Criptografia Aplicada, 2024/2025

RSA— and related subjects

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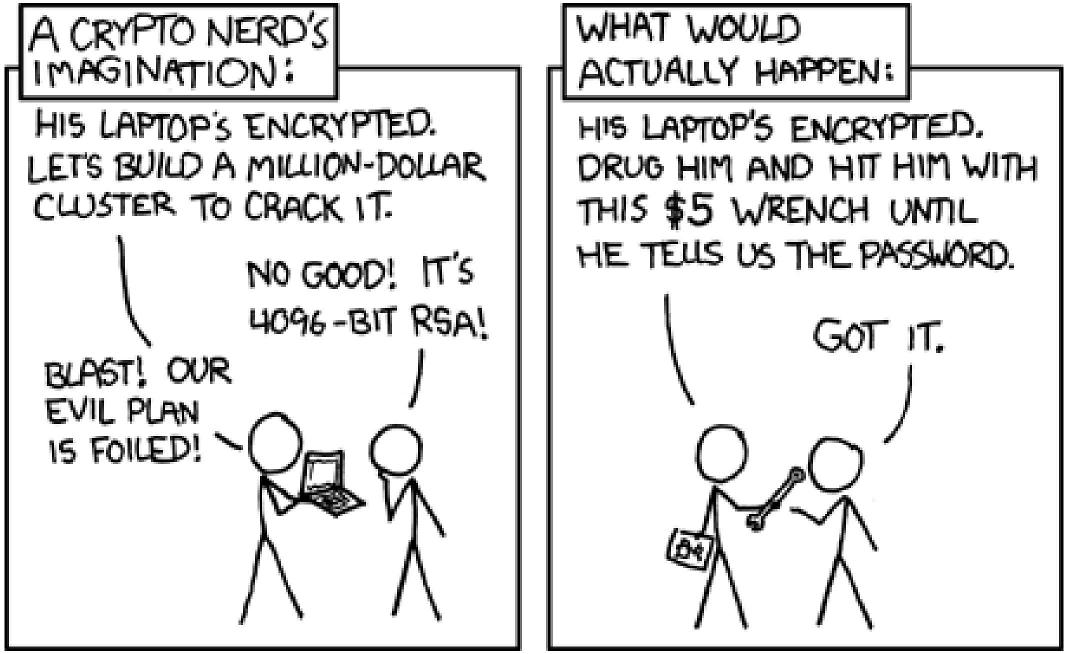
The Magic Words are Squeamish Ossifrage

Guess who contributed a modest amount of computation time to this collaborative effort.

Security (spoiler)


A CRYPTO NERD'S IMAGINATION:
HIS LAPTOP'S ENCRYPTED. LET'S BUILD A MILLION-DOLLAR CLUSTER TO CRACK IT.
NO GOOD! IT'S 4096-BIT RSA!
BLAST! OUR EVIL PLAN IS FOILED!
WHAT WOULD ACTUALLY HAPPEN:
HIS LAPTOP'S ENCRYPTED. DRUG HIM AND HIT HIM WITH THIS $5 WRENCH UNTIL HE TELLS US THE PASSWORD.
GOT IT.

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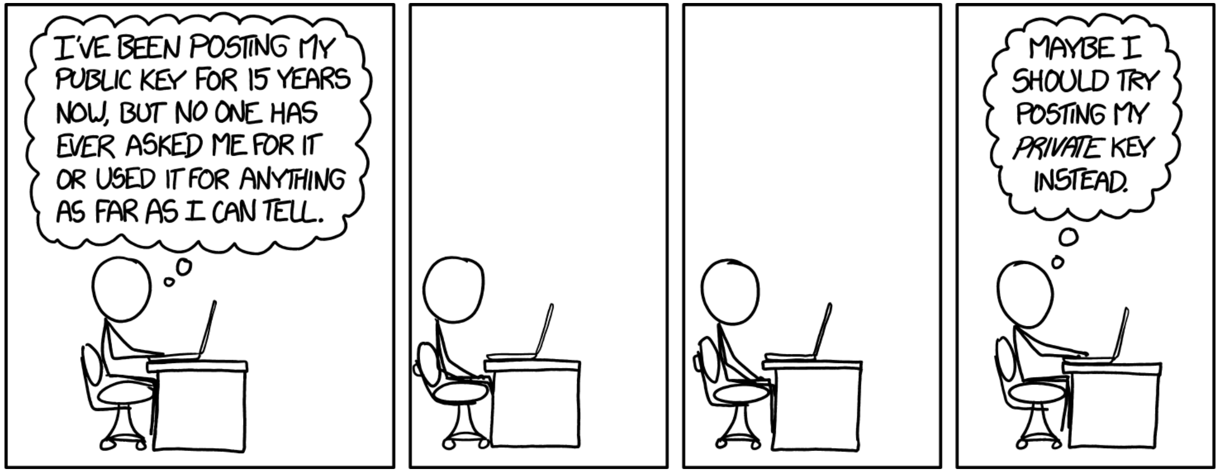
Goals

- Public-key cryptography
- Sharing secrets
- Doing things without leaking information

Public Key (spoiler)


I'VE BEEN POSTING MY PUBLIC KEY FOR 15 YEARS NOW, BUT NO ONE HAS EVER ASKED ME FOR IT OR USED IT FOR ANYTHING AS FAR AS I CAN TELL.
MAYBE I SHOULD TRY POSTING MY *PRIVATE* KEY INSTEAD.

Means

- Number theory.
- In particular, modular arithmetic. Why? Because:
  - we will be performing computations with integers;
  - we get exact results (there is no need to worry about roundoff errors, which may differ on different computing devices);
  - modular arithmetic can be done efficiently on almost all computing devices;
  - and last, but by no means least, because there exist many number theoretic theorems that have cryptographic applications.

Mathematics is the queen of the sciences and number theory is the queen of mathematics.

Carl Friedrich Gauss (1777–1855)

The Theory of Numbers has always been regarded as one of the most obviously useless branches of Pure Mathematics. The accusation is one against which there is no valid defence; and it is never more just than when directed against the parts of the theory which are more particularly concerned with primes. A science is said to be useful if its development tends to accentuate the existing inequalities in the distribution of wealth, or more directly promotes the destruction of human life. The theory of prime numbers satisfies no such criteria. Those who pursue it will, if they are wise, make no attempt to justify their interest in a subject so trivial and so remote, and will console themselves with the thought that the greatest mathematicians of all ages have found in it a mysterious attraction impossible to resist.

Godfrey Harold Hardy (1877–1947)

Programming languages you may use

- C, in particular the GNU MP library, also known as `libgmp`
- C++, using also the GNU MP library, but with classes and arithmetic operator overloading!
- Python
- Java, in particular the `BigInteger` class
- pari-gp (get it here), because it has everything we will need
- SageMath (get it here), because it has everything we will need and its interface uses the Python programming language (but it is a big download)

Modular arithmetic

notation	meaning
$m \mid n$	m divides n.
$m \nmid n$	m does not divide n.
$n^k \parallel m$	n^k divides m but n^{k+1} does not (k is the valuation of m at n).
$n \equiv r \pmod{m}$	$m \mid (n - r)$, that is, because m divides $n - r$, n and r have the same remainder when divided by m.
$\lfloor x \rfloor$	floor function: largest integer not larger than x.
$n \bmod m$	(binary operator) remainder of n when divided by m (m is called the modulus, which we assume here to be a positive integer). Equal to $n - m\lfloor \frac{n}{m} \rfloor$. Note that $0 \leqslant r < m$. In `C`, `Python`, `Java`, and `pari-gp`, it can be computed using the % binary operator (applied to unsigned integers).
$\gcd(a, b)$	greatest common divisor of a and b.
$\operatorname{lcm}(a, b)$	least common multiple of a and b; equal to $\dfrac{ab}{\gcd(a, b)}$.
$\mathbb{Z}_m$	set of equivalence classes modulo m; slightly abusing the mathematical notation for equivalence classes, $\mathbb{Z}_m = \{\, 0, 1, \ldots, m - 1 \,\}$.

Modular arithmetic examples

- $1 \mid 10$, $5 \mid 20$, $7 \mid 7$, $11 \mid 44$, $3 \nmid 5$
- $17 \equiv 7 \pmod{10}$, $27 \equiv 17 \pmod{10}$, $27 \equiv 7 \pmod{10}$
- $\lfloor 1.1 \rfloor = 1$, $\lfloor 7/3 \rfloor = 2$, $\lfloor -1.1 \rfloor = -2$
- $17 \bmod 6 = 5$, $7 \bmod 6 = 1$, $(17 \times 7) \bmod 6 = (5 \times 1) \bmod 6 = 5$
- $\gcd(15, 25) = 5$, $\gcd(7, 6) = 1$, when n is a positive integer, $\gcd(n, n + 1) = 1$
- $\operatorname{lcm}(15, 25) = 75$, $\operatorname{lcm}(7, 6) = 42$
- modulo m, the set of the integers — $\mathbb{Z}$ — is partitioned into m equivalence classes; we can choose as representative for each equivalence class an integer from the set $\mathbb{Z}_m$; for example, for $m = 5$, we have (the representative is underlined)

equivalence class with representative 0: $\ldots, -5, \underline{0}, 5, 10, \ldots$
equivalence class with representative 1: $\ldots, -4, \underline{1}, 6, 11, \ldots$
equivalence class with representative 2: $\ldots, -3, \underline{2}, 7, 12, \ldots$
equivalence class with representative 3: $\ldots, -2, \underline{3}, 8, 13, \ldots$
equivalence class with representative 4: $\ldots, -1, \underline{4}, 9, 14, \ldots$

The binary mod operator we defined in the previous slide computes this representative.

More modular arithmetic examples

Tables for addition (on the left) and multiplication (on the right) modulo 7.

$+$

$a\backslash b$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

$\times$

$a\backslash b$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

- All elements of $\mathbb{Z}_7$ have a symmetric value; given any a it is also possible to find b, which is unique, such that $a + b \equiv 0 \pmod{m}$. This is so for any modulus.

- In this case all non-zero elements of $\mathbb{Z}_7$ have inverses. However, this is not general. An element a of $\mathbb{Z}_m$ has an inverse if and only if $\gcd(a, m) = 1$. The inverse of a, if it exists, is the (unique in $\mathbb{Z}_m$) b such that $ab \equiv 1 \pmod{m}$. We say that $a^{-1} \equiv b \pmod{m}$. When the modulus is a prime number, as is the case here, only 0 does not have an inverse.

Modular arithmetic in C

- Addition, for small integers:

```
long add_mod(long a,long b,long m)
{ // assuming that 0 <= a,b < m, return (a+b) mod m, 0 < m < (1L << 62)
  long r = a + b;
  if(r >= m)
    r -= m;
  return r;
}
```

- Addition, for arbitrary precision integers (using the GNU MP library):

```
#include <gmp.h>
void add_mod(mpz_t r,mpz_t a,mpz_t b,mpz_t m)
{ // assuming that 0 <= a,b < m, compute r = (a+b) mod m
  mpz_add(r,a,b); // r = a+b
  if(mpz_cmp(r,m) >= 0)
    mpz_sub(r,r,m); // r -= m
}
```

Modular arithmetic exercises

Use a program (and perhaps brute force) to compute:

- $(1122334455 \times 6677889900) \bmod 349335433$
- $3^{-1} \bmod 7$. This one does not require a program but do it anyway, it can be used to check if your program is working properly.
- $4^{-1} \bmod 7$. Neither does this one.
- $3^{-1} \bmod 10$. Neither does this one.
- $271828^{-1} \bmod 314159$. Just to warm up.
- $271828183^{-1} \bmod 314159265$. Now we are cooking!
- $2718281828459^{-1} \bmod 3141592653590$. Can you handle this one?
- $27182818284590452353602875^{-1} \bmod 31415926535897932384626434$. Is the teacher sane?

The greatest common divisor

- Let p_k be the k-th prime number, so that $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, and so on.
- Each positive integer can the factored into prime factors in a unique way (this is the fundamental theorem of arithmetic).
- Let $a = \prod_{k=1}^{\infty} p_k^{a_k}$, where a_k is the number of times p_k divides a. Since a is a finite number, almost all of the a_k values will be zero.
- Likewise for b, let $b = \prod_{k=1}^{\infty} p_k^{b_k}$.
- Then,

  $$\gcd(a, b) = \prod_{k=1}^{\infty} p_k^{\min(a_k, b_k)}$$

  and

  $$\operatorname{lcm}(a, b) = \prod_{k=1}^{\infty} p_k^{\max(a_k, b_k)}$$

- If $\gcd(a, b) = 1$ then a and b are said to be relatively prime (or coprime).
- The greatest common divisor can be generalized to polynomials with integer coefficients!

The greatest common divisor (algorithm)

Assume that $a \geqslant 0$ and that $b \geqslant 0$. Then:

- $\gcd(a,b) = \gcd(b,a)$, and so $\gcd(a,b) = \gcd\bigl(\max(a,b), \min(a,b)\bigr)$. Thus, by exchanging a with b if necessary, we may assume that $a \geqslant b$.
- as any positive integer divides 0 we have $\gcd(a,0) = a$ for $a > 0$. The mathematicians say that $\gcd(0,0) = 0$, and so we can say that $\gcd(a,0) = a$ as long as $a \geqslant 0$.
- If $a \geqslant b$ then $\gcd(a,b) = \gcd(a-b,b)$. We can keep subtracting b from (the updated) a until it becomes smaller than b, and so $\gcd(a,b) = \gcd(a \bmod b, b) = \gcd(b, a \bmod b)$.

These observations give rise to the following so-called Euclidean algorithm (coded in C, but it can easily be translated to another programming language):

```
long gcd(long a,long b)
{
  while(b != 0) { long c = a % b; a = b; b = c; } return a;
}
```

The GNU MP library has a function, mpz_gcd, for this; pari-gp does this with the gcd function.

The greatest common divisor (example)

Goal: to compute $\gcd(273, 715)$.

- Step 1: $\gcd(273, 715) = \gcd(715, 273)$.
- Step 2: $\gcd(715, 273) = \gcd(715 - 2 \times 273, 273) = \gcd(169, 273)$.
- Step 3: $\gcd(169, 273) = \gcd(273, 169) = \gcd(273 - 169, 169) = \gcd(104, 169)$.
- Step 4: $\gcd(104, 169) = \gcd(169, 104) = \gcd(169 - 104, 104) = \gcd(65, 104)$.
- Step 5: $\gcd(65, 104) = \gcd(104, 65) = \gcd(104 - 65, 65) = \gcd(39, 65)$.
- Step 6: $\gcd(39, 65) = \gcd(65, 39) = \gcd(65 - 39, 39) = \gcd(26, 39)$.
- Step 7: $\gcd(26, 39) = \gcd(39, 26) = \gcd(39 - 26, 26) = \gcd(13, 26)$.
- Step 8: $\gcd(13, 26) = \gcd(26, 13) = \gcd(26 - 2 \times 13, 13) = \gcd(0, 13)$.
- Step 9: $\gcd(0, 13) = \gcd(13, 0) = 13$.

It is known that the computational complexity of computing $\gcd(a, b)$ is $\mathcal{O}\big(\log \max(a, b)\big)$.
Compute $\gcd(1538099040171999308, 1505213291912594821)$.

The extended Euclidean algorithm (computation of the modular inverse)

- The Euclidean algorithm generates a finite integer sequence that begins with a and b and proceeds by doing modular reductions on consecutive terms of the sequence until zero is reached. For example, for $\gcd(105, 40)$ the sequence is $105, 40, 25, 15, 10, 5, 0$, so the answer is 5.

- But it is possible to do more!

- Let the sequence begin with $x_0 = a$ and $x_1 = b$. At any time, let $x_k = s_k a + t_k b$. So, $s_0 = t_1 = 1$, and $s_1 = t_0 = 0$.

- The next term of the sequence is given by $x_k = x_{k-2} \bmod x_{k-1}$. Let $q_k = \left\lfloor \frac{x_{k-2}}{x_{k-1}} \right\rfloor$. Then,

$$x_k = x_{k-2} - q_k x_{k-1}, \qquad s_k = s_{k-2} - q_k s_{k-1}, \quad \text{and} \quad t_k = t_{k-2} - q_k t_{k-1}.$$

- We have to stop when $x_k = 0$, at which time $\gcd(a, b) = x_{k-1}$. But here we know more:

$$x_{k-1} = s_{k-1} a + t_{k-1} b.$$

If $\gcd(a, b) = 1$ then $x_{k-1} = 1$, and this formula allows us to compute easily

$$a^{-1} \bmod b = s_{k-1} \bmod b \qquad \text{and} \qquad b^{-1} \bmod a = t_{k-1} \bmod a.$$

Be aware that some of the s_k's or t_k's may be negative integers.

The extended Euclidean algorithm (example)

Goal: apply the extended Euclidean algorithm to compute $\gcd(77, 54)$.

- The following table illustrates the computations done by the extended Euclidean algorithm.

k	x_k	q_k	s_k	t_k
0	77		1	0
1	54		0	1
2	23	1	1	-1
3	8	2	-2	3
4	7	2	5	-7
5	1	1	-7	10
6	0	7	54	-77

- Because $x_6 = 0$, the information we seek corresponds to the row with $k = 5$. We have $\gcd(77, 54) = 1$, $77^{-1} \bmod 54 = -7 \bmod 54 = 47$, and $54^{-1} \bmod 77 = 10$.

The GNU MP library has a function, `mpz_gcdext`, for this; `pari-gp` also has a function, `gcdext`, for this. Let $a = 830150497265848419$ and $b = 472332647410202896$. Compute $a^{-1} \bmod b$ and $b^{-1} \bmod a$.

Curiosity: efficient computation of several modular inverses

Goal: to compute $a_k^{-1} \bmod m$, for $k = 1, \ldots, n$.

- Let $A_1 = a_1 \bmod m$, and for $k = 2, 3, \ldots, n$, let $A_k = (A_{k-1} a_k) \bmod m$, i.e., let A_k be the product of the first k numbers we wish to invert:

$$A_k = \prod_{i=1}^{k} a_i \bmod m.$$

- Note that for $k = 2, 3, \ldots, n$, we have

$$(*) \quad a_k^{-1} \equiv A_{k-1} A_k^{-1} \pmod{m} \qquad \text{and} \qquad A_{k-1}^{-1} \equiv a_k A_k^{-1} \pmod{m}.$$

- Compute A_n^{-1}. If it does not exist then at least one of the a_k does not have an inverse. Otherwise, for $k = n, n-1, \ldots, 2$, use equations (*) to compute $a_k^{-1} \bmod m$ and $A_{k-1}^{-1} \bmod m$. Finally, $a_1^{-1} = A_1^{-1}$, so we are done!

- The entire computation requires $(k+1) + 2(k-1) = 3(k-1)$ multiplications and one inversion. So, this "trick" is useful when the cost of a modular inversion is higher than the cost of three modular multiplications, which is usually the case because a modular inversion requires $\mathcal{O}(\log m)$ operations using the extended Euclidean algorithm.

Linear maps

- When working modulo m it suffices to work with integers in the range $0, 1, \ldots, m-1$, i.e., it suffices to work with $\mathbb{Z}_m$.
- Let

  $$f(x; m, a) = (ax) \bmod m$$

  be the linear map $x \mapsto (ax) \bmod m$ from $\mathbb{Z}_m$ into itself.
- Recall that a function $f(x)$ is said to be linear iff $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ for all α, β, x, and y.
- For example, for $m = 4$ the linear map with $a = 2$ (on the left) is not invertible, but the linear map with $a = 3$ (on the right) is invertible.

map for $m = 4$ and $a = 2$
$0 \mapsto 0$
$1 \mapsto 2$
$2 \mapsto 0$
$3 \mapsto 2$

map for $m = 4$ and $a = 3$
$0 \mapsto 0$
$1 \mapsto 3$
$2 \mapsto 2$
$3 \mapsto 1$

Linear maps (continuation)

- Why are we interested in inverting the map? Because the map scrambles the elements of $\mathbb{Z}_m$ and we may be interested in unscrambling them (think in cryptographic terms).
- So, what is the inverse map?
- It turns out that the inverse map, if it exists, is also a linear map.
- More specifically, the inverse map of $f(x, m, a \bmod m)$ is $f(x, m, a^{-1} \bmod m)$, where $a^{-1} \bmod m$ is the modular inverse of $a \bmod m$. Indeed, if $y = f(x; m, a) = ax \bmod m$ then $x = a^{-1}y \bmod m$.
- Since the modular inverse of a modulo m only exists when $\gcd(a, m) = 1$ the linear map is invertible if and only if $\gcd(a, m) = 1$.
- Keep in mind that we wish to devise a way to encrypt information by providing public data to do so (in this case it would be m and a).
- Alas, this way of scrambling information is very easy to unscramble, so useless from a cryptography point of view. Affine maps, of the form $x \mapsto (ax + b) \bmod m$, are not better.
- Modular multiplication scrambles the information but it is easy to undo if we known m and a. What about modular exponentiation?

Linear maps (a failed cryptosystem)

The Merkle-Hellman knapsack cryptosystem keeps the following information secret:

- a set $W = \{ w_1, w_2, \ldots, w_n \}$ of n positive integers, such that w_k is a super-increasing sequence, i.e., $w_k > \sum_{i=1}^{k-1} w_i$ for $2 \leqslant k \leqslant n$ (each term is greater that the sum of the previous terms),
- a modulus m such that $m > \sum_{i=1}^{n} w_i$,
- a scrambling integer a such that $\gcd(a, m) = 1$,

and publishes the following information:

- set $W' = \{w'_1, w'_2, \ldots, w'_n\}$, where $w'_i = (a w_i) \bmod m$, for $1 \leqslant i \leqslant n$.

Actually, it is much better to publish a random permutation of W'. (Homework: why?). To send a message composed by the n bits α_k, $1 \leqslant k \leqslant n$, compute and send

$$C = \sum_{k=1}^{n} \alpha_k w'_k.$$

This is a hard knapsack problem (in this case a subset sum problem). To decipher transform it into a trivial knapsack problem by computing $a^{-1}C \bmod m$, which is equal to $\sum_{k=1}^{n} \alpha_k w_k$ and so can be solved by a greedy algorithm because the terms of the sequence are super-increasing.

Linear maps (Merkle-Hellman knapsack example)

The following example shows the Merkle-Hellman cryptosystem in action.

- Secret data: $W = \{1, 3, 5, 12, 22, 47\}$, $m = 100$, and $a = 13$.
- Public data: $W' = \{13, 39, 65, 56, 86, 11\}$,
- Unencrypted message to be sent: $A = \{0, 0, 1, 1, 0, 1\}$.
- Encrypted message sent: $C = 0 \times 13 + 0 \times 39 + 1 \times 65 + 1 \times 56 + 0 \times 86 + 1 \times 11 = 132$.
- To decrypt compute $132 \times 13^{-1} \bmod 100 = 32 \times 77 \bmod 100 = 64$ and then reason as follows [greedy algorithm for the easy subset sum problem]:

  1. 47 must be used to form the sum because $64 > 47$. Hence $\alpha_6 = 1$. The rest of the sum is $64 - 47 = 17$.
  2. 22 cannot be used to form the sum because $17 < 22$. Hence $\alpha_5 = 0$.
  3. 12 must be used to form the sum because $17 > 12$. Hence $\alpha_4 = 1$. The rest of the sum is $17 - 12 = 5$.
  4. As so on. In this particular case, the next iteration finishes the deciphering process.

Now, imagine a set W with hundreds of elements and numbers with hundreds of bits. Looks hard, doesn't it? With 100 elements there are 2^{100} cases to try. Meet-in-the middle techniques reduce this to about 50×2^{50} cases but need to store 2^{50} numbers, which is already impractical. With thousands of terms, the original ideia appeared to be reasonable. Alas, lattice reduction techniques can be used to solve this problem in a reasonable amount of time. See The Rise and Fall of Knapsack Cryptosystems by Andrew Odlyzko.

Fermat's little theorem

- The elements of $\mathbb{Z}_m$ that have an inverse are called the units of $\mathbb{Z}_m$. The set containing all these units is denoted by $\mathbb{Z}_m^*$. When m is a prime number, $\mathbb{Z}_m^* = \{\,1, 2, \ldots, m-1\,\}$.

- Euler's totient function $\varphi(m)$ counts how many integers in $\mathbb{Z}_m$ are relatively prime to m, i.e., it counts the number of elements of $\mathbb{Z}_m^*$. It can be computed using the formula

$$\varphi(m) = m \prod_{p|m} \left(1 - \frac{1}{p}\right),$$

  where the product is over the distinct prime factors of m.

- $\varphi(m)$ can be computed in pari-gp with the `eulerphi` function.

- Let $P = \prod_{k \in \mathbb{Z}_m^*} k$. Clearly, P has to be relatively prime to m because each of its factors is relatively prime to m. (When m is prime then $P + 1 \equiv 0 \pmod{m}$ — that's Wilson's theorem — but we will not use this fact here.)

- Now assume that $a \in \mathbb{Z}_m^*$, i.e., that $\gcd(a, m) = 1$, and let us now consider what the map $f(x; m, a)$ does to the elements of $\mathbb{Z}_m^*$.

- It scrambles them! Because everything is relatively prime to m, $\mathbb{Z}_m^*$ is mapped into itself. Since $au \equiv av \pmod{m}$ implies $u \equiv v \pmod{m}$, different elements have different images (a bijection, also known as a one-to-one map).

Fermat's little theorem (continuation)

- So, since the map $x \mapsto ax \bmod m$ when applied to $\mathbb{Z}_m^*$ just reorders its elements (take a look at the multiplication modulo 7 in a previous slide), it follows that

  $$Q \equiv \left(\prod_{k\in\mathbb{Z}_m^*} ak\right) \equiv \left(a^{\varphi(m)} \prod_{k\in\mathbb{Z}_m^*} k\right) \equiv (a^{\varphi(m)}P) \pmod{m},$$

  but also that (because of the reordering!)

  $$Q \equiv P \pmod{m}.$$

- Since $\gcd(P,m) = 1$, $P^{-1} \bmod m$ exists, and so we can say that, for any $a \in \mathbb{Z}_m^*$, we have (this is Fermat's little theorem)

  $$a^{\varphi(m)} \equiv 1 \pmod{m}.$$

- For a prime number p we have $\varphi(p) = p - 1$, and Fermat's little theorem takes the form

  $$a^{p-1} \equiv 1 \pmod{p}, \qquad \text{for all } a \text{ with } \gcd(a,p) = 1.$$

  We can take care of the case $a \equiv 0 \pmod{p}$ by multiplying both sides by a:

  $$a^p \equiv a \pmod{p}, \qquad \text{for all } a.$$

Fermat's little theorem (examples)

- Let's see what happens for three distinct values of m (all exponentiations are done modulo m):

$m = 7$, $e = \varphi(m) = 6$:

k	k^e	k^{e+1}
0	0	0
1	1	1
2	1	2
3	1	3
4	1	4
5	1	5
6	1	6

(The values of k for which $\gcd(k, m) = 1$ have a gray background.)

$m = 10$, $e = \varphi(m) = 4$:

k	k^e	k^{e+1}
0	0	0
1	1	1
2	6	2
3	1	3
4	6	4
5	5	5
6	6	6
7	1	7
8	6	8
9	1	9

$m = 12$, $e = \varphi(m) = 4$:

k	k^e	k^{e+1}
0	0	0
1	1	1
2	4	8
3	9	3
4	4	4
5	1	5
6	0	0
7	1	7
8	4	8
9	9	9
10	4	4
11	1	11

- What happens for $m = 2 \times 3 \times 5$?

- It looks like $a^{\varphi(m)+1} \equiv a \pmod{m}$ when m does not have repeated prime factors!

- The Chinese remainder theorem, explained next, will help us prove this.

Chinese remainder theorem

- Suppose that you know that $x \equiv a \pmod{m}$ and that $x \equiv b \pmod{n}$.
- From the first condition x has to be equal to $a + km$ for some integer k.
- But $a + km \equiv b \pmod{n}$, and so $k \equiv m^{-1}(b - a) \pmod{n}$. The modular inverse exists for sure if $\gcd(m, n) = 1$, which we assume is the case here.
- Therefore, we know that $k = ln + c$ for some integer l, where $c = m^{-1}(b - a) \bmod n$. Note that $c = 0$ when $b = a$.
- Finally, we get $x = a + cm + lmn$, i.e., $x \equiv a + cm \pmod{mn}$.
- If $b = a$ things are simpler: $x = a + lmn$, i.e., $x \equiv a \pmod{mn}$.
- It is possible to reach the same conclusion more quickly:
$$x \equiv a(n^{-1} \bmod m)n + b(m^{-1} \bmod n)m \pmod{mn}.$$
- In general, if we know that $x \equiv a_k \pmod{m_k}$, for $1 \leqslant k \leqslant K$, with the moduli m_k pairwise coprime (i.e., $\gcd(m_i, m_j) = 1$ when $i \neq j$) then, with $M = \prod_{k=1}^{K} m_k$ and $M_k = M/m_k$, we have
$$x \equiv \sum_{k=1}^{K} a_k (M_k^{-1} \bmod m_k) M_k \pmod{M}.$$

Chinese remainder theorem (problems)

Solve the following systems of congruences:

$$\begin{cases} x \equiv 0 \pmod{8} \\ x \equiv 1 \pmod{9} \end{cases}$$

$$\begin{cases} x \equiv 0 \pmod{8} \\ x \equiv 8 \pmod{16} \\ x \equiv 3 \pmod{5} \end{cases}$$

$$\begin{cases} x \equiv 2 \pmod{3} \\ x \equiv 2 \pmod{5} \\ x \equiv 2 \pmod{7} \end{cases}$$

$$\begin{cases} x \equiv 1 \pmod{2} \\ x \equiv 2 \pmod{3} \\ x \equiv 4 \pmod{5} \\ x \equiv 6 \pmod{7} \\ x \equiv 10 \pmod{11} \\ x \equiv 12 \pmod{13} \end{cases}$$

$$\begin{cases} x \equiv 12345 \pmod{2718281828} \\ x \equiv 67890 \pmod{3141592653} \end{cases}$$

- Hint: pari-gp groks the Chinese remainder theorem (chinese function). For example, the first problem can be solved in pari-gp by

```
chinese(Mod(0,8),Mod(1,9))
```

Fermat's little theorem (revisited)

- Let p be any prime number. By Fermat's little theorem we know that

$$x^{\varphi(p)} \equiv x^{p-1} \equiv 1 \pmod{p}, \qquad \text{when } \gcd(x,p) = 1.$$

- It follows that for any integers r and x we have

$$x^{r(p-1)+1} \equiv x \pmod{p}.$$

  For $x \equiv 0 \pmod{p}$ this is obvious. For the other cases use Fermat's little theorem to adjust the exponent: $x^{r(p-1)+1} \equiv x\,(x^{p-1})^r \equiv x\,(1)^r \pmod{p}$.

- Now consider a second prime, q, different from p. We also have, for any integer s,

$$x^{s(q-1)+1} \equiv x \pmod{q}.$$

- Let t be the least common multiple of $p-1$ and $q-1$. If follows that

$$x^{t+1} \equiv x \pmod{p} \qquad \text{and} \qquad x^{t+1} \equiv x \pmod{q}.$$

- By the Chinese remainder theorem this implies that

$$x^{t+1} \equiv x \pmod{pq}.$$

Fermat's little theorem (conclusion)

- The previous result can be generalized to K primes.
- Let $p_1, p_2, \ldots, p_K$ be K distinct primes. Here, p_1 is not necessarily the first prime (two) and so on.
- Let P be their product: $P = \prod_{k=1}^{K} p_k$.
- Let $\lambda(P)$ be the so-called Carmichael function, given by

$$\lambda(P) = \lambda(p_1 p_2 \cdots p_K) = \operatorname{lcm}(p_1 - 1, p_2 - 1, \ldots, p_K - 1).$$

- Then, for any integers k and x, we have

$$\begin{cases} x^{k\lambda(P)+1} \equiv x \pmod{P}, & \text{always,} \\ x^{\lambda(P)} \equiv 1 \pmod{P}, & \text{when } \gcd(x, P) = 1. \end{cases}$$

- This result is often presented with $\lambda(P)$ replaced by $\varphi(P) = \prod_{k=1}^{K}(p_k - 1)$. That is not really wrong but it is not the best possible result, because when P a product of odd primes, $\varphi(P)/\lambda(P)$ is an integer larger than 1.
- This means that in a modular exponentiation we may reduce the exponent modulo $\lambda(P)$ when $\gcd(x, P) = 1$. When $\gcd(x, P) \neq 1$ things are more complicated.

Modular exponentiation

- The modular exponentiation $a^b \bmod m$ can be done recursively using the following two observations:

  $$a^{2n+0} \bmod m = (a^2)^n \bmod m$$

  and

  $$a^{2n+1} \bmod m = a(a^2)^n \bmod m.$$

  If follows that it can be done using $\mathcal{O}(\log n)$ modular multiplications.

- Example:

  $$\begin{aligned}
  &13^{21} \bmod 71 = 13 \times (13^2)^{10} \bmod 71 = 13 \times 27^{10} \bmod 71,\\
  &27^{10} \bmod 71 = (27^2)^5 \bmod 71 = 19^5 \bmod 71,\\
  &19^5 \bmod 71 = 19 \times (19^2)^2 \bmod 71 = 19 \times 6^2 \bmod 71,\\
  &6^2 \bmod 71 = 36 \bmod 71,
  \end{aligned}$$

  backsubstituting. . .

  $$\begin{aligned}
  &19^5 \bmod 71 = 19 \times 36 \bmod 71 = 45 \bmod 71,\\
  &27^{10} \bmod 71 = 45 \bmod 71,\\
  &13 \times 27^{10} \bmod 71 = 17 \bmod 71,\\
  &13^{21} \bmod 71 = 17 \bmod 71 = 17.
  \end{aligned}$$

Modular exponentiation (another way)

- Let the exponent n, with $N+1$ bits, be represented in base-2 as follows:

$$n = \sum_{k=0}^{N} n_k 2^k.$$

- Then,

$$a^n \bmod m = a^{\sum_{k=0}^{N} n_k 2^k} \bmod m = \prod_{k=0}^{N} a^{n_k 2^k} \bmod m.$$

- Using the example of the previous slide, we have $n = 21 = 10101_2$, so $N = 4$. Thus,

k	a^{2^k}	use in the final product?
0	13	yes
1	27	no; note that $27 = 13^2 \bmod 71$
2	19	yes; note that $19 = 27^2 \bmod 71$
3	6	no; in general, each number is the square of the previous number
4	26	yes

So, $13^{21} \bmod 71 = 13 \times 19 \times 26 \bmod 71 = 17$.

- Compute $12345^{67890} \bmod 123456789$.

Modular exponentiation (a slightly better way)

- It is possible to do slightly better (Brauer's algorithm). Let the exponent n, with $d+1$ base-B digits, be represented in base-B as follows:

  $$n = \sum_{k=0}^{d} n_k B^k = n_0 + B\Big(n_1 + B\big(n_2 + B(\ldots + n_d)\big)\Big).$$

  The last equality is the Horner's rule to evaluate a polynomial. Note that $0 \leqslant n_k < B$. (Usually, B is a power of 2.)

- Then, $a^n \bmod m$ can be evaluated using the following sequence of steps:

  $$\begin{array}{rlll} r_0 = a^{n_d} \bmod m & r_1 = r_0^B & r_2 = a^{n_{d-1}} r_1 \bmod m & r_3 = r_2^B \\ r_4 = a^{n_{d-2}} r_3 \bmod m & r_5 = r_4^B & \cdots & \cdots \\ r_{2d} = r_{2d-1} a^{n_0} \bmod m & & & \end{array}$$

- When $B = 8$, the 8 possible values of $a^{n_k} \bmod m$ can be precomputed and stored — in an interleaved way to avoid side-channel attacks — in memory.

first word of a^0	first word of a^1	first word of a^2	first word of a^3	first word of a^4	first word of a^5	first word of a^6	first word of a^7
second word of a^0	second word of a^1	second word of a^2	second word of a^3	second word of a^4	second word of a^5	second word of a^6	second word of a^7
...	...	...	...	...	...	...	...

- To explore further: addition chains.

Modular exponentiation with large numbers (error detection)

- Some large scale computations do many modular exponentiations of large integers (GIMPS, PrimeGrid). Whay if, due to, say, a memory error or a stray cosmic ray, the computation produces a wrong result?

- There exists a very cheap (in terms of computational complexity) way to detect with very high probability an error in this kind of computations.

- Suppose we wish to compute $c = a^b \bmod m$. The idea is to use a "small" number r. say a 64-bit number, and compute $d = a^b \bmod (mr)$. The value of c can then be quickly computed as $c = d \bmod m$. The value of $d \bmod r$ can be compared to $(a \bmod r)^b \mod r$, which requires only 64-bit arithmetic and is thus fast. If these two last values do not agree, then there was an error during the computation and it has to be redone.

- As we will see later, this error checking is useful in the context of RSA deciphering (using the Chinese remainder theorem), because an error in that computation can be used to factor the RSA modulus (if so, game over. . .).

Modular exponentiation, Fermat's little theorem and time locks

- Suppose you wish to apply a time lock to a given document, that is, you want to give a document to a person, but you wish that that person can only open the document some time in the future.

- One way to do it is the following:
  - Select two large primes p and q, a random number r between 2 and $pq - 2$, and a number t that is related to the time delay; if it is possible to do x modular squaring operations modulo pq per second with the fastest hardware in the entire planet, and if you wish a delay of at least s seconds, set $t = xs$.
  - Encrypt the document with a key derived from the value of $r^{2^t} \mod (pq)$, which you can compute quickly by reducing the exponent to $2^t \bmod \operatorname{lcm}(p-1, q-1)$. Even better: do the computation modulo p and modulo q and use the Chinese remainder theorem!
  - Send the document, the key derivation details, r, pq, and t.
  - Throw away p and q.
  - Without p and q, anyone wishing to read the document will be forced to either factor pq, which is hard, or do t modular squarings, which take at least t/x seconds.

Fast modular multiplication

A modular multiplication requires a remainder operation, which is a slow operation if the modulus is a general integer. For example, contemporary processors can multiply two 64-bit integers, producing a 128-bit result, with a latency of 3 or 4 clock cycles. But, dividing a 128-bit integer by a 64-bit integer, producing a 64-bit quotient and a 64-bit remainder, is considerably slower (tens of clock cycles). [For more information about how many clock cycles elementary arithmetic operations take on Intel/AMD processors, take a look at Agner Fog's instruction tables.]

If the modulus is a power of two, say 2^n, the remainder operation is very fast; the remainder is just the last n bits of the number being remaindered. In 1985, Peter Montgomery came up with a beautiful way to explore this to efficiently perform general remaindering operations without performing an expensive division. His ideia is useful when many modular multiplications are performed in tandem; that is exactly what happens in a modular exponentiation.

Peter Montgomery's paper, Modular Multiplication Without Trial Division, explains how that can be done. For details, see next slide or search for "Montgomery modular multiplication" in the internet.

Montgomery multiplication (REDC algorithm)

To perform arithmetic modulo m choose $b > m$ such that $\gcd(m, b) = 1$ and such that divisions by b are very efficient (a power of two is an obvious choice). Let m' be such that $mm' = -1 \bmod b$. For $0 \leqslant x < bm$ let

$$y = x + m(xm' \bmod b).$$

Then $0 \leqslant y < 2bm$. Also, $y \bmod m = x \bmod m$ and $y \bmod b = 0$. So

$$xb^{-1} \bmod m = yb^{-1} \bmod m = (y/b) \bmod m.$$

The last equality follows because y is a multiple of b. But $y/b < 2m$, so we have

$$xb^{-1} \bmod m = \begin{cases} y/b, & \text{if } y/b < m; \\ y/b - m, & \text{if } y/b \leqslant m. \end{cases}$$

To take advantage of this, transform u to $\bar{u} = (ub) \bmod m$; likewise for v, etc. It follows that

$$\begin{aligned} w &= (u \pm v) \bmod m & \rightarrow \quad & \bar{w} = (\bar{u} \pm \bar{v}) \bmod m \\ w &= (uv) \bmod m & \rightarrow \quad & \bar{w} = (\bar{u}\bar{v}b^{-1}) \bmod m. \end{aligned}$$

With $x = \bar{u}\bar{v}$, the computation of $\bar{u}\bar{v}b^{-1} \bmod m$ can be performed efficiently using the formula for $xb^{-1} \bmod m$ given above. This is explained in detail in the next page.

Montgomery multiplication (details)

For a processor with 64-bit words, i.e., $b = 2^{64}$, let the modulus m be odd and smaller that 2^{63}. Given $0 \leqslant \bar{u} < m$ and $0 \leqslant \bar{v} < m$, to compute $(\bar{u}\bar{v}b^{-1}) \bmod m$:

- compute $x = \bar{u}\bar{v} = r_1 b + s_1$; on Intel/AMD processors, this is a single assembly instruction that produces a 128-bit result split among two registers, one holding r_1 and another holding s_1
- compute $t_1 = xm' \bmod b = s_1 m' \bmod b$; on Intel/AMD processors, this is again a single assembly instruction (here only the 64 lower order bits are necessary)
- compute $t_2 = mt_1 = r_2 b + s_2$; again, one instruction
- add $x + t_2$; this requires two addition instructions (the second with carry)
- the result we seek is the 64-most significant bits of this addition.

If the result is larger than or equal to b, be must subtract b; two more instructions. On a chain of multiplications, for $m < 2^{62}$, this last step can be performed only at the end!

It is possible to extend this idea to multi-word integers, also known as multi-precision integers.

```
asm volatile (
    "movq  %[r_mu],%[rax]"
    "mulq  %[r_mv]"           // rdx:rax = mu*mv
    "movq  %[rax],%[r_lo]"
    "movq  %[rdx],%[r_hi]"    // hi:lo = x = mu*mv
    "mulq  %[r_mi]"           // rax = x * mi mod 2^64
    "mulq  %[r_m]"            // rdx:rax = m * (x * mi mod 2^64)  [128 bits]
    "addq  %[rax],%[r_lo]"    // lo is now zero
    "adcq  %[rdx],%[r_hi]"    // hi is now smaller than 2*m and is the result we want
  : [rdx]  "=&d" (tmp1),   [rax]  "=&a" (tmp2),   [r_hi] "=&r" (hi),   [r_lo] "=&r" (lo)
  : [r_mu] "r"   (mu),     [r_mv] "r"   (mv),     [r_m]  "r"   (m),    [r_mi] "r"   (mi)
  : "cc"
);
```

Montgomery multiplication (example)

To illustrate how the Montgomery multiplication works, we will use $b = 10^3$. Let us compute $123 \times 456 \bmod 789$. We have,

- $m = 731$, and so $m' = 891$; note that $mm' = 702999$, which means that $mm' \bmod b = -1$.
- $u = 123$, and so $\bar{u} = (ub) \bmod m = 705$.
- $v = 456$, and so $\bar{v} = (vb) \bmod m = 747$.
- $\bar{u}\bar{v} = 526635$, so $\bar{u}\bar{v}m' = 469231785$, and so $\bar{u}\bar{v}m' = 469231785 \bmod b = 785$. This can be computed with smaller numbers as follows: $\big((\bar{u}\bar{v}) \bmod b)m'\big) \bmod b = 635m' \bmod b = 565785 \bmod b = 785$.
- $\bar{u}\bar{v} + m(\bar{u}\bar{v}m' \bmod b) = 705 \times 747 + 789 \times 785 = 1146000$.
- The result we seek, $\bar{w}$, is then $1146 - m = 357$.
- To get to w we need to compute $\bar{w}b^{-1} \bmod m$, which is similar to what was done above (with one of the multiplicands set to one). Or, since this is usually a terminal operation and so done only once, we could compute $b^{-1} \bmod m$ and do the operation using normal arithmetic. Since $b^{-1} \bmod m = 703$, we finally get $357 \times 703 \bmod m = 69$, which is the correct result.

Multiplicative order

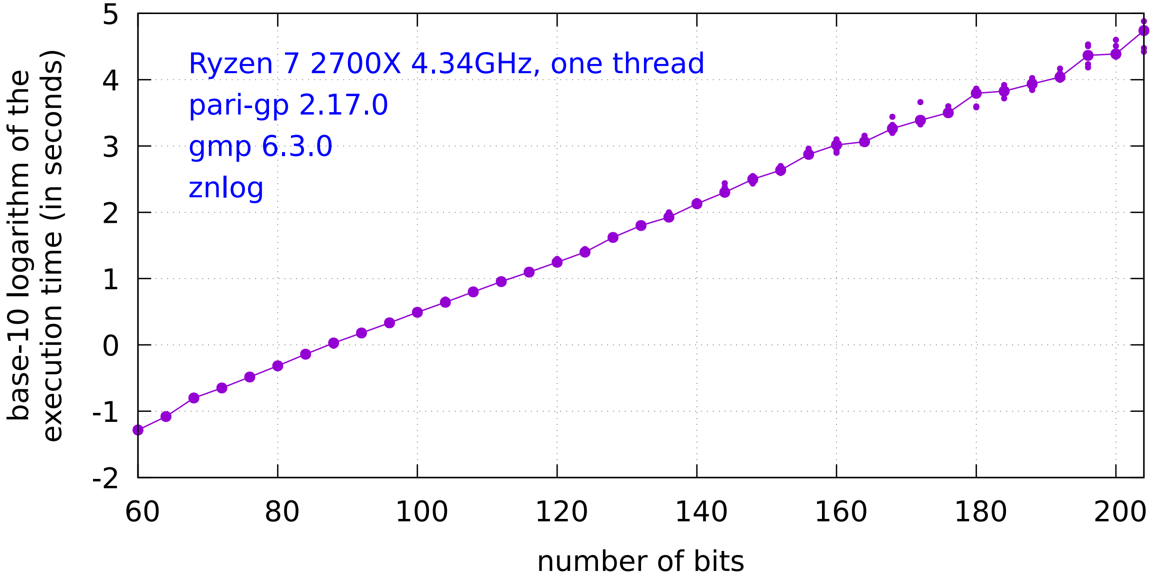
- Fermat's little theorem says that $x^{\lambda(m)} \equiv 1 \pmod{m}$ for any $x \in \mathbb{Z}_m^*$.
- For a given $x \in \mathbb{Z}_m^*$ what is the least exponent o such that $x^o \bmod m = 1$?
- This least exponent is called the order of x modulo m (the function `znorder` computes this in `pari-gp`).
- The order has to be a divisor of $\lambda(m)$.
- For a prime number p, $\lambda(p) = \varphi(p) = p - 1$.
- It turns out that there are $\varphi(p-1)$ elements of $\mathbb{Z}_p^*$ with maximal order $p-1$. These elements are called primitive roots.
- `pari-gp` has a function, `znprimroot`, to compute one of them.
- They generate $\mathbb{Z}_p^*$ multiplicatively. In particular, let r be one primitive root. Then, for $k = 0, 1, 2, \ldots, p-2$, $r^k \bmod p$ takes all values of $\mathbb{Z}_p^*$ (without repetitions).
- We can therefore speak of logarithms (modulo p), with respect to base r. The logarithm of $a = r^x \bmod p$ in base r is obviously x. This so-called discrete logarithm problem is currently very hard to solve when p is large.

The discrete logarithm problem for $\mathbb{Z}_p^*$

- Given a prime p, a primitive root r of p, and a, find x such that

$$a \equiv r^x \bmod p.$$

- This is a hard problem if $p - 1$ has large prime factors:



Pari-gp code used to compute the figure data:

```
do_one(b,seed)={ my(q,p,r,a,x,dt,e);
  setrand(seed);
  while(1,q=precprime(random([5*2^(b-4),2^(b-1)]));
    p=2*q+1; if(isprime(p),break(1);););
  r=znprimroot(p); printf("# %d %d\n",p,lift(r));
  a=random([floor(p/10),floor(9*p/10)]); x=r^a;
  dt=getabstime(); e=znlog(x,r); dt=getabstime()-dt;
  if(e!=a,quit(1);); return(0.001*dt); };

do_many(b,nt=5)={ my(dt);
  dt=vecsort(vector(nt,k,do_one(b,100*b+k)));
  printf("%3d",b); for(k=1,nt,printf(" %.3f",dt[k]););
  printf("\n"); return(vecmax(dt)); };

printf("# version=%d\n",version());
forstep(b=60,500,4,if(do_many(b)>90000.0,break(1);););
quit();
```

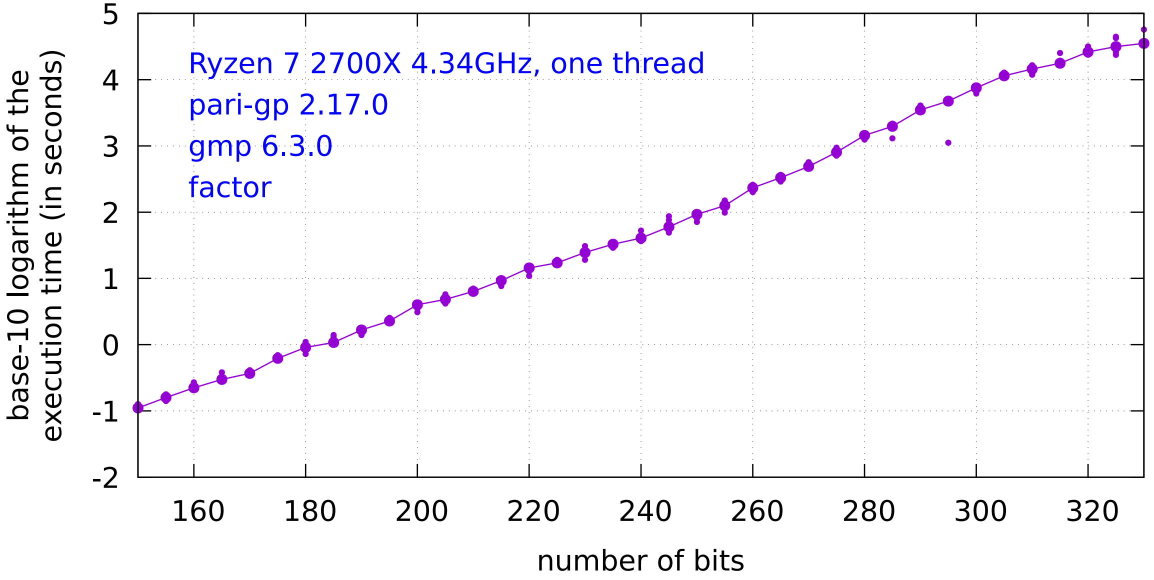
- But, if $p - 1$ only has small factors, the discrete logarithm problem is easy:

```
while(1,p=1+prod(k=1,160,prime(1+random(25)));\
  if(isprime(p),break(););); r=znprimroot(p); % p has about 800 bits
znlog(r^(10^50),r)                              % milliseconds...
```

- The same problem but with elliptic curves

The integer factorization problem

- Given an integer n, find its factors.
- This problem has countless applications. It is needed, for example, so solve efficiently the discrete logarithm problem.
- This is a hard problem if n has large factors:



Pari-gp code used to compute the figure data:

```
do_one(b,seed)={ my(dt,p,n,f);
  setrand(seed); default(realbitprecision,b+10);
  p=vector(2); p[1]=nextprime(random([2^(b/2-4),2^(b/2)]));
  p[2]=precprime(random(2^(b-3)*[5,8])/p[1]); p=vecsort(p);
  n=p[1]*p[2]; printf("# %d %d %d\n",n,p[1],p[2]);
  dt=getabstime(); f=factor(n); dt=getabstime()-dt;
  if(f[1,1]!=p[1] || f[2,1]!=p[2],quit(1););
  return(0.001*dt); };

do_many(b,nt=5)={ my(dt);
  dt=vecsort(vector(nt,k,do_one(b,100*b+k)));
  printf("%3d",b); for(k=1,nt,printf(" %.3f",dt[k]););
  printf("\n"); return(vecmax(dt)); };

printf("# version=%d\n",version());
forstep(b=150,500,5,if(do_many(b)>90000.0,break(1);););
quit();
```

- However, if all factors are small the problem is easy. Moreover, even with large factors, if $n = pq$ with p/q very close to a rational number with small numerator and denominator, the problem is not so hard (Lehmer factorization).

Primality tests

- One way to prove that a given number m is prime is to find one of its primitive roots.
- Choose a random a between 2 and $m-2$.
- If $gcd(a, m) \neq 1$, then m is not prime. Better yet, the greatest common divisor allow us to partially factor m.
- By Fermat's little theorem we know that $a^{m-1} \equiv 1 \bmod m$. If this is not so, then definitely m is not prime.
- Furthermore, when m is an odd number, we must have either $a^{(m-1)/2} \equiv 1 \bmod m$ or $a^{(m-1)/2} \equiv -1 \bmod m$.
- Now, it can be shown that a is a primitive root modulo m if, for every prime divisor d of $m-1$, we have $a^{(m-1)/d} \bmod m \neq 1$. If a satisfies these conditions then the order of a modulo m must be $m-1$, and thus m must be prime.
- If not, try another a.
- These exist composite numbers, called Carmichael numbers, for which $a^{m-1} \bmod m = 1$ for all a which are relatively prime to m. For these numbers, $\lambda(m) \mid (m-1)$.

The Miller-Rabin primality test

- Goal: to test if the odd number n, with $n > 3$, is a prime number or not.
- Result of the test: either n is definitely not prime or it may be prime (a probable prime number); in the second case, the probability that the test fails to identify a composite number is at most 0.25.
- How it is done (to increase the confidence on the result, do steps 2 to 6 several times):
  1. Let $n-1$ be written as $n-1 = 2^r d$, with d an odd number (so r is as large as possible).
  2. Select at random an integer a uniformly distributed in the interval $2 \leqslant a \leqslant n-2$.
  3. If $\gcd(a, n) \neq 1$, then n is definitely a composite number.
  4. Compute $x_0 = a^d \bmod n$. If $x_0 = 1$ or $x_0 = n-1$ then n is a probable prime.
  5. Otherwise, for $k = 1, 2, \ldots, d-1$, compute $x_k = x_{k-1}^2 \bmod n$. If $x_k = n-1$ then n is a probable prime.
  6. Finally, if we get here, say that n is definitely a composite number (Fermat's little theorem failed to be true because $a^{(m-1)/2} \equiv \pm 1 \bmod m$).
- The composite numbers that pass this test (meaning that the algorithm above says that they are probable primes) for a given a are called base-a strong pseudo-primes.

The Diffie-Hellman key exchange protocol

- Alice and Bob have never met but wish to exchange a secret key (perhaps to be used in the initialization stage of a symmetric-key cipher algorithm).

- They agree, on a public channel, on a prime p and on a primitive root r modulo p. (Choose a prime number with an easy to find factorization of $p-1$, say a safe prime.)

- Alice generates a random number α between 2 and $p-2$, or, better yet, between, say, $p^{0.8}$ and $p-p^{0.8}$, and sends Bob the integer $A = r^\alpha \bmod p$. She keeps α only to herself.

- Likewise, Bob generates a random number β between 2 and $p-2$, and sends Alice the integer $B = r^\beta \bmod p$. He keeps β only to himself.

  For extra protection, make sure $\gcd(\alpha, p-1) = \gcd(\beta, p-1) = 1$. This forces A and B to be primitive roots.

- Alice computes $S = B^\alpha \bmod p = r^{\beta\alpha} \bmod p = r^{\alpha\beta} \bmod p$.

- Bob computes $S = A^\beta \bmod p = r^{\alpha\beta} \bmod p$.

- They have arrived at the same number, which is their shared secret. They can now discard A, B, α, β, and p (do not reuse p many times).

- Anyone eavesdropping their communications (Eve?, Mallory?) has to either infer α from A or β from B. This is known as the discrete logarithm problem, which is currently a very hard problem to solve.

Diffie-Hellman key exchange protocol (exercises)

- Let $p = 101$ and $r = 2$. Alice chooses $\alpha = 52$, and so $A = 97$. Bob chooses $\beta = 46$ and so $B = 82$. Confirm that their common secret is $S = 58$.
- Let $p = 3141601$ and $r = 26$. Alice chooses $\alpha = 2437429$, and so $A = 1282989$. Bob chooses $\beta = 2988228$ and so $B = 2426580$. Their common secret is $S = 1669355$. Try to find α given A and to find β given B.
- Let $p = 31415926541$ and $r = 10$. Alice chooses $\alpha = 29770170945$, and so $A = 5728872032$. Bob chooses $\beta = 23956179675$ and so $B = 22727460975$. Their common secret is $S = 26991399064$. Try to find α given A and to find β given B.
- Let $p = 3141592653589793239$ and $r = 6$. Alice chooses

  $$\alpha = 2459372999633886947, \qquad \text{and so} \qquad A = 2408130236552768716.$$

  Bob chooses

  $$\beta = 2502449096145193611, \qquad \text{and so} \qquad B = 434542471090467423.$$

  Their common secret is $S = 1267222359226852228$. Can you find by yourself α given A and β given B? Hint: `pari-gp` does this with the `znlog` function.

Diffie-Hellman key exchange protocol (man-in-the-middle attack)

- Mallory, being a powerful individual, can intercept and replace all messages between Alice and Bob.
- Here's how he can compromise the Diffie-Hellman key exchange protocol.
- Mallory intercepts all messages coming from Alice in the Diffie-Hellman key exchange protocol and impersonates Bob. At the end of the key-exchange protocol he will share a secret key with Alice.
- Likewise, Mallory intercepts all messages coming from Bob in the Diffie-Hellman key exchange protocol and impersonates Alice. At the end of the key-exchange protocol he will share a secret key with Bob (different from the one he shares with Alice).
- From this point on, he decrypts all messages between them, using the appropriate shared secret key, and re-encrypts them using the other shared secret key. He may even modify the messages.
- But Alice and Bob can counter this if they send their messages in two or more distinct parts in an interlocked fashion (this assumes that decoding can only be performed after all parts have been received). Also they can, and should, authenticate themselves to the other.

ElGamal public key cryptosystem

- Alice and Bob agree on a large prime number p and on an element g of $\mathbb{F}_p^*$ with a large prime order

- Alice chooses a private key a, with $1 < a < p - 1$, and publishes $A = g^a \bmod p$.

- Bob chooses a random ephemeral key k.

- He uses Alice's public key A to compute $c_1 = g^k \bmod p$ and $c_2 = mA^k \bmod p$, where m is the plaintext.

- He then sends (c_1, c_2) to Alice.

- To recover the plaintext m, Alice computes $m = (c_1^a)^{-1} c_2 \bmod p$. This works because $(c_1^a)^{-1} c_2 = g^{-ak} m g^{ak} = m \bmod p$.

- An eavesdropper has to find k from c_1 (discrete logarithm problem).

- A middle-man can easily manipulate c_2; for example, to replace m by $2m$ all that is necessary is to replace c_2 by $2c_2 \bmod p$.

- This public key cryptosystem, implemented exactly as above, has some security problems.

The Rivest-Shamir-Adleman cryptosystem

- The Rivest-Shamir-Adleman cryptosystem (or RSA for short), invented in 1977 on the MIT (but previously invented in 1973 by Clifford Cocks and kept classified by the GCHQ), is based on the observation (Fermat's little theorem) that when N is the product of two distinct prime numbers, i.e., $N = pq$, then for any x and any k we have

  $$x^{k\lambda(N)+1} \equiv x \pmod{N}.$$

- In particular, the transformation

  $$y = x^e \bmod N$$

  can be undone using the transformation

  $$x = y^d \bmod N$$

  provided that

  $$ed \equiv 1 \pmod{\lambda(N)},$$

  i.e., provided that $e = d^{-1} \bmod \lambda(N)$.

Rivest-Shamir-Adleman cryptosystem (continuation)

- The key observation is that this is easy to do only when $\lambda(N)$ is known.
- In turn, $\lambda(N)$ can be computed easily only when the factorization of N is known: $\lambda(N) = \lambda(pq) = \mathrm{lcm}(p-1, q-1)$.
- Since the factorization of a large number is considered to be a hard problem — for example RSA-250 was factored in 2020 using about 2700 core years — given N and e it is hard to compute d, and thus to recover y given x.
- Also, one may try do find the decryption exponent d directly by solving a discrete logarithm problem, which is also a hard problem.
- It is thus possible to publish N and e without revealing too much information.
- So, anyone using the RSA public key cryptosystem publishes hers/his own N and e.
- Sending a ciphered message to someone entails using that person's public modulus (N) and exponent (e)
- About the choice of the primes p and q:
  1. They should be random (do not reuse primes!)
  2. $p-1$ and $q-1$ should not have small prime factors

Rivest-Shamir-Adleman cryptosystem (continuation)

- Alice wants to send a message M to Bob.
- First, she fetches Bob's public encryption data: a modulus N_{bob} and an encryption exponent e_{bob}.
- Then, she computes the ciphered message $C = M^{e_{\text{bob}}} \bmod N_{\text{bob}}$, and sends it to Bob.
- Bob knows that $N_{\text{bob}} = p_{\text{bob}} q_{\text{bob}}$ (the secret information that only he knows), and so he can compute d_{bob}, the decryption exponent, such that $e_{\text{bob}} d_{\text{bob}} \equiv 1 \pmod{\lambda(N_{\text{bob}})}$.
- Using d_{bob} he can decipher C: $M = C^{d_{\text{bob}}} \bmod N_{\text{bob}}$.
- This works because

$$C^{d_{\text{bob}}} \bmod N_{\text{bob}} = M^{e_{\text{bob}} d_{\text{bob}}} \bmod N_{\text{bob}} = M^{k\lambda(N_{\text{bob}})+1} \bmod N_{\text{bob}} = M$$

- Note that the decryption can be done more efficiently using the Chinese remainder theorem. Instead of doing one modular exponentiation modulo N do, perhaps in parallel, two modular exponentiations, one modulo p and another modulo q, and at the end combine them using the Chinese remainder theorem. However, be aware of side-channel attacks...

Rivest-Shamir-Adleman cryptosystem (signing)

- The RSA cryptosystem can do even more: it is possible to ensure that the message came from a specified sender (that makes virtually impossible to forge a properly signed message)
- Main idea: Alice computes a message digest (hash) S of the message she wants to send to Bob and enciphers it using her own modulus and private decryption exponent:

$$S_{\text{alice}} = S^{d_{\text{alice}}} \bmod N_{\text{alice}}$$

- Bob can recover S using Alice's public data:

$$S_{\text{alice}}^{e_{\text{alice}}} \bmod N_{\text{alice}} = S^{e_{\text{alice}} d_{\text{alice}}} \bmod N_{\text{alice}} = S^{k\lambda(N_{\text{alice}})+1} \bmod N_{\text{alice}} = S$$

- So, Bob decodes the message Alice sent him, computes its message digest, and compares it with the S obtained from the S_{alice} data. If they match it is almost certain that it was indeed Alice that has sent the message. Otherwise, someone else was trying to impersonate Alice.
- For this to actually work, Bob has to trust Alice's public data. So, that public data has to be signed by a party trusted by everyone. Homework: Find out how certification chains and certification authorities work.

Rivest-Shamir-Adleman cryptosystem (blind signature)

- A signing entity publishes its public exponent e and its modulus n; it signs a message m it receives by computing $c = m^d \bmod n$, where d is its decryption exponent. The original message can be recoved by computing $m = c^e \bmod n$.

- Alice wants that public entity to sign her message m_a, but does not want enyone to know m_a. (Sending m_a directly to be signed is a very bad idea, because it could be decrypted from the signature by using only public information.)

- So, she chooses a random r such that $\gcd(r, n) = 1$ and sends $m = m_a r^e$ to be signed.

- She receives $c = m_a^d r^{ed} \bmod n = m_a r \bmod n$, and so she can compute the signature as $c_a = c r^{-1} \bmod n = m_a^d \bmod n$.

- She can now encrypt c_a with her own private key and send it, together with her encrypted m, to the intended recipient of the message.

- This method is unsafe. At the very least, apply a padding scheme with a reasonably large random field to the message that is to be signed.

Rivest-Shamir-Adleman cryptosystem (big example)

In August 1977, in his Scientific American Mathematical games column, Martin Gardner posed the following RSA challenge.

- Character encoding: space is 00, `A` to `Z` are 01 to 26. Other two digits combinations are illegal.
- The plain text is obtained by concatenating the two digits of each character encoding; the result is a large base-10 integer M.
- The plain text was then encoded using the modulus

$$\begin{aligned}N{=}&114381625757888867669235779976146612010218296721242362562561842935706935245733897830597123563958705058989075147599290026879543541\end{aligned}$$

  and the exponent $e = 9007$. The encoded message is $C = M^e \bmod N$, given by

$$\begin{aligned}C{=}&96869613754622061477140922254355882905759991124574319874695120930816298225145708356931476622883989628013391990551829945157815154.\end{aligned}$$

- What is M?

Rivest-Shamir-Adleman cryptosystem (solution of the big example)

- It took more than 10 years until $N = pq$ was factored (in 1977 it was estimated that the factorization would take much more time!):

  $$p{=}\mathbf{3490529510847650949147849619903898133417764638493387843990820577}$$

  and

  $$q{=}\mathbf{32769132993266709549961988190834461413177642967992942539798288533}.$$

- That made possible the computation of $d = e^{-1} \bmod \operatorname{lcm}(p-1, q-1)$;

  $$d{=}\mathbf{20912395050161373690941936346810195773046184093006090879304842322045608569697121472257875853682203172258717888678557376735780271}.$$

- Once d was known, M was recovered from $M = C^d \bmod N$:

  $$M{=}\mathbf{200805001301070903002315180419000118050019172105011309190800151919090618010705}.$$

- $20 \rightarrow T$, $08 \rightarrow H$, $05 \rightarrow E$, and so on. (The complete decryption is in the first slide.)
- Any decryption exponent of the form $d + k \operatorname{lcm}(p-1, q-1)$ works. Try a few values of k to find the exponent with the smallest sideways addition (population count).

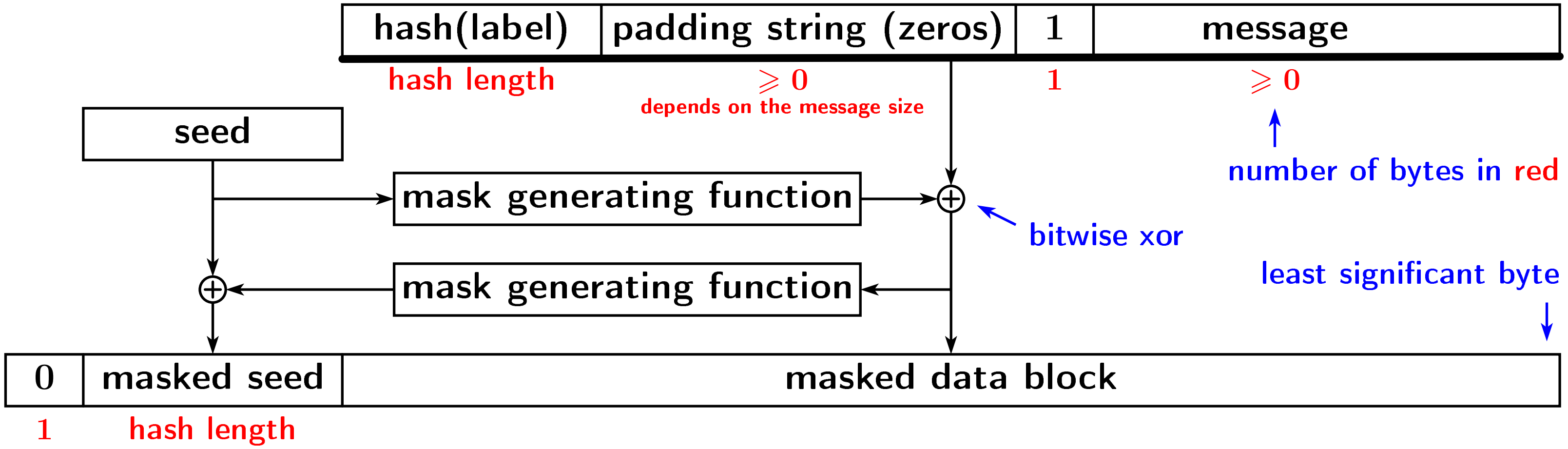
Rivest-Shamir-Adleman cryptosystem (padding)

Data source: section 7 of RFC 8017 (PKCS #1 RSA Cryptography Specifications Version v2.2).

- PKCS #1 v1.5 — avoid:

0	2	padding string (random non-zero bytes)	0	message
1	1	$\geqslant 8$, depends on the message size	1	$\geqslant 0$

- Optimal Asymmetric Encryption Padding (OAEP) — use:



hash(label)

padding string (zeros)

1

message

hash length

$\geqslant 0$

depends on the message size

1

$\geqslant 0$

number of bytes in red

seed

mask generating function

bitwise xor

mask generating function

least significant byte

0

masked seed

masked data block

1

hash length

Finite fields

- It is now time to generalize the modular arithmetic concept.
- In the so-called finite fields we do arithmetic on integers modulo a prime number p and we work with polynomials with coefficients in $\mathbb{Z}_p$.
- There is one extra twist: we also work modulo a polynomial!
- So our modular arithmetic will have two different aspects:
  - all integer arithmetic is done modulo a prime number p, and
  - all polynomial arithmetic is done modulo a polynomial of degree d.
- Not all polynomials of degree d can be used as the modulus: only those that are irreducible can be used. Just like a prime number, a polynomial is irreducible modulo p if it is not possible to factor it modulo p.
- The irreducibility of the polynomial modulus is fundamental. It ensures that the only polynomial of degree smaller than that of the modulus polynomial that does not have an inverse is the zero polynomial (and that is a fundamental property of a field).
- Complex numbers can be thought of as polynomials $ax + b$ of degree one for which arithmetic is done modulo $x^2 + 1$. Indeed, $x^2 \bmod (x^2 + 1) \equiv -1$, and so whenever x^2 appears is an expression it can be replaced by -1.

Finite fields (more info)

- The modulus polynomial can (and should) be a monic polynomial; the leading coefficient of a monic polynomial is one.

- Indeed, let $P(x)$ be the modulus polynomial, and let $A(x)$ be any polynomial. Then $A(x) \bmod P(x)$ is the remainder $R(x)$ of the division of $A(x)$ by $P(x)$. We have $A(x) = Q(x)P(x) + R(x)$, where $Q(x)$ is the quotient:

$$\begin{array}{l|l} A(x) & P(x) \\ \cline{2-2} R(x) & Q(x) \end{array}$$

- Now, if we replace $P(x)$ by $\alpha P(x)$, where α belongs to $\mathbb{Z}_p^*$ — recall that all integer arithmetic is done modulo p and that all elements of $\mathbb{Z}_p^*$ are invertible — then we have $A(x) = \big(\alpha^{-1}Q(x)\big)\big(\alpha P(x)\big) + R(x)$, so the remainder is the case no matter how α was selected.

- When the (irreducible) modulus polynomial has degree k the finite field is usually denoted by $\mathbb{F}_{p^k}$ or by $\mathrm{GF}(p^k)$; in publications involving finite fields, p^k is often replaced by the easier to write q (if so, q has to be the power of a prime).

- For the particular case $k = 1$ we have that $\mathbb{F}_p$ is the same as $\mathbb{Z}_p$.

Finite fields (example)

- Let us work with the prime $p = 5$.
- Let us work with the irreducible polynomial (modulo 5):

  $$P(x) = x^3 + x^2 + 3x + 4.$$

  This irreducible polynomial was found using the following `pari-gp` code (tutorial):

  ```
  x = ffgen([5,3]);
  x.mod
  ```

  (`x.p` gives the integer modulus, in this case 5).
- Each element of the finite field $\mathbb{F}(5^3)$ is a polynomial of the form

  $$a_2x^2 + a_1x + a_0$$

  where $a_0, a_1, a_2 \in \mathbb{F}_5$.
- Addition and subtraction of polynomials is done in the usual way (modulo 5).
- Multiplications is done in the usual way, but replacing x^3 by $-x^2-3x-4$, i.e., by $4x^2+2x+1$. (Why?)

Finite fields (example)

- Continuing the example of the previous slide, the quotient of the division of x^3 by $P(x)$ is $Q(x) = 1$, and so $x^3 \bmod P(x) = x^3 - P(x) = -x^2 - 3x - 4 = 4x^2 + 2x + 1$.
- In pari-gp, this can be confirmed by doing

```
x^3
```

- Here is a larger example:

$$\begin{array}{ccccccc|cccc}
 & \mathbf{1x^5} & \mathbf{4x^4} & \mathbf{3x^3} & \mathbf{0x^2} & \mathbf{1x^1} & \mathbf{3x^0} & \mathbf{1x^3} & \mathbf{1x^2} & \mathbf{3x^1} & \mathbf{4x^0} \\ \cline{8-11}
- & \mathbf{1x^5} & \mathbf{1x^4} & \mathbf{3x^3} & \mathbf{4x^2} & & & \mathbf{1x^2} & \mathbf{3x^1} & \mathbf{2x^0} & \\ \cline{2-5}
 & \mathbf{0x^5} & \mathbf{3x^4} & \mathbf{0x^3} & \mathbf{1x^2} & & & & & & \\
 & - & \mathbf{3x^4} & \mathbf{3x^3} & \mathbf{4x^2} & \mathbf{2x^1} & & & & & \\ \cline{3-6}
 & & \mathbf{0x^4} & \mathbf{2x^3} & \mathbf{2x^2} & \mathbf{4x^1} & & & & & \\
 & & - & \mathbf{2x^3} & \mathbf{2x^2} & \mathbf{1x^1} & \mathbf{3x^0} & & & & \\ \cline{4-7}
 & & & \mathbf{0x^3} & \mathbf{0x^2} & \mathbf{3x^1} & \mathbf{0x^0} & & & &
\end{array}$$

pari-gp confirmation (the modulo arithmetic is done automatically):

```
x^5+4*x^4+3*x^3+x+3
```

Finite fields (useful algorithms)

- The Euclidean algorithm works!
- In particular, the extended Euclidean algorithm can be used to compute inverses.
- The modular exponentiation algorithm also works.
- Since in the finite field $\mathbb{F}_q$ — recall that $q = p^k$ — we have

  $$a^q = a \qquad \text{for all } a \in \mathbb{F}_q$$

  (this is similar to Fermat's little theorem), the inverse can also be computed using

  $$a^{-1} = a^{q-2}$$
- Note that the exponents can be reduced modulo $q - 1$.
- The factorization of $q - 1$ is something that is useful to know.

Finite fields (primitive elements)

- The invertible elements (the units) of $\mathbb{F}_q$ form a set, denoted by $\mathbb{F}_q^*$.
- For a finite field we have $\mathbb{F}_q^* = \mathbb{F}_q \backslash \{0\}$.
- The order of an element of $\mathbb{F}_q^*$ is the smallest exponent o for which $a^o = 1$.
- The order has to divide $q - 1$, which is the number of elements of $\mathbb{F}_q^*$.
- A primitive element has maximal order.
- Repeated multiplication by the same primitive element generates $\mathbb{F}_q^*$; when that happens the field is said to be a multiplicative cyclic group.
- There exist $\varphi(q-1)$ primitive elements: if r is a primitive element then r^e will also be a primitive element if and only if $\gcd(e, q-1) = 1$.
- Therefore, there exist lots of primitive elements if q is large, so finding one is easy (if the factorization of $q - 1$ is known, see next slide).
- In pari-gp we can compute a primitive element using the function `ffprimroot()`.

Finite fields (one way to find a primitive polynomial)

Just like in standard modular arithmetic, r is a primitive element of $\mathbb{F}_q^*$ if and only if

- $r^{q-1} = 1$, and
- for every prime divisor d of $q - 1$, we have $r^{(q-1)/d} \neq 1$.

One way to find an irreducible polynomial is to find one of the primitive roots of the finite field it generates. So, to compute an irreducible polynomial of degree k when we are working modulo p — finite field $\mathbb{F}_q$ with $q = p^k$ — do the following:

1. Choose a monic polynomial of degree k.
2. Choose a desired primitive element r, say, $r = x$ (that choice is particularly useful, read the slide about cyclic redundancy checksums).
3. Check if r is a primitive element.
4. If so, the polynomial is irreducible, and we are done.
5. If not, then the polynomial may be irreducible but r is not a primitive element or it is not irreducible; go back to the beginning and try another polynomial.
6. Since there exist $\varphi(q-1)$ primitive elements when the polynomial is irreducible, and there exist $\frac{1}{k}\sum_{d|k}\mu(d)p^d$ irreducible polynomials of degree k modulo p, this procedure finds one of them in a reasonable amount of time.

Applications of finite fields

The Diffie-Hellman key exchange protocol can be trivially extended to finite fields.

- Unique difference: Alice and Bob, instead of agreeing on a prime and on one of its primitive roots, have to agree on a finite field (prime p, irreducible monic polynomial of degree k) and on one of its primitive elements.
- Anyone wishing to infer the shared secret has to solve the discrete logarithm problem, in this case for finite fields.

Elliptic curves (discussed later in this course) also work in finite fields.

Shamir's secret sharing scheme also works in finite fields (discussed later in this course).

- Unique difference: the coefficients of the polynomials, instead of belonging to $\mathbb{F}_p$, belong to $\mathbb{F}_{p^k}$. Nice! Here we have polynomials whose coefficients are other polynomials (in another variable and subjected to modulo arithmetic in two distinct ways!)

Finite fields and Cyclic Redundancy Checksums (CRCs)

- The so-called cyclic redundancy checksum (CRC) is a way to compute a "signature" of a data set (used at the hardware level as a simple way to perform error detection).

- The data is transformed into a polynomial, and the CRC is just the remainder of that polynomial when divided by a known polynomial.

- Usually, the modulus polynomial is an irreducible polynomial having x as one of its primitive elements (a so-called primitive polynomial).

- Furthermore, this is done with $p = 2$, i.e., in the finite field $\mathbb{F}_{2^k}$. This is so because in the base field, $\mathbb{F}_2 = \mathbb{Z}_2$, addition and multiplication are particularly simple: addition is the exclusive-or binary logic operator and multiplication is the and binary logic operator.

- They are not useful (i.e., unsafe) in cryptographic applications as a way to compute message hashes (due to its linear nature, it is trivial to forge a message having a specific message hash).

- But they can be used as a hash function in an implementation of a hash table.

Elliptic curves

- Now we get to play with some weird stuff.
- We will do addition is a strange way.
- The addition operator $+$ is a binary operator; it takes two elements of a group and produces a third element of the same group.
- Addition properties (in any group):

  commutative law $x + y = y + x$

  associative law $x + (y + z) = (x + y) + z$
- Idea: suppose we have a plane curve with the following property: any straight line intersects it in exactly three points, counted with multiplicity. If so, the addition of two points on that line can be the third point!
- To make this work, it is necessary to treat the point at infinity as a legitimate point (use homogeneous coordinates, also known as projective coordinates). The point at infinity is the neutral element, and so it plays a fundamental role.
- Three intersection points $\Rightarrow$ cubic equation.

Elliptic curves (cubic equation)

- The cubic equations we will consider have in the following form (Weierstrass parameterization):

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

- Both x and y belong to a field F (or are the point at infinity).
- With a change of variables (which in some cases cannot be done due to divisions by zero), the equation above can be put in the so-called Weierstrass form

$$(*) \quad y^2 = x^3 + ax + b.$$

- The so-called discriminant of the curve E, whose points satisfy equation (*), is the quantity

$$\Delta(E) = -16(4a^3 + 27b^2).$$

To avoid degenerate curves this discriminant cannot be zero.

Elliptic curves (homogeneous coordinates)

- In homogeneous coordinates we add a third coordinate: z.
- (x, y) becomes (X, Y, Z).
- (X, Y, Z), for any $Z \neq 0$, corresponds to the two-dimensional point $(\frac{X}{Z}, \frac{Y}{Z})$ — it's an equivalence class.
- $Z = 0$ represents the "points at infinity"; X and Y then specify the direction.
- For an elliptic curve in Weierstrass form, $y^2 = x^3 + ax + b$, for very large x we have $y \approx \pm x^{3/2}$.
- So, very far from the origin, y will be considerably larger than x.
- The homogeneous coordinates of the point at infinity (any non-zero Y will do, so we normalize it to be 1) are $(0, 1, 0)$.
- The equation of an elliptic curve in Weierstrass form in homogeneous coordinates is $Y^2Z = X^3 + aXZ^2 + bZ^3$.

Elliptic curves (pari-gp)

- In pari-gp, the general curve

  $$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

  can be specified using the command

  ```
  E=ellinit([a1,a2,a3,a4,a6]);
  ```

- In pari-gp, the special curve

  $$y^2 = x^3 + ax + b$$

  can obviously be specified using the command

  ```
  E=ellinit([0,0,0,a,b]);
  ```

  The shortcut

  ```
  E=ellinit([a,b]);
  ```

  can also be used.

Elliptic curves over finite fields (pari-gp)

- In pari-gp it is also possible to specify the field over which all computations will be performed.
- This is specified in a second (optional) argument to `ellinit`.
- If this second argument
  - ⋆ is missing or is the integer 1, the field will be $\mathbb{Q}$
  - ⋆ is the integer p, a prime number, or is a `Mod(*,p)`, the field will be $\mathbb{F}_p$
  - ⋆ is the value returned by `ffgen([p,k])`, the field will be the finite field $\mathbb{F}_{p^k}$
  - ⋆ is a real number, the field will be $\mathbb{C}$

  It may also be a more exotic object.
- The number of points on the elliptic curve can be computed using the `ellcard` function.
- The order of a point (number of times we have to add the point to itself until we get the point at infinity), can be computed using the `ellorder` function.
- In the specific case of the field $\mathbb{F}_p$ the number of points on the elliptic curve can also be computed using the more efficient `ellsea` function. It is known that the number of points N of the curve satisfies the so called Hasse bound

$$\bigl|N-(p+1)\bigr| \leqslant 2\sqrt{p}.$$

Playing with elliptic curves (pari-gp)

- To get a list of official pari-gp tutorials consult this web page.
- In particular, read the elliptic curves tutorial.
- Better yet, read the elliptic curves over finite fields tutorial.
- You can also look at the list of functions related to elliptic curves.
- Let's play!

```
/* find an elliptic curve of the form y^2=x^3+x+1 */
/* over Fp which has a prime number of points     */
forprime(p=2^100,oo,E=ellinit([1,1],p);\
         q=ellsea(E);if(isprime(q),break;);); /* 1 minute */
E.p            /* print the modulus                        */
q=ellsea(E)    /* print the number of points of the curve */
G=E.gen;G=G[1] /* get a generator (there is only one)     */
ellorder(E,G)
```

Elliptic curves (adding distinct finite points — geometric interpretation)

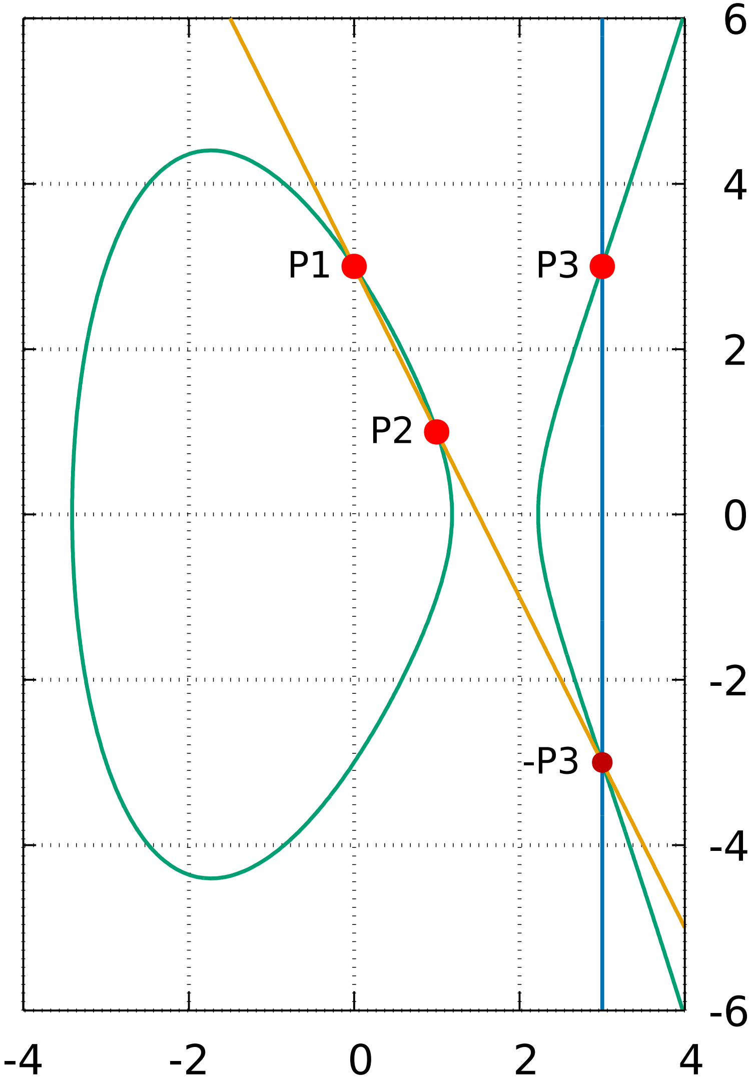
- elliptic curve over $\mathbb{Q}$:

$$y^2 = x^3 - 9x + 9$$

- pari-gp code:

```
E=ellinit([0,0,0,-9,9]);
P1=[0,3];
P2=[1,1];
ellisoncurve(E,P1)  /* 1     */
ellisoncurve(E,P2)  /* 1     */
P3=elladd(E,P1,P2)  /* [3,3] */
```

- Draw the line that passes through P_1 and P_2
- That line intersects the elliptic curve at a third point: $-P_3$
- Reflect it on the x axis to get the sum of P_1 and P_2



Elliptic curves (adding the point at infinity)

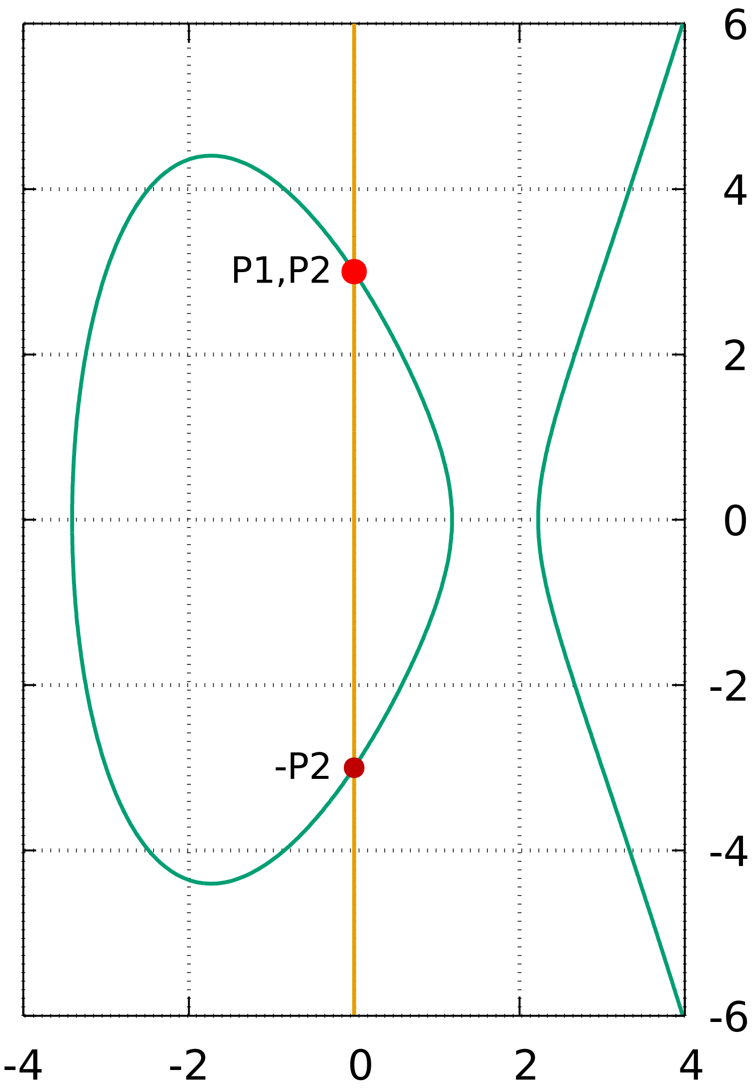
- elliptic curve over $\mathbb{Q}$:

$$y^2 = x^3 - 9x + 9.$$

- pari-gp code (the point at infinity is represented by [0]):

```
E=ellinit([-9,9]);
P1=[0,3];
ellisoncurve(E,P1)  /* 1     */
ellisoncurve(E,[0]) /* 1     */
P2=elladd(E,P1,[0]) /* [0,3] */
```

- The point of infinity is the neutral element (the zero).
- Adding to a point the point at infinity (intersection with a vertical line) leaves it unchanged.
- Adding a point to its symmetric (its reflection on the x axis) gives rise to the point at infinity.



Elliptic curves (doubling — geometric interpretation)

- elliptic curve over $\mathbb{Q}$:

$$y^2 = x^3 - 9x + 9.$$

- pari-gp code:

```
E=ellinit([-9,9]);
P1=[0,3];
ellisoncurve(E,P1)
P2=ellmul(E,P1,2);
/* same as P2=elladd(E,P1,P1); */
```

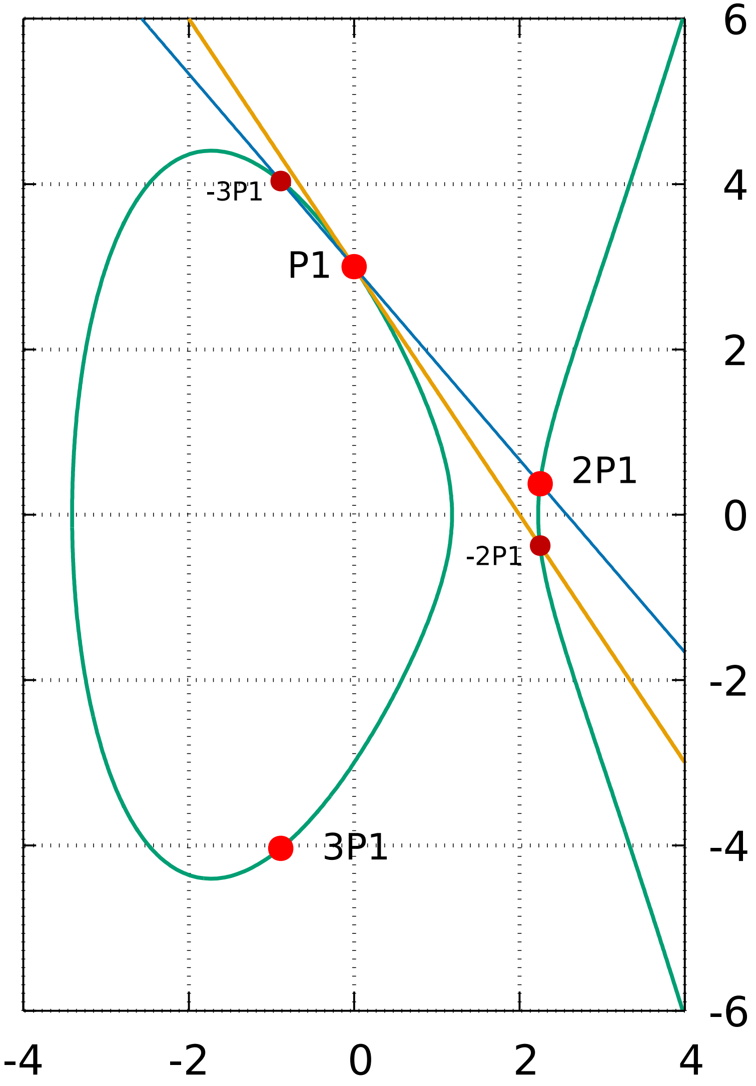
We have

$$2P1 = \left(\frac{9}{4}, \frac{3}{8}\right)$$

$$3P1 = \left(-\frac{8}{9}, -\frac{109}{27}\right)$$

$$4P1 = \left(\frac{1017}{16}, -\frac{32397}{64}\right)$$

$$5P1 = \left(\frac{7848}{12769}, \frac{2775711}{1442897}\right)$$



Elliptic curves (adding two points — some formulas)

- The equation of a straight line that passes through two distinct points (x_1, y_1) and (x_2, y_2) is

$$(x - x_1)(y_2 - y_1) = (x_2 - x_1)(y - y_1).$$

- It can be put in the form $Ax + By + C = 0$.
- If the inverse of B exists (i.e., the line is not a vertical line), then we can say that

$$y = Dx + E.$$

- Putting this in the cubic equation $y^2 = x^3 + ax + b$ gives rise to a polynomial equation of third degree in x of the general form

$$x^3 + \alpha x^2 + \beta c + \gamma = 0.$$

- It has three solutions. Two of them must be x_1 and x_2. The third one is the x coordinate of the point we are looking for. Explicit formulas are cumbersome. See next page or, for example, Complete addition formulas for prime order elliptic curves.
- When working with rational numbers ($\mathbb{Q}$), because the sum of the roots is $-\alpha$, it follows that this third root must also be a rational number!

Elliptic curves (adding two points — explicit formulas)

- Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be two points of the elliptic curve $y^2 = x^3 + ax + b$. Let O denote the point at infinity.

- Then

$$P + Q = \begin{cases} P, & \text{if } Q = O \text{, otherwise} \\ Q, & \text{if } P = O \text{, otherwise} \\ O, & \text{if } x_2 = x_1 \text{ and } y_2 = -y_1 \text{, otherwise} \\ (x_3, y_3) \end{cases}$$

  where (when applicable)

$$x_3 = m^2 - x_1 - x_2, \qquad y_3 = m(x_1 - x_3) - y_1,$$

  and where

$$m = \begin{cases} \dfrac{y_2 - y_1}{x_2 - x_1}, & P \neq Q; \\ \dfrac{3x_1^2 + a}{2y_1}, & P = Q. \end{cases}$$

- Using homogeneous coordinates it is possible to avoid almost all divisions.

- Warning! Possible side-channel attack (timing attack).

Multiplication by an integer (adding a point to itself several times)

- We can now define the mathematical operation that is useful for cryptographic purposes: multiplication of a point by an integer.

- This corresponds to adding a point with itself several times.

- In terms of cryptographic applications this corresponds roughly to the modular exponentiation done in finite fields.

- Example: to compute, say, $11P$ we can proceed as follows:

  1. $11 = 1 + 2 + 8$
  2. compute $2P = P + P$
  3. compute $4P = (2P) + (2P)$
  4. compute $8P = (4P) + (4P)$
  5. finally, compute $11P = (1P) + (2P) + (8P)$

- This multiplication algorithm is similar in spirit to the algorithm presented in the modular exponentiation slides.

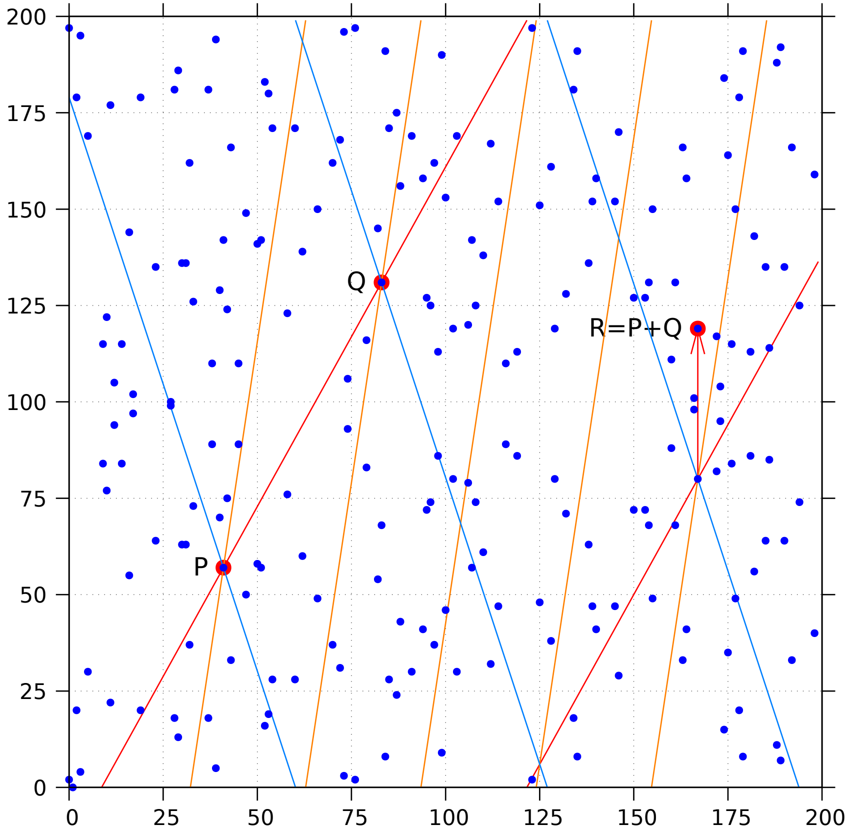
- Hard problem (on some elliptic curves): given P and kP find k.

Elliptic curves (aspect of the “curve” on a finite field)

- elliptic curve over $\mathbb{F}_{199}$:

$$y^2 = x^3 - 5x + 4.$$

- it has 218 points (including the point “at infinity”):



```
p=199;
E=ellinit([-5,4],p);
N=ellsea(E)            /* 218 */
P=Mod([41,57],p);
ellisoncurve(E,P)      /* 1   */
Q=Mod([83,131],p);
ellisoncurve(E,Q)      /* 1   */
R=elladd(E,P,Q);
lift(Q[1])             /* 167 */
lift(Q[2])             /* 119 */
```

Can we do RSA-like things with elliptic curves?

- No (here we do not have any hidden secret):

```
bits=100;
p=nextprime(random([2^(bits-1)+1,2^bits-1]));
E=ellinit([0,0,0,1,1],p);
P=random(E);
o=ellorder(E,P);
k=0;while(gcd(k,o)!=1,k=random([2,o-2]);); /* public multiplier */
Q=ellmul(E,P,k);
kInv=lift(1/Mod(k,o)); /* private multiplier used for decoding  */
R=ellmul(E,Q,kInv)     /* we recover P                          */
```

- Finite fields must have a modulus that is the power of a prime number, so hard-to-factor moduli cannot be used.
- We would need a point in an elliptic curve for which it would be extremely difficult to compute its order without knowing the “secret”.
- However, we can do Diffie-Hellman-like things! Stay tuned.
- We can also scramble the information we want to encrypt using the coordinates of a point of the elliptic curve, as explained in the next page.

The Menezes-Vanstone elliptic curve cryptosystem

A simple way to construct a public-key cryptosystem using elliptic curves is the following:

- Choose an elliptic curve on $\mathbb{F}_p$,
- Choose a point P of that elliptic curve. Let n be the order of that point; n should be large and have large factors.
- Choose a private (secret) integer α such that $2 \leqslant \alpha \leqslant p - 2$.
- Publish the elliptic curve details (formula and p), P, and $Q = \alpha P$.
- To encrypt (m_1, m_2), with $0 \leqslant m_1, m_2 < p$, choose a random k such that $1 \leqslant k < n$, and compute $C_0 = kP$ and $C_1 = kQ = (x_1, y_1)$. If x_1 or y_1 is zero or if C_1 is the point at infinity, then choose another k and recalculate. Send C_0, $m_1 x_1 \bmod p$, and $m_2 y_1 \bmod p$.
- To decrypt, compute $\alpha C_0 = \alpha k P = kQ = C_1 = (x_1, y_1)$. The rest is easy because x_1 and y_1 have modular inverses modulo p.

To break this cryptosystem one has to find the secret α starting from the public P and Q. That is precisely the discrete-logarithm problem on elliptic curves, which is hard if n is large and has large factors (there is more information about this hard problem latter on).

If you want to know more about elliptic curves

- Edwards curves (alternative parameterization of elliptic curves) — paper about them
- “Safe” elliptic curves
- Curve 25519, wikipedia

Diffie-Hellman using elliptic curves

- We can now explain how the Diffie-Hellman key exchange protocol (i.e., secret sharing) scheme can be done using elliptic curves.

- Alice and Bob agree on an elliptic curve and on a point P — of large order — of that elliptic curve.

- Alice chooses a private random integer k_A and sends $k_A P$ to Bob.

- Bob chooses a private random integer k_B and sends $k_B P$ to Alice.

- The shared secret S is the point $k_A k_B P$; Alice and Bob can compute it easily using the private information they have and the information each received from the other one.

- A third party will have to attempt to compute k_A from the information Alice sent to Bob (over a possibly compromised channel) or to compute k_B from the information Bob sent to Alice. This can be a very hard problem (discrete logarithm for elliptic curves).

Let's play some more with elliptic curves in pari-gp

- Let's see how long it takes to compute k given P and kP:

```
#
bits=50;
p=nextprime(random([2^(bits-1)+1,2^bits-1]));
E=ellinit([0,0,0,1,1],p);
P=random(E);
o=ellorder(E,P)    /* a few seconds for 200 bits! */
k=random([2,o-2])
Q=ellmul(E,P,k);
elllog(E,Q,P)      /* grows very fast with the number of bits */
```

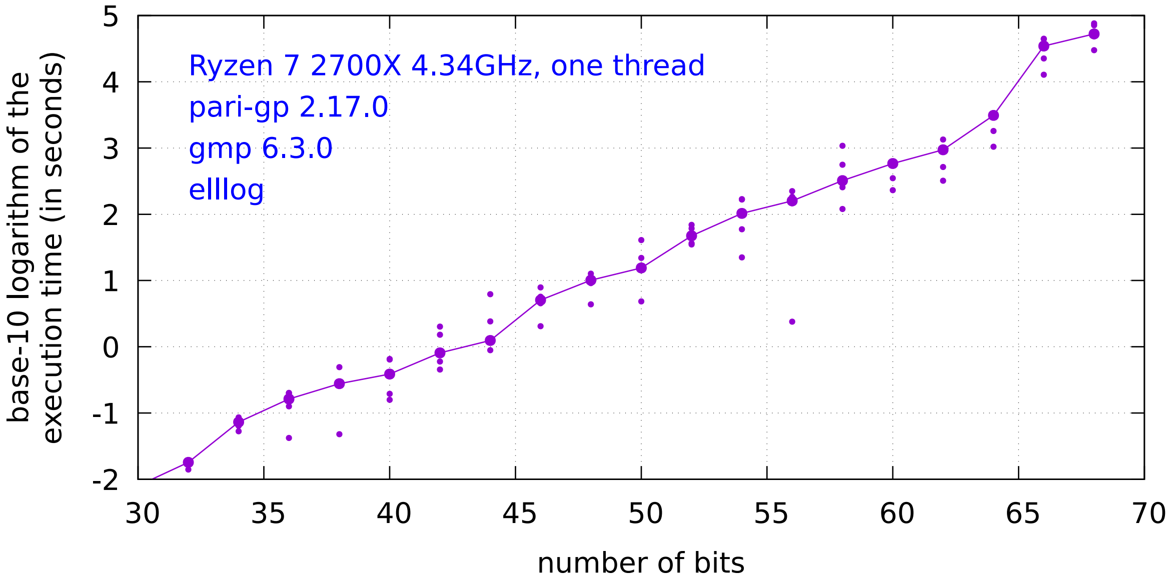
- That was fast. The Diffie-Hellman key exchange protocol with this number of bits can be broken very easily. How about a larger number of bits?

The discrete logarithm problem for elliptic curves

- Given points P and Q on an elliptic curve, find k such that

$$Q = kP.$$

- This is a hard problem if the order of P has large prime factors:



Pari-gp code used to compute the figure data:

```
do_one(b,seed)={ my(p,E,q,G,k1,P,dt,k2);
  setrand(seed); p=random([5*2^(b-3),2^b]);
  while(1,p=precprime(p-1); E=ellinit([1,1],p);
    q=ellsea(E); if(isprime(q)==1,break(1);););
  printf("# %d %d %d\n",b,p,q);
  G=E.gen; G=G[1]; k1=random([floor(q/10),floor(9*q/10)]);
  P=ellmul(E,G,k1); dt=getabstime();
  k2=elllog(E,P,G); dt=getabstime()-dt;
  if(k2!=k1,quit(1);); return(0.001*dt); };

do_many(b,nt=5)={ my(dt);
  dt=vecsort(vector(nt,k,do_one(b,100*b+k)));
  printf("%3d",b); for(k=1,nt,printf(" %.3f",dt[k]););
  printf("\n"); return(vecmax(dt)); };

printf("# version=%d\n",version());
forstep(b=30,500,2,if(do_many(b)>90000.0,break(1);););
quit();
```

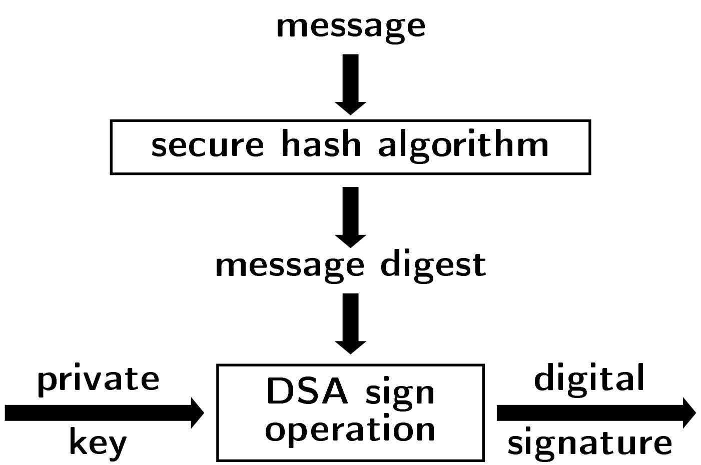
- However, it turns out that the problem is easy if the order of P does not have large prime factors or if the number of points of the elliptic curve over $\mathbb{F}_p$ is p itself (a prime-anomalous elliptic curve). Those cases must be avoided.

- The same problem but with plain modular arithmetic

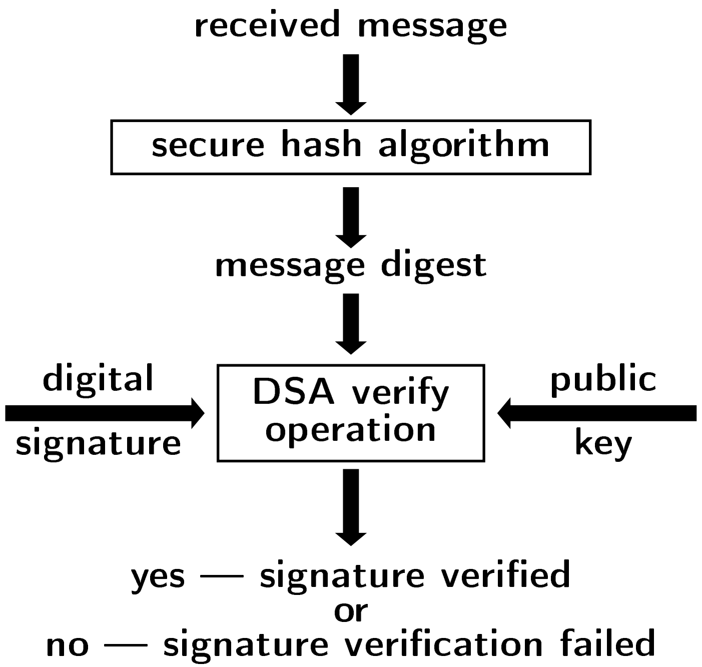
Digital signatures (DSA and ECDSA)

The 1994 Digital Signature Standard (DSS, FIPS PUB 186), explains how to perform a digital signature (DSA). It uses ideas we have already encountered, viz., modular exponentiation and Fermat's little theorem. The following figure explains the top level flow of information for signing (on the left), and for verifying a signature (on the right).

Signature generation



Signature verification



This workflow remains valid in the lattest version of the standard (FIPS 186-5, February 2023).

Digital signatures (DSA parameters and variables)

DSA parameters and variables:

- A large prime p, with exactly L bits, i.e., $2^{L-1} < p < 2^L$.
- A large prime q that divides $q - 1$ with exactly N bits, considerably smaller than L.
- $g = h^{(p-1)/q} \bmod p$, where h is any integer with $2 \leqslant h \leqslant p-2$ such that $h^{(p-1)/q} \bmod p \neq 1$; this implies that the order of g modulo p is exactly q,
- an integer x, with $2 \leqslant x \leqslant q - 2$ — it is best to use one such that $\gcd(x, p - 1) = 1$,
- $y = g^x \bmod p$,
- an integer k, with $2 \leqslant k \leqslant q - 2$,
- an integer H, which is the numerical value of the message digest of the message being signed. (The standard states that this has to be truncated to L bits but that is not really necessary.)

The integers p, q and g can be made public and can be the same for a group of users. They can also be published by a signing authority. The integer x is the private key of a given user, and y is the corresponding public key. The integer k must be regenerated for each signature. Repeating k in the signatures of two different messages allows anyone to recover the secret x. DSA signatures are now obsolete.

Digital signatures (DSA sign and verify operations)

To sign a message with message digest H, do:

- Generate a new random k. Compute $r = (g^k \bmod p) \bmod q$, and $s = \big(k^{-1}(H + xr)\big) \bmod q$. If either r or s is zero then choose another k and recalculate.
- The signature is the pair of numbers (r, s).

To verify a signature, do:

- Make sure $0 < r < q$ and $0 < s < q$. Reject the signature if not.
- Compute $w = s^{-1} \bmod q$, $u_1 = Hw \bmod q$, and $u_2 = rw \bmod q$.
- Finally, compute $v = (g^{u_1} y^{u_2} \bmod p) \bmod q$.
- If $v = r$ accept the signature. If not, reject it.

Why it works:

- We have $v = (g^{Hw} g^{xrw} \bmod p) \bmod q = (g^{w(H+rx)} \bmod p) \bmod q$.
- We also have $w = k(H + rx)^{-1} \bmod q$.
- Since g has order q, and so we must have $g^q \bmod p = 1$, when the base is g exponents can be reduced modulo q and so $g^{w(H+rx)} \bmod p = g^k \bmod p$. Modulo q this is precisely r.

Why repeating k is very bad (all computations are done modulo q):

- $r_1 = r_2 = r = g^k$
- $s_1 k = H_1 + xr$
- $s_2 k = H_2 + xr$
- $(s_1 - s_2)k = H_1 - H_2$
- $k = (s_1 - s_2)^{-1}(H_1 - H_2)$
- $x = g^{-k}(s_1 k - H_1)$
- game over!

Idem if $k_2 = ak_1 + b$ when a and b are known.

Digital signatures (ECDSA parameters and variables)

ECDSA domain parameters and variables:

- An elliptic curve E.
- A point G on E of order n; n must be a (large) prime number.
- a private key d, such that $2 \leqslant q \leqslant n - 2$, and a corresponding public key $Q = dG$.
- like the DSA, an integer k, with $2 \leqslant k \leqslant n - 2$, and an integer H, which is the numerical value of the message digest of the message being signed.

To sign a message with message digest H, for an elliptc curve in $\mathbb{F}_p$, do:

- Generate a new random k. Compute $r = (kG)_x \bmod n$, which is the x coordinate of kG modulo n, and compute $s = \bigl(k^{-1}(H + rd)\bigr) \bmod n$. If either r or s is zero then choose another k and recalculate. The signature is the pair of numbers (r, s).

To verify a signature, do:

- Make sure $0 < r < n$ and $0 < s < n$. Reject the signature if not.
- Compute $u = Hs^{-1} \bmod n$ and $v = rs^{-1} \bmod n$, and then compute $R = uG + vQ$. Reject the signature if R is the point at infinity. Accept the signature if $(R)_x \bmod n = r$. If not, reject it. Homework: prove that this works. Warning: a bad signature for which $R = -kG$ is also accepted as valid (same x coordinate)!

Secret sharing

Problem:

- n persons what to share a secret.
- Any group of t persons can recover the secret.
- Obviously, $n \geqslant 1$ and $1 \leqslant t \leqslant n$.
- On a computer program, the secret will ultimately be an integer.

How to do it:

- A trusted central entity prepares and distributes part of the secret (a secret share) to each person.

Hurdle to overcome:

- Knowing $t - 1$ shares of the secret must not give any information about the secret.

Resilience to tampering:

- To make the secret unrecoverable, $n - t + 1$ secret shares have to be corrupted.

Secret sharing (how to do it when $t = n$)

- Let the secret be the integer S, and let it have k bits.
- Let the first $n-1$ shares of the secret, s_1 to s_{n-1}, be random integers with k bits.
- Let the last share of the secret be the exclusive-or of the secret with all the other shares of the secret ($\oplus$ denotes here the bit-wise exclusive-or binary operator):

$$s_n = S \oplus s_1 \oplus s_2 \oplus \cdots \oplus s_{n-1}.$$

- To recover the secret it is only necessary to perform an exclusive-or of all secret shares:

$$S = s_1 \oplus s_2 \oplus \cdots \oplus s_n.$$

- Knowledge of $n-1$ secret shares does not give any information about the secret.
- It is possible to replace the bit-wise exclusive-or operations by addition and subtractions modulo m. In this case, the first $n-1$ secret shares are random integers from 0 to $m-1$, and the last secret share is $(S - s_1 - s_2 - \cdots s_{n-1}) \bmod m$. To recover the secret it is only necessary to add all secret shares (modulo m, of course).
- Using modulo arithmetic, it is also possible to replace addition by multiplication, but only if the secret shares, and the secret itself, are oprime to the modulus. (Why?)

Secret sharing (how to do it when $t \leqslant n$, Blakley)

Blakley's secret sharing scheme:

- The secret is a point P in a t-dimensional space.
- Each share of the secret is a linear equation (with t unknowns) that has P has one of its solutions.
- Putting together t equations allows us to find P.
- It is necessary to ensure that the system of equations has a unique solution for all possible $C^n_t = \frac{n!}{t!(n-t)!}$ possible combinations of t equations chosen from the n equations, and that is cumbersome (it is necessary to ensure that the rank of the $n \times t$ matrix that stores the right-hand side of all linear equations is exactly t).
- Each share of the secret is a tuple of $t+1$ numbers.
- Improved security: the secret is kept only in one of the coordinates of the point P.
- Modular arithmetic should be used. (Why?).
- The unknowns may also be points on an elliptic curve (the elliptic curve should have an appropriate number of points, say, a prime number).

Secret sharing ($t \leqslant n$, Blakley example)

The following example illustrates the need of modular arithmetic. Let $t = 3$, and let the secret be the last coordinate of the three-dimensional coordinate $(23, -17, 40)$. The secret shares were designed so that the coordinates have absolute value smaller that 100, as were the coefficients of the linear equations (but but the independent terms). Three of the secret shares are

$$\begin{aligned} 93x + 37y - 6z &= 1270 \\ 83x + 39y - 69z &= -1514 \\ 79x + 48y - 86z &= -2439 \end{aligned}$$

Now, imagine that the first two intities attempt to recover the secret. From the first secret share we have

$$x = \frac{1270 - 37y + 6z}{93}$$

and applying that to the second secret share we get

$$556y - 5919z = -246212,$$

i.e., (remember, the secret is z)

$$z = \frac{556y + 246212}{5919}.$$

Since z must be an integer it is easy to find it! With modular arithmetic, each y value gives rise to a valid z value.

Secret sharing (another way to do it when $t \leqslant n$, Shamir)

Shamir's secret sharing scheme:

- The secret in the independent coefficient a_0 of a polynomial of degree $t-1$,

$$A(x) = \sum_{k=0}^{t-1} a_k x^k.$$

- Each secret share in the pair $\big(x_k, A(x_k)\big)$.
- It is necessary to ensure that distinct values of x_k are used.
- Again, modular arithmetic should be used (why?).
- Each share of the secret is a tuple of only 2 numbers.

Things to think about:

- Can we do it using square matrices for the a_k coefficients?
- And how about for the a_k coefficients and for the x_k values?

Secret sharing (polynomial interpolation)

Given points (x_k, y_k), for $k = 0, 1, \ldots, n$, with $x_i \neq x_j$ for $i \neq j$, compute the unique polynomial of degree n that passes through these points.

- Newton's interpolation formula:

$$P_0(x) = y_0,$$

and, for $k = 1, 2, \ldots, n$,

$$P_k(x) = P_{k-1}(x) + \bigl(y_k - P_{k-1}(x_k)\bigr)\frac{(x - x_0)\cdots(x - x_{k-1})}{(x_k - x_0)\cdots(x_k - x_{k-1})}.$$

- Lagrange's interpolation formula:

$$P_n(x) = \sum_{k=0}^{n} y_k \prod_{\substack{i=0\\ i\neq k}}^{n} \frac{x - x_i}{x_k - x_i}.$$

If arithmetic modulo p is used we must have $x_i \not\equiv x_j \pmod{p}$ for $i \neq j$. If so, all modular inverses needed by Newton's or Lagrange's intepolation formulas exist.

Secret sharing (one more, Mignotte and Asmuth–Bloom)

Let us do it using the Chinese remainder theorem!

- In the Mignotte threshold secret sharing scheme n coprime integers $m_1 < m_2 < \cdots < m_n$ are chosen such that $\beta - \alpha$ is positive and large, where $\alpha = \prod_{k=1}^{t-1} m_{n+1-k}$ is the product of the $t-1$ largest integers and where $\beta = \prod_{k=1}^{t} m_k$ is the product of the t smallest integers.

- The secret is a randomly chosen integer s belonging to the interval $]\alpha, \beta[$. The secret shares are the tuples $(s \bmod m_k, m_k)$ for $1 \leqslant k \leqslant n$.

- To uniquely recover the secret using the Chinese remainder theorem, the product of the moduli of the secret shares being used has to be larger that s. This is ensured because $s < \beta$. On the other hand, $t-1$ shares cannot be used to recover the secret, and this is ensured because $s > \alpha$.

- Example ($n = 6$, $t = 3$, $s = 774157997$):

(s_1, m_1)	(s_2, m_2)	(s_3, m_3)	(s_4, m_4)	(s_5, m_5)	(s_6, m_6)
$(997, 1000)$	$(613, 1001)$	$(471, 1003)$	$(565, 1007)$	$(729, 1009)$	$(923, 1011)$

  In this example $\alpha = 1020099$ and $\beta = 1004003000$. The two shares with the largest moduli are not enough to expose the secret, because using them it is only possible to infer that $s = 922955 + 1020099$ for some integer $0 \leqslant k < \beta/\alpha \approx 984.22$. However, if the secret is just $s \bmod m_0$ with $m_0 < \beta/\alpha$ coprime with the other moduli, no information is leaked about that smaller secret. That is precisely the Asmuth–Bloom secret sharing scheme.

Quadratic residues

- Let n be a positive integer and let a be an integer such that $\gcd(a, n) = 1$.
- a is said to be a quadratic residue modulo n if and only if there exists a x such that

$$x^2 \equiv a \pmod{n}.$$

- When n is a prime number ($n = p$) there exist three cases:
  1. either a is a multiple of p, or
  2. a is a quadratic residue, or
  3. a is a not a quadratic residue (a quadratic nonresidue).
- The Legendre symbol $\left(\frac{a}{p}\right)$ captures this as follows

$$\left(\frac{a}{p}\right) = \begin{cases} 0, & \text{if } p \text{ divides } a, \\ +1, & \text{if } p \text{ does not divide } a \text{ and } a \text{ is a quadratic residue modulo } p, \\ -1, & \text{if } p \text{ does not divide } a \text{ and } a \text{ is a quadratic nonresidue modulo } p. \end{cases}$$

Quadratic residues (Legendre symbol)

- For $p > 2$ the Legendre symbol satisfies the equation

$$\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} \pmod{p}.$$

  (Recall that from Fermat's little theorem we know that $a^{p-1} \equiv 1 \pmod{p}$ when p does not divide a.)

- So, $\left(\frac{a}{p}\right) = \left(\frac{a \bmod p}{p}\right)$ and, if a is not divisible by p, $\left(\frac{a^2}{p}\right) = 1$.
- In particular, it is possible to prove that

$$\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}, \quad \text{that} \quad \left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}, \quad \text{and that} \quad \left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right).$$

- If q is an odd prime, we also have (this is the famous law of quadratic reciprocity)

$$\left(\frac{q}{p}\right) = (-1)^{\frac{(p-1)(q-1)}{4}} \left(\frac{p}{q}\right).$$

- These properties allow us to easily compute the Legendre symbol for any a and p (if the factorization of a is known).

Quadratic residues (Legendre symbol computation)

- Example: Compute $\left(\frac{-14}{73}\right)$.
- $\left(\frac{-14}{73}\right) = \left(\frac{-1}{73}\right)\left(\frac{2}{73}\right)\left(\frac{7}{73}\right)$
- $\left(\frac{-1}{73}\right) = (-1)^{36} = +1.$
- $\left(\frac{2}{73}\right) = (-1)^{666} = +1.$
- $\left(\frac{7}{73}\right) = (-1)^{108}\left(\frac{73}{7}\right) = \left(\frac{3}{7}\right).$
- $\left(\frac{3}{7}\right) = (-1)^{3}\left(\frac{7}{3}\right) = -\left(\frac{1}{3}\right) = -1.$ (Obviously, $\left(\frac{1}{p}\right) = +1.$)
- So, putting it all together, we have $\left(\frac{-14}{73}\right) = -1$
- pari-gp agrees (in pari-gp the Legendre symbol can be computed with the kronecker function):

```
kronecker(-14,73)  /* returns -1 */
```

Quadratic residues (Jacobi symbol)

- The Jacobi symbol is an extension of the Legendre symbol to the case where the modulus is not a prime number.
- Let $n = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$.
- The Jacobi symbol $\left(\frac{a}{n}\right)$ — yes, it is denoted in exactly the same way as the Legendre symbol — is given by

  $$\left(\frac{a}{n}\right) = \left(\frac{a}{p_1}\right)^{m_1} \left(\frac{a}{p_2}\right)^{m_2} \cdots \left(\frac{a}{p_k}\right)^{m_k}.$$

  (The right-hand side of this formula uses Legendre symbols!)
- Its properties are similar to those of the Legendre symbol, but we also have

  $$\left(\frac{a}{mn}\right) = \left(\frac{a}{m}\right)\left(\frac{a}{n}\right).$$
- If $\left(\frac{a}{n}\right) = -1$ then a is not a quadratic residue modulo nm. But, if $\left(\frac{a}{n}\right) = +1$ then a may, or may not, be a quadratic residue modulo nm.

Quadratic residues (counts)

- For a prime p, the number of integers belonging to the set $\{1, 2, \ldots, p-1\}$ that are quadratic residues is exactly $(p-1)/2$.

```
a(n)=local(v,s);v=vector(eulerphi(n));s=0;\
     for(k=1,n,if(gcd(k,n)==1,s=s+1;v[s]=k;););return(v);
qr(n)=local(v);v=a(n); /* number of true quadratic residues */\
     return(length(Set(vector(length(v),k,(v[k]^2)%n))));
tf(n)=local(v,c);v=a(n); /* number of true or fake quadratic residues */\
     return(sum(k=1,length(v),kronecker(v[k],n)==1));
f(n)=return([eulerphi(n),qr(n),tf(n)]);
f(101) /* returns [100,50,50] */
f(103) /* returns [102,51,51] */
f(107) /* returns [106,53,53] */
f(109) /* returns [108,54,54] */
```

- How about composite numbers that are the product of two distinct prime numbers?

```
f(11*13) /* returns [120,30, 60] --- half are fakes! */
f(11*17) /* returns [160,40, 80] --- half are fakes! */
f(11*19) /* returns [180,45, 90] --- half are fakes! */
f(13*17) /* returns [192,48, 96] --- half are fakes! */
f(13*19) /* returns [216,54,108] --- half are fakes! */
f(17*19) /* returns [288,72,144] --- half are fakes! */
```

Quadratic residues (square roots)

- Let p be a prime number of the form $4k+3$ and let a be a quadratic residue modulo p, i.e., $\left(\frac{a}{p}\right)=+1$.
- Then a has two square roots.
- They are given by the formula $r=\pm a^{\frac{p+1}{4}} \bmod p$. **This is so because if a is a quadratic residue we must have, by Fermat's little theorem, that $a^{\frac{p-1}{2}}=1 \bmod p$, and so $a^{\frac{p+1}{2}}=a \bmod p$. But $(p+1)/2$ is an even number so the square roots can be computed easily as stated above.**
- If n is the product of two primes p and q of the form $4k+3$ and if a is a quadratic residue modulo n then a will have four square roots. They can be easily computed using the Chinese remainder theorem. Two of them will have a Jacobi symbol of $+1$ and two will have a Jacobi symbol of -1 (recall that $\left(\frac{-1}{p}\right)=-1$ for primes of this form).
- Example:

```
p=11; q=19; n=p*q;  /* the modulus */
r=20;               /* one of the the square roots */
a=lift(Mod(r^2,n)); /* the square, a=191 */
rp=lift(Mod(a,p)^((p+1)/4)); /* the square roots modulo p are +rp and -rp */
rq=lift(Mod(a,q)^((q+1)/4)); /* the square roots modulo q are +rq and -rq */
f(rp,rq)=r=lift(chinese(Mod(rp,p),Mod(rq,q))); /* r is a square root modulo n */ \
         printf("r=%3d, (r/p)=%+d, (r/q)=%+d r^2=%3d\n",r,kronecker(r,p),kronecker(r,q),r^2%n);
f(+rp,+rq); /* r= 20, (r/p)=+1, (r/q)=+1 r^2=191 */
f(+rp,-rq); /* r= 75, (r/p)=+1, (r/q)=-1 r^2=191 */
f(-rp,+rq); /* r=134, (r/p)=-1, (r/q)=+1 r^2=191 */
f(-rp,-rq); /* r=189, (r/p)=-1, (r/q)=-1 r^2=191 */
```

Quadratic residues (square roots and factorization)

- Let n be the product of two primes.
- Let a be a quadratic residue modulo n.
- Then, it will have four square roots.
- Let x and y be two of then.
- Then $x^2 = y^2 \bmod n$, i.e., $x^2 - y^2 = (x-y)(x+y) = 0 \bmod n$.
- If $y = x$ or $y = -x$, then the above equation gives us nothing.
- Otherwise, we can factor n. Just compute $\gcd(x-y,n)$ and $\gcd(x+y,n)$.
- Example (continuation of the code of the previous slide):

```
p=11; q=19; n=p*q;
r1=20; r2=75; r3=134; r4=189; /* square roots of 191 */
gcd(r1-r2,n);                 /* 11                  */
gcd(r1+r2,n);                 /* 19                  */
```

- pari-gp can only compute square roots when the modulus is prime:

```
sqrt(Mod(5,11))  /* ok (because 5 is a quadratic residue)    */
sqrt(Mod(9,14))  /* error (because the modulus is not prime) */
```

Zero-Knowledge

- A cryptographic protocol is said to be zero knowledge if it does not disclose any data.
- For example, someone wants to retrieve an item of information somebody else has, but she/he does not want to disclose which item she/he wants. The one of two oblivious transfer protocol can do that when there are two items of information.
- In another example, two parties want to generate a random number is a fair and verifiable way. The coin flipping protocol does that.
- In a zero-knowledge proof of identity one party (the prover) proves to another party (the verifier) that she/he knows a secret without revealing any information about it. Usually, the proof is probabilistic, i.e., the zero-knowledge proof has several rounds. The larger the number or rounds, the smaller the probability of an impostor faking the proof.

One of two oblivious transfer

- Alice holds two items of information, say m_0 and m_1.
- Bob wants to know one of these two items of information, but does not want Alice to know which one he wants.
- This problem is known as the oblivious transfer problem (in the case, one of two).
- It can be solved in several ways. We will do it here using RSA techniques.
- N is Alice's public RSA modulus and e is the corresponding public exponent; d is the corresponding private decryption exponent.
- At Bob's request, Alice generates two random messages x_0 and x_1 (random numbers smaller than N) and sends then to Bob.
- Bob wants m_b, where $b \in \{0, 1\}$. So, Bob generates a random k and computes and sends to Alice $v = (x_b + k^e) \bmod N$.
- Alice computes $m_0' = m_0 + (v - x_0)^d \bmod N$ and $m_1' = m_1 + (v - x_1)^d \bmod N$ and sends both to Bob. **Either $(v - x_0)^d \bmod N$ or $(v - x_1)^d \bmod N$ will be equal to k, but Alice has no way of knowing which one is the case.**
- Bob computes $m_b = m_b' - k \bmod N$. **He can not infer m_{1-b} from m_{1-b}'.**

Coin flipping

- Alice and Bob want to flip a coin (by telephone in Manuel Blum's 1981 paper) to decide who wins (in Blum's paper, who gets the car after a divorce).
- Actually, each one flips a coin and if they came out equal (two heads or two tails) Bob wins.
- How can this be done fairly and without cheating when the two are far apart?
- Using computers: square roots of quadratic residues!

1. One of the two, say Bob, selects two large random primes p and q of the form $4k+3$ (Blum primes!) and then computes $n = pq$. He then sends n to Alice.
2. Alice chooses a random b and sends $a = b^2 \bmod n$ to Bob.
3. Bob computes the 4 square roots $\pm x$ and $\pm y$ of a, chooses one of then, let us call it r, and sends it to Alice.
4. Alice checks if $\pm r = b$. If so, then Bob wins. If not, he looses.
5. Alice proves her claims by disclosing b. (Observe that if Alice does not like the outcome she may simply do not finish the execution of this protocol, but that would be cheating.)

- To flip m coins, do step 2 m times, then step 3 m times and so on. It has to be done in this way because as soon as Alice receives the square roots from Bob she will likely be able to factor n (and so be able to change her choice of the b).

Zero-knowledge proofs of identity (main idea)

- Let us introduce two new protagonists:

  1. Peggy, who wishes to prove to Victor that she knows a secret
  2. Victor, who wishes to verify that Peggy knows the secret

- The proof will be based on challenge-response pairs and it will be probabilistic in nature.
- The probability that an impersonator is accepted (false proof) decreases as more challenge-response pairs are used.
- One of the first published ways to do it uses (again) the hardness of factoring large integers.
- Again, the underlying problem is computing square roots modulo $n = pq$.

Zero-knowledge proofs of identity (Feige-Fiat-Shamir scheme)

Preparatory steps (disclosure of public information):

- Peggy chooses a large number n that is the product of two primes of the form $4k+3$ (such a number is called a Blum number). **The interesting thing here is that -1 is not a quadratic residue modulo n but its Jacobi symbol has value $+1$; $x^2 \equiv -1 \pmod{pq}$ implies $x^2 \equiv -1 \pmod{p}$ and $x^2 \equiv -1 \pmod{q}$, so -1 can only be a quadratic residue modulo pq if it is a quadratic residue modulo both p and q, which is not the case here because -1 is not a quadratic residue for primes of the form $4k+3$.**

- She also chooses k large random numbers $S_1, S_2, \ldots, S_k$ coprime to n.

- Finally, she also chooses each I_j (randomly and independently) as $\pm S_j^{-2} \bmod n$. **The interesting thing here is that no matter which choice was made we always have $\left(\frac{I_j}{n}\right) = +1$, so without computing square roots an external observer cannot determine which choice was made. The S_j are witnesses of the quadratic character of the I_j.**

- She publishes n and the $I = I_1, I_2, \ldots, I_k$ (but keeps $S = S_1, S_2, \ldots, S_k$ secret).

Instead of publishing n herself, Peggy could have used any Blum integer computed by a trusted entity (the factors of n are not used anywhere in this scheme.)

Zero-knowledge proofs of identity (Feige-Fiat-Shamir scheme)

To generate and verify a proof of identity, Peggy and Victor execute the following T times (the higher T is the harder it will be to fake the proof of identity):

- Peggy chooses a random R and sends to Victor $X = \pm R^2 \bmod n$. Here she also chooses the sign, either $+$ or $-$ randomly, so X is, or isn't a quadratic residue. (Remember, zero knowledge leaked!)

- [The challenge] Victor send to Peggy the random vector of bits $E = E_1, E_2, \ldots, E_k$; each E_j is either 0 or 1.

- [The reply] Peggy computes and sends to Victor $Y = \pm R \prod_{E_j=1} S_j \mod n$; here, again, she chooses the sign in a random way.

- [The verification] Victor checks if $X = \pm Y^2 \prod_{E_j=1} I_j \mod n$, and rejects immediately the proof if this is not so.

- Anyone trying to impersonate Peggy (Eve?) could try to guess the E_j — let the guesses be E'_j — them precompute the next round of the protocol by selecting a random Y and by presenting $X = \pm Y^2 \prod_{E'_j=1} I_j$ when so requested. The probability of success of this cheating attempt is 2^{-k} per round (so, 2^{-kT} overall).

Schnorr Non-interactive Zero-Knowledge Proof (RFC 8235)

Interactive proof of knowledge using discrete logarithms:

- Let p be a large prime, q a large factor of $p-1$, and g an element of $\mathbb{Z}_p^*$ of multiplicative order q (the first positive solution of $g^x = 1 \bmod p$ is $x = q$); p, q and g are public.
- Alice (the prover) publishes her public key $A = g^a \bmod p$; $a \in [2, q-2]$ is her private key.
- To prove she really knowns a, she chooses a random $v \in [0, q-1]$, computes $V = g^v \bmod p$, and sends V to Bob.
- Bob chooses a challenge $c \in [0, 2^t - 1]$, where t is the bit length of the challenge and sends c to Alice.
- Alice computes $r = v - ac \bmod q$ and sends it to Bob.
- Bob checks that $A \in [1, p-1]$, that $A^q = 1 \bmod p$, and that $V = g^r A^c \bmod p$.

Non-interactive zero-knowledge proof:

- Apply a Fiat-Shamir transformation to define c, thus avoiding interaction with Bob as he will be able to compute it himself.
- Using the Fiat-Shamir transformation c will be the output of a secure cryptographic hash function that hashes g, V, A, and other public information (say, Alice's public user id).

This protocol can be adapted to use arithmetic in elliptic curves. See the RFC for details.

Homomorphic encryption

- Idea: do some useful operation, or operations, using only encrypted data
- Example: in the RSA cryptosystem with unpadded messages, multiplication of the ciphertexts corresponds to multiplication of the plaintexts.
- Using a lot of processing power (and using somewhat cumbersome methods), it is possible to apply an arbitrary function (a logic function described by a boolean circuit) to the encrypted data (to know more, search for fully homomorphic encryption schemes and lattice-based cryptography).

Homomorphic encryption (Paillier cryptosystem)

- Choose two large primes p and q. Ensure that p is not a factor of $q-1$, and vice-versa.
- Compute $n = pq$ and $\lambda = \operatorname{lcm}(p-1, q-1)$.
- Select a random integer g in the interval $]0, n^2[$ that is coprime to n.
- Compute $u = g^\lambda \bmod n^2$. It must be of the form $u = 1 + vn$ (Fermat's little theorem).
- Compute $\mu = v^{-1} \bmod n$.
- The public key is (n, g).
- The private key is (λ, μ).
- To encrypt the plaintext m, with $0 \leqslant m < n$, select a random r such that $0 < r < n$ that is coprime to n, and compute the ciphertext $c = g^m r^n \bmod n^2$.
- To decrypt, compute $x = c^\lambda \bmod n^2$. Again, it must be of the form $x = 1 + sn$. Then $m = s\mu \bmod n$.
- Let $r^\lambda \bmod n^2 = 1 + tn$. The decryption works because (all operations are modulo n^2)

  $(1+nx)^k = 1 + \frac{k}{1}nx + \frac{k(k-1)}{2}(nx)^2 + \cdots$

  $$x = 1 + sn = g^{m\lambda} r^{n\lambda} = (1+vn)^m (1+tn)^n = 1 + mnv$$

  and so $s = mv \bmod n$, i.e., $m = s\mu \bmod n$.
- If m_1 gives rise to c_1 and m_2 to c_2 (different r can be used here) then $c_1^{k_1} c_2^{k_2}$ is a valid encryption of $k_1 m_1 + k_2 m_2$ (multiplication of the cyphertexts corresponds to addition of the plaintexts). This is the homomorphic property.

Lattice-based cryptography (what is a lattice)

What is a lattice?

- In geometry a lattice is the infinite set of points obtained from an integer linear combination of linearly independent vectors.

- Mathematically, if the vectors are denoted by $\bar{v}_1$, $\bar{v}_2$, ..., $\bar{v}_k$ and the coefficients of the linear combination by a_1, a_2, ..., a_k, the points of the lattice are given by $p = \sum_{i=1}^{k} a_i \bar{v}_i$. This can also be written as

  $$p = \underbrace{\begin{bmatrix} \bar{v}_1 & \cdots & \bar{v}_k \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix}}_{a} = Va.$$

  The coordinates of each $\bar{v}_i$ can be real numbers or more exotic things but each a_i must be an arbitrary integer (i.e., $a_i \in \mathbb{Z}$).

- Adding lattice points always gives rise to another lattice point: $Va + Vb = V(a + b)$. The lattice points thus form a group under addition. Multiplying a lattice point by an integer also gives rise to a lattice point: $k(Va) = V(ka)$.

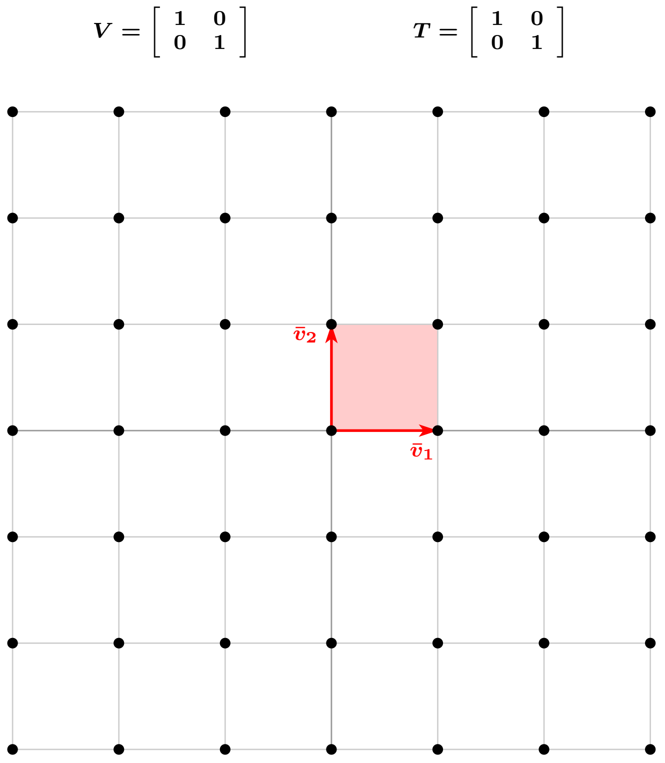
- The lattice basis is not unique. Let T be a square matrix with integer coefficients and determinant ±1. We have $p = Va = (VT^{-1})(Ta)$, so VT^{-1} is also a basis for the lattice. Note that Ta is an invertible linear transformation for vectors with integer elements. It transform any vector of integers into another vector of integers and vice versa.

Lattice-based cryptography (illustration)

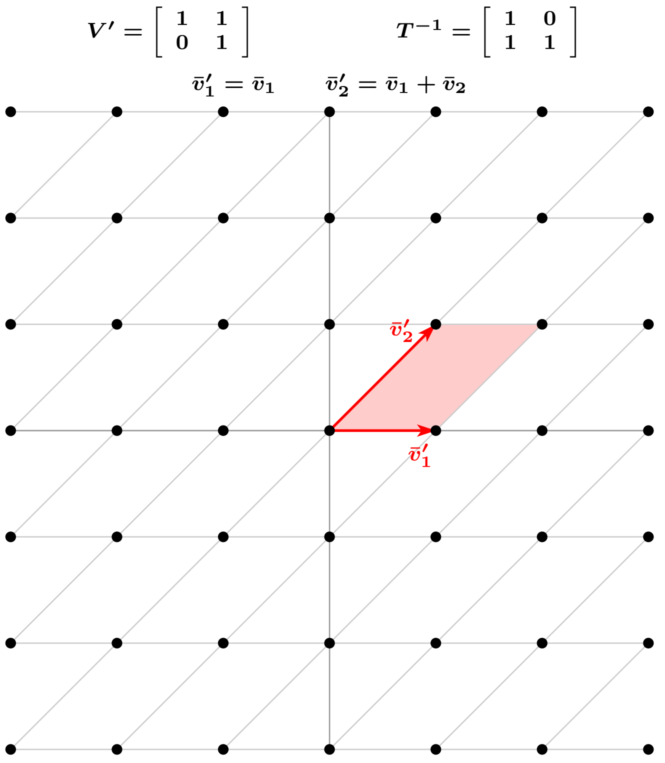
Some extra definitions:

- The fundamental region of a lattice is given by $\sum_{i=1}^{k} a_i \bar{v}_i$ with $0 \leqslant a_i < 1$ for all i.
- The size of the fundamental region is given by $|\det(V)|$. It does not depent of the basis used to define the lattice.

A bidimensional example ($k = 2$):


$V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
$T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
$\bar{v}_2$
$\bar{v}_1$

How about $v_1 = \begin{bmatrix} 7 \\ 13 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 8 \\ 15 \end{bmatrix}$?


$V' = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
$T^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$
$\bar{v}_1' = \bar{v}_1$
$\bar{v}_2' = \bar{v}_1 + \bar{v}_2$
$\bar{v}_2'$
$\bar{v}_1'$

Lattice-based cryptography (hard problems)

Two hard problems (there are more):

- The Shortest Vector Problem (SVP) asks for the shortest non-zero vector that is the difference of two distinct lattice points. This is the same as asking for the point that is closest to the origin of the lattice (ignoring the origin itself, of course).
- The Closest vector Problem (CVP) asks for the lattice vector that is closest to a given vector. As in the previous item, this can be formulated as asking for the lattice point that is closest to a given point.
- These two problems as believed to be NP-hard.
- SVP challenges.

Lattice-based cryptography (lattice basis reduction)

- Given the vectors that describe a lattice, can we find a "better" basis?
- By "better" we mean smaller coefficients and as near orthogonal vectors as we can make them.
- The Lenstra–Lenstra–Lovász (LLL) algorithm gives a reasonable solution to this problem. The block Korkine-Zolotarev algorithm (BKZ) is another possibility.
- These algorithms have many uses. For example, the LLL algorithm was used to break the Merkle-Hellman knapsack cryptosystem described earlier. Pari-gp has the function `qflll` to compute it.
- For example, the basis on the left hand side has a fundamental region of size 366. Because of this small size, we can expect that a basis with small vectors exist. That is the case (on the right-hand size).

$$\boldsymbol{V} = \begin{bmatrix} -71 & 386 & 517 & -142 \\ 662 & 696 & 943 & -249 \\ 719 & 826 & 986 & -360 \\ 533 & -252 & 832 & 659 \end{bmatrix} \quad \Rightarrow \texttt{qflll(V,3)} \Rightarrow \quad \boldsymbol{V}' = \begin{bmatrix} 0 & 3 & -4 & 3 \\ 1 & -1 & -3 & 0 \\ 3 & 2 & 2 & -1 \\ 1 & -2 & 1 & 5 \end{bmatrix}$$

Lattice-based cryptography (learning with errors)

- Solving the system of equations $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ is easy when we have at least as many equations as unknowns.

- If we add unknown noise, things can become more difficult: $\boldsymbol{A}\boldsymbol{x} + \boldsymbol{e} = \boldsymbol{b}$ is much harder to solve when $\boldsymbol{e}$ is unknown. (We will have much more equations than unknowns, so a brute force search over all possible error vectors will uncover the solution if the number of equations is sufficiently large).

- However, if all numbers are real numbers and we have many more equations that unknowns, a least squares solution is usually good. It is given by $\boldsymbol{x} = \boldsymbol{A}^{+}\boldsymbol{b}$, where $\boldsymbol{A}^{+}$ is the pseudo-inverse of $\boldsymbol{A}$, and it will usually be close to the true solution. (In `matlab` or `octave`, we just do `x=pinv(A)*b`).

- Fortunately, if we switch to modular arithmetic, the pseudo-inverse is useless and the problem becomes indeed very hard.

- It will be necessary to try all possible values of the error vector until a consistent system of equations is found (and solved).

- As an alternative, since when $\boldsymbol{x}$ is a vector with integer entries $\boldsymbol{A}\boldsymbol{x}$ is a lattice point, what we want is the $\boldsymbol{x}$ vector for which the lattice point $\boldsymbol{A}\boldsymbol{x}$ is closest to the point $\boldsymbol{b}$, and that is also hard.

Lattice-based cryptography (learning with errors, Regev)

- This can be turned into a public key encryption method (Regev, 2009):

  1. Choose a modulus q, a secret size n and a pool size m. All arithmetic will be done modulo q.
  2. Choose n random integer values s_1, s_2, …, s_n. They are the secret key.
  3. Choose m random integer vectors a_i, each with n elements.
  4. Compute the vector b with m elements, where $b_i = b'_i + e_i$, e_i is a small error term, and where $b'_i = \sum_{j=1}^{n} s_j a_{ij}$; a_{ij} is the j-th element of a_i. We should have $|e_i| < \lfloor \frac{q}{4m} \rfloor$. Each element of b corresponds to one equation (with error) of the system of equations.
  5. Publish the a_i and b. They are the public key.
  6. To encrypt a single bit B, select at random a subset S of the set $1, 2, \ldots, m$, and send $u = \sum_{i \in S} a_i$ and $v = B\lfloor \frac{q}{2} \rfloor + \sum_{i \in S} b_i$.
  7. To decrypt, compute $v - \sum_{i=1}^{n} s_i u_i$. If it is closer to zero than to $q/2$, a zero was sent, otherwise a 1 was sent.
  8. This works because $\sum_{i=1}^{n} s_i u_i = \sum_{i \in S} b'_i$, and because the absolute sum of the errors cannot be larger than $q/4$.

- The number of bits of the public key can be drastically reduced if the vectors a_i are chosen in a special way: all but the first are computed from the previous one using a predefined formula. This gives rise to the Ring Learning With Errors (RLWE) method (more information).

Lattice-based cryptography (learning with errors, pari-gp)

- The following pari-gp code implements a simple LWE public key scheme.

```
q=2^10;                                                                    /* the modulus */
n=5;                                                                       /* secret size */
m=50;                                                                      /* sample size */
max_e=floor((5*q)/(12*m));                                                 /* maximum error */
s=vector(n,i,random([0,q]));                                               /* the secret */
A=vector(m,i,vector(n,j,random([0,q])));                                   /* the public matrix */
b=vector(m,i,(sum(j=1,n,s[j]*A[i][j])+random(round([-max_e,max_e+1])))%q); /* the public vector */

encode(bit)=my(c1,c2);c1=vector(n,i,0);c2=bit*floor(q/2);\
            for(i=1,random(round([2*m/5,3*m/5])),j=1+random(m);c1+=A[j];c2+=b[j];);return([c1,c2]%q);
decode(c)=return((c[2]-sum(i=1,n,s[i]*c[1][i]))%q);
```

- It is possible to modify the method to send more that one bit at a time; it will necessary, though, the reduce the amplitude of the errors.
- Also, instead of the offsets 0 and $\left\lfloor\frac{q}{2}\right\rfloor$, the offsets $\left\lfloor\frac{q}{4}\right\rfloor$ and $\left\lfloor\frac{3q}{4}\right\rfloor$ could have been chosen.
- Instead of an uniform distribution, Regev suggested the use of a normal distribution.

Quantum-resistant cryptography

Quantum computers and cryptography.

- Quantum-resistant cryptography, also known as Post-Quantum Cryptography (PQC), aims to develop algorithms that remain safe when quantum computers become powerful enough to tackle integer factorization and discrete logarithm problems (if ever, see this 2018 essay against quantum computing).
- In August 2024, three Post-Quantum Cryptography standards were published by NIST: FIPS 203 (key-encapsulation mechanism), FIPS 204 (lattice-based digital signatures), and FIPS 205 (stateless hash-based digital signatures).
- What NSA said about it in 2021: link.

But, is quantum computing feasible?

- What the USA National Academy of Sciences has to say about quantum computing in 2018: link.
- What an article in Nature said about it in 2023: link. (Executive summary: for now, absolutely nothing. But researchers and firms are optimistic about the applications.) However, see also this 2024 IEEE article (link).
- And what a Communications of the ACM article said about it also in 2023: link. (Just take a look at the key insights. . .)

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