Aveiro Summer School
Arithmetic Statistics
Exercise sheet 1

Alex Bartel

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1. Let (a, b, c) be a binary quadratic form, and let d be its discriminant. Show that if $d < 0$, then all integers that (a, b, c) represents have the same sign (either all positive or all negative), while if $d > 0$, then the form represents both positive and negative integers. In the former case, the signs of a and c are necessarily the same. If one has $d < 0$ and $a, c > 0$, the form is called *positive definite*; if one has $d < 0$ and $a, c < 0$, the form is *negative definite*; if one has $d > 0$, then the form is called *indefinite*.

2. A positive definite binary quadratic form (a, b, c) is called *reduced* if

   - either $c > a$ and $-a < b \leq a$,
   - or $c = a$ and $0 \leq b \leq a$.

   Show that if (a, b, c) is a reduced form, with discriminant $d < 0$, then one has $|b| \leq a \leq \sqrt{|d|/3}$. Deduce that given $d < 0$, there are only finitely many reduced forms of discriminant d.

3. For each of the following values for d, list all reduced forms of discriminant d: -3, -4, -7, -8, -15, -23.

4. Consider the following *reduction algorithm*. Let (a, b, c) be a positive definite binary quadratic form, so that in particular we have $a, c > 0$. Apply one of the following operations if possible:

   (A) if $c < a$, apply the matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ to obtain the equivalent form $(c, -b, a)$.

   (B) if $|b| > a$, apply the matrix $\left(\begin{smallmatrix} 1 & 0 \\ s & 1 \end{smallmatrix}\right)$ for suitable $s \in \mathbb{Z}$ to obtain the equivalent form (a, b', c'), where $b' = b + 2as$. Choose s such that $|b'| \leq a$.

   Replace the original form by the new one, and continue repeating these steps until neither condition is satisfied. When none of these steps are possible, perform a final "cleanup" step:

(A') if $a = c$, then use operation (A), if necessary, to ensure that $b \geq 0$.

(B') if $b = -a$, then use operation (B) with $s = 1$ to preserve a and change b to $+a$.

Show that this algorithm results in a reduced form that is equivalent to the original one. Deduce from this and Excercise 2 that for every $d < 0$, there exist only finitely many equivalence classes of binary quadratic forms of discriminant d, i.e. that the class number of d is finite.

5. For each of the following forms (a, b, c), find a reduced form equivalent to it:

   (a) $(12, -14, 9)$;

   (b) $(30, -25, 6)$;

   (c) $(21, -11, 3)$.

6. It is also true that every positive definite binary quadratic form is equivalent to a *unique* reduced one, so that the class number of d is equal to the number of reduced forms of discriminant d. Here is a sketch proof.

   (a) Let (a, b, c) be a positive definite reduced form. Show that a is the smallest positive integer properly represented by (a, b, c).

   (b) Show that b is the unique integer satisfying $|b| \leq a$ and ($b \geq 0$ if $a = c$) that appears as the xy-coefficient among all forms $(a, \ldots)$ that are equivalent to (a, b, c).

   (c) Deduce from these two assertions that every positive definite binary quadratic form is equivalent to a unique reduced one.

7. For each of the following forms f and g, decide whether f and g are equivalent:

   (a) $f(x, y) = 4x^2 + 3xy + y^2$, $g(x, y) = 2x^2 + 5xy + 4y^2$;

   (b) $f(x, y) = 7x^2 - 5xy + 2y^2$, $g(x, y) = 5x^2 + 7xy + 4y^2$.