

An extension of Euclid-Euler Theorem to certain α -perfect numbers

Gabriel Simões Cardoso

gabriel.cardoso@ua.pt

Department of Mathematics, University of Aveiro

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Motivation

Euclid-Euler Theorem

N is an even perfect number if and only if $N = 2^{p-1}(2^p - 1)$, where $2^p - 1$ is a Mersenne prime number.

Proof.

(Euler) Suppose $N = 2^a m$ and $\sigma(N) = 2N$. Then

$$2^{a+1}m = \sigma(N) = (2^{a+1} - 1)\sigma(m).$$

Since $\gcd(2^{a+1} - 1, 2^{a+1}) = 1$, then there exists a positive integer k such that $m = k(2^{a+1} - 1)$ and $\sigma(m) = k2^{a+1}$. Since k and m divide m , and $\sigma(m) = k + m$, then $2^{a+1} - 1$ is a prime number and $k = 1$. Now, $2^{a+1} - 1$ can only be a prime number if $a + 1$ is a prime number p . Therefore, $N = 2^{p-1}(2^p - 1)$. □

Index function

Let σ to be the sum of divisors function and

$$I(N) = \frac{\sigma(N)}{N}$$

to be the *index function* of N . Clearly, $I(N)$ is multiplicative. For p a prime number and a a non-negative integer, the index function $I(p^a)$ is monotonically increasing in a but monotonically decreasing in p . Also, for any prime number p and any positive integer a , we have

$$\frac{p+1}{p} \leq I(p^a) < \frac{p}{p-1}. \quad (1)$$

α -perfect

We say N is an α -perfect number if $I(N) = \alpha$.

Generalizing Euler's method

Let α be a rational number and $N > 1$ be an α -perfect number, then $I(N) = \alpha$. Using (1) and the multiplicativity of $I(N)$, there exist positive integers r and m , prime numbers p_i , and positive integers a_i , with $1 \leq i \leq r$; such that

$$N = m \prod_{i=1}^r p_i^{a_i}, \quad (2)$$

$$\alpha \prod_{i=1}^r \frac{p_i - 1}{p_i} \leq 1, \quad (3)$$

and

$$\gcd\left(m, \prod_{i=1}^r p_i^{a_i}\right) = 1.$$

Therefore,

$$\alpha m \prod_{i=1}^r p_i^{a_i} = \sigma(N) = \sigma(m) \prod_{i=1}^r \frac{p_i^{a_i+1} - 1}{p_i - 1}.$$

Therefore,

$$\alpha m \prod_{i=1}^r p_i^{a_i} = \sigma(N) = \sigma(m) \prod_{i=1}^r \frac{p_i^{a_i+1} - 1}{p_i - 1}.$$

Hence, for some integers k and d , we have

$$\sigma(m) = \frac{\beta k}{d} \prod_{i=1}^r p_i^{a_i+1} \quad (4)$$

and

$$m = \frac{k}{d} \prod_{i=1}^r (p_i^{a_i+1} - 1), \quad (5)$$

where

$$\beta = \alpha \prod_{i=1}^r \frac{p_i - 1}{p_i} \leq 1$$

and

$$d = \gcd \left(\beta \prod_{i=1}^r p_i^{a_i+1}, \prod_{i=1}^r (p_i^{a_i+1} - 1) \right).$$

We will try to find a lower bound for $\sigma(m)$, by summing divisors of m that are explicitly indicated in

$$m = \frac{k}{d} \prod_{i=1}^r (p_i^{a_i+1} - 1). \quad (6)$$

A comparison of the lower bound with

$$\sigma(m) = \frac{\beta k}{d} \prod_{i=1}^r p_i^{a_i+1} \quad (7)$$

will give us contradictions or conditions on the form of N . From now on, we will always consider $\beta = 1$.

Differences between powers of 2 and 3

Lemma

The only solutions of the diophantine equation

$$2^a - 3^b = -1 \quad (8)$$

are $(1, 1)$ and $(3, 2)$. Also, the only solutions of the diophantine equation

$$2^a - 3^b = 2^c - 1, \quad (9)$$

are $(2, 1, 1)$, $(4, 2, 3)$, and $(a, 0, a)$, for all $a \in \mathbb{Z}$.

Board!

Differences between powers of 2 and 3

Proof: We start by solving the equation $|2^a - 3^b| = 1$. Since 2 is a primitive root of 3^b , for any $b \geq 1$ then the solutions of $2^x \equiv \pm 1 \pmod{3^b}$ are $x = 3^{b-1}k$, for $k \geq 0$. But, for $b \geq 3$ we have

$$2^{3^{b-1}} > 2 \cdot 3^b.$$

Therefore, $b \in \{0, 1, 2\}$ and so we obtain the solutions $(a, 0, a)$ and $(2, 1, 1)$ of

$$2^a - 3^b = 2^c - 1.$$

Also, we obtain the solutions $(1, 1)$ and $(3, 2)$ of

$$2^a - 3^b = -1.$$

Differences between powers of 2 and 3

We have

$$2^a - 3^b = 2^c - 1.$$

Suppose $c \geq 2$. If c is even, then $3 \mid 2^c - 1$ and we obtain a contradiction. If c is odd, we have $2^a \equiv 1 \pmod{3}$ so a is even. Therefore, there exists a positive integer a' such that $a = 2a'$. As $c = 2c' + 1$, for some positive integer c' , we have $2^{2a'} - 3^b = 2^{2c'+1} - 1$. So $(-1)^{b+1} \equiv -1 \pmod{4}$. Hence, b is even and we have $b = 2b'$, for some non-negative integer b' .

Then we have

$$2^{2a'} - 3^{2b'} = 2^{2c'+1} - 1.$$

Therefore,

$$(-1)^{a'} - (-1)^{b'} \equiv 2 \cdot (-1)^{c'} - 1 \pmod{5}.$$

$$(-1)^{a'} - (-1)^{b'} \equiv 2 \cdot (-1)^{c'} - 1 \pmod{5}.$$

If a' and b' have the same parity we have a contradiction.

If a' is odd and b' is even we have a contradiction as well. Therefore, a' is even and b' is odd. So we have $4 \mid a$ and $b \equiv 2 \pmod{4}$, which implies

$$1 - (-1) \equiv 2 \cdot (-1)^{c'} - 1 \pmod{5}.$$

Hence, c' is odd and $c \equiv 3 \pmod{4}$. Since $c > 1$, then $a > c$. Therefore, we have $2^c(2^{a-c} - 1) = 3^b - 1$. Hence, $2^c \parallel 3^b - 1$.

Since $b \equiv 2 \pmod{4}$, then $2^3 \parallel 3^b - 1$. Hence, $c = 3$ and we can rewrite

$$2^a - 3^b = 2^c - 1$$

as

$$\left(2^{\frac{a}{2}} - 3^{\frac{b}{2}}\right) \left(2^{\frac{a}{2}} + 3^{\frac{b}{2}}\right) = 7.$$

Therefore, $a = 4$ and $b = 2$. Hence, if $c \geq 2$, the only solution of

$$2^a - 3^b = 2^c - 1$$

is $(4, 2, 3)$.

3-perfect numbers divisible by 6

Theorem

Suppose N is a 3-perfect number of the form $N = 2^a 3^b m$, where $a, b \geq 1$ and $\gcd(6, m) = 1$. Then $a = 1$ and $b \neq 1$, or $a \neq 1$ and $b = 1$.

Board!

Proof: Let $N = 2^a 3^b m$ such that $a, b \geq 1$, $\gcd(6, m) = 1$, and $\sigma(N) = 3N$. Then

$$3N = 2^a 3^{b+1} m = \sigma(N) = (2^{a+1} - 1) \frac{3^{b+1} - 1}{2} \sigma(m).$$

Therefore,

$$\frac{\sigma(m)}{m} = \frac{2^{a+1} 3^{b+1}}{(2^{a+1} - 1)(3^{b+1} - 1)}.$$

Let $d = \gcd(2^{a+1} 3^{b+1}, (2^{a+1} - 1)(3^{b+1} - 1))$. It is easy to see that $d = 2^s 3^t$, where $1 \leq s \leq a + 1$ and $0 \leq t \leq b + 1$.

Since

$$\gcd\left(\frac{2^{a+1}3^{b+1}}{2^s 3^t}, \frac{(2^{a+1}-1)(3^{b+1}-1)}{2^s 3^t}\right) = 1,$$

then

$$\sigma(m) = \frac{2^{a+1}3^{b+1}}{2^s 3^t}k \text{ and } m = \frac{2^{a+1}-1}{3^t} \frac{3^{b+1}-1}{2^s}k,$$

for some positive integer k .

Let us consider the following three cases, which will establish the claim.

Case A: Suppose that $t \neq 0$ and let

$$M = \max\left(\frac{2^{a+1}-1}{3^t}, \frac{3^{b+1}-1}{2^s}\right).$$

Then we have

$$\begin{aligned} \frac{\sigma(m)}{k} &= \frac{2^{a+1}3^{b+1}}{2^s 3^t} \\ &= \frac{((2^{a+1}-1)+1)((3^{b+1}-1)+1)}{2^s 3^t} \end{aligned}$$

3-perfect numbers divisible by 6

$$\begin{aligned} &= \frac{2^{a+1} - 1}{3^t} \frac{3^{b+1} - 1}{2^s} + \frac{2^{a+1} - 1}{3^t} \frac{1}{2^s} + \frac{3^{b+1} - 1}{2^s} \frac{1}{3^t} + \frac{1}{2^s 3^t} \\ &< \frac{m}{k} + \frac{M}{2} + \frac{M}{3} + 1 \\ &< \frac{m}{k} + M + 1. \end{aligned}$$

Therefore,

$$\sigma(m) < m + Mk + k. \quad (10)$$

By definition of M , we have $m = uMk$, with $u \in \left\{ \frac{2^{a+1}-1}{3^t}, \frac{3^{b+1}-1}{2^s} \right\}$. By definition of s and t , we have $u \in \mathbb{Z}$. Then $Mk \mid m$. If $Mk \neq m$ and $M \neq 1$, then m , Mk , and k are different divisors of m . Thus,

$$\sigma(m) \geq m + Mk + k. \quad (11)$$

By combination of inequalities (10) and (11), we have a contradiction.

3-perfect numbers divisible by 6

If $Mk = m$ or $M = 1$, then

$$\frac{2^{a+1} - 1}{3^t} = 1 \text{ or } \frac{3^{b+1} - 1}{2^s} = 1.$$

Therefore, $2^{a+1} - 3^t = 1$ or $3^{b+1} - 2^s = 1$. We have $a = 1$ or $b = 1$ (by the solutions of the diophantine equations presented before).

Case B: Suppose that $t = 0$ and

$$2^{a+1} - 1 \neq \frac{3^{b+1} - 1}{2^s}.$$

Let

$$M' = \min \left(2^{a+1} - 1, \frac{3^{b+1} - 1}{2^s} \right).$$

If $M' = 1$ then

$$2^{a+1} - 1 = 1 \text{ or } \frac{3^{b+1} - 1}{2^s} = 1.$$

3-perfect numbers divisible by 6

Therefore, $2^{a+1} = 2$ or $3^{b+1} - 2^s = 1$. As $a, b \geq 1$, we conclude that $b = 1$ (by the solutions of the diophantine equations presented before). If $M' \neq 1$ then

$$m, (2^{a+1} - 1)k, \frac{3^{b+1} - 1}{2^s}k, \text{ and } k,$$

are different divisors of m . Therefore,

$$\begin{aligned}\sigma(m) &\geq m + (2^{a+1} - 1)k + \frac{3^{b+1} - 1}{2^s}k + k \\ &> (2^{a+1} - 1) \frac{3^{b+1} - 1}{2^s}k + \frac{2^{a+1} - 1}{2^s}k + \frac{3^{b+1} - 1}{2^s}k + \frac{k}{2^s} \\ &= \sigma(m),\end{aligned}$$

where the strict inequality results from $s \geq 1$. So, we obtain a contradiction.

3-perfect numbers divisible by 6

Case C: Suppose that $t = 0$ and

$$2^{a+1} - 1 = \frac{3^{b+1} - 1}{2^s}.$$

Then $2^{a+1+s} - 3^{b+1} = 2^s - 1$. We have $(a, b) \in \{(0, 0), (0, 1)\}$ (by the solutions of the diophantine equations presented before). Since $a, b \geq 1$ we obtain a contradiction.

Hence, we must have $a = 1$ or $b = 1$.

Fermat numbers result

Let $F_n = 2^{2^n} + 1$ denote the n -th Fermat number, for any non-negative integer n . The next result states some properties of Fermat numbers that will be used later.

Lemma

Let F_n be the n -th Fermat number. Then

- 1 $F_{n-1} \mid F_n^{2^n} - 1$, for $n \geq 1$;
- 2 $\text{ord}_{2^{2^n+s}}(F_n) = 2^s$, for $n \geq 1$ and s a non-negative integer.

Board!

Proof of statement 1: Let $n \geq 1$. Clearly, $F_n = (F_{n-1} - 1)^2 + 1$. We have that

$$\begin{aligned} F_n^{2^n} - 1 &= \left((F_{n-1} - 1)^2 + 1 \right)^{2^n} - 1 \\ &= -1 + \left((F_{n-1}^2 - 2F_{n-1}) + 2 \right)^{2^n} \end{aligned}$$

$$\begin{aligned}
&= -1 + \sum_{j=0}^{2^n} \binom{2^n}{j} (F_{n-1}^2 - 2F_{n-1})^j 2^{2^n-j} \\
&= -1 + tF_{n-1} + 2^{2^n} \\
&= tF_{n-1} + F_0F_1 \cdots F_{n-1},
\end{aligned}$$

for some integer t . Hence, $F_{n-1} \mid F_n^{2^n} - 1$.

Proof of statement 2: The statement is clearly true for $s = 0$. We prove by induction that

$$F_n^{2^{s-1}} \equiv 1 + 2^{2^n+s-1} \pmod{2^{2^n+s}}, \quad (12)$$

for any $s \geq 1$.

Since $F_n \equiv F_n 2^{2^n+1}$, the congruence (12) is valid for $s = 1$. Now, suppose (12) is valid for s . Then there exists an integer t such that

$$\begin{aligned}
F_n^{2^s} &= \left(F_n^{2^{s-1}}\right)^2 \\
&= \left(1 + \left(2^{2^n+s-1} + 2^{2^n+s}t\right)\right)^2
\end{aligned}$$

$$\begin{aligned}
& (1 + (2^{2^n+s-1} + 2^{2^n+s}t))^2 \\
&= 1 + 2^{2^n+s} + 2^{2^n+s+1}t + 2^{2^{n+1}+2s-2} + 2^{2^{n+1}+2s}t^2 + 2^{2^{n+1}+2s}t \\
&\equiv 1 + 2^{2^n+s} \pmod{2^{2^n+s+1}}.
\end{aligned}$$

Therefore, Eq. (12) is true for any $s \geq 1$. So we have

$$\text{ord}_{2^{2^n+s}}(F_n) \neq 2^{s-1}.$$

Since $\text{ord}_{2^{2^n+s}}(F_n)$ is a power of 2 and

$$F_n^{2^s} \equiv 1 \pmod{2^{2^n+s}},$$

then statement 2. is obtained.

Before applying Euler's method we still need two technical results about some exponential diophantine equations. Our proof of these results is inspired by Styer's work (see references).

Lemma

Let $n \geq 1$ and F_n be the n -th Fermat number. Suppose there exists a non-negative integer s and a prime number q_n , such that

$$q_n \mid F_n^{2^s} - 1 \quad (13)$$

and

$$\text{ord}_{F_n}(2) \mid \text{ord}_{q_n}(2). \quad (14)$$

Then the exponential diophantine equation

$$2^a - F_n^b = 2^c - 1, \quad (15)$$

has no solutions (a, b, c) with $b \geq 1$ and $c \geq 2^n + s$.

Board!

Proof: Suppose there exists a non-negative integer s and a prime number q_n satisfying Cond. (13) and (14). Also, suppose that exists a solution (a, b, c) of Eq. (15) with $b \geq 1$ and $c \geq 2^n + s$. Since $b \geq 1$, we have $a > c$. Therefore, we have

$$F_n^b \equiv 1 \pmod{2^{2^n+s}}.$$

By previous Lemma, we have $\text{ord}_{2^{2^n+s}}(F_n) = 2^s$. Hence, $2^s \mid b$. Now, Cond. (13) and Eq. (15) imply $2^a - 1 \equiv 2^c - 1 \pmod{q_n}$. Let $t = \text{ord}_{q_n}(2)$. Then $a \equiv c \pmod{t}$. Hence, there exists a' such that $a = c + a't$. But, by Eq. (14), $\text{ord}_{F_n}(2) \mid t$, which implies

$$2^c - 1 \equiv 2^a \equiv 2^c (2^t)^{a'} \equiv 2^c \pmod{F_n}.$$

Since the previous congruence has no solutions, we obtain a contradiction. Thus, we obtain the stated result.

Lemma - exponential diophantine equations (board!)

Let F_n be the n -th Fermat number and consider the following exponential diophantine equations

$$2^a - F_n^b = -1 \quad (16)$$

and

$$2^a - F_n^b = 2^c - 1. \quad (17)$$

Then

(a) when $n \in \{1, 2, 3, 4\}$, Equation (16) only holds for $(a, b) = (2^n, 1)$;

(b) when $n \in \{2, 3, 4\}$, Equation (17) only holds for

$$(a, b, c) \in \{(a, 0, a) \mid a \in \mathbb{Z}\} \cup \{(2^n + 1, 1, 2^n)\};$$

(c) when $n = 1$, Equation (17) only holds for

$$(a, b, c) \in \{(a, 0, a) \mid a \in \mathbb{Z}\} \cup \{(3, 1, 2), (7, 3, 2), (5, 2, 3)\}.$$

Theorem

Let F_n be the n -th Fermat number. Then

- 1 if exists N such that $\frac{\sigma(N)}{N} = \frac{F_1}{2}$ and $F_1 \mid N$, then $2^4 \parallel N, F_1^2 \parallel N$, and $31^2 \mid N$.
- 2 if $n \in \{2, 3, 4\}$ then $\frac{2F_n}{F_n-1}$ is a F_n -abundancy outlaw.

Proof: Let $n \in \{1, 2, 3, 4\}$ and $F_n = 2^{2^n} + 1$. We can write $N = 2^a F_n^b m$ such that $a, b \geq 1$, $\gcd(2F_n, m) = 1$, and

$$\sigma(N) = \frac{2F_n}{F_n-1} N.$$

Then

$$\frac{2F_n}{F_n-1} = \frac{\sigma(N)}{N} = \frac{2^{a+1}-1}{2^a} \frac{F_n^{b+1}-1}{F_n^b(F_n-1)} \frac{\sigma(m)}{m}.$$

$\frac{2F_n}{F_n-1}$ -perfect numbers divisible by F_n

Therefore,

$$\frac{\sigma(m)}{m} = \frac{2^{a+1} F_n^{b+1}}{(2^{a+1} - 1)(F_n^{b+1} - 1)}.$$

Let

$$d = \gcd\left(2^{a+1} F_n^{b+1}, (2^{a+1} - 1)(F_n^{b+1} - 1)\right).$$

As $F_n - 1 \mid F_n^{b+1} - 1$, then $d = 2^s F_n^t$, where $2^n \leq s \leq a + 1$ and $0 \leq t \leq b + 1$. Since

$$\gcd\left(\frac{2^{a+1} F_n^{b+1}}{2^s F_n^t}, \frac{(2^{a+1} - 1)(F_n^{b+1} - 1)}{2^s F_n^t}\right) = 1,$$

then

$$\sigma(m) = \frac{2^{a+1} F_n^{b+1}}{2^s F_n^t} k \text{ and } m = \frac{2^{a+1} - 1}{F_n^t} \frac{F_n^{b+1} - 1}{2^s} k,$$

for some positive integer k .

$\frac{2F_n}{F_n-1}$ -perfect numbers divisible by F_n

Let us consider the following three cases, which will establish the claim.

Case A: Suppose that $t \neq 0$ and let

$$M = \max \left(\frac{2^{a+1} - 1}{F_n^t}, \frac{F_n^{b+1} - 1}{2^s} \right).$$

Then we have that

$$\begin{aligned} \frac{\sigma(m)}{k} &= \frac{2^{a+1} F_n^{b+1}}{2^s F_n^t} \\ &= \frac{2^{a+1} - 1}{F_n^t} \frac{F_n^{b+1} - 1}{2^s} + \frac{2^{a+1} - 1}{F_n^t} \frac{1}{2^s} + \frac{F_n^{b+1} - 1}{2^s} \frac{1}{F_n^t} + \frac{1}{2^s F_n^t} \\ &< \frac{m}{k} + \frac{M}{F_n - 1} + \frac{M}{F_n} + 1 \\ &< \frac{m}{k} + M + 1. \end{aligned}$$

Therefore,

$$\sigma(m) < m + Mk + k. \quad (18)$$

If $Mk \neq m$ and $M \neq 1$, then m has at least the divisors m , Mk , and k .

Thus,

$$\sigma(m) \geq m + Mk + k. \quad (19)$$

By combination of inequalities (18) and (19) we have a contradiction.

If $Mk = m$ or $M = 1$, then

$$\frac{2^{a+1} - 1}{F_n^t} = 1 \text{ or } \frac{F_n^{b+1} - 1}{2^s} = 1.$$

Thus $2^{a+1} - F_n^t = 1$ or $F_n^{b+1} - 2^s = 1$. The only solutions of these equations are $(a, t) = (0, 0)$ and $(b, s) = (0, 2^n)$, by the previous Lemma. Hence, we obtain a contradiction because $a, b \geq 1$.

$\frac{2F_n}{F_n-1}$ -perfect numbers divisible by F_n

Case B: Suppose $t = 0$ and

$$2^{a+1} - 1 \neq \frac{F_n^{b+1} - 1}{2^s}.$$

Let

$$M' = \min \left(2^{a+1} - 1, \frac{F_n^{b+1} - 1}{2^s} \right).$$

If $M' = 1$ then

$$2^{a+1} - 1 = 1 \text{ or } \frac{F_n^{b+1} - 1}{2^s} = 1.$$

Thus $2^{a+1} = 2$ or $F_n^{b+1} - 2^s = 1$. As $a, b \geq 1$, by the previous Lemma, we have a contradiction.

If $M' \neq 1$ then m has at least the divisors

$$m, (2^{a+1} - 1)k, \frac{F_n^{b+1} - 1}{2^s}k \text{ and } k.$$

Therefore,

$$\begin{aligned}\sigma(m) &\geq m + (2^{a+1} - 1)k + \frac{F_n^{b+1} - 1}{2^s}k + k \\ &> (2^{a+1} - 1) \frac{F_n^{b+1} - 1}{2^s}k + \frac{2^{a+1} - 1}{2^s}k + \frac{F_n^{b+1} - 1}{2^s}k + \frac{k}{2^s} \\ &= \sigma(m),\end{aligned}$$

where the strict inequality results from $s \geq 2^n$. Hence, we obtain a contradiction.

Case C: Suppose $t = 0$ and

$$2^{a+1} - 1 = \frac{F_n^{b+1} - 1}{2^s}.$$

Then $2^{a+1+s} - F_n^{b+1} = 2^s - 1$.

$\frac{2F_n}{F_n-1}$ -perfect numbers divisible by F_n

$$2^{a+1+s} - F_n^{b+1} = 2^s - 1$$

By the previous Lemma, we have

$$(F_n, a, b, s) \in \{(5, -1, -1, 1), (5, 0, 0, 2), (5, 1, 1, 3), (5, 4, 2, 2), (17, 0, 0, 4), \\ (257, 0, 0, 8), (65537, 0, 0, 16)\}$$

Since $a, b \geq 1$, we only have solutions for $F_n = 5$ and then

$$(F_n, a, b) = (5, 1, 1) \text{ or } (F_n, a, b) = (5, 4, 2).$$

If $(F_n, a, b) = (5, 1, 1)$ then

$$\frac{\sigma(N)}{N} = \frac{5}{2} = \frac{3}{2} \cdot \frac{6}{5} \cdot \frac{\sigma(m)}{m}.$$

Therefore, $9 \mid m$. But then

$$\frac{\sigma(N)}{N} = \frac{5}{2} \geq \frac{3}{2} \cdot \frac{6}{5} \cdot \frac{13}{9} > \frac{5}{2}.$$

Hence, $(F_n, a, b) = (5, 4, 2)$ and so $31^2 \mid N$.

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