show that there is a hyperelliptic realization of the motive. That is, there exists a hyperelliptic curve $\mathcal{C}_{5,3}^+(t)$ defined over $\mathbb{Q}(t)$ that gives rise to a Frey representation, when considered over K . The representation evaluated at $t_0 = -a^5/c^3$ coincides with the representation $\rho_{\mathfrak{p}}^+$ appearing in Theorem 1. The following result summarizes the results and the obstructions obtained while studying solutions of (1) for the signature $(5, p, 3)$.

Theorem A. Set $\mathcal{P} := \{2, 3, 5, 7, 11, 13, 19, 29, 31, 41, 61, 71, 79, 89, 101, 109\}$. Let p be a prime number not in $\mathcal{P}$ and let (a, b, c) be a primitive solution to

$$x^5 + y^p + z^3 = 0$$

Set $t_0 := -a^5/c^3$. Then one of the following holds:

- (1) The residual Galois representation $\bar{\rho}_{\mathfrak{p}}^+$ is reducible.
- (2) The Galois representation $\rho_{\mathfrak{p}}^+$ is congruent modulo p to the Galois representation associated to a Hilbert modular form of $\mathbb{Q}(\sqrt{5})$ with complex multiplication by $\mathbb{Q}(\sqrt{-3})$ or by $\mathbb{Q}(\zeta_{15})/\mathbb{Q}(\sqrt{5})$.
- (3) The Galois representation $\rho_{\mathfrak{p}}^+$ is congruent modulo p to $\rho_{C_{5,3}^+(t_0), \mathfrak{p}}$, where $t_0 \in \{-\frac{1}{8}, \frac{9}{8}\}$ (these two values correspond to ghost solutions).

The set $\mathcal{P}$ in Theorem A arises from a standard elimination step (based on ideas due to Mazur, as explained in §7.4, using prime ideals of $\mathbb{Q}(\sqrt{5})$ of norm up to 400). With the current standard tools of the modular method, we do not expect that this set can be reduced further (see Remark 7.15).

In theory the first case of Theorem A can be discarded when p is large enough (as it is expected that only small primes can produce reducible representations). Assuming some extra hypotheses on the solution we can produce an explicit bound C such that the residual representation $\bar{\rho}_{\mathfrak{p}}^+$ is irreducible for all primes $p > C$ (see Corollary 7.7). This allows to prove the following corollary of Theorem A.

Theorem B. Let $\mathcal{P} := \{2, 7, 11, 13, 19, 29, 31, 41, 61\}$. Then if p is a prime number not in $\mathcal{P}$, there are no primitive solutions (a, b, c) to

$$x^5 + y^p + z^3 = 0$$

with b odd and $3 \mid b$ or $5 \mid b$.

The hypothesis b being odd is used to prove that the residual representation $\bar{\rho}_{\mathfrak{p}}^+$ is absolutely irreducible, while the hypothesis $3 \mid b$ or $5 \mid b$ is used to rule out the second case in Theorem A, since it implies that the Hilbert modular form attached to a solution has special local type at 3 or at $(\sqrt{5})$. Consequently, on the one hand the form does not have complex multiplication, and on the other hand it does not lie in the space of the two newforms appearing in the third case of Theorem A.

It is expected that Frey varieties without complex multiplication have residual Galois image as large as possible. More concretely, the following is expected to hold (see [Dar00, Conjecture 4.1]).

Conjecture 1. Let K be a totally real field and L a number field. There exists a constant $C(L, K)$, depending only on L and K , such that for any abelian variety A/L of GL_2 -type with

$$\mathrm{End}_L(A) \otimes \mathbb{Q} = \mathrm{End}_{\bar{L}}(A) \otimes \mathbb{Q} \simeq K,$$

and all primes $\mathfrak{p}$ in K above rational primes p of norm greater than $C(L, K)$, the image of the mod p representation associated to A contains $\mathrm{SL}_2(\mathbb{F}_{\mathfrak{p}})$, where $\mathbb{F}_{\mathfrak{p}}$ denotes the residue field of K at $\mathfrak{p}$.

We can prove that any non-trivial solution corresponds to a GL_2 -type abelian variety without complex multiplication. Assuming Conjecture 1 we get the following stronger result.

Theorem C. *Let $\mathcal{P}$ be as in Theorem A. Assume that Conjecture 1 holds for $L = K = \mathbb{Q}(\sqrt{5})$ with constant C . Then if p is a prime number satisfying that $p \notin \mathcal{P}$ and $p > C$, there are no non-trivial primitive solutions (a, b, c) to*

$$x^5 + y^p + z^3 = 0$$

with $3 \nmid c$.

The condition $3 \nmid c$ is used to eliminate the two forms coming from ghost solutions. We can prove (see Theorem 7.18) that when $3 \nmid c$ the local type of a solution does not match the one of a ghost solution. Then Theorem B follows from Theorem A.

To the best of our knowledge, Theorems B and C represent the first Diophantine result for an infinite family of signatures involving three distinct prime exponents. It also constitutes the first successful application of the hypergeometric-motives approach within the modular method.

Notation. Throughout this paper we use the following notation. Let F denote a number field.

- We denote by Gal_F the absolute Galois group $\text{Gal}(\overline{F}/F)$.
- If $c \in F$, we denote by θ_c the character of Gal_F corresponding by class field theory to the extension $F(\sqrt{c})/F$.
- If $\mathfrak{n}$ is a prime ideal of F , we denote by $N(\mathfrak{n})$ its norm.
- If $\mathcal{F}$ is a newform, we denote by $K_{\mathcal{F}}$ its coefficient field.

Electronic resources. The Magma [BCP97] and PARI/GP [PAR25] programs used for the computations in this article are posted in [PV25], which also includes a list of the programs, their descriptions, output transcripts, timings, and the machines used.

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2. HYPERGEOMETRIC MOTIVES

The foundations of the theory of hypergeometric motives is due to Katz ([Kat90, Kat96]); we recommend the reader to look at the nice survey [RRV22] for a friendly introduction to rational hypergeometric motives. For Diophantine applications, we are mostly interested in rank 2 motives which are defined over totally real number fields, as studied in [GPV]. The

goal of the present section is not to give a general review of the theory, but to present a concise account of its main properties (whose proofs are given in [GPV]).

Let us consider the generalized Fermat equation

$$(1) \quad x^q + y^p + z^r = 0,$$

where $q \neq r$ and $\min\{q, r, p\} > 2$. Without loss of generality, we assume that $q > r$. Darmon's innovative idea (developed in [Dar00]) is to attach a varying family of residual representations, called *Frey representations*, to the signature (q, p, r) , such that each particular solution (a, b, c) to (1) gives rise to a specialization of this family. In [GP24] the authors applied a similar strategy, replacing Darmon's Frey representations with hypergeometric motives. More concretely, to the signature (q, p, r) they attach the two hypergeometric motives

$$\mathcal{H}^+(t) := \mathcal{H}\left(\left(\frac{1}{r}, -\frac{1}{r}\right), \left(\frac{1}{q}, -\frac{1}{q}\right) \middle| t\right), \quad \mathcal{H}^-(t) := \mathcal{H}\left(\left(\frac{1}{2r}, -\frac{1}{2r}\right), \left(\frac{1}{q}, -\frac{1}{q}\right) \middle| t\right),$$

for t the varying parameter.

2.1. Geometric properties. Rank 2 hypergeometric motives are a motivic incarnation of Gauss hypergeometric function ${}_2F_1(a, b; c|t)$. The function satisfies an ordinary differential equation, so has attached a *monodromy representation*, i.e. $\rho : \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, x_0) \rightarrow \mathrm{GL}_2(\mathbb{C})$. If γ_i denotes a loop in counterclockwise direction around the point i , for $i = 0, 1, \infty$, then the monodromy representation for $\mathcal{H}^+(t)$ and $\mathcal{H}^-(t)$ at γ_i has matrices conjugate to

$$(4) \quad M_0^+ := \begin{pmatrix} \zeta_q & 0 \\ 0 & \zeta_q \end{pmatrix}, \quad M_1^+ := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad M_\infty^+ := \begin{pmatrix} \zeta_r & 0 \\ 0 & \zeta_r^{-1} \end{pmatrix},$$

and

$$(5) \quad M_0^- := \begin{pmatrix} -\zeta_q & 0 \\ 0 & -\zeta_q \end{pmatrix}, \quad M_1^- := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad M_\infty^- := \begin{pmatrix} \zeta_r & 0 \\ 0 & \zeta_r^{-1} \end{pmatrix},$$

respectively, where $\zeta_r := \exp(\frac{2\pi i}{r})$. A result of Levelt proves that one can chose a basis so that the representation in fact has image in $\mathrm{GL}_2(\mathbb{Q}(\zeta_{qr}))$.

For every prime ideal $\mathfrak{p}$ of K , the monodromy representation extends to a continuous representation

$$(6) \quad \rho_{\mathfrak{p}}^{\pm} : \mathrm{Gal}_{\overline{\mathbb{Q}}(t)} \rightarrow \mathrm{GL}_2(K_{\mathfrak{p}}),$$

with the property that the image of the inertia group around the point $0, 1, \infty$ is generated (up to conjugation) by the matrix $M_0^{\pm}, M_1^{\pm}$ and $M_\infty^{\pm}$ respectively.

For N a positive integer, let $\mathbb{Q}(\zeta_N)^+$ denote the totally real subfield of the cyclotomic extension $\mathbb{Q}(\zeta_N)$. The motives $\mathcal{H}^{\pm}(t)$ are defined over the totally real number field $K := \mathbb{Q}(\zeta_q)^+ \cdot \mathbb{Q}(\zeta_r)^+$ (the composition of both fields). Then they have attached Galois representations

$$\rho_{\mathfrak{p}}^{\pm} : \mathrm{Gal}_{K(t)} \rightarrow \mathrm{GL}_2(K_{\mathfrak{p}})$$

that extend the representations (6) (a property that characterizes the motive up to a twist).

2.2. Construction of the motive. Let us recall an explicit construction of the motives that will be useful for later purposes. Set

$$A^+ := q - r, \quad B^+ := -q - r, \quad C^+ := q + r, \quad D^+ := -r.$$

Consider Euler's curve given by the equation

$$(7) \quad \mathcal{C}^+ : y^{qr} = x^{A^+} (1 - x)^{B^+} (1 - tx)^{C^+} t^{D^+}.$$

The curve $\mathcal{C}^+$ has an action of the group μ_{qr} of qr -th roots of unity (via $\zeta_{qr} \cdot (x, y) = (x, \zeta_{qr} y)$). Over the cyclotomic field $F = \mathbb{Q}(\zeta_{qr})$, the motive $\mathcal{H}^+(t)$ corresponds to the ζ_{qr} -eigenspace for this action on the new part of $\text{Jac}(\mathcal{C}^+)$.

To define the motive over K we need to consider quotients of $\mathcal{C}^+$. The curve possesses three involutions denoted by ι_{-1} , ι_j and ι_i , defined as follows:

$$(8) \quad \iota_{-1}(x, y) = \left(\frac{1}{tx}, \frac{1}{y} \right).$$

Let j be an integer satisfying that $j \equiv 1 \pmod{r}$ and $j \equiv -1 \pmod{q}$. Let μ, ν be positive integers so that

$$\frac{j}{r} = \frac{1}{r} + \mu, \quad \frac{j}{q} = -\frac{1}{q} + \nu.$$

Then by [GPV, Lemma 8.10] the map

$$(9) \quad \iota_j(x, y) = \left(\frac{x - 1}{tx - 1}, \frac{x^{\mu-\nu} (1 - x)^{-\mu-\nu} (1 - tx)^{\mu+\nu} t^{-\nu}}{y^j} \right)$$

is an involution. There is a similar definition for their composition: let i be an integer satisfying $i \equiv -1 \pmod{r}$ and $i \equiv 1 \pmod{q}$. Let $\mu := \frac{i+1}{r}$, $\nu := \frac{i-1}{q}$. Then by [GPV, Lemma 8.11], the map

$$(10) \quad \iota_i(x, y) = \left(\frac{tx - 1}{t(x - 1)}, \frac{x^{\mu-\nu} (1 - x)^{-\mu-\nu} (1 - tx)^{\mu+\nu} t^{-\nu}}{y^i} \right)$$

is an involution corresponding to their composition. Let $\mathcal{D}^+$ be the quotient of $\mathcal{C}^+$ by $\langle \iota_{-1}, \iota_j \rangle$. Then the *new* part of $\text{Jac}(\mathcal{D}^+)$ has an action of $\mathcal{O}_K$, the ring of integers of K , and it is of GL_2 -type over K (see [GPV, Proposition 8.12]). A Tate twist of its new part gives a realization of the motive $\mathcal{H}^+(t)$ over K .

A similar construction holds for the motive $\mathcal{H}^-(t)$. Set

$$A^- := q - 2r, \quad B^- := -q - 2r, \quad C^- := q + 2r, \quad D^- := -2r.$$

and define Euler's curve

$$(11) \quad \mathcal{C}^- : y^{2pq} = x^{A^-} (1 - x)^{B^-} (1 - tx)^{C^-} t^{D^-}.$$

The same quotient as before produces a realization of the motive $\mathcal{H}^-(t)$.

2.3. Specializations. Let (a, b, c) be a putative primitive solution to (1), and set $t_0 = -\frac{a^q}{c^r}$. We will use the following notation:

$$(12) \quad \mathfrak{h}^+ := \mathcal{H}^+(t_0), \quad \mathfrak{h}^- := \mathcal{H}^-(t_0) \otimes \theta_c,$$

where θ_c is the quadratic character of $\text{Gal}_{\mathbb{Q}}$ corresponding to the extension $\mathbb{Q}(\sqrt{c})/\mathbb{Q}$ (a similar notation will be used for its restriction to Gal_K). The specialized motive $\mathfrak{h}^{\pm}$ satisfies that its coefficient field (i.e. the field generated by Frobenius traces) is contained in K , the field of definition of the motive [GPV, Theorem 8.2]. Then for each prime ideal $\mathfrak{p}$ of K there exists a Galois representation

$$(13) \quad \rho_{\mathfrak{p}}^{\pm} : \text{Gal}_K \rightarrow \text{GL}_2(K_{\mathfrak{p}})$$

attached to the motive $\mathfrak{h}^{\pm}$. The following are some properties of $\mathfrak{h}^+$.

Theorem 2.1. *Let $\mathfrak{l}$ be a prime ideal of K . Then*

- (1) *The motive $\mathfrak{h}^+$ is unramified at $\mathfrak{l}$ if $\mathfrak{l} \nmid qrb$.*
- (2) *It has multiplicative reduction at primes $\mathfrak{l} \nmid qr$ dividing b .*
- (3) *The residual representation of $\rho_{\mathfrak{p}}^+$ is unramified at all primes not dividing pqr .*
- (4) *The residual representation of $\rho_{\mathfrak{p}}^+$ is finite at all primes dividing p .*
- (5) *If either $q \mid a$ and $r \mid c$ then $\mathfrak{h}^+$ is modular.*

Proof. See [GP24], Theorems 4.1, 4.2, 4.3 and 5.7 respectively. $\square$

Remark 2.2. As explained in [GP24, Remark 6], the modularity statement is true up to a Tate twist. For the parameters studied in the present article, $\mathfrak{h}^{\pm}(1)$ has Hodge-Tate weights matching a parallel weight 2 Hilbert modular form.

The motive $\mathfrak{h}^-$ satisfies analogous properties.

Theorem 2.3. *Let $\mathfrak{l}$ be a prime ideal of K . Then*

- (1) *The motive $\mathfrak{h}^-$ is unramified at $\mathfrak{l}$ if $\mathfrak{l} \nmid 2qrb$.*
- (2) *It has multiplicative reduction at primes $\mathfrak{l} \nmid 2qr$ dividing b .*
- (3) *The residual representation of $\rho_{\mathfrak{p}}^-$ is unramified at all primes not dividing $2pqr$.*
- (4) *The residual representation of $\rho_{\mathfrak{p}}^-$ is finite at all primes dividing p .*
- (5) *If either $r > 5$ and $q \mid a$ or $q > 5$ and $r \mid c$ then $\mathfrak{h}^-$ is modular.*

Proof. See [GP24], Theorems 4.1, 4.2, 4.3 and 5.8 respectively. $\square$

When $r = 3$ we can improve the previous modularity result.

Theorem 2.4. *Let $r = 3$. Then*

- (1) *If $q \geq 5$, the motive $\mathfrak{h}^+$ is modular for all non-zero specialization of the parameter t .*
- (2) *If $q \geq 11$, the motive $\mathfrak{h}^-$ is modular for all non-zero specialization of the parameter t .*

Proof. See [GP24], Theorems 6.2 and 6.3]. $\square$

Remark 2.5. Unfortunately the strategy used in [GP24] (suggested in [Dar00]) does not allow to prove modularity of all specializations of the motive $\mathfrak{h}^-$ when $q = 5$. The reason is that for some specializations of the parameter, the representation modulo 5 is reducible. Let us explain the situation with more detail. By [GPV, Theorem 10.3], the motive

$\mathcal{H}((\frac{1}{6}, -\frac{1}{6}), (\frac{1}{5}, -\frac{1}{5}) | t)$ is congruent modulo $(\sqrt{5})$ to the rational motive $\mathcal{H}((\frac{1}{6}, -\frac{1}{6}), (1, 1) | t)$. The latter corresponds to the rational elliptic curve (see [GP24, §2.1.4])

$$E_t : y^2 + xy = x^3 - \frac{t}{432}.$$

The curve is modular for all rational specializations t_0 of the parameter, so if its residual representation modulo 5 is irreducible, the motive $\mathcal{H}((\frac{1}{6}, -\frac{1}{6}), (\frac{s}{5}, -\frac{s}{5}) | t_0)$ is modular as well. The problem is that the residual representation of E_{t_0} modulo 5 is not always irreducible. For example, taking $t_0 = \frac{27}{16}$, E_{t_0} matches the elliptic curve with LMFDB label 176-b3, which has a rational 5-subgroup. Similarly, the value $t_0 = -\frac{11}{16}$ corresponds to the elliptic curve with label 11-a3, which has a 5-rational point.

2.4. Traces of Frobenius elements. Let N be a positive integer (in our case $N^+ = qr$ and $N^- = 2qr$). Let λ be a prime ideal of $F = \mathbb{Q}(\zeta_N)$ not dividing N with residue field $\mathbb{F}_{\ell^n}$. Following [Wei52], define the character

$$(14) \quad \chi_\lambda : (\mathbb{Z}[\zeta_N]/\lambda)^\times \rightarrow \mathbb{C}^\times$$

as follows: for x an integer prime to λ , let $\chi_\lambda(x)$ be the N -th root of unity congruent to $x^{(\ell^n-1)/N}$ modulo λ . If $\lambda \mid x$ set $\chi_\lambda(x) = 0$. By construction, the character χ_λ has order N .

Fix an additive character ψ on $\mathbb{F}_{\ell^n}$. For $\omega \in \widehat{\mathbb{F}_{\ell^n}^\times}$, a character of $\mathbb{F}_{\ell^n}^\times$, denote by $g(\psi, \omega)$ the Gauss sum

$$(15) \quad g(\psi, \omega) = \sum_{x \in \mathbb{F}_{\ell^n}^\times} \omega(x) \psi(x).$$

Definition 2.6 (Finite hypergeometric sum). For $a, b, c, d \in \mathbb{Z}[\frac{1}{N}]$ and $t_0 \in \mathbb{F}_{\ell^n}$, define the finite hypergeometric series

$$H_\lambda((a, b), (c, d) | t_0) = \frac{1}{1 - \ell^n} \sum_{\omega \in \widehat{\mathbb{F}_{\ell^n}^\times}} \frac{g(\psi, \chi_\lambda^{-aN} \omega) g(\psi, \chi_\lambda^{cN} \omega^{-1})}{g(\psi, \chi_\lambda^{-aN}) g(\psi, \chi_\lambda^{cN})} \frac{g(\psi, \chi_\lambda^{-bN} \omega) g(\psi, \chi_\lambda^{dN} \omega^{-1})}{g(\psi, \chi_\lambda^{-bN}) g(\psi, \chi_\lambda^{dN})} \omega(t_0).$$

It is not hard to verify that the definition does not depend on the choice of the additive character ψ . Set:

$$(16) \quad \begin{aligned} a^+ &= \frac{1}{r}, & b^+ &= -\frac{1}{r}, & c &= \frac{1}{q}, & d &= -\frac{1}{q}, & N^+ &= qr. \\ a^- &= \frac{1}{2r}, & b^- &= -\frac{1}{2r}, & N^- &= 2qr. \end{aligned}$$

For $\mathfrak{p}$ a prime ideal of F , denote also by $\rho_{\mathfrak{p}}^\pm$ the restriction of (13) to Gal_F .

Theorem 2.7. *Let t_0 be a rational number. Let $\mathfrak{p}$ be a prime ideal of F and let λ be a prime ideal of F not dividing $N(\mathfrak{p}) \cdot N^\pm$. Let Frob_λ be a Frobenius element of Gal_F . Then if $v_\lambda(t_0(t_0 - 1)) = 0$, the representation $\rho_{\mathfrak{p}}^\pm$ is unramified at λ and*

$$\text{Tr } \rho_{\mathfrak{p}}^\pm(\text{Frob}_\lambda) = H_\lambda((a^\pm, b^\pm), (c, d) | t_0).$$

Proof. See [GPV, Theorems 4.8 and 7.23]. □

3. CONDUCTOR AT POTENTIALLY WILD PRIMES

We begin by recalling some basic properties of the conductor of a Galois representation. Let K be a number field and let $\rho : \text{Gal}_K \rightarrow \text{GL}_2(\overline{\mathbb{Q}_p})$ be a Galois representation. Let $\mathfrak{q}$ be a prime ideal of K whose residual characteristic is prime to p and let $\mathfrak{n}_{\mathfrak{q}}$ denote the $\mathfrak{q}$ -th valuation of Artin's conductor of ρ . Then

$$\mathfrak{n}_{\mathfrak{q}} = \mathfrak{n}_{\mathfrak{q}}^{\text{tame}} + \mathfrak{n}_{\mathfrak{q}}^{\text{wild}},$$

where $\mathfrak{n}_{\mathfrak{q}}^{\text{tame}}$, called the *tame* conductor, is the codimension of the subspace fixed by the inertia group $I_{\mathfrak{q}}$, and $\mathfrak{n}_{\mathfrak{q}}^{\text{wild}}$, the *wild* or *Swan* conductor, is the sum of the codimensions of the spaces fixed by the higher ramification groups (see for example Chapter VI, §2 of [Ser79]). It is easy to verify that if $\mathfrak{n}_{\mathfrak{q}}^{\text{wild}} \neq 0$ then $\mathfrak{n}_{\mathfrak{q}}^{\text{tame}} \neq 0$.

The *potentially wild primes* for the motive $\mathfrak{h}^{\pm}$ (i.e. the prime ideals $\mathfrak{l}$ of K for which $\mathfrak{n}_{\mathfrak{l}}^{\text{wild}}$ might be non-zero) are those prime ideals of $K = \mathbb{Q}(\zeta_q)^+ \cdot \mathbb{Q}(\zeta_r)^+$ dividing $N^{\pm}$, i.e. for $\mathfrak{h}^+$ is the set of primes dividing qr while for $\mathfrak{h}^-$ is the set of primes dividing $2qr$ (see (16)).

The goal of the present section is to compute the conductor of the motives $\mathfrak{h}^{\pm}$ at potentially wild primes. In doing so, we follow the strategy used in [GP24] and [CV25a, Section 6.4], namely relate our motive to a hyperelliptic curve (as defined by Darmon in [Dar00] and whose definition we recall) and use the theory of clusters (as developed in [DDMM23]) to compute its local conductor.

Let $h(x)$ denote the minimal polynomial of $\zeta_r + \zeta_r^{-1}$ and consider the odd function

$$(17) \quad f(x) = (-1)^{\frac{r-1}{2}} x h(2 - x^2).$$

For t a variable, consider the following two curves defined in [Dar00, Equations (3) and (4)].

$$(18) \quad C_r^-(t) : y^2 = f(x) + 2 - 4t, \quad C_r^+(t) : y^2 = (x + 2)(f(x) + 2 - 4t).$$

Proposition 3.1. *Let $t_0 \in \mathbb{Q} \setminus \{0, 1\}$. Then*

- *The hypergeometric motive $\mathcal{H}((\frac{1}{r}, -\frac{1}{r}), (1, 1)|t_0)$ is isomorphic to a rank 2 motive of the curve $C_r^+(t_0)$.*
- *The hypergeometric motive $\mathcal{H}((\frac{1}{2r}, -\frac{1}{2r}), (1, 1)|t_0)$ is isomorphic to the quadratic twist by θ_{-1} of a rank 2 motive of the curve $C_r^-(t_0)$.*

Proof. See Corollaries A.5 and A.11 of [GP24]. □

For later purposes, let us recall the following congruences between hypergeometric motives.

Proposition 3.2. *Let $\mathfrak{q}$ be a prime ideal of $F = \mathbb{Q}(\zeta_{N^{\pm}})$ dividing q . Let $t_0 \in \mathbb{Q} \setminus \{0, 1\}$. Then the following congruences hold.*

- *The motive $\mathcal{H}^+(t_0) \equiv \mathcal{H}((\frac{1}{r}, -\frac{1}{r}), (1, 1)|t_0) \pmod{\mathfrak{q}}$.*
- *The motive $\mathcal{H}^-(t_0) \equiv \mathcal{H}((\frac{1}{2r}, -\frac{1}{2r}), (1, 1)|t_0) \pmod{\mathfrak{q}}$.*

Proof. See [GPV, Theorem 10.3]. □

Remark 3.3. The previous congruences only hold over the field F . Since the extension F/K is biquadratic, we may deduce that, over K , a similar congruence holds up to a quadratic twist. Furthermore, the twist is independent of the parameter t (see [Pac25, Corollary 2.2]), so it can be easily computed for each signature (see Lemma 7.3).

3.1. Local inertial type. The last two propositions suggest that we should start studying the reduction type of Darmon's curves $C_r^\pm(t)$ at primes dividing r and its “local inertial type”. Let $K/\mathbb{Q}_\ell$ be a local field, and let $\rho : \text{Gal}_K \rightarrow \text{GL}_2(\overline{\mathbb{Q}_p})$ be a Galois representation, with $p \neq \ell$.

Definition 3.4. The *local inertial type* (or *local type* for short) of ρ is the isomorphism class of the restriction of ρ to the inertia subgroup.

To characterize local types, it is customary to study representations of the Weil-Deligne group. Instead of recalling the definition of the Weil-Deligne group (see §4.1 of [Tat79]) we recall the definition of its representations. A Weil-Deligne representation consists of a 2-dimensional complex representation ρ of the Weil group W_K together with a monodromy operator $N \in M_2(\mathbb{C})$. Let $\omega_1 : W_{\mathbb{Q}_\ell} \rightarrow \mathbb{C}^\times$ be the unramified character sending Frobenius to $\|\ell\|_\ell$. For $\ell \neq 2$, there are three different local types for a Weil-Deligne representation ρ with trivial Nebentypus:

- (1) **Principal Series:** $N = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\rho = \chi \oplus \chi^{-1}\omega_1^{1-k}$ for some quasi-character $\chi : W(\mathbb{Q}_p)^{\text{ab}} \rightarrow \mathbb{C}^\times$.
- (2) **Special:** $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and the representation ρ equals $\omega_1^r \begin{pmatrix} \chi\omega_1 & 0 \\ 0 & \chi \end{pmatrix}$ for some quasi-character $\chi : W_K \rightarrow \mathbb{C}^\times$.
- (3) **Supercuspidal:** $N = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\rho = \text{Ind}_{W_E}^{W_K} \varkappa$, where E is a quadratic extension of K , and $\varkappa : W_E^{\text{ab}} \rightarrow \mathbb{C}^\times$ is a quasi-character which does not factor through the norm map with a quasi-character of W_K^{ab} .

In many instances, the local type is preserved under a congruence between two Galois representations (see e.g. [GPV23, Proposition 1.1]).

3.2. Conductor at primes dividing r . For the reader's convenience, we start with the case $r = 3$, since in this case the curves have genus 1 (the local theory of elliptic curves is better understood).

3.2.1. The case $r = 3$. The curve C_3^+ and the quadratic twist by -1 of the curve C_3^- are elliptic curves given by the equations

$$(19) \quad E_3^+(t) : y^2 + 3xy + ty = x^3, \quad E_3^-(t) : y^2 = x^3 - 3x + 4t - 2.$$

The discriminant and j -invariant of each curve are

$$\begin{aligned} \Delta(E_3^+(t)) &= -3^3 t^3 (t-1), & j(E_3^+(t)) &= \frac{3^3 (8t-9)^3}{t^3 (t-1)}, \\ \Delta(E_3^-(t)) &= -2^8 3^3 t (t-1), & j(E_3^-(t)) &= -\frac{2^4 3^3}{t(t-1)}. \end{aligned}$$

Following standard notations, when the curve has potential good reduction, we denote by e the degree of the minimal extension of $\mathbb{Q}_p^{\text{ur}}$ (the maximal unramified extension of $\mathbb{Q}_p$) where it acquires good reduction.

Proposition 3.5. *Let $t_0 \in \mathbb{Q} \setminus \{0, 1\}$. Then Table 3.1 contains the conductor exponent and the local type of the elliptic curve $E_3^\pm(t_0)$ at the prime $p = 3$. When $v_3(t_0) < 0$, we assume that $3 \mid v_3(t_0)$ and we use the notation $t_0 = 3^{v_3(t_0)} t'_0$.*

Condition	$v_3(\text{cond}(E_3^+(t_0)))$	$v_3(\text{cond}(E_3^-(t_0)))$	e	Type
$v_3(t_0) > 3$	1	2	–	special
$v_3(t_0 - 1) > 3$	2	2	–	special
$t_0 \equiv 5 \pmod{9}$	2	2	4	supercuspidal
$t_0 \equiv 2, 8 \pmod{9}$	3	3	12	supercuspidal
$t'_0 \equiv \pm 2 \pmod{9}$	2	2	4	supercuspidal
$t'_0 \not\equiv \pm 2 \pmod{9}$	3	3	12	supercuspidal

TABLE 3.1. Local information of $E_3^\pm(t_0)$ at 3.

Proof. In the first two cases, the j -invariant of the curve has negative valuation at 3, so we just need to determine whether the curve has additive or multiplicative reduction at 3. For $E_3^-(t_0)$ it is additive and for $E_3^+(t_0)$ is multiplicative in the first case and additive in the second one. To compute the third entry, Tate's algorithm shows that the curve has Kodaira type III at 3, so $e = 4$ by [Kra90, Théorème 1]. The local type can be computed using [DFV22, Table 1]. The Kodaira type for the fourth row is II, hence by [Kra90, Théorème 1] $e = 12$. The last two statements follow from a similar argument. $\square$

Corollary 3.6. *Let $q > 3$ be a prime number. Let $K = \mathbb{Q}(\zeta_q)^+$ and let $\mathfrak{r}$ be a prime ideal of K dividing 3. Let (a, b, c) be a non-trivial solution to (1) with $p > 2$. Set $t_0 = -\frac{a^q}{c^3}$. Then the possible values for the conductor exponent of $\mathfrak{h}^\pm$ at $\mathfrak{r}$ are given in Table 3.1 (when $3 \mid c$, we use the notation $c = 3^{v_3(c)}c_0$).*

$v_{\mathfrak{r}}(\text{cond}(\mathfrak{h}^\pm))$	Condition	$v_{\mathfrak{r}}(\text{cond}(\mathfrak{h}^\pm))$	Condition
1, 2	$3 \mid a$	0, 1, 2	$3 \mid b$
2	$4c^3 \equiv a^q \pmod{9}$	2	$3 \mid c, \pm 2c_0^3 \equiv a^q \pmod{9}$
3	$c^3 \equiv a^q \pmod{9}$	3	$3 \mid c, \pm c_0^3 \equiv a^q \pmod{9}$
3	$7c^3 \equiv a^q \pmod{9}$	3	$3 \mid c, \pm 4c_0^3 \equiv a^q \pmod{9}$

TABLE 3.2. Conductor exponent of $\mathfrak{h}^\pm$ at 3.

Proof. The field $K = \mathbb{Q}(\zeta_q)^+$ is the field of definition of the motive $\mathfrak{h}^\pm$, while $E_3^\pm(t_0)$ is defined over $\mathbb{Q}$. Let $\rho_{E_3^\pm, q} : \text{Gal}_{\mathbb{Q}} \rightarrow \text{GL}_2(\mathbb{Z}_q)$ be the q -adic Galois representations attached to $E_3^\pm := E_3^\pm(t_0)$.

Start considering the values t_0 where the local type of $E_3^\pm$ is supercuspidal at 3 (as detailed in Table 3.1). Since $q > 3$, the image of a decomposition group D_3 for the residual representation $\bar{\rho}_{E_3^\pm, q}$ is also dihedral, so its residual tame conductor equals 2. Since $K/\mathbb{Q}$ is unramified at 3, the same is true for the residual representation $\bar{\rho}_{E_3^\pm, q}|_{\text{Gal}_K}$.

Let $\mathfrak{q}$ be a prime in K above q . By Propositions 3.1 and 3.2, and Remark 3.3, the $\mathfrak{q}$ -adic residual representation of $\mathfrak{h}^\pm$ is isomorphic to $\bar{\rho}_{E_3^\pm, q}|_{\text{Gal}_K}$, up to a quadratic twist. Hence, its tame conductor is also 2. The result then follows from the fact that the Swan conductor is preserved under congruences (see e.g. [Wie08, Proposition 3.1.42]).

When $3 \nmid ab$ the curve $E_3^\pm$ has (potentially) multiplicative reduction at 3, hence its Swan conductor is 0, so the conductor exponent of $\mathfrak{h}^\pm$ is at most 2. Recall that $t_0 = -\frac{a^q}{c^3}$, so if $3 \nmid a$,

$$v_3(\Delta(E_3^\pm)) \equiv 3 \not\equiv 0 \pmod{q},$$

so the residual representation $\bar{\rho}_{E_3^\pm, q}$ is ramified at 3 (by the theory of Tate's curves), giving a lower bound of 1 to the tame conductor of $\mathfrak{h}^\pm$. $\square$

Remark 3.7. A little more can be said when the congruences of Proposition 3.2 hold over K (i.e. no twist is needed, see Remark 3.3). If $E_3^\pm(t_0)$ is special with conductor valuation 2, its residual conductor also has valuation 2 (since the local type is special with the character χ quadratic and ramified) hence in this case $v_{\mathfrak{r}}(\text{cond}(\mathfrak{h}^\pm)) = 2$.

Remark 3.8. The isomorphism of residual representations between $E_3^\pm := E_3^\pm(t_0)$ and $\mathfrak{h}^\pm$ allows us to compute the conductor at $\mathfrak{r}$ of $\mathfrak{h}^\pm$ from that of the elliptic curve $E_3^\pm$, as done in the Corollary 3.6. However, this is not enough to compute its local type, since even when the field extension $K/\mathbb{Q}$ is unramified at 3, the local type is not necessarily preserved under restriction. Looking at Table 3.1, we get:

- If the local type is special, it is indeed preserved under base extension.
- When the local type is supercuspidal and $e = 12$, the local representation at 3 is induced from a quadratic ramified extension of $\mathbb{Q}_3$ (see e.g. [Pac13, Corollary 3.1]), hence the local type is indeed preserved.
- When the local type is supercuspidal and $e = 4$, the local type corresponds to the induction of an order 4 character from $\mathbb{Q}_9$ (the unramified quadratic extension of $\mathbb{Q}_3$) [DFV22, Proposition 4.2.1]. If $K_{\mathfrak{r}}$ contains $\mathbb{Q}_9$ then the local type of the restriction is a ramified principal series; otherwise, it is a supercuspidal representation. Note that $K_{\mathfrak{r}}/\mathbb{Q}_3$ is unramified of order equal to $\text{ord}_q(3^2)$, hence contains $\mathbb{Q}_9$ precisely when $\text{ord}_q(3)$ is divisible by 4.

Even when the local type of $\mathfrak{h}^\pm$ at $\mathfrak{r}$ is not determined by the congruence, it is still the case that when $3 \nmid ab$ it has potentially good reduction at $\mathfrak{r}$.

3.2.2. The case $r \geq 5$. Let (a, b, c) be a non-trivial solution to (1) and set $t_0 = -\frac{a^q}{c^r}$. Let as before $K = \mathbb{Q}(\zeta_q)^+$ and let $\mathfrak{r}$ be a prime ideal of K dividing r . We need to understand the semistable reduction of the curves $C_r^\pm(t_0)$ in (18) over $K_{\mathfrak{r}}$. To get an integral equation, define

$$(20) \quad F(x) = c^r f\left(\frac{x}{c}\right) + 2c^r + 4a^q,$$

for $f(x)$ as in (17), and consider the two hyperelliptic curves

$$\begin{aligned} C_r^-(a, b, c) : y^2 &= F(x), \\ C_r^+(a, b, c) : y^2 &= (x + 2c)F(x). \end{aligned}$$

Lemma 3.9. *The following relations hold:*

- (1) $C_r^-(a, b, c)$ is isomorphic to $C_r^-(t_0)$ over $\mathbb{Q}(\sqrt{c})$.
- (2) $C_r^+(a, b, c)$ is isomorphic to $C_r^+(t_0)$ over $\mathbb{Q}$.

Proof. The first statement follows from the change of variables $(x, y) \rightarrow (x/c, y/\sqrt{c^r})$, while the second one by $(x, y) \rightarrow (x/c, y/c^{\frac{r+1}{2}})$. $\square$

Let $J_r^\pm(t_0)$ be the Jacobians of $C_r^\pm(t_0)$ over K . By [TTV91], the abelian variety $J_r^\pm(t_0)$ is of GL_2 -type over K . Therefore, it has an associated compatible system of 2-dimensional Galois representations $\{\rho_{J_r^\pm(t_0), \mathfrak{l}} : \text{Gal}_K \rightarrow \text{GL}_2(K_{\mathfrak{l}})\}$ indexed by prime ideals $\mathfrak{l}$ of K . By [BCDF25, Theorem 3.13], its Nebentypus is trivial (i.e. the representation $\rho_{J_r^\pm(t_0), \mathfrak{l}}$ has determinant the

cyclotomic character restricted to Gal_K). Let $N_r^\pm(t_0)$ denote the conductor of the family. Then (see e.g. [ACIK⁺24, Proposition A.12]),

$$(21) \quad \left(\frac{r-1}{2}\right) v_{\mathfrak{r}}(N_r^\pm(t_0)) = v_{\mathfrak{r}}(\text{cond}(J_r^\pm(t_0))).$$

Keeping the previous notation, if $J_r^\pm(t_0)$ has potential good reduction, we denote by e the degree of the minimal extension of $\mathbb{Q}_r^{\text{ur}}$ where it acquires good reduction.

Proposition 3.10. *Keeping the previous notation, assume that $p, q > 3$. Then Table 3.3 contains the valuation of $N_r^\pm(t_0)$ at the prime ideal $\mathfrak{r}$ together with its local type.*

$F(x)$ reducible in $\mathbb{Q}_r$	$r \bmod 4$	$r \mid a$	$r \mid b$	$v_{\mathfrak{r}}(N_r^-(t_0))$	$v_{\mathfrak{r}}(N_r^+(t_0))$	e	Type
–	–	✓	×	2	1	–	special
–	–	×	✓	2	2	–	special
✓	1	×	×	2	2	4	principal series
✓	3	×	×	2	2	4	supercuspidal
×	–	×	×	3	3	$4r$	supercuspidal

TABLE 3.3. Local information of $J_r^\pm(t_0)$ at $\mathfrak{r}$.

Proof. Let $\mathcal{J}_r^\pm(a, b, c)$ denote the Jacobian of $C_r^\pm(a, b, c)$ respectively. By Lemma 3.9 the abelian variety $\mathcal{J}_r^+(a, b, c)$ is isomorphic to $J_r^+(t_0)$, while $\mathcal{J}_r^-(a, b, c)$ is isomorphic to $J^-(t_0) \otimes \theta_c$, where θ_c is the quadratic character of the extension $\mathbb{Q}(\sqrt{c})/\mathbb{Q}$ (and by abuse of notation its restriction to Gal_K). We claim that it is enough to prove the stated results for the varieties $\mathcal{J}_r^\pm(a, b, c)$. In the $+$ case, the claim is clear, while for the $-$ curve, note that the character θ_c is ramified at a prime ideal $\mathfrak{p}$ if and only if $\mathfrak{p} \mid c$, in which case $\mathfrak{p} \nmid ab$ (by the primitive assumption). But the values of the last four rows of the table are invariant under quadratic twist.

(1) The curve $C_r^-(a, b, c)$: it is a particular instance of the family of hyperelliptic curves $C(z, s)$ studied in [CV25a] (see equation (3) in loc. cit.) specialized at $z = c^2$ and $s = 2c^r + 4a^q$. Its cluster picture (over $K_{\mathfrak{r}}$) is given in [CV25a, Corollary 3.5]. In the notation of [CV25a],

$$(22) \quad \Delta = s^2 - 4z^r = 16a^qb^p,$$

so $r \mid \Delta$ if and only if $r \mid ab$, and in that case $v_{\mathfrak{r}}(\Delta) \geq 3$ (by the assumption $p, q > 2$).

If $r \mid ab$, $\mathcal{J}_r^-(a, b, c)$ has potential multiplicative reduction at $\mathfrak{r}$, by [DDMM23, Theorem 1.9]. The stated conductor exponent formula follows from [CV25a, Theorem 5.4]. Moreover, $C_r^-(a, b, c)/K_{\mathfrak{r}}$ is a quadratic ramified twist of a semistable curve, by [CV25a, Corollary 5.5].

Assume then that $r \nmid ab$. By [DDMM23, Theorem 1.9], $\mathcal{J}_r^-(a, b, c)$ has potentially good reduction at $\mathfrak{r}$. Its conductor exponent is computed in [CV25a, Theorem 5.4], so it remains to compute the last two columns of the table. Let K' be the splitting field of $F(x)$ (described in [CV25a, Proposition 4.5]). By [CV25a, Theorem 4.11], the ramification index of $K'_{\mathfrak{r}}/K_{\mathfrak{r}}$ equals 1 if $F(x)$ is reducible over $\mathbb{Q}_r$ and r otherwise.

Let $M/K'_{\mathfrak{r}}$ be any finite extension with ramification index equal to 4 (so the ramification index of $M/K_{\mathfrak{r}}$ coincides with the values of e given in Table 3.3). Then [DDMM23, Theorem 1.9] implies that $\mathcal{J}_r^-(a, b, c)$ has good reduction over M , and that there is no extension of

$K_{\mathfrak{r}}$ with ramification index $\leq e$ where $\mathcal{J}_r^-(a, b, c)$ obtains good reduction. This information is enough to compute the local type. If $F(x)$ is reducible, $e = 4$, so the local type is either principal series or supercuspidal (induced by a character of order 4 of the unramified quadratic extension of $K_{\mathfrak{r}}$). The first case occurs precisely when there is a totally ramified degree 4 abelian extension of $K_{\mathfrak{r}}$, which by local class field theory, is equivalent to the condition $4 \mid \#\mathbb{F}_{\mathfrak{r}}^\times = r - 1$. If $F(x)$ is irreducible, the conductor exponent is 3, so the local type is that of a supercuspidal representation induced from a ramified quadratic extension of $K_{\mathfrak{r}}$ (see [Pac13, Corollary 3.1]).

(2) The curve $C_r^+(a, b, c)$: by [ACIK⁺24, Remark 2.10], the Swan part of $C_r^+(a, b, c)$ equals that of $C_r^-(a, b, c)$ (this is a particular instance of congruences of hypergeometric motives, since by [GPV, Theorem 10.3] the motives $\mathcal{H}((\frac{1}{r}, -\frac{1}{r}), (1, 1)|t_0)$ and $\mathcal{H}((\frac{2+r}{2r}, -\frac{2+r}{2r}), (1, 1)|t_0)$ are congruent modulo 2, and the latter is a Galois conjugate of the hypergeometric motive $\mathcal{H}((\frac{1}{2r}, -\frac{1}{2r}), (1, 1)|t_0)$). Then, by [CV25a, Equation (39)] and (21), the Swan component equals

$$\mathfrak{n}_{\mathfrak{r}}^{\text{wild}}(\rho_{\mathcal{J}_r^+, \mathfrak{l}}) = \begin{cases} 1 & \text{if } r \nmid ab \text{ and } F(x) \text{ is irreducible,} \\ 0 & \text{otherwise,} \end{cases}$$

where $\{\rho_{\mathcal{J}_r^+, \mathfrak{l}}\}$ is the compatible system of 2-dimensional Galois representations attached to $\mathcal{J}_r^+(a, b, c)$. To compute the tame conductor, note that the curve $C_r^+(a, b, c)$ is a particular instance of the family of hyperelliptic curves $C_r^+(s)$ defined in [Azo25, Definition 3.2], whose cluster picture is given in [Azo25, Theorem 4.21]. Then [DDMM23, Theorem 1.9] implies that $\mathcal{J}_r^+(a, b, c)$ has multiplicative reduction at $\mathfrak{r}$ if $r \mid a$ and additive reduction otherwise, finishing the conductor computation.

The proof of the stated results on the last two columns of the table for $\mathcal{J}_r^+(a, b, c)$ follows from similar arguments as the ones used in the previous case. $\square$

Let $\mathfrak{r}$ be a prime ideal of $K = \mathbb{Q}(\zeta_q)^+ \cdot \mathbb{Q}(\zeta_r)^+$ dividing r and let $v_{\mathfrak{r}}(\mathfrak{h}^-)$ denote the $\mathfrak{r}$ -th valuation of the motive's conductor.

Corollary 3.11. *Assume that $p, q > 2$. Then, the values of the conductor exponent of $\mathfrak{h}^\pm$ at $\mathfrak{r}$ are given in Table 3.4. Moreover, if $q \nmid \frac{r-1}{2}$, the case 0 can be rule out in the first row.*

$F(x)$ reducible in $\mathbb{Q}_r$	$r \mid a$	$r \mid b$	$v_{\mathfrak{r}}(\mathfrak{h}^-)$	$v_{\mathfrak{r}}(\mathfrak{h}^+)$
–	✓	×	0, 1, 2	0, 1, 2
–	×	✓	0, 1, 2	0, 1, 2
✓	×	×	2	2
×	×	×	3	3

TABLE 3.4. Conductor exponent values of $\mathfrak{h}^\pm$ at $\mathfrak{r}$.

Proof. Recall that $J_r^\pm(t_0)$ is a rational variety that become of GL_2 -type over K , and that $\mathfrak{h}^\pm$ is defined over K . Note that by Propositions 3.1 and 3.2 we have a congruence over $F = \mathbb{Q}(\zeta_{N^\pm})$ between $\mathfrak{h}^\pm$ and $J_r^\pm(t_0)$. Moreover, by Remark 3.3, they are congruent over K up to a quadratic twist. Hence, the proof mimics that of Corollary 3.6, replacing the results of Proposition 3.5 by those of Proposition 3.10.

To prove the last statement, it is sufficient to prove that the residual representation $\bar{\rho}_{\mathcal{J}_r^\pm, \mathfrak{q}}$ is ramified at $\mathfrak{r}$ if $q \nmid \frac{r-1}{2}$ and $r \mid a$, where $\mathcal{J}_r^\pm = \text{Jac}(C_r^\pm(a, b, c))$. This follows from the theory of Mumford curves (see e.g. [BCDF25, Theorem 7.6]), since

$$v_{\mathfrak{r}}(\Delta(C^\pm(a, b, c))) \equiv rv_{\mathfrak{r}}(r) \equiv \frac{r(r-1)}{2} \not\equiv 0 \pmod{q},$$

where the first congruence follows from [CV25a, Lemma 3.1] in the $-$ case, and from [Azo25, Corollary 4.12] in general. $\square$

Remark 3.12. As in Remark 3.7, if Proposition 3.2 holds over K (or up to a quadratic unramified twist), then the exponent equals 2 for the first two rows (except the case $r \mid b$ for the motive $\mathfrak{h}^+$) because a special local type of conductor valuation 2 and trivial Nebentypus has residual representation of conductor valuation 2.

3.2.3. Irreducibility of $F(x)$. Let $F(x)$ be as in (20). According to Corollaries 3.6 and 3.11 the conductor exponent of $J_r^\pm(t_0)$ sometime depends on the irreducibility of $F(x)$. The following result, inspired by [CK22, Proposition 7.23], provides a criterion in this direction.

Lemma 3.13. *For (a, b, c) a solution to (1), let $d(a, c)$ be the constant term of $g(x) = F(x - 2c^r - 4a^q)$. Suppose $r \nmid ab$. Then, $v_r(d(a, c)) \geq 1$ and*

- (1) *If $v_r(d(a, c)) \geq 2$, $F(x)$ is reducible over $\mathbb{Q}_r$.*
- (2) *If $v_r(d(a, c)) = 1$, $F(x)$ is irreducible over $\mathbb{Q}_r$.*

Furthermore, the case depends only on the congruence of (a, b, c) modulo r^2 .

Proof. By [BCDF25, Lemma 2.7],

$$F(x) = \sum_{k=0}^{\frac{r-1}{2}} (-1)^k c_k c^{2k} x^{r-2k} + 2c^r + 4a^q, \quad \text{where } c_k = \frac{r}{r-k} \binom{r-k}{k}.$$

Since $r \mid c_k$ for all $0 < k$, Fermat's Little Theorem implies that $r \mid d(a, c)$. Clearly $F(x)$ is irreducible if and only if $g(x)$ is irreducible. When $v_r(d(a, c)) = 1$ the polynomial $g(x)$ is irreducible over $\mathbb{Q}_r$ by Eisenstein's criterion.

Let us assume that $v_r(d(a, c)) = 2$, and consider the previous curve

$$C := C_r^-(a, b, c) : y^2 = F(x).$$

By [CV25a, Lemma 3.1], its discriminant equals $\Delta_C = (-1)^{\frac{r-1}{2}} 2^{2(r-1)} r^r \Delta^{\frac{r-1}{2}}$, where $\Delta = 16a^q b^p$ by (22).

Let $\mathfrak{r}$ be a prime in K above r , and $M/K_{\mathfrak{r}}$ be an extension with ramification index equal to 4. We claim that C/M has good reduction. Then Table 3.3 implies that $F(x)$ is reducible in $\mathbb{Q}_r$.

Let π be a uniformizer in M and v its normalized valuation so that $v(\pi) = 1$. Then $v(r) = 2(r-1)$. Since $r \nmid ab$, we have $v(\Delta_C) = v(r^r) = 2r(r-1)$. The change of variable $x \rightarrow x - 2c^r - 4a^q$ leads to the model

$$y^2 = g(x) = x^r + \sum_{j=1}^{r-1} b_j x^j + d(a, c),$$

where $r \mid b_j$ for all $1 \leq j \leq r-1$. Now, applying the change of variables $x \rightarrow \pi^2 x$, $y \rightarrow \pi^r y$ we obtain the model

$$C' : y^2 = x^r + \sum_{j=1}^{r-1} \frac{b_j}{\pi^{2(r-j)}} x^j + \frac{d(a, c)}{\pi^{2r}}.$$

The assumption $v_r(d(a, c)) \geq 2$ implies that C' is integral. By [Loc94, §1], $\Delta_{C'} = \Delta_C / \pi^{2r(r-1)}$, which is a unit so C' is a model with good reduction over M .

It follows from the proof that the result depends only on the congruence of the parameters (a, b, c) modulo r^2 . $\square$

3.3. Conductor at primes dividing q . This case can be deduced from the previous one using the following two properties.

- For $a, b, c, d \in \mathbb{Q}$,

$$(23) \quad \mathcal{H}((a, b), (c, d)|t) \simeq \mathcal{H}((-c, -d), (-a, -b)|t^{-1}),$$

as proven in [GPV, Proposition 7.28].

- For $a, b \in \mathbb{Q}$, the motive $\mathcal{H}((a + \frac{1}{2}, -a + \frac{1}{2}), (b + \frac{1}{2}, -b + \frac{1}{2})|t)$ is isomorphic to the quadratic twist by $\sqrt{t}$ of the motive $\mathcal{H}((a, -a), (b, -b)|t)$ (see [GP24, Proposition 2.11]).

Equation (23) implies that

$$\mathfrak{h}^+ \simeq \mathcal{H}\left(\left(\frac{1}{q}, -\frac{1}{q}\right), \left(\frac{1}{r}, -\frac{1}{r}\right) \middle| -\frac{c^r}{a^q}\right).$$

Similarly,

$$\begin{aligned} \mathfrak{h}^- &\simeq \mathcal{H}\left(\left(\frac{1}{q}, -\frac{1}{q}\right), \left(\frac{1}{2r}, -\frac{1}{2r}\right) \middle| -\frac{c^r}{a^q}\right) \\ &\simeq \mathcal{H}\left(\left(\frac{2+q}{2q}, -\frac{(2+q)}{2q}\right), \left(\frac{(r+1)}{2r}, -\frac{(r+1)}{2r}\right) \middle| -\frac{c^r}{a^q}\right) \otimes \theta_{-ac}. \end{aligned}$$

The latter is Galois conjugate to

$$\mathcal{H}\left(\left(\frac{1}{2q}, -\frac{1}{2q}\right), \left(\frac{1}{r}, -\frac{1}{r}\right) \middle| -\frac{c^r}{a^q}\right) \otimes \theta_{-ac}.$$

Therefore, the conductor exponent of $\mathcal{H}^\pm(-a^q/c^r)$ at $\mathfrak{q}$ can be obtained from Corollaries 3.6 and 3.11 by exchanging $a \leftrightarrow c$ and $q \leftrightarrow r$ (noting that the quadratic twist does not affect the conductor exponents), proving the following result.

Proposition 3.14. *Assume that $p, r > 2$. Let $\mathfrak{q}$ be a prime ideal of $K = \mathbb{Q}(\zeta_q)^+ \cdot \mathbb{Q}(\zeta_r)^+$ dividing q . Then, the values of the conductor exponent of $\mathfrak{h}^\pm$ at $\mathfrak{q}$ are given in Table 3.5. Moreover, if $r \nmid \frac{q-1}{2}$, the case 0 can be ruled out in the first row.*

$F(x)$ reducible in $\mathbb{Q}_q$	$q \mid c$	$q \mid b$	$v_q(\mathfrak{h}^-)$	$v_q(\mathfrak{h}^+)$
–	✓	×	0, 1, 2	0, 1, 2
–	×	✓	0, 1, 2	0, 1, 2
✓	×	×	2	2
×	×	×	3	3

TABLE 3.5. Conductor exponent values of $\mathfrak{h}^\pm$ at $\mathfrak{q}$.

3.4. Conductor at primes dividing 2. Let $\mathfrak{q}_2$ be a prime in $K = \mathbb{Q}(\zeta_q)^+ \cdot \mathbb{Q}(\zeta_r)^+$ above 2. The conductor at $\mathfrak{q}_2$ of $\mathfrak{h}^\pm$ is computed in [GP24]. The prime $\mathfrak{q}_2$ is a tame prime for $\mathfrak{h}^+$, so its conductor and local type information follows from Theorem 2.9 in loc. cit. The conductor valuation of $\mathfrak{h}^-$ is the following.

Proposition 3.15. *Let (a, b, c) a non-trivial solution to (1), where $q \geq 5$ and $r \geq 3$. The conductor exponent of $\mathfrak{h}^-$ at $\mathfrak{q}_2$ is one of the following*

$$\begin{cases} 0, 1, 2 & \text{if } 2 \mid a, \\ 5 & \text{if } 2 \mid b \text{ and } a^q \equiv 3c^r \pmod{4}, \\ 6 & \text{if } 2 \mid c \text{ and } 2 \nmid v_2(c), \\ 4 & \text{if } 2 \mid c, 2 \mid v_2(c) \text{ and } a \equiv 3 \pmod{4}, \\ 0, 1, 2 & \text{if } 2 \mid c, 2 \mid v_2(c) \text{ and } a \equiv 1 \pmod{4}. \end{cases}$$

Proof. See [GP24, Theorem 7.2] for $t_0 = -a^q/c^r$. □

4. MOTIVES COMING FROM TRIVIAL SOLUTIONS

Recall that a solution to (1) is called trivial if one of its coordinates is zero. There are six different trivial primitive solutions, namely

$$(0, \pm 1, \mp 1), (\pm 1, 0, \mp 1), (\pm 1, \mp 1, 0).$$

They correspond to the specialization $t_0 = 0, 1, \infty$, respectively. Given $a, b, c, d \in \mathbb{Q}$, the specialization of the motive $\mathcal{H}((a, b), (c, d)|t)$ at any of these three points yields a “singular motive”. If the monodromy matrix at the point t_0 has infinite order, then the singular motive does not correspond to any automorphic form (as occurs in Fermat’s Last Theorem, where a trivial solution corresponds to a singular elliptic curve with “multiplicative reduction”). However, when the monodromy matrix has finite order, it does correspond to a motive with complex multiplication, as we now prove. First, let us recall the following necessary definition.

Definition 4.1. The parameters $(a, b), (c, d)$ are called *generic* if no element of the set $\{a - c, a - d, b - c, b - d\}$ is an integer.

Theorem 4.2. *Let $(a, b), (c, d)$ be generic parameters such that $a + b$ and $c + d$ are integers, but $c - d$ is not. Let N be their least common denominator. Then $\mathcal{H}((a, b), (c, d)|0)$ is a motive with complex multiplication unramified outside primes dividing N .*

If λ is a prime ideal of F not dividing N , then the trace of Frob_λ acting on $\mathcal{H}((a, b), (c, d)|0)$ equals

$$(24) \quad -\chi_\lambda(-1)^{(d-b)(N(\lambda)-1)} N(\lambda)^\varepsilon \left(J(\chi_\lambda^{(d-b)N}, \chi_\lambda^{(b-c)N}) + \chi_\lambda(-1)^{(b-c)N} J(\chi_\lambda^{(d-c)N}, \chi_\lambda^{(a-d)N}) \right),$$

where

$$\varepsilon = \begin{cases} -1 & \text{if } a, c \notin \mathbb{Z}, \\ 0 & \text{otherwise,} \end{cases}$$

χ_λ is as in (14) and $J(\cdot, \cdot)$ denotes the usual Jacobi sum.

Proof. The proof mimics that of [GPV, Theorem A.1]. Let $A = (d - b)N$, $B = (b - c)N$, $C = (a - d)N$ and $D = dN$. Euler's curve is defined by the equation

$$\mathcal{C} : y^N = x^A(1 - x)^B(1 - tx)^C t^D.$$

Write $t = z^N$, so the curve can be written in the form

$$(25) \quad \mathcal{C} : y^N = x^A(1 - x)^B(1 - z^N x)^C.$$

The model reduces modulo z to the curve

$$\mathcal{C}_1 : y^N = x^A(1 - x)^B.$$

Since the parameters are generic, $N \nmid A$ and $N \nmid B$. The hypothesis on $c - d$ implies that $N \nmid A + B = N(d - c)$, hence $\dim(\text{Jac}(\mathcal{C}_1)^{\text{new}}) = \frac{\varphi(N)}{2} > 0$ (see formula (16) of [Arc03]). The curve $\mathcal{C}_1$ is a component of the stable model of $\mathcal{C}$.

Set $x_1 = z^N x$, and $y_1 = z^{A+B} y$. Then $\mathcal{C}$ transforms into

$$y_1^N = x_1^A(z^N - x_1)^B(1 - x_1)^C.$$

Its reduction modulo z equals

$$\mathcal{C}_2 : y_1^N = (-1)^B x_1^{A+B}(1 - x_1)^C.$$

This is another component of genus $\frac{\varphi(N)}{2}$. Both curves have complex multiplication, and their number of points can be computed using [GPV, Theorem 7.23]. $\square$

Remark 4.3. Using (23) we get a similar result for the specialization at $t_0 = \infty$.

Applying the result to the motive $\mathfrak{h}^\pm$ we get that its specialization at the point $t_0 = 1$ (corresponding to the trivial solution $(\pm 1, 0, \mp 1)$) will not correspond to any Hilbert modular form, but its specialization at the point $t_0 = 0$ and $t_0 = \infty$ does. Let us present a good estimate for the conductor of the attached Galois representation.

Lemma 4.4. *Let M denote the conductor of the specialization of $\mathcal{H}^\pm$ at $t_0 = 0$. Let ε_2 (respectively $\varepsilon_r, \varepsilon_q$) denote the valuation of M at a prime ideal of K dividing 2 (respectively dividing r and q). Then*

$$\varepsilon_2 = \begin{cases} 0 & \text{for } \mathcal{H}^+, \\ 0, 1, 2 & \text{for } \mathcal{H}^-, \end{cases} \quad \varepsilon_r \in \{0, 1, 2\}, \quad \varepsilon_q = \begin{cases} 2 & \text{if } v_q(4^{q-1} - 1) \geq 2, \\ 3 & \text{if } v_q(4^{q-1} - 1) = 1. \end{cases}$$

Proof. In §3 we have studied the conductor of $\mathcal{H}^\pm$ at values $t = -a^q/c^r$, where (a, b, c) is a solution to (1). The value $t_0 = 0$ corresponds to $(a, b, c) = (0, \pm 1, \mp 1)$. Since in this case $2r \mid a$, the values of ε_2 and ε_r follow from Proposition 3.15 and Corollaries 3.6 and 3.11, respectively. To compute the value of ε_q we may use, as in the proof of Proposition 3.14, that $(c, b, a) = (\mp 1, \pm 1, 0)$ is a solution to $z^r + y^p + x^q = 0$; that is, we must interchange

$a \leftrightarrow c$ and $q \leftrightarrow r$ in Corollaries 3.6 and 3.11. This implies $\varepsilon_q \in \{2, 3\}$ and its value depends on the irreducibility of

$$(26) \quad \begin{aligned} F(x) &= a^q f\left(\frac{x}{a}\right) + 2a^q + 4c^r \\ &= x^q \mp 4, \end{aligned}$$

since $a = 0$ and $c = \mp 1$. Then, the result follows by noting that $F(x)$ is irreducible over $\mathbb{Q}_q$ if and only if $v_q(4^{q-1} - 1) = 1$, by Lemma 3.13. $\square$

Similarly, for $t_0 = \infty$ the following holds.

Lemma 4.5. *Let M denote the conductor of the specialization of $\mathcal{H}^\pm$ at $t_0 = \infty$. Let ε_2 (respectively $\varepsilon_r, \varepsilon_q$) denote the valuation of M at a prime ideal of K dividing 2 (respectively dividing r and q). Then*

$$\varepsilon_2 = \begin{cases} 0 & \text{for } \mathcal{H}^+, \\ 0, 1, 2 & \text{for } \mathcal{H}^-, \end{cases} \quad \varepsilon_q \in \{0, 1, 2\}, \quad \varepsilon_r = \begin{cases} 2 & \text{if } v_r(4^{r-1} - 1) \geq 2, \\ 3 & \text{if } v_r(4^{r-1} - 1) = 1. \end{cases}$$

Proof. Follows from Lemma 4.4, applying (23). $\square$

Remark 4.6. As in some of the results in the previous section, the value of ε_r in Lemma 4.4 could be more restrictive under some conditions. For example, if $q \nmid \frac{r-1}{2}$, or if Proposition 3.2 holds over K (see Remark 3.12). The same applies to the values of ε_q in Lemma 4.5.

5. AN OBSTRUCTION TO THE METHOD: GHOST SOLUTIONS

The existence of a non-trivial primitive solution to (1) typically prevents a successful application of the modular method. As briefly explained in the introduction, there is another obstruction which, to our knowledge, has not previously been observed in the literature, is inherent to Darmon's program, and which we refer to as *ghost solutions*.

As clearly explained in [Dar00], Darmon's program consists of constructing, from a hypothetical solution to the generalized Fermat equation, a number field (obtained as the fixed field of a Frey representation) with a small ramification set. A ghost solution is a rational parameter $t_0 \neq 0$ for which the conductor of the specialized motive $\mathcal{H}^\pm(t_0)$ has the same small ramification set. To illustrate the phenomenon, let us consider Fermat's equation with signature (p, p, p) . To a putative solution (a, b, c) one attaches the Frey elliptic curve

$$(27) \quad E_t : y^2 = x(x-1)(1-tx),$$

specialized at $t_0 = -\frac{a^p}{c^p}$. Denote by $\rho_{t_0, p}$ the Galois representation attached to E_{t_0} twisted by the quadratic character θ_c . Then the residual representation $\bar{\rho}_{t_0, p}$ is unramified outside 2 (as expected) and is the reduction of the Galois representation associated to a modular form whose level is only supported at 2.

For p an odd prime, consider the Diophantine equation

$$x^p + 2 + z^p = 0.$$

Its set of solutions consists of the point $(-1, -1)$. Specializing (27) at $t_0 = -\frac{-1}{-1} = -1$ we obtain the elliptic curve 32-a3, a curve with good reduction outside 2. Moreover, it has conductor $N_1 = 32$. Similarly, if we consider the equations

$$2 + y^p + z^p = 0, \quad x^p + y^p + 2 = 0,$$

the point $(-1, -1)$ is always a solution, and corresponds (respectively) to the specializations $t_0 = 2$ (a curve isomorphic to 32-a3) and $t_0 = \frac{1}{2}$ (corresponding to the elliptic curve 64-a3). Their conductors are $N_2 = 32$ and $N_3 = 64$, respectively. The three specializations obtained do not come from real solutions of Fermat's equation, but from a variant of it (which motivates the terminology *ghost solution*).

It is easy to verify, using Tate's algorithm, that if (a, b, c) is a solution to Fermat's equation of signature (p, p, p) and $t_0 = -\frac{a^p}{c^p}$, then the conductor of the specialized Frey's elliptic curve E_{t_0} might have valuation 5 or 6 at 2. There is no clear reason why the elliptic curves 32-a3 and 64-a3 (coming from ghost solutions) should not be related to genuine solutions, so the modular method "theoretically" fails to prove Fermat's Last Theorem. However, if we impose the extra hypothesis on b being even, then $N = 2$ and the other spaces of modular forms do not need to be considered. It is a remarkable coincidence that we can impose a parity condition on b , as this is only true when the original equation has symmetries, which is not the case for an arbitrary generalized Fermat equation.

A similar phenomenon occurs while studying the generalized Fermat equation (1).

Definition 5.1. A *ghost solution* for the exponents (q, r) is a tuple $(a, c, s, \ell, m, n, u, v) \in \mathbb{Z}^2 \times \mathbb{Z}_{\geq 0}^6$, with $ac \neq 0$, satisfying

$$(2) \quad q^s r^\ell a^q \pm q^m r^n + q^u r^v c^r = 0.$$

If $(a, c, s, \ell, m, n, u, v)$ is a ghost solution and we set $t_0 := -\frac{a^q q^{s-u} r^{\ell-v}}{c^r}$, then a similiary argument as in Theorem 2.1 implies that the motive $\mathfrak{h}^+$ is unramified outside prime ideals of K dividing qr .

Example 5.2. If $q > 3$ is odd, the tuple $(-1, -2, 0, 0, 0, 2, 0, 0)$ is a ghost solution for exponents $(q, 3)$, since

$$(-1)^q + 3^2 + (-2)^3 = 0.$$

It corresponds to the specialization $t_0 = -\frac{1}{8}$. The tuple $(1, -2, 0, 2, 0, 0, 0, 0)$, satisfying

$$3^2 + (-1)^q + (-2)^3 = 0,$$

corresponds to the ghost solution for exponents $(q, 3)$ with specialization $\frac{9}{8}$.

Similarly, if $q > 2$ is odd, $(-1, 3, 0, 0, 0, 3, 0, 0)$ is a ghost solution for exponents $(q, 2)$, since

$$(-1)^q + (-2)^3 + 3^2 = 0.$$

It corresponds to the specialization $t_0 = \frac{1}{9}$. Similarly, the tuple $(-1, 3, 0, 3, 0, 0, 0, 0)$ corresponding to the equation

$$2^3(-1)^3 - 1 + 3^2 = 0$$

gives the ghost solution with specialization $\frac{8}{9}$. These tuples are *Catalan's solutions*, since they are related to solutions of Catalan's equation

$$x^n - y^m = 1,$$

whose complete set of solutions is $\{(3^2, 2^3)\}$ as proved by Mihăilescu in [Mih04].

The modular method has good chances to prove non-existence of solutions precisely when the specialization of $\mathcal{H}^+(t)$ at any ghost solution is a motive whose conductor exponents are not the ones described in Tables 3.2, 3.4 and 3.5. Otherwise, using current techniques, we can only expect to obtain partial results.

Remark 5.3. While working with the motive $\mathcal{H}^-(t)$ we need to also allow powers of 2 in (2).

5.1. Conductors of Catalan's ghost solutions. Let $\mathfrak{q}$ be the prime ideal of $K = \mathbb{Q}(\zeta_q)^+$ dividing q and let $f(x)$ be the polynomial defined in (17). Throughout this section we denote $\mathcal{H}(t) = \mathcal{H}\left(\left(\frac{1}{4}, -\frac{1}{4}\right), \left(\frac{1}{q}, -\frac{1}{q}\right) \middle| t\right)$ for the exponents $(q, 2)$ and $\mathcal{H}^+(t) = \mathcal{H}\left(\left(\frac{1}{3}, -\frac{1}{3}\right), \left(\frac{1}{q}, -\frac{1}{q}\right) \middle| t\right)$ for the exponents $(q, 3)$.

Lemma 5.4. *If $q \geq 5$, the specialization of $\mathcal{H}(t)$ at $t_0 = \frac{1}{9}$ has conductor $2^6 \cdot \mathfrak{q}^{\varepsilon_q}$, with*

$$\varepsilon_q = \begin{cases} 2 & \text{if } F(x) \text{ is reducible over } \mathbb{Q}_q, \\ 3 & \text{if } F(x) \text{ is irreducible over } \mathbb{Q}_q, \end{cases}$$

where $F(x) = f(x) + 34$.

Proof. The proof mimics that of Corollary 3.11. We know that $\mathcal{H}(t)$ is defined over the totally real field K . By (23) and [GP24, Theorem 9.3], over $F = \mathbb{Q}(\zeta_{4q})$, it is congruent modulo 2 to

$$(28) \quad \tilde{\mathcal{H}}(t) = \mathcal{H}\left(\left(\frac{1}{q}, -\frac{1}{q}\right), (1, 1) \middle| \frac{1}{t}\right).$$

Then, by Proposition 3.1, the conductor at $\mathfrak{q}$ of $\tilde{\mathcal{H}}(\frac{1}{9})$ equals that of the 2-dimensional Galois representation attached to $J_q^+(9)$. The latter follows from Proposition 3.10, noting that t_0 corresponds to the solution $(a, b, c) = (-1, -2, 3)$, and using (26) for the formula of $F(x)$. It is worth noting that Proposition 3.10 can be applied, since the hypothesis $p, q > 3$ is only used to compute the conductor in the two first rows, which is not our case.

By Remark 3.3, over K , the motive $\mathcal{H}(t)$ might be isomorphic not to $\tilde{\mathcal{H}}(t)$, but to a ramified quadratic twist of it. But the fact that $e = 4$ or $4r$ in Proposition 3.10 implies that conductor is invariant under ramified quadratic twists, because the local type is either supercuspidal, or a ramified principal series given by an order 4 character and its inverse.

Let $\mathfrak{n}$ be a prime ideal of K dividing 2. To compute the conductor at $\mathfrak{n}$, we use its congruence (over F) modulo $\mathfrak{q}$ with the motive

$$\mathcal{H}\left(\left(\frac{1}{4}, -\frac{1}{4}\right), (1, 1) \middle| t\right).$$

The latter motive corresponds (see [Coh11] and also [GP24, Section 3.3]) to the elliptic curve

$$(29) \quad E_2(t) : y^2 + xy = x^3 + \frac{t}{64}x.$$

The conductor exponent of E_2 specialized at $t_0 = \frac{1}{9}$ at 2 is 6, as can be easily computed by running Tate's algorithm. Furthermore, by [DFV22, Table 1], its local type corresponds to the quadratic twist by θ_8 of a supercuspidal representation obtained by inducing a character of order 4 from a ramified quadratic extension. Since 2 is unramified in $K/\mathbb{Q}$, the twisted curve has the same local type at $\mathfrak{n}$ over K , and the local type is preserved by the congruence, because q is odd. A similar argument proves that the result is also true over K . $\square$

Lemma 5.5. *If $q \geq 5$, the specialization of $\mathcal{H}(t)$ at $t_0 = \frac{8}{9}$ has conductor $2^5 \cdot \mathfrak{q}^{\varepsilon_q}$, with*

$$\varepsilon_q = \begin{cases} 2 & \text{if } F(x) \text{ is reducible over } \mathbb{Q}_q, \\ 3 & \text{if } F(x) \text{ is irreducible over } \mathbb{Q}_q, \end{cases}$$

where $F(x) = 2f(x/2) - 5 \cdot 2^{q-1}$.

Proof. As in the proof of Lemma 5.4, we have that $\mathcal{H}(t)$ is congruent modulo 2 to $\tilde{\mathcal{H}}(t)$, defined in (28). Hence, to determine the value of ε_q we need to study the conductor of $C_q^+(\frac{9}{8})$ at $\mathfrak{q}$. As in Lemma 3.9, it is easy to see that $C^+(\frac{9}{8})$ and $C^-(\frac{9}{8})/\mathbb{Q}(\sqrt{2})$ are isomorphic to the hyperelliptic curves

$$\mathcal{C}^+ : y^2 = (x+4)(2f(x/2) - 5 \cdot 2^{q-1}), \quad \mathcal{C}^- : y^2 = 2f(x/2) - 5 \cdot 2^{q-1},$$

respectively. Thus, it suffices to study the conductor of $\mathcal{C}^+$ at $\mathfrak{q}$. This can be done identically as in the proof of Proposition 3.10, noting that $\mathcal{C}^-$ equals the curve $C(z, s)$ defined in [CV25a, Equation (3)], for $z = 2^2$ and $s = -5 \cdot 2^{q-1}$. To finish the computation of ε_q , the same argument as in the proof of Lemma 5.4 holds.

To compute the conductor exponent at some prime $\mathfrak{n}$ dividing 2, we can again use that $\mathcal{H}(t)$ is congruent modulo $\mathfrak{q}$ to the motive corresponding to the elliptic curve $E_2(t)$, defined in (29). The conductor exponent at 2 of E_2 specialized at $t_0 = \frac{8}{9}$ is 5 and, by [DFV22, Table 1], its local type is given by a supercuspidal representation obtained by inducing a character of order 4 from a ramified quadratic extension. Then the result follows as in Lemma 5.4. $\square$

Lemma 5.6. *If $q \geq 5$, the specialization of $\mathcal{H}^+(t)$ at $t_0 = -\frac{1}{8}$ has conductor $3^3 \cdot \mathfrak{q}^{\varepsilon_q}$, with*

$$\varepsilon_q = \begin{cases} 2 & \text{if } F(x) \text{ is reducible over } \mathbb{Q}_q, \\ 3 & \text{if } F(x) \text{ is irreducible over } \mathbb{Q}_q, \end{cases}$$

where $F(x) = f(x) - 34$.

Proof. The argument to compute the value of ε_q follows as in Lemma 5.4. That is, we can apply Proposition 3.10 via a congruence modulo 3 (with the same subtlety of working over K instead of F). Note that t_0 corresponds to the solution $(a, b, c) = (-1, 3, 2)$, so the formula of $F(x)$ follows from (26). In this case, the formula for $F(x)$ follows from (26), using that $(a, b, c) = (-1, 3, 2)$.

Let $\mathfrak{n}$ be a prime ideal of K dividing 3. Note that Proposition 3.5 does not apply because $v_3(1/t_0 - 1) = 2$. By Proposition 3.2, $\mathcal{H}^+(t)$ is congruent modulo $\mathfrak{q}$ to the elliptic curve $E_3^+(t)$ defined in (19). Using Tate's algorithm, it is easy to see that the conductor of $E_3^+(-1/8)$ at 3 is 3, so its local type corresponds to a supercuspidal representation obtained from the induction of an order 6 character of a ramified quadratic extension $\mathbb{Q}_3$ (see Table 1 of [DFV22]). Since $q > 3$, $K/\mathbb{Q}$ is unramified at 3, so $\mathcal{H}^+(-1/8)$ has the same local type and conductor exponent at $\mathfrak{n}$ as the elliptic curve. $\square$

Lemma 5.7. *Suppose $q \geq 5$, $q \notin \{19, 37\}$. Then, the specialization of $\mathcal{H}^+(t)$ at $t_0 = \frac{9}{8}$ has conductor $3^3 \cdot \mathfrak{q}^{\varepsilon_q}$, with*

$$\varepsilon_q = \begin{cases} 2 & \text{if } F(x) \text{ is reducible over } \mathbb{Q}_q, \\ 3 & \text{if } F(x) \text{ is irreducible over } \mathbb{Q}_q, \end{cases}$$

where $F(x) = 3f(x/3) - 14 \cdot 3^{q-2}$.

Proof. As in the proof of Lemma 5.4, we have that $\mathcal{H}^+(t)$ is congruent modulo 3 to $\tilde{\mathcal{H}}(t)$, defined in (28). Hence, to determine the value of ε_q we need to study the conductor of $C_q^+(\frac{8}{9})$ at $\mathfrak{q}$. As in Lemma 3.9, it is easy to see that $C^+(\frac{8}{9})$ and $C^-(\frac{8}{9})/\mathbb{Q}(\sqrt{3})$ are isomorphic to the hyperelliptic curves

$$\mathcal{C}^+ : y^2 = (x+6)(3f(x/3) - 14 \cdot 3^{q-2}), \quad \mathcal{C}^- : y^2 = 3f(x/3) - 14 \cdot 3^{q-2},$$

respectively. Note that $\mathcal{C}^-$ equals the curve $C(z, s)$ defined in [CV25a, Equation (3)], for $z = 3^2$ and $s = -14 \cdot 3^{q-2}$. Following the notation in [CV25a], Δ (see equation (22)) is given by

$$\Delta = z^q - 4s^2 = -3^{2q-4} \cdot 19 \cdot 37.$$

Therefore, if $q \neq 19, 37$, we get that $q \nmid \Delta$. The rest of the proof to compute ε_q follows as in the previous lemmas, which repeat the arguments in Proposition 3.10.

The conductor exponent at a prime dividing 3 follows in the same way as Lemma 5.6. $\square$

5.2. Specializations with Complex Multiplication. We have seen in §4 that the specializations $t_0 = 0, \infty$ (arising from trivial solutions) correspond to abelian varieties with complex multiplication (hence Hilbert modular forms with complex multiplication).

Proposition 5.8. *Let $q > r$ be odd primes. Let (a, b, c) be a non-trivial solution to (1) with $p \geq 3$ and set $t_0 = -\frac{a^q}{c^r}$. If the motive $\mathfrak{h}^+$ has complex multiplication, then there exists $m, n \in \mathbb{Z}_{\geq 0}$ such that $(a, c, 0, 0, mp, np, 0, 0)$ is a ghost solution for the exponents (q, r) .*

Proof. Abelian varieties with complex multiplication have potentially good reduction at all primes, by [ST68, Corollary 1]. By Theorem 2.1, all primes dividing b and prime to qr are primes of multiplicative reduction so b is supported only at the primes q and r . If $b = \pm q^m r^n$, then we have the relation

$$a^q \pm q^{mp} r^{np} + c^r = 0,$$

so $(a, c, 0, 0, mp, np, 0, 0)$ is a ghost solution. $\square$

Remark 5.9. In general we do not expect ghost solutions to have complex multiplication, but we do not know how to prove this fact. This problem is related to that of computing the set of values for which the specialized motive has complex multiplication (as explicitly done in §5 of [BW88]; unfortunately it depends on a result of [Wol88] which is known to have a gap in its proof).

5.3. Computing some sets of ghost solutions. It is not clear to us how, given the exponents (q, r) , one can compute all tuples $(a, c, s, \ell, m, n, u, v)$ satisfying (2). What is easy to do is to run some numerical experiments, searching for solutions to (2) in boxes and compute for the solutions found the conductor of the respective 2-dimensional Galois representation. When the conductor lies within the set of conductors coming from hypothetical solutions to (1) then the modular method will fail to prove complete results on non-existence of solutions. Otherwise, our computations (if complete) allow to determine which exponents of the generalized Fermat equation should be studied if we expect the modular method to work (always conditional on the validity of Conjecture 1 used to discard trivial solutions). Assuming that our numerical experiments succeed in finding all ghost solutions, we deduce that for any pair of exponents (q, r) with $3 < r < q \leq 13$ either there are no ghost solutions, or their conductor do not match the conductor of a real solution. We alert the reader that the dimension of the space of Hilbert modular forms for exponents different from the one studied in the present article (namely $q = 5$ and $r = 3$) is too large, making the newform decomposition infeasible with our electronic resources.

We present the ghost solutions obtained for different exponents (q, r) in the range $3 \leq r < q \leq 13$ except the case $(5, 3)$ that will be studied in detail later. For each value of (q, r) , let $\mathfrak{q}$ (respectively $\mathfrak{r}$) be a prime ideal of K dividing q (respectively r). We include in the tables the valuation at $\mathfrak{q}$ and $\mathfrak{r}$ of the conductor of the attached 2-dimensional representation.

5.3.1. *The case $(q, r) = (7, 3)$.* We found the following solutions.

t_0	$-\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{64}$	$\frac{7}{8}$	$\frac{9}{8}$	$\frac{63}{64}$	$\frac{128}{125}$
$(v_{\mathfrak{q}}(N), v_{\mathfrak{r}}(N))$	$(3, 3)$	$(3, 6)$	$(3, 6)$	$(3, 9)$	$(3, 3)$	$(3, 9)$	$(4, 3)$

Only Catalan's solutions appear in spaces of modular forms associated with solutions to (1).

5.3.2. *The case $(q, r) = (7, 5)$.* We found the following two solutions.

t_0	$\frac{7}{32}$	$\frac{25}{32}$
$(v_{\mathfrak{q}}(N), v_{\mathfrak{r}}(N))$	$(3, 9)$	$(3, 6)$

None of them lie in a space coming from solutions to the generalized Fermat equation.

5.3.3. *The case $(q, r) = (11, 3)$.* We found the following four solutions.

t_0	$-\frac{3}{8}$	$-\frac{1}{8}$	$\frac{9}{8}$	$\frac{11}{8}$
$(v_{\mathfrak{q}}(N), v_{\mathfrak{r}}(N))$	$(4, 8)$	$(3, 3)$	$(3, 3)$	$(4, 13)$

The two middle values do lie in an allowed conductor, so the classical elimination step will fail to prove non-existence of solutions.

5.3.4. *The case $(q, r) = (11, 5)$.* We found a unique solution corresponding to $t_0 = \frac{2048}{2673}$. Its conductor valuation at a prime ideal dividing 5 is 2, while its conductor valuation at a prime dividing 11 is 8. Then it does not lie in any space related to solutions to (1).

5.3.5. *The case $(q, r) = (11, 7)$.* We found the following two solutions.

t_0	$\frac{7}{128}$	$\frac{121}{128}$
$(v_{\mathfrak{q}}(N), v_{\mathfrak{r}}(N))$	$(6, 3)$	$(6, 13)$

Both conductors do not match the conductor of a newform coming from a solution to (1).

5.3.6. *The case $(q, r) = (13, 3)$.* We found only the two solutions.

t_0	$-\frac{1}{8}$	$\frac{9}{8}$
$(v_{\mathfrak{q}}(N), v_{\mathfrak{r}}(N))$	$(2, 3)$	$(3, 3)$

They both have conductors matching conductors of real solutions to the generalized Fermat equation.

Notice that when $r = 3$ and $t_0 = -\frac{1}{8}$ or $t_0 = \frac{9}{8}$, the conductor exponent at $\mathfrak{q}$ and $\mathfrak{r}$ matches the value computed in Lemma 5.6 and Lemma 5.7, respectively.

We did not find any solution for the exponents $(q, r) = (13, 5), (13, 7)$ nor $(13, 11)$.

6. SOME RESULTS ON RESIDUAL IMAGES

Keeping the previous notation, let $\mathfrak{p}$ be a prime ideal of $K = \mathbb{Q}(\zeta_q)^+ \cdot \mathbb{Q}(\zeta_r)^+$, and let $\rho_{\mathfrak{p}}^{\pm}$ denote the $\mathfrak{p}$ -th Galois representation attached to the motive $\mathfrak{h}^{\pm}$, as in (13). As explained in the introduction, the third step of the modular method (needed to apply a level lowering result) depends on the residual representation $\bar{\rho}_{\mathfrak{p}}^{\pm}$ being absolutely irreducible. Since K is a totally real field, this is equivalent to being irreducible. In this direction, let us recall the following result, which is [BCDF17, Corollary 1].

Theorem 6.1. *Let K be a totally real field, $\mathfrak{q}$ a prime ideal of K and g a positive integer. Then there exists a constant $C(K, g, \mathfrak{q})$ such that for all primes $p > C(K, g, \mathfrak{q})$ and all g -dimensional abelian varieties A/K satisfying:*

- (i) *A has potentially good reduction at $\mathfrak{q}$,*
 - (ii) *A is semistable at the primes of K dividing p ,*
 - (iii) *A is of GL_2 -type with multiplications by some totally real field F ,*
 - (iv) *all endomorphisms of A are defined over K (i.e. $\mathrm{End}_K(A) = \mathrm{End}_{\bar{K}}(A)$),*
- the residual representation $\bar{\rho}_{A, \mathfrak{p}}$ is irreducible.*

Let $N^{\pm}$ be as in (16). The following corollary is an application of Theorem 6.1 to $\bar{\rho}_{\mathfrak{p}}^{\pm}$.

Corollary 6.2. *There exists a constant $C(q, r)$ such that if (a, b, c) is a primitive solution to (1) satisfying:*

- (i) *b is divisible by a prime $\ell \nmid N^{\pm}$,*
- (ii) *$r \nmid ab$ or $q \nmid bc$.*

Then, for every $p > C(q, r)$ and all $\mathfrak{p} \mid p$ in K , the residual representation $\bar{\rho}_{\mathfrak{p}}^{\pm}$ is absolutely irreducible.

Proof. Let $\mathfrak{q}$ and $\mathfrak{r}$ be prime ideals of K dividing q and r respectively. Let

$$C(q, r) := \max\{C(K, \varphi(qr), \mathfrak{q}), C(K, \varphi(qr), \mathfrak{r}), q, r\},$$

where the constants are those appearing in Theorem 6.1.

Let (a, b, c) be a primitive solution to (1) and let A be the new part of $\mathrm{Jac}(\mathcal{C}^{\pm}(t_0))$. It is an abelian variety of GL_2 -type over K , of genus $\varphi(qr) = (q-1)(r-1)$. The variety A is isogenous to the direct sum of the motive $\mathfrak{h}^{\pm}$ and its Galois conjugates. We need to verify that A satisfies the hypotheses of Theorem 6.1.

Suppose that $r \nmid ab$. Then Remark 3.8 (and its analogue for primes larger than 3) implies that A has potentially good reduction at $\mathfrak{r}$. When $q \nmid bc$ a similar reasoning applies to the prime ideal $\mathfrak{q}$, proving condition (i).

Let $\mathfrak{p}$ be a prime ideal of K dividing $p \neq q, r$. Theorems 2.1 and 2.3 imply that if $\mathfrak{p} \nmid b$, A has good reduction at $\mathfrak{p}$, otherwise, it has multiplicative reduction. In both cases it is semistable, so condition (ii) is satisfied.

The first hypothesis implies that there exists a prime $\ell \neq q, r$ (and also $\ell \neq 2$ in the case of $\mathfrak{h}^{-}$) dividing b . Let $\mathfrak{l}$ be a prime in K dividing ℓ . By Theorem 2.1 (respectively Theorem 2.3) the prime $\mathfrak{l}$ is of multiplicative reduction for $\mathfrak{h}^{+}$ (respectively $\mathfrak{h}^{-}$), so the motive (and A) does not have complex multiplication and all its endomorphisms are defined over K . The result now follows from Theorem 6.1 and our definition of $C(q, r)$. $\square$

Remark 6.3. The first condition is used to ensure that A does not have complex multiplication (see Proposition 5.8). There are only finitely many specializations coming from non-trivial primitive solutions of the motive $\mathcal{H}^\pm(t)$ where the variety has complex multiplication (see [BW88] and [EY03]). Then for p large enough (with a bound that is ineffective) the first condition can be removed.

7. THE EQUATION $x^5 + y^p + z^3 = 0$

In this section we apply the strategy developed in [GP24] and the results of the previous sections to study primitive solutions to the equation

$$(3) \quad x^5 + y^p + z^3 = 0.$$

7.1. A hyperelliptic realization of the motive. In §2 we explained how to obtain a curve whose Jacobian realizes our motive $\mathcal{H}^\pm$. For $q = 5$ and $r = 3$, the quotient of Euler's curve $\mathcal{C}^+$ by the three described involutions has genus 2, hence is hyperelliptic and an explicit equation for it can be computed. Euler's curve (7) has equation

$$(30) \quad \mathcal{C}^+(t) : y^{15} = x^2(1-x)^{-8}(1-tx)^8 t^{-3}.$$

The curve has genus 14. The old contribution to its Jacobian comes from the curves

$$(31) \quad \mathcal{C}_5(t) : y^5 = x^2(1-x)^{-8}(1-tx)^8 t^{-3} \quad \text{and} \quad \mathcal{C}_3(t) : y^3 = x^2(1-x)^{-8}(1-tx)^8 t^{-3},$$

of genus 4 and 2 respectively. The *new* part of $\text{Jac}(\mathcal{C}^+)$ is an abelian variety of dimension 8 with an action of $\mathbb{Z}[\zeta_{15}]$ (of GL_2 -type over $F = \mathbb{Q}(\zeta_{15})$). To compute quotients of curves by involutions in **Magma** we need to use a projective model. To keep degrees as small as possible, we change variables $y \rightarrow y \cdot (1-x)$ and consider the projective model

$$\mathcal{C}^+(t) : y^{15} z^2 = x^2(z-x)^7(z-tx)^8 t^{-3}.$$

Keeping the notation of §2, an involution of the curve (in the new variables) is given by the map

$$\iota_{-1}(x, y, z) = \left(\frac{yz}{t}, \frac{(x-z)(z-tx)}{t}, xy \right).$$

We can define the involution in **Magma** with the following commands:

```
FF<t> := FunctionField(Rationals());
A<x,y,z> := ProjectiveSpace(FF,2);
C := Curve(A,y^15*z^2-x^2*(z-x)^7*(z-t*x)^8/t^3);
phiAmb := map<A->A|[y*z/t,(x-z)*(z-t*x)/t,x*y]>;
phi := Restriction(phiAmb,C,C);
```

Let $j = 4$ (so $j \equiv -1 \pmod{5}$ and $j \equiv 1 \pmod{3}$). Then the map (9) in the new variables is given by

$$\iota_4(x, y, z) = \left(\frac{x-z}{tx-z}, \frac{(t-1)(z-x)^2(tx-z)x}{ty^4}, 1 \right).$$

It is another involution of $\mathcal{C}^+$. We define it in **Magma** by

```
phiAmb4 := map<A->A|[(x-z)/(t*x-z),(t-1)*(z-x)^2*(t*x-z)*x/t/y^4,1]>;
phi4 := Restriction(phiAmb4,C,C);
```

Then we can compute the quotient of $\mathcal{C}^+$ by both involutions (with a little patience, since the computation takes around 10 minutes).

```

Mp := iso<C->C|DefiningEquations(phi),DefiningEquations(phi)>;
Mp4 := iso<C->C|DefiningEquations(phi4),DefiningEquations(phi4)>;
G := AutomorphismGroup(C,[Mp4,Mp]);
D, map := CurveQuotient(G);

```

The output (available in [PV25]) is a hyperelliptic curve isomorphic to

$$(32) \quad \mathcal{C}_{5,3}^+(t) : y^2 = 5x^6 - 12x^5 + 10tx^3 + t^2,$$

with discriminant given by $\Delta(\mathcal{C}_{5,3}^+(t)) = 2^{20}3^65^5t^8(t-1)^2$.

Its Jacobian over $K = \mathbb{Q}(\sqrt{5})$ has $\mathbb{Q}(\sqrt{5})$ as its algebra of endomorphisms. There is a remarkable coincidence here: the quotient of $\mathcal{C}^+$ by the group generated by the two involutions has genus 2. In general, only the new part of the quotient has genus $[K : \mathbb{Q}]$, but in this case there is no contribution from the old parts (corresponding to the quotients of the curves $\mathcal{C}_3$ and $\mathcal{C}_5$ in (31) by the same involutions). This is not the case for the curve $\mathcal{C}^-$. We can compute the quotient curve, but it is a curve of genus 4. It is not clear to us how to get an equation for the new part of it.

In the recent paper [CV25b], the authors study the local conductor at the even place of the genus 2 hyperelliptic curve

$$C(t) : y^2 + y(x^3 + t(1-t)^2) = 2t(1-t)^2x^3 + 3t^2(1-t)^3x + t^2(1-t)^4.$$

The curve is related to solutions of the generalized Fermat equation

$$(33) \quad x^3 + y^5 = z^p.$$

Concretely, if (a, b, c) is a solution to (33), then specializing at $t_0 = a^3/c^p$ gives the curve

$$y^2 + y(x^3 + b) = 2bx^3 + 3ax + b^2,$$

which seems to be suitable for the modular method. The hyperelliptic curve $C(t)$ was obtained through a reverse engineering process, and none of its expected properties were not addressed by the authors

Proposition 7.1. *The hyperelliptic curves $\mathcal{C}_{5,3}^+((t-1)/t)$ and $C(t)$ are isomorphic.*

Proof. Applying the change of variables $(x, y) \rightarrow (1/x, y/x^3)$ for the curve $\mathcal{C}_{5,3}^+(t)$, followed by the change of variable $y \rightarrow 2y + (tx^3 + 1)$, we get the curve

$$y^2 + y(tx^3 + 1) = 2tx^3 - 3x + 1.$$

Substituting t by $(t-1)/t$ and applying the change of variables $(x, y) \rightarrow (-x/(1-t), y/(t(1-t)^2))$ we obtain the model $C(t)$. $\square$

Remark 7.2. Theorem 2.7 gives a formula to compute the trace of a Frobenius element $\text{Frob}_{\mathfrak{p}}$ for $\mathfrak{p}$ a prime of $\mathbb{Q}(\zeta_{15})$ not dividing 15 (not for primes in $K = \mathbb{Q}(\sqrt{5})$). Equation (32) allows us to compute Frobenius traces over K .

To lower the number of possibilities for the conductor of $\mathcal{H}^+$, let us prove the veracity of Proposition 3.2 over K .

Lemma 7.3. *Let $t_0 \in \mathbb{Q} \setminus \{0, 1\}$. Then the following congruences hold over K .*

- The motive $\mathcal{H}^+(t_0) \equiv \mathcal{H}((1, 1)(\frac{1}{5}, -\frac{1}{5})|t_0) \pmod{3}$.
- The motive $\mathcal{H}^+(t_0) \equiv \mathcal{H}((\frac{1}{3}, -\frac{1}{3}), (1, 1)|t_0) \pmod{(\sqrt{5})}$.

Proof. By Remark 3.3 the statement is true up to a quadratic twist by a character of $\text{Gal}(\mathbb{Q}(\zeta_{15})/\mathbb{Q}(\sqrt{5}))$, i.e. up to one of the following characters:

- θ_{-3} , corresponding to the extension $\mathbb{Q}(\sqrt{-3}, \sqrt{5})/\mathbb{Q}(\sqrt{5})$;
- θ_{ζ_5} , corresponding to the extension $\mathbb{Q}(\zeta_5)/\mathbb{Q}(\sqrt{5})$;
- $\theta_{6\sqrt{5}+15}$, corresponding to the third subextension obtained by adding to $\mathbb{Q}(\sqrt{5})$ a square root of $6\sqrt{5} + 15$.

Since the twist is independent of t , it is enough to prove the claim for a particular specialization of the parameter, say $t = 3$. Recall that $\mathcal{H}^+(3)(1)$ (the Tate twist of the hypergeometric motive; see Remark 2.2) matches the hyperelliptic curve

$$\mathcal{C}_{5,3}^+(3) : y^2 = 5x^6 - 12x^5 + 30x^3 + 9.$$

By (23) we have that $\mathcal{H}((1, 1), (\frac{1}{5}, -\frac{1}{5})|3) \simeq \mathcal{H}((\frac{1}{5}, -\frac{1}{5}), (1, 1)|\frac{1}{3})$. By Proposition 3.1, the latter corresponds to Darmon's hyperelliptic curve

$$C_5^+(1/3) : y^2 = x^6 + 6x^5 - 45x^4 - 270x^3 + 405x^2 + 2592x + 972.$$

The values of the characters θ_{-3} , θ_{ζ_5} , and $\theta_{6\sqrt{5}+15}$ at a prime ideal $\mathfrak{l}$ of K dividing $\{7, 11, 29\}$ are, respectively, $(+, -, -)$, $(-, +, -)$, and $(-, -, +)$. Therefore, if we prove that the congruence holds for $\text{Frob}_{\mathfrak{l}}$ at each such $\mathfrak{l}$, then it cannot hold for any of the possible twists, yielding the desired result.

The characteristic polynomials $P_{\mathfrak{l}}$ of $\text{Frob}_{\mathfrak{l}}$ for both curves $\mathcal{C}_{5,3}^+(3)$ and $C_5^+(1/3)$ are given in Table 7.1 (they were computed using **Magma**).

$\mathfrak{l}$	$P_{\mathfrak{l}}(\mathcal{C}_{5,3}^+(3))$	$P_{\mathfrak{l}}(C_5^+(1/3))$
7	$x^4 + 10x^3 + 123x^2 + 490x + 2401$	$x^4 - 20x^3 + 198x^2 - 980x + 2401$
$(\frac{1+3\sqrt{5}}{2})$	$x^4 - x^3 + 21x^2 - 11x + 121$	$x^4 + x^3 + 21x^2 + 11x + 121$
$(\frac{11+\sqrt{5}}{2})$	$x^4 - 2x^3 + 14x^2 - 58x + 841$	$x^4 + 5x^3 + 53x^2 + 145x + 841$

TABLE 7.1. Characteristic polynomials of Frobenius.

Recall that we need to add a Tate twist to the second column or, equivalently, replace x by $N(\mathfrak{l})x$. The twisted polynomials of the second column match those of the third column modulo 3, proving the first statement. To prove the second statement, we use that, by Proposition 3.1, $\mathcal{H}((\frac{1}{3}, -\frac{1}{3}), (1, 1)|3)$ corresponds to the elliptic curve

$$C_3^+(3) \simeq E_3^+(3) : y^2 + 3xy + 3y = x^3.$$

The characteristic polynomial of the Frobenius endomorphism $\text{Frob}_{\mathfrak{l}}$ acting on $E_3^+(3)$ for the same prime ideals $\mathfrak{l}$ as before are

$$x^2 + 10x + 49, \quad x^2 - 3x + 11, \quad x^2 + 6x + 29.$$

The square of each of these polynomials matches the twisted polynomials in the second column of Table 7.1 modulo 5. See the script in [PV25] for a verification of these computations. $\square$

Proposition 7.4. *Let $\ell \in \{3, 5\}$ and $t \in \mathbb{Q} \setminus \{0, 1\}$. If $v_{\ell}(t-1) > 3$ the Jacobian of $\mathcal{C}_{5,3}^+(t)$ has potentially multiplicative reduction at ℓ . Moreover, if $v_{\ell}(t-1) = 0$ and also $v_3(17t+108) \leq 3$ when $\ell = 3$, then it has potentially good reduction at ℓ .*

Proof. The Igusa invariants of $\mathcal{C}_{5,3}^+(t)$ (see [Igu60, page 623]) are given by

$$\begin{aligned} J_2 &= -1200t^2, \\ J_4 &= -480000t^4, \\ J_6 &= 43520000t^6 + 276480000t^5, \\ J_8 &= -70656000000t^8 - 82944000000t^7, \\ J_{10} &= 2388787200000t^{10} - 4777574400000t^9 + 2388787200000t^8. \end{aligned}$$

Set $\varepsilon = 1$ if $\ell = 5$ and $\varepsilon = 3$ if $\ell = 3$. By Theorem 1 of [Liu93], the Jacobian of $\mathcal{C}_{5,3}^+(t)$ has potentially good reduction if and only if one of the two following conditions holds:

- (I) The values $J_{2i}^\varepsilon J_{10}^{-i}$ are integral over $\mathbb{Q}_\ell$ for $1 \leq i \leq 5$,
- (V) All the following conditions hold: $I_4^\varepsilon I_{2\varepsilon}^{-2} \in \ell\mathbb{Z}_\ell$, $J_{10}^\varepsilon I_{2\varepsilon}^{-5} \in \ell\mathbb{Z}_\ell$, $I_{12}^\varepsilon I_{2\varepsilon}^{-6} \in \ell\mathbb{Z}_\ell$, $I_4^{3\varepsilon} J_{10}^{-\varepsilon} I_{2\varepsilon}^{-1} \in \mathbb{Z}_\ell$ and $I_{12}^\varepsilon J_{10}^{-\varepsilon} I_{2\varepsilon}^{-1} \in \mathbb{Z}_\ell$, where $I_4 = J_2^2 - 24J_4$, $I_2 = J_2/12$, $I_6 = J_6$ and $I_{12} = -2^3 J_4^3 + 3^2 J_2 J_4 J_6 - 3^3 J_6^2 - J_2^2 J_8$.

The first case corresponds to a genus 2 curve with good reduction, while the second case corresponds to a genus 2 curve whose special fiber consists of two elliptic curves with complex multiplication meeting at a single point (so the curve has bad reduction but its Jacobian does not).

The values of (I) are respectively

$$-\frac{5^5 t^2}{3(t-1)^2}, \quad -\frac{5^{10} t^4}{3^7(t-1)^4}, \quad \frac{5^5 t(17t+108)^5}{3^{18}(t-1)^6}, \quad -\frac{5^{10} t^3(23t+27)^5}{3^{19}(t-1)^8}, \quad 1.$$

The values of (V) for $\ell = 3$ are respectively

$$\frac{3^{12} 5^4 t^2}{(17t+108)^2}, \quad \frac{3^{18}(t-1)^6}{5^5 t(17t+108)^5}, \quad \frac{3^{27}(t-1)^3(9t+16)^3}{(17t+108)^6}, \quad \frac{3^{18} 5^{17} t^7}{(17t+108)(t-1)^6}, \quad \frac{3^9 5^5 t(9t+16)^3}{(17t+108)(t-1)^3}.$$

For $\ell = 5$, they equal

$$6^4, \quad -\frac{2^{10} 3^6 (t-1)^2}{5^5 t^2}, \quad \frac{2^{12} 3^9 (t-1)(9t+16)}{5^4 t^2}, \quad -\frac{2^2 3^6 5^5 t^2}{(t-1)^2}, \quad -\frac{2^2 3^3 5(9t+16)}{t-1}.$$

All of these computations can be checked in [PV25]. Conditions (I) and (V) do not hold at $\ell = 3, 5$ if $v_\ell(t-1) > 3$, proving the first claim. If $v_\ell(t-1) = 0$ then condition (I) holds for $\ell = 5$, while condition (V) holds for $\ell = 3$, since $v_3(17t+108) \leq 3$ by hypothesis. $\square$

Remark 7.5. If (a, b, c) is a hypothetical solution to (3) and $t_0 = -\frac{a^5}{c^3}$ then it is always the case that $v_3(17t_0 + 108) \leq 3$. Indeed, if $17t_0 + 108 = \frac{3^n \alpha}{\beta}$, with $n \geq 4$ and $3 \nmid \beta$ then $t_0 = \frac{3^3(3^{n-3}\alpha - 4\beta)}{17\beta}$. Since the numerator is a fifth power, $3 \mid \beta$, giving a contradiction. This implies that the last condition of Proposition 7.4 is automatically satisfied for our Diophantine applications.

Corollary 7.6. *Let (a, b, c) be a non-trivial primitive solution to (3) with $p > 3$ and set $t_0 = -\frac{a^5}{c^3}$. Then the Jacobian of the specialization $\mathcal{C}_{5,3}^+(t_0)$ does not have complex nor quaternionic multiplication.*

Proof. It follows from Proposition 5.8 that b can only be divisible by the primes 3 and 5. If $3 \mid b$ (respectively $5 \mid b$) then Proposition 7.4 and Remark 7.5 implies that $\text{Jac}(\mathcal{C}_{5,3}^+(t_0))$ has potential multiplicative reduction at 3 (respectively 5), so it cannot have complex nor

quaternionic multiplication (see [Jor86, §4]). Then $b = \pm 1$, and $(\mp a, \mp c)$ must be a solution to Catalan's equation

$$x^5 - (-z)^3 = 1,$$

which does not have any non-trivial solution (as proved in [Mih04]). $\square$

The existence of a hyperelliptic model allows us to prove a stronger irreducibility result.

Corollary 7.7. *Let ℓ be a fixed prime. There exists a bound $C(\ell)$ such that the following holds. If (a, b, c) is a non-trivial primitive solution to (3) satisfying that $\ell \nmid b$ then for all primes $p > C(\ell)$ and all $\mathfrak{p} \mid p$ in K , the residual representation of $\rho_{\mathfrak{p}}^{\pm}$ is absolutely irreducible. Explicitly, $C(2) = 13$, $C(3) = 41363281$ and $C(5) = 335809$.*

Proof. The proof follows the one given in Corollary 6.2. By Corollary 7.6, the Jacobian $J = \text{Jac}(\mathcal{C}_{5,3}^+(t_0))$ does not have complex multiplication, so the first hypothesis of Corollary 6.2 can be removed (see Remark 6.3). Let $\mathfrak{l}$ be a prime in K above ℓ . Taking $\mathfrak{l}$ in Theorem 6.1 we need to verify that the variety J has potentially good reduction at $\mathfrak{l}$. Since $\ell \nmid b$, this follows from Theorem 2.1 (1) if $\ell \neq 3, 5$ and from Proposition 7.4 if $\ell = 3, 5$. The explicit value of $C(\ell)$ can be computed following the proof of Theorem 2 in [BCDF17]. Let M denote a field where J/K attains good reduction at $\mathfrak{l}$, and let f denote the inertial degree of M/K . Then according to loc. cit.,

$$(34) \quad p \mid \prod_{a_J} \text{Res}(X^{c_J \cdot h_K} - 1, x^2 - a_J x + \mathcal{N}(\mathfrak{l})^f),$$

where Res denotes the resultant, c_J is the inertial exponent of J , h_K denotes the class group of K and the product runs over all the possible local factors of J over M .

In our case, $h_K = 1$. If $\ell = 3, 5$ then $f = 1, 2$ and a priori $c_J = 4, 12$ for $q = 3$ and $c_J = 4, 20$ for $q = 5$. Otherwise, $\ell \neq 3, 5$ and so, since J has good reduction over K , we have $f = c_J = 1$.

Suppose $\ell = 2$. Then, an easy computation in **Magma** shows that $a_J \in \{-1, -8\}$. Then, the product in (34) equals 6084 and so $C(2) = 13$.

Suppose then that $\ell = 3, 5$ and that the residual representation decomposes like

$$\bar{\rho}_{\mathfrak{p}} \sim \psi_1 \chi_p \oplus \psi_2,$$

where χ_p denotes the cyclotomic character. Then ψ_1, ψ_2 have conductor dividing $3 \cdot (\sqrt{5})$. The ray class group of K of modulus $3(\sqrt{5})\infty_1\infty_2$, where ∞_i denote the two places at infinity, is isomorphic to $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Hence ψ_i has order at most 4, for $i = 1, 2$. Then (following the proof of [BCDF17, Theorem 2]) we can take $c_J = 4$ in (34). To compute all possible values of a_J in (34), recall that $a_J \in \mathbb{Z}[\frac{1+\sqrt{5}}{2}]$. Write $a_J = \frac{\alpha+\beta\sqrt{5}}{2}$. Then the Hasse-Weil bound (our curve has genus 2) over $\mathbb{F}_{\ell}$ and $\mathbb{F}_{\ell^2}$, where $\ell = \mathcal{N}(\mathfrak{l})$, gives the inequalities

$$|\alpha| \leq 4\sqrt{\ell}, \quad \left| \frac{\alpha^2 + 5\beta^2}{2} - 4\ell \right| \leq 4\ell.$$

This gives the bounds for $f = 1$: $C(3) = 8969$, $C(5) = 809$ and for $f = 2$: $C(3) = 41363281$, $C(5) = 335809$. The code to compute these values is included in [PV25]. $\square$

7.2. Spaces of Hilbert modular forms. Let $K = \mathbb{Q}(\sqrt{5}) = \mathbb{Q}(\zeta_5)^+$. Let (a, b, c) be a primitive solution to

$$(3) \quad x^5 + y^p + z^3 = 0.$$

Keeping the notation of §2, let $\rho_{\mathfrak{p}} := \rho_{\mathfrak{p}}^+ : \text{Gal}_K \rightarrow \text{GL}_2(K_{\mathfrak{p}})$ be the Galois representation in (13) attached to the specialization of the motive $\mathcal{H}^+$ at the point $t_0 = -\frac{a^5}{c^3}$. Define $a_0, c_0 \in \mathbb{Z}$ by $a_0 = a5^{-v_5(a)}$ and $c_0 = c3^{-v_3(c)}$. Let $\mathfrak{p}$ be a prime in K above p .

Theorem 7.8. *Suppose that $\bar{\rho}_{\mathfrak{p}}$ is irreducible. Then, there exists a Hilbert newform $\mathcal{F} \in S_2(\Gamma_0(3^{\varepsilon_3} \cdot (\sqrt{5})^{\varepsilon_5}))$ over K such that*

$$(35) \quad \bar{\rho}_{\mathfrak{p}} \simeq \bar{\rho}_{\mathcal{F}, \mathfrak{p}},$$

where $\mathfrak{P}$ is a prime ideal of the coefficient field $K_{\mathcal{F}}$ dividing p . In addition,

(i) The value ε_3 satisfies

$$\varepsilon_3 = \begin{cases} 2 & \text{if } 3 \mid ab, \\ 2 & \text{if } 3 \mid c \text{ and } \pm 2c_0^3 \equiv a^5 \pmod{9}, \\ 2 & \text{if } 3 \nmid abc \text{ and } 4c^3 \equiv a^5 \pmod{9}, \\ 3 & \text{otherwise.} \end{cases}$$

(ii) The value ε_5 satisfies

$$\varepsilon_5 = \begin{cases} 2 & \text{if } 5 \mid bc, \\ 2 & \text{if } 5 \mid a \text{ and } \frac{c^3}{a_0^3} \equiv 6, 12, 28 \pmod{25}, \\ 2 & \text{if } 5 \nmid abc \text{ and } \frac{c^3}{a} \equiv 6, 8, 17, 19 \pmod{25}, \\ 3 & \text{otherwise.} \end{cases}$$

Proof. By Theorem 2.4 and [BCDF25, Theorem 3.13], we have that $\bar{\rho}_{\mathfrak{p}}$ is modular with trivial Nebentypus. Since K is totally real, $\bar{\rho}_{\mathfrak{p}}$ is absolutely irreducible. Hence, by level lowering for Hilbert modular forms [BD14], we have that (35) holds for some newform $\mathcal{F}$ of parallel weight 2, trivial Nebentypus over K and level equals the conductor of $\bar{\rho}_{\mathfrak{p}}$. By Theorem 2.1 the latter is only supported at the prime ideals (3) and $(\sqrt{5})$.

To compute the value ε_3 we use Corollary 3.6 and Remark 3.7 (Lemma 7.3 says that there is no twist involved in the congruence). When $3 \mid a$, the curve $\mathcal{C}_{5,3}^+(t_0)$ has potential good reduction (by Proposition 7.4 and Remark 7.5) so the conductor exponent must be 2. The value of ε_5 when $5 \mid bc$ follows from Corollary 3.11 (and Remark 3.12) and the result when $5 \mid a$ follows from Proposition 7.4. To prove the last two cases we need to characterize when the polynomial $F(x)$ is irreducible over $\mathbb{Q}_5$. By Lemma 3.13 there are finitely many cases to consider, so we run all cases with a script available in [PV25]. $\square$

Table 7.2 contains information on the spaces of cuspidal Hilbert modular forms of parallel weight 2 and level $\Gamma_0(3^i \cdot (\sqrt{5})^j)$ for $i, j \in \{2, 3\}$. We included an extra column with the number of newforms whose coefficient field equals $\mathbb{Q}(\sqrt{5})$. The script used to compute the entries of the table is available at [PV25].

Level	Dimension	# newforms	$K_{\mathcal{F}} = K$
$3^2 \cdot (\sqrt{5})^2$	45	14	4
$3^3 \cdot (\sqrt{5})^2$	405	111	11
$3^2 \cdot (\sqrt{5})^3$	225	35	21
$3^3 \cdot (\sqrt{5})^3$	2025	112	44

TABLE 7.2. Dimensions of spaces of Hilbert modular forms.

7.3. Ghost solutions. Recall from Definition 5.1 that ghost solutions are tuples of the form $(a, c, s, \ell, m, n, u, v)$, $ac \neq 0$, satisfying

$$(36) \quad 5^s 3^\ell x^5 \pm 5^m 3^n + 5^u 3^v z^3 = 0,$$

that might not be solutions to (3). Following §5, in this case we set $t_0 := -\frac{a^5 5^{s-u} 3^{\ell-v}}{c^3}$. Doing a computer search we found the list of solutions given in Table 7.3 (which is probably complete). The table includes the conductor of the 2-dimensional representation obtained from this specialization.

t_0	Conductor	t_0	Conductor	t_0	Conductor	t_0	Conductor	t_0	Conductor
$-\frac{32}{343}$	$3^4 \cdot (\sqrt{5})^2$	$\frac{9}{8}$	$3^3 \cdot (\sqrt{5})^3$	$-\frac{1}{8}$	$3^3 \cdot (\sqrt{5})^3$	$\frac{25}{24}$	$3^5 \cdot (\sqrt{5})^7$	$-\frac{1}{24}$	$3^5 \cdot (\sqrt{5})^3$
$\frac{32}{5}$	$3^2 \cdot (\sqrt{5})^5$	$\frac{3}{8}$	$3^4 \cdot (\sqrt{5})^5$	$\frac{32}{27}$	$3^3 \cdot (\sqrt{5})^5$	$\frac{5}{8}$	$3^4 \cdot (\sqrt{5})^7$	$\frac{1024}{1029}$	$3^5 \cdot (\sqrt{5})^5$

TABLE 7.3. Ghost solutions.

The values $t_0 = -\frac{1}{8}$ and $t_0 = \frac{9}{8}$ (coming from Catalan's solution) are the only ones that correspond to newforms in $S_2(\Gamma_0(3^3 \cdot (\sqrt{5})^3))$. As we will prove, they correspond to the two newforms in the space that cannot be discarded using Mazur's method, which is the standard strategy to eliminate possible newforms satisfying (35). Note that the study of ghost solutions allows us to find an equation for the hyperelliptic curve attached to these two newforms. This might be useful while applying other elimination procedures, like the symplectic argument.

7.4. Discarding solutions: Mazur's method. To discard forms not related to solutions, we apply an idea due to Mazur that we now recall. Let $\mathcal{F}$ be one of the newforms in the above described spaces. For $\mathfrak{l}$ a prime ideal of $K = \mathbb{Q}(\sqrt{5})$, denote by $a_{\mathfrak{l}}(\mathcal{F})$ the $\mathfrak{l}$ -th Fourier coefficient of $\mathcal{F}$.

The underlying idea of the method is to understand solutions to (3) modulo ℓ (for ℓ a rational prime not dividing $15p$). They will impose congruence conditions on the Fourier coefficient $a_{\mathfrak{l}}(\mathcal{F})$ for prime ideals $\mathfrak{l}$ of K dividing ℓ . More concretely, let

$$S_{\ell} = \{(\tilde{a}, \tilde{b}, \tilde{c}) \in \mathbb{F}_{\ell}^3 \setminus \{(0, 0, 0)\} : \tilde{a}^5 + \tilde{b}^p + \tilde{c}^3 = 0\}.$$

Any primitive solution (a, b, c) to (3) reduces to an element in S_{ℓ} . Let $S_{\ell}^{\times}$ denote the subset of S_{ℓ} made of elements where none of their entries are zero. Then a solution (a, b, c) belongs to one of the three cases:

Case (1). Its reduction $(\tilde{a}, \tilde{b}, \tilde{c}) \in S_{\ell}^{\times}$.

Case (2). Either $\tilde{a} = 0$ or $\tilde{c} = 0$, but $\tilde{b} \neq 0$.

Case (3). $\tilde{b} = 0$.

7.4.1. *Case (1)*: The first case corresponds to values $t_0 = -\frac{a^5}{c^3} \in \mathbb{P}^1(\mathbb{Q}) \setminus \{0, 1, \infty\}$ whose mod ℓ reduction $\tilde{t}_0$ belongs to $\mathbb{F}_\ell \setminus \{0, 1\}$. We compute for each $\tilde{t}_0 \in \mathbb{F}_\ell \setminus \{0, 1\}$ the value $a_{\mathfrak{l}}(\mathfrak{h}^+)$ corresponding to the trace of $\text{Frob}_{\mathfrak{l}}$ acting on $\mathfrak{h}^+$, using the code written by Fernando Rodriguez Villegas in Pari/GP (available at the GitHub repository <https://github.com/frvillegas/frvmath>). If the newform $\mathcal{F}$ is attached to a solution, then

$$(37) \quad a_{\mathfrak{l}}(\mathcal{F}) \equiv a_{\mathfrak{l}}(\mathfrak{h}^+) \pmod{\mathfrak{p}},$$

where $\mathfrak{p}$ is a prime ideal dividing p in the composition of $K_{\mathcal{F}}$ and K .

Remark 7.9. There is a caveat here: Villegas' code computes the trace of a Frobenius element as an ℓ -adic number. The number is in fact algebraic (see [GPV, Corollary 6.19]). In practice, we compute the number to enough precision that allows us to recognize it as a root of a degree 2 polynomial (without distinguishing between the value and its Galois conjugate).

Remark 7.10. For the particular signature $(5, p, 3)$ we have a genus 2 curve giving rise to the hyperelliptic motive so we can compute its Euler factor using Magma. However, we deliberately choose to use the aforementioned code to compute traces of hypergeometric motives, because our implementation works for any signature, opening the door to future applications with signatures for which no Frey hyperelliptic curve is known.

We say that a prime ℓ *eliminates* Case (1) of a newform $\mathcal{F}$ if there exists a prime $\mathfrak{l} \mid \ell$ of K such that the left-hand side of (37) is different from the right-hand side for all possible specializations $\tilde{t}_0 \in \mathbb{F}_\ell \setminus \{0, 1\}$. In this case we get an explicit bound for the exponents p for which a solution might exist, since $\mathfrak{p}$ must divide the (non-zero) value $a_{\mathfrak{l}}(\mathcal{F}) - a_{\mathfrak{l}}(\mathfrak{h}^+)$.

We say that a prime ℓ *fails* Case (1) for $\mathcal{F}$ if for each prime ideal $\mathfrak{l} \mid \ell$ there is a value $\tilde{t}_0$ such that both sides coincide. Catalan's solutions described in §5 correspond to two Hilbert newforms in $S_2(\Gamma_0(3^3 \cdot (\sqrt{5})^3))$ that give for each prime ℓ two values $\tilde{t}_0$ (the reduction of $-\frac{1}{8}$ or $\frac{9}{8}$ modulo ℓ) for which Case (1) fails.

Remark 7.11. If the coefficient field $K_{\mathcal{F}}$ is different from K , then there is a prime ideal $\mathfrak{l}$ that eliminates Case (1) of $\mathcal{F}$. This argument applies to all three cases, so while proving non-existence of solutions to (3) for p sufficiently large, it is enough to restrict to newforms $\mathcal{F}$ satisfying $K_{\mathcal{F}} = K$. This justifies the last column of Table 7.2.

Example 7.12. Take $\ell = 11$, a split prime in K and let $\mathfrak{l} = (\frac{-1+3\sqrt{5}}{2})$. We need to compute the minimal polynomial of $11 \cdot a_{\mathfrak{l}}(\mathfrak{h}^+)$ (the Tate twist of the motive, as explained in Remark 2.2) for $\tilde{t}_0 = 2, \dots, 6$. Here is how we compute the value for $\tilde{t}_0 = 2$ with a 10 digits precision.

```
? hgm(2, [1/5, -1/5], [1/3, -1/3], 11, 1, 10)
% 1 = 9*11^-1 + 7 + 4*11^2 + 11^3 + 4*11^4 + 2*11^5 + 3*11^6 + 5*11^7 + 7*11^8
+ 0(11^9)
```

The input of the routine is as follows: the value $\tilde{t}_0$, the parameters, the prime ℓ , the inertial degree of ℓ in K (in this case 1) and the precision. We can recognize this value as an algebraic integer as follows.

```
? algdep(hgm(2, [1/5, -1/5], [1/3, -1/3], 11, 1, 10)*11, 2)
%2 = x + 5
```

There is a caveat here: the script that we used does not depend on the prime ideal $\mathfrak{l}$, but of its norm! The underlying reason is that the finite hypergeometric sum is really a number in $K = \mathbb{Q}(\sqrt{5})$. If we want to describe it as an element in $\mathbb{Q}_{11}$ we need to choose a prime ideal

of K dividing 11. Different choices give algebraic conjugate values, so while recognizing the 11-adic number as an algebraic number, we do not distinguish between the values at both primes dividing 11, but in practice this is still good enough for the elimination procedure.

Applying the algorithm to the other values of $\tilde{t}_0$ we obtain the values in Table 7.4.

t_0	$11a_l(\mathfrak{h}^+)$	t_0	$11a_l(\mathfrak{h}^+)$	t_0	$11a_l(\mathfrak{h}^+)$
2	$x + 5$	5	$x - 2$	8	$x^2 + 2x - 19$
3	$x^2 - 5$	6	$x^2 + 4x - 1$	9	$x^2 + 3x + 1$
4	$x^2 - x - 1$	7	$x^2 + 5x - 5$	10	$x^2 + x - 11$

TABLE 7.4. Characteristic polynomial of $11a_l(\mathfrak{h}^+)$.

The space $S_2(\Gamma_0(3^2 \cdot (\sqrt{5})^2))$ has 14 (Galois orbits of) newforms whose l -th eigenvalue equals: $0, -2, 0, 4, -2, 0, \pm 3\sqrt{2}, \pm 2\sqrt{2}, \frac{\pm 9 + \sqrt{5}}{2}, \frac{\pm 3 + 3\sqrt{5}}{2}$. These values are different from the roots of the polynomials in Table 7.4, proving that the form cannot correspond to a Case (1) solution if $p > 281$. Using more values for the prime ℓ we get a better bound.

7.4.2. Case (2): The second case corresponds to a prime ℓ dividing ac (so t_0 reduces to 0 or ∞ modulo ℓ). Since $\ell \nmid b$, by Theorem 2.1, the motive $\mathfrak{h}^+$ has good reduction at any prime ideal of K dividing ℓ , but Euler's curve has bad reduction at it. Using the formula given in [GPV, Appendix A] we can compute the trace of a Frobenius element for a prime ideal λ of $F = \mathbb{Q}(\zeta_{15})$ dividing ℓ . Write $t_0 = \ell^{v_\ell(t_0)} \tilde{t}_0$. Then if t_0 reduces to 0 modulo ℓ , the trace of Frob_λ is given by the formula

$$(38) \quad -\mathcal{N}(\lambda)^{-1} \cdot (\chi_\lambda(\tilde{t}_0)^{-3} J(\chi_\lambda^2, \chi_\lambda^{-8}) + \chi_\lambda(\tilde{t}_0)^3 J(\chi_\lambda^{-6}, \chi_\lambda^8)),$$

where χ_λ is as in (14) and $J(\cdot, \cdot)$ denotes the usual Jacobi sum. If t_0 reduces to ∞ , then the trace of Frob_λ equals

$$(39) \quad -\mathcal{N}(\lambda)^{-1} \cdot (\chi_\lambda(\tilde{t}_0)^{-5} J(\chi_\lambda^2, \chi_\lambda^8) + \chi_\lambda(\tilde{t}_0)^5 J(\chi_\lambda^{10}, \chi_\lambda^{-8})).$$

As in the previous case, the value of $\tilde{t}_0 \bmod \ell$ is a priori any element of $\mathbb{F}_\ell^\times$.

Remark 7.13. The character χ_λ^3 has order 5 so (38) takes at most five different values. Similarly, the character χ_λ^5 has order 3, so (39) takes at most 3 different values.

Example 7.14. [PV25] contains a **Pari/GP** file with different scripts. One of them allows to compute the set of possible values for (38) and (39). Take as in the previous example $\ell = 11$ and let us show how to compute (38). The five different values are computed by

```
? Degenerate0(11)
%1 = [x^2 - 4*x - 316, x^2 + x - 101, x^2 - 19*x - 61, x^2 - 19*x + 59, x^2 + 41*x + 419]
```

Note that this gives the characteristic polynomial of the possible values of the Frobenius coefficient over the field F . We can compute the possible values over K using **Magma** scripts for the hyperelliptic curve $\mathcal{C}_{5,3}^+$. Concretely, let $t = 11^5$ and $l = (4 + \sqrt{5})$. Then we get

```
_<x> := PolynomialRing(Rationals());
K := NumberField(x^2-5);
O := Integers(K);
Id := Factorisation(11*O)[1][1];
EulerFactor(BaseExtend(HyperellipticCurve(5*x^6-12*x^5+10*11^5*x^3+11^10),K),Id);
121*x^4 - 44*x^3 + 6*x^2 - 4*x + 1
```

Over the field K the degree 4 polynomial factors in the form

$$(11x^2 - (2 + 2\sqrt{5})x + 1)(11x^2 - (2 - 2\sqrt{5})x + 1),$$

so the value $a_{\mathfrak{l}}(\mathfrak{h})$ corresponds (after normalizing by multiplication by $N(\mathfrak{l}) = 11$) to one of the values $2 \pm 2\sqrt{5}$. The prime $\mathfrak{l}$ has inertial degree 2 over F , so the trace of Frobenius at a prime dividing λ equals $a_{\lambda} = a_{\mathfrak{l}}^2 - 22$, which is a root of $x^2 - 4x - 316$ (the first polynomial).

To discard Case (2) solutions, we compute all possible values (38) and (39) using our script and compare them to the values $a_{\mathfrak{l}}(\mathcal{F})$ for each newform $\mathcal{F}$. If they are different, then the ideal $\mathfrak{p}$ must divide their difference.

7.4.3. Case (3): This case corresponds to the usual *level lowering* situation, namely the representation attached to the motive $\mathfrak{h}^+$ has multiplicative reduction at the prime ideal $\mathfrak{l}$ but the residual $\mathfrak{p}$ -th adic one does not. If $\mathcal{F}$ is an eigenform in this situation, the following congruence must hold

$$(40) \quad a_{\mathfrak{l}}(\mathcal{F}) \equiv \pm(N(\mathfrak{l}) + 1) \pmod{\mathfrak{p}}.$$

The described procedure allows us to prove the following result.

Theorem A. *Set $\mathcal{P} := \{2, 3, 5, 7, 11, 13, 19, 29, 31, 41, 61, 71, 79, 89, 101, 109\}$. Let p be a prime number not in $\mathcal{P}$ and let (a, b, c) be a primitive solution to*

$$(3) \quad x^5 + y^p + z^3 = 0.$$

Set $t_0 := -a^5/c^3$. Then one of the following holds:

- (1) *The residual Galois representation $\bar{\rho}_{\mathfrak{p}}^+$ is reducible.*
- (2) *The Galois representation $\rho_{\mathfrak{p}}^+$ is congruent modulo p to the Galois representation associated to a Hilbert modular form of $\mathbb{Q}(\sqrt{5})$ with complex multiplication by $\mathbb{Q}(\sqrt{-3})$ or by $\mathbb{Q}(\zeta_{15})/\mathbb{Q}(\sqrt{5})$.*
- (3) *The Galois representation $\rho_{\mathfrak{p}}^+$ is congruent modulo p to $\rho_{\mathcal{C}_{5,3}^+(t_0), \mathfrak{p}}$, where $t_0 \in \{-\frac{1}{8}, \frac{9}{8}\}$ (these two values correspond to ghost solutions).*

Proof. Let $\rho_{\mathfrak{p}}^+$ be the representation of $\mathcal{H}^+$ specialized at t_0 , as in (13). Suppose that $\bar{\rho}_{\mathfrak{p}}^+$ is irreducible. Then by Theorem 7.8, there exists a Hilbert newform $\mathcal{F} \in S_2(\Gamma_0(3^i \cdot (\sqrt{5})^j))$, with $i, j \in \{2, 3\}$, such that $\bar{\rho}_{\mathfrak{p}} \simeq \bar{\rho}_{\mathcal{F}, \mathfrak{p}}$, where $\mathfrak{P}$ is a prime in $K_{\mathcal{F}}$ dividing p .

We apply an implementation in **Magma** of the method previously described as follows: first we chose a set of primes to apply the elimination procedure. The stated result corresponds to the choice all rational primes ℓ not dividing 30 having a prime ideal $\mathfrak{l}$ in K dividing it with $N(\mathfrak{l}) \leq 400$. For each prime $\mathfrak{l}$ we compute the possible value of $a_{\mathfrak{l}}(\mathcal{H}^+(t))$ for all values of t (including $t = 0, \infty$) with the **Pari/GP** command

```
? MagmaInput([7, 11, 13, 17, 19, 29, 31, 41, 59, 61, 71, 79, 89, 101, 109, 131,
139, 149, 151, 179, 181, 191, 199, 211, 229, 239, 241, 251, 269, 271, 281, 311,
331, 349, 359, 379, 389])
```

This creates a file called “Data.txt” (available in [PV25]) needed to run the elimination script in **Magma**. In **Magma**, we load the file and run the following command.

```

> load "Data.txt";
> time TheoremA(2,2,Data);
The newforms [ 3, 9, 12 ] cannot be discarded using primes in [ 7, 11, 13, 17,
19, 29, 31, 41, 59, 61, 71, 79, 89, 101, 109, 131, 139, 149, 151, 179, 181, 191,
199, 211, 229, 239, 241, 251, 269, 271, 281, 311, 331, 349, 359, 379, 389 ].
The rest of the newforms can be discarded for p outside [ 2, 3, 5, 11, 19, 29 ]
Time: 57.710

```

This proves that except for the forms indexed by 3, 9, 12, all other newforms in the space $S_2(\Gamma_0(3^2 \cdot (\sqrt{5})^2))$ can be discarded if $p \notin \{2, 3, 5, 11, 19, 29\}$. We are led to prove that the three forms that cannot be eliminated in fact have complex multiplication. It is not hard to verify that the first form has complex multiplication by $\mathbb{Q}(\sqrt{-3})$ and the other two ones by $\mathbb{Q}(\zeta_{15})/K$. The way to verify this is simple: take the first form $\mathcal{F}$, and compute its quadratic twist by $\sqrt{-3}$. If it equals $\mathcal{F}$ then it has complex multiplication by $\sqrt{-3}$. Otherwise, it should correspond to either another newform in the same space, or a newform in a smaller space. The space of Hilbert modular forms of level $(\sqrt{5})^2$ is the zero space, while the space of level $3 \cdot (\sqrt{5})^2$ has three newforms. By looking at the Fourier coefficient at a prime ideal $\mathfrak{l}$ dividing 11 it is easy to verify that none of them corresponds to a twist of $\mathcal{F}$ (because $a_{\mathfrak{l}}(\mathcal{F}) = 0$, while the others are nonzero). Similarly, looking at the Fourier coefficients of newforms in $S_2(\Gamma_0(3^2 \cdot (\sqrt{5})^2))$, it is easy to verify that if $\mathfrak{l}$ is a prime ideal dividing 79 then $a_{\mathfrak{l}}(\mathcal{F}) = 16$ while no other form has eigenvalue ± 16 at $\mathfrak{l}$. A similar computation proves the result for the other two forms.

The output for the other three spaces (including the running time) is available at [PV25]. We summarize the needed information:

- (1) All newforms in $S_2(\Gamma_0(3^2 \cdot (\sqrt{5})^3))$ except the ones labeled $\{1, 7, 11, 12, 13, 16, 21\}$ can be discarded when $p \notin \{2, 3, 5, 7, 11, 13, 19, 29, 31\}$.
- (2) All newforms in $S_2(\Gamma_0(3^3 \cdot (\sqrt{5})^2))$ except the ones labeled $\{64, 65, 69, 73, 77, 78, 79\}$ can be discarded when $p \notin \{2, 3, 5, 7, 11, 19, 29, 41, 61\}$.
- (3) All newforms in $S_2(\Gamma_0(3^3 \cdot (\sqrt{5})^3))$ except the ones labeled $\{22, 39\}$ can be discarded when $p \notin \{2, 3, 5, 7, 11, 13, 19, 29, 31, 41, 61, 71, 79, 89, 101, 109\}$.

The seven newforms that cannot be eliminated in (1) have complex multiplication by $\mathbb{Q}(\sqrt{-3})$. One of them corresponds to the trivial solution $(\pm 1, \mp 1, 0)$, as proved in Lemma 4.5 (see also Remark 4.6), and another is its quadratic twist by $\mathbb{Q}(\zeta_{15})$. The seven forms in (2) have complex multiplication by $\mathbb{Q}(\zeta_{15})$. One of them arises from the trivial solution $(0, \pm 1, \mp 1)$, by Lemma 4.4 and Remark 4.6, and another is its quadratic twist by $\sqrt{-3}$. Finally, the two newforms in (3) that cannot be discarded correspond to the stated ghost solutions. $\square$

Remark 7.15. While studying other families of exponents (like the family (p, p, r)) once a finite set of primes $\mathcal{P}$ is obtained, one might try to use an idea due to Kraus (as explained in [Kra98]) to prove non-existence of solutions for primes in $\mathcal{P}$. Kraus' idea consists on given a particular value of p , chose a prime ℓ satisfying that $p \mid \ell - 1$. Then one expects that not all values $\tilde{t}_0 \in \mathbb{F}_{\ell} \setminus \{0, 1\}$ in Case (1) come from elements of S_{ℓ} , so computing the particular set S_{ℓ} gives an improvement on the elimination procedure. Take for example the generalized Fermat equation with signature $(7, 7, 5)$, and take $\ell = 71$. Then the only values for t_0 in Case (1) are $\{4, 12, 19, 31, 36, 41, 53, 60, 68\}$ (there are only 9 possible values instead of 69).

It looks like Kraus' idea does not give any improvement while working with three different prime exponents. We computed for many pairs (p, ℓ) , with $15p \mid \ell - 1$, the set S_ℓ and the corresponding value $\tilde{t}_0$ for the exponents $(5, p, 3)$. In all cases the resulting set was the whole $\mathbb{F}_\ell \setminus \{0, 1\}$.

The second stated result follows readily from Theorem A.

Theorem B. *Let $\mathcal{P} = \{2, 7, 11, 13, 19, 29, 31, 41, 61\}$. Then if p is a prime number not in $\mathcal{P}$, there are no primitive solutions (a, b, c) to*

$$x^5 + y^p + z^3 = 0$$

with b odd and $3 \mid b$ or $5 \mid b$.

Proof. Let (a, b, c) be a solution to the equation satisfying that $3 \mid b$ or $5 \mid b$ and $p > 13$. Since b is odd, the representation $\bar{\rho}_{\mathfrak{p}}^+$ is absolutely irreducible (by Corollary 7.7 since $C(2) = 13$). By Proposition 7.4 (and Remark 7.5) the local type at 3 or at $(\sqrt{5})$ (depending on whether $3 \mid b$ or $5 \mid b$) is special. Then we can lower the level at all primes of K dividing abc , keeping the same local type as $\rho_{\mathfrak{p}}^+$ at primes dividing $3 \cdot (\sqrt{5})$. The local type being special implies that there exists a Hilbert newform $\mathcal{F}$ without complex multiplication in $S_2(\Gamma_0(3^{\varepsilon_3} \cdot (\sqrt{5})^{\varepsilon_5}))$ congruent to $\rho_{\mathfrak{p}}^+$, with $\varepsilon_3 = 2$ or $\varepsilon_5 = 2$. This contradicts Theorem A, since the fact that $\mathcal{F}$ does not have complex multiplication rules out the second case, and the fact that one exponent is 2 rules out the third one. The set $\mathcal{P}$ is the one obtained while discarding forms in the spaces with $\varepsilon_3 = 2$ or $\varepsilon_5 = 2$. Note that we can also remove primes 3 and 5 from $\mathcal{P}$, since those cases were solved in [Bru00, Poo98]. $\square$

7.5. On the Local type at 3. Let $p > 3$ be a prime number and let $\mathfrak{p}$ be a prime ideal of K dividing p . The goal of this section is to compute the local type (see Definition 3.4) at 3 of the Galois representation $\rho_{\mathfrak{p}} := \rho_{\mathfrak{p}}^+$ (for different specializations of the parameter) in order to rule out the third case in Theorem A by imposing a local condition on the solution. Let $\mathbb{Q}_9$ denote the quadratic unramified extension of $\mathbb{Q}_3$.

Let $\rho_i : \text{Gal}_K \rightarrow \text{GL}_2(K_{\mathfrak{p}})$, $i = 1, 2$, denote the $\mathfrak{p}$ -adic Galois representations attached to $\mathfrak{h}_1 := \mathcal{H}^+(-1/8)$ and $\mathfrak{h}_2 := \mathcal{H}^+(9/8)$ respectively.

Lemma 7.16. *The local type of ρ_i (for $i = 1, 2$) at the prime ideal 3 is supercuspidal, given by the induction of a character of conductor 3 from a quadratic ramified extension L of $\mathbb{Q}_9$.*

Proof. Since the conductor exponent at 3 of $\mathfrak{h}_i$ (for $i = 1, 2$) is 3 and its Nebentypus is trivial, its local type is supercuspidal, obtained as the induction of a character of conductor 3 from a quadratic ramified extension L of $\mathbb{Q}_9$ (see for example [Pac13, Corollary 3.1]). $\square$

Lemma 7.17. *If $\bar{\rho}_{\mathfrak{p}} \simeq \bar{\rho}_i$, for $i = 1, 2$, then the local type of $\rho_{\mathfrak{p}}$ at 3 is also supercuspidal, induced from the same quadratic extension L .*

Proof. Follows from Lemma 7.16 and [GPV23, Proposition 1.1]. $\square$

The following theorem gives sufficient conditions to prove $\bar{\rho}_{\mathfrak{p}} \not\simeq \bar{\rho}_i$, for $i = 1, 2$.

Theorem 7.18. *Let (a, b, c) be a primitive solution to (3) with $p > 3$ satisfying that $3 \nmid c$. Set $t_0 = -\frac{a^5}{c^3}$. Then $\mathcal{H}^+(t_0)$ is not congruent modulo p to $\mathcal{H}^+(-1/8)$ nor to $\mathcal{H}^+(9/8)$.*

Proof. Let K be a finite extension of $\mathbb{Q}_p$. Let $\mathcal{C}$ be a hyperelliptic curve defined over K by an equation

$$\mathcal{C} : y^2 = F(x).$$

Then by [DDMM23, Theorem 1.9], if $\text{Jac}(\mathcal{C})$ has potentially good reduction, it attains good reduction over an at most quadratic ramified extension of $K(\mathcal{R})$, the splitting field of $F(x)$ (where $\mathcal{R}$ is the set of roots of $F(x)$, following the authors' notation). In our case, from (32) we have $F(x) = 5x^6 - 12x^5 + 10tx^3 + t^2$ and $K = \mathbb{Q}_9$.

For $t_0 = -\frac{1}{8}$ we obtain the polynomial $5x^6 - 12x^5 - \frac{5}{4}x^3 + \frac{1}{64}$, giving the same extension as the *reduced* polynomial $x^6 + 3x^4 + 9$ (a reduction algorithm is implemented in [JR06]). The isomorphisms of fields can also be verified with the Pari/GP implementation of Panayi's algorithm available in [JR06]. Its Galois closure $\mathbb{Q}_9(\mathcal{R})$ has Galois group D_{12} , the dihedral group with 12 elements, and ramification degree 6. To verify whether the Jacobian of $\mathcal{C}_{5,3}^+(-1/8)$ attains good reduction over $\mathbb{Q}_9(\mathcal{R})$ or a quadratic extension of it, we can compute its cluster picture in Magma using Tim Dokchitser's implementation.

```
> ClusterPicture(5*x^6 - 12*x^5 - 5/4*x^3 + 1/64,3);
((1,2,3),(4,5,6)) F[[7,8]] d=[5/6,5/6,0]
```

Note that over $\mathbb{Q}_9(\mathcal{R})$ it is true that all distances are integral (since $e = 6$), but the last condition of the semistability criterion ([DDMM23, Definition 1.7]) is not satisfied. By Lemma 7.16, the local inertial type of $\text{Jac}(\mathcal{C}_{5,3}^+(-1/8))$ at 3 is the induction of an order 6 character $\varkappa$ from one of the quadratic ramified extensions of $\mathbb{Q}_9$. There are two such extensions, namely $L_1 = \mathbb{Q}_9(\sqrt{3})$ and $L_2 = \mathbb{Q}_9(\sqrt{3\alpha})$, where $\alpha^2 = -1$ in $\mathbb{Q}_9$.

Then the field $\mathbb{Q}_9(\mathcal{R})$ is the fixed field of the local Galois representation obtained from the induction of the character $\varkappa^2$. We can compute the degree 12 polynomial corresponding to $\mathbb{Q}_9(\mathcal{R})/\mathbb{Q}_3$ in three different ways:

- by a direct computation (compute the compositum of the polynomial with itself and compute its 3-adic factors),
- search the database of David Roberts and John Jones [JR06] (see also [RJK⁺25]) for degree 12 extensions and eliminate all of them but one,
- use the fact that our inertial type satisfies the properties listed in the introduction of [CMF25], so the representation appears as the 4-torsion of an elliptic curve (see Theorem 1.1 of loc. cit) that is computed by the authors.

Using any of these three approaches, the Galois closure is obtained by adding to $\mathbb{Q}_3$ a root of the degree 12 polynomial

$$P = x^{12} + 3x^{11} + 3x^{10} - 6x^9 + 3x^8 + 9x^7 + 9x^4 + 9x^3 + 9.$$

Using the polynomial P it is easy to verify that $L_1 \subset \mathbb{Q}_9(\mathcal{R})$. We mimic the same computation for $t_0 = \frac{9}{8}$ and get that the reduced polynomial also equals $x^6 + 3x^4 + 9$. Then the two ghost solutions share the same degree 12 extension.

Let (a, b, c) be a solution to (3) with $3 \nmid c$. According to Table 3.2 it must be the case that either $t_0 \equiv -1 \pmod{9}$ or $t_0 \equiv 2 \pmod{9}$. In both cases, the reduced polynomial equals $x^6 + 3x^4 + 6x^3 + 9x^2 + 63x + 9$, and its Galois closure corresponds to the field extension obtained by adding a root of $Q = x^{12} + 6x^{11} + 21x^{10} + 36x^9 + 30x^8 + 36x^7 + 3x^6 + 36x^5 + 27x^4 - 9x^2 + 36$. But the extension obtained from P is not isomorphic to that obtained from Q . $\square$

Remark 7.19. When $3 \mid c$, it occurs sometimes that the local type of ρ_p matches that of the ghost solutions.

Theorem C. *Let $\mathcal{P}$ be as in Theorem A. Assume that Conjecture 1 holds for $L = K = \mathbb{Q}(\sqrt{5})$ with constant C . Then if p is a prime number satisfying that $p \notin \mathcal{P}$ and $p > C$, there are no non-trivial primitive solutions (a, b, c) to*

$$x^5 + y^p + z^3 = 0$$

with $3 \nmid c$.

Proof. Let (a, b, c) be a putative non-trivial solution to (3) with $3 \nmid c$. Set $t_0 = -a^5/c^3$. By Corollary 7.6, the Jacobian of $\mathcal{C}_{5,3}^+(t_0)$ has a prime of potentially multiplicative reduction.

If the Jacobian of $\mathcal{C}_{5,3}^+(t_0)$ is isogenous over $\mathbb{C}$ to the product of two elliptic curves, then either the curves do not have complex multiplication (in which case Conjecture 1 is proven) or they do have complex multiplication and so all bad primes have potentially good reduction, leading to a contradiction. Therefore, we can assume that the Jacobian of $\mathcal{C}_{5,3}^+(t_0)$ is absolutely simple and does not have complex nor quaternionic multiplication. Then Conjecture 1 implies that if $p > C$, $\bar{\rho}_p$ cannot be reducible nor lie in the normalizer of a Cartan group. Theorem A then implies that $\bar{\rho}_p$ is isomorphic to $\bar{\rho}_1$ or $\bar{\rho}_2$, which contradicts Theorem 7.18. $\square$

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