5 The AK model

"…a level effect can appear as a growth effect for long periods of time, since adjustments in real economies may take place over decades". [Sachs and Warner].

Learning Goals:

- Understand why getting rid of diminishing returns one can obtain unceasing growth via factor accumulation.
- Distinguish the case with endogenous savings.
- Review different models were simple factor accumulation can generate endogenous growth.
- Acknowledge the empirical challenges raised by the abandonment of diminishing returns.

5.1 Introduction

Along the previous chapters, we learned that, under diminishing returns, factor accumulation cannot, by itself, explain the tendency for per capita income to grow over time. For this reason, a sustained growth of per capita income can only be achieved in the neoclassical model by postulating an exogenous rate of technological progress.

In this chapter it is shown that, by getting rid of diminishing returns on capital accumulation, one can obtain continuous growth of per capita income *without* the need to postulate an exogenous rate of technological progress. The AK model differs critically from the Solow model in that it relies on a production function that is linear in the stock of capital. In this model, per capita income grows continuously in the equilibrium, without any tendency to stabilize. In that model, a rise in the saving rate produces a permanent increase in the growth rate of per capita income. This contrasts with the Solow model, where a rise in the saving rate only delivers only a "level effect".

The pitfall of the AK model is that the assumption of diminishing returns is very fundamental in economic thinking. Hence, it cannot be abandoned without a well-motivated story. Some of the sections below describe alternative theories and models that have been proposed in the literature to motivate moving away from diminishing returns assumption.

Modern formulations of the AK model have expanded the interpretation of capital to include human capital and technology. While none of these models should be seen as *the true model*, they all offer alternative explanations for why diminishing returns do not always occur. The AK model can be primarily interpreted as a toy version of more complex models of endogenous growth.

Sections 5.2 describes the AK model in its simpler formulation. Section 5.3 extends the AK model to the case of endogenous savings. Section 5.4 reviews alternative models that emulate the AK model. Section 5.5 addresses the empirical evidence on the relationship between savings on economic growth. Section 5.6 concludes.

5.2 The basic AK model

5.2.1 Getting rid of diminishing returns

Consider a closed economy where the population growth rate, the savings rate and the depreciation rate are all constant over time. The novelty relative to the Solow model is that the production is linear in K:

$$Y_t = AK_t \ , \ \mathrm{A} > 0 \qquad (5.1)$$

In (5.1), the parameter A stands for the level of technology (or *aggregate efficiency*), and is assumed constant.

In light of (5.1), production depends only on capital and there are no diminishing returns. The reader may be suspicious of this formulation: after all, does it make sense to model production without work? In fact, we do not need to assume that labour has no role: in the following sections, we will review alternative models that, while taking into account the role of labour in production, emulate versions of production function (5.1). For now, however, let's stick to this simple formulation.

Dividing (5.1) by *N*, one obtains a linear relationship between per capita income and capital per worker:

$$y_t = Ak_t \qquad (5.2)$$

The remaining equations of the model are the same as in the basic Solow model:

$$Y_t = C_t + S_t \qquad (2.5)$$

$$sY_t = I_t \qquad (2.7)$$

$$\dot{K}_t = I_t - \delta K_t \qquad (2.8)$$

$$n = \dot{N}_t / N_t \qquad (2.9)$$

From (2.7), (2.8), (2.9) and (5.2), we obtain the dynamics of the capital labour ratio:

$$\dot{k}_t = sAk_t - (n + \delta)k_t \qquad (5.3)$$

This equation is similar to (2.14), with the only difference that now $\beta=1$. This small difference has an important implication: since both terms on the right-hand side of (5.3) are linear in k, only by an exceptional coincidence of parameters would this expression be equal to zero. Hence, the general case in the AK model is that there is no steady state.

Dividing (5.3) by k, one obtains the equation that describes the growth rate of capital per worker in this economy:

$$\frac{\dot{k}}{k} = sA - (n + \delta)$$

Since output is linear in K, the growth rate of capital per worker is also the growth rate of per capita income. That is:

$$\gamma = sA - (n + \delta) \qquad (5.4)$$

This equation states that the growth rate of per capita income rises with total factor productivity (A) and the saving rate (s) and declines with the depreciation of the capital-labour ratio (n and δ). As long as $sA > (n + \delta)$, per capita income will expand forever, at a constant growth rate. Note that this conformity with the real-world facts is achieved without the need to postulate any exogenous technological progress.

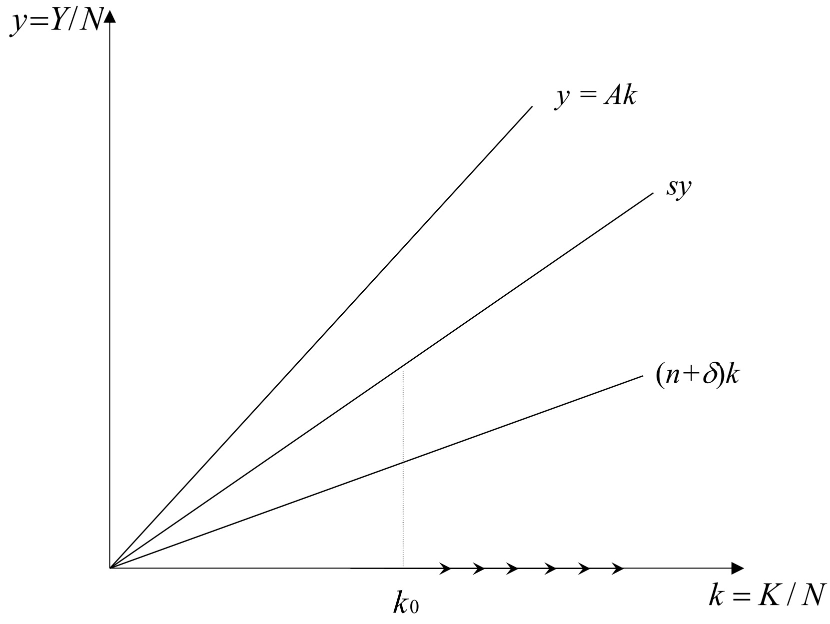
Because the growth rate of per capita income in (5.4) is influenced by the other parameters, instead of being given, the model is categorized as of *endogenous growth*.

5.2.2 A Graphical Illustration

Figure 5.1 describes the dynamics of the AK model. The horizontal axis measures the capital labour ratio (k). The vertical axis measures output per capita (y). The top line shows the production function in the intensity form, (5.2); the middle line corresponds to gross savings per capita (the first term in the right hand side of 5.3); the lower of the three lines is the break-even investment line (the second term in the right hand side of 5.3).

Since the production function is now linear in k, the locus representing gross savings never crosses the break-even investment line (compare with Figure 2.3). This means that, as long as $sA>n+\delta$, per capita output will grow forever, without any tendency to approach an equilibrium level.

Figure 5.1. The AK model



When the production function is linear, the curve describing per capita savings never crosses the break-even investment line. Hence, the capital-labour ratio and per capita output will expand without limits.

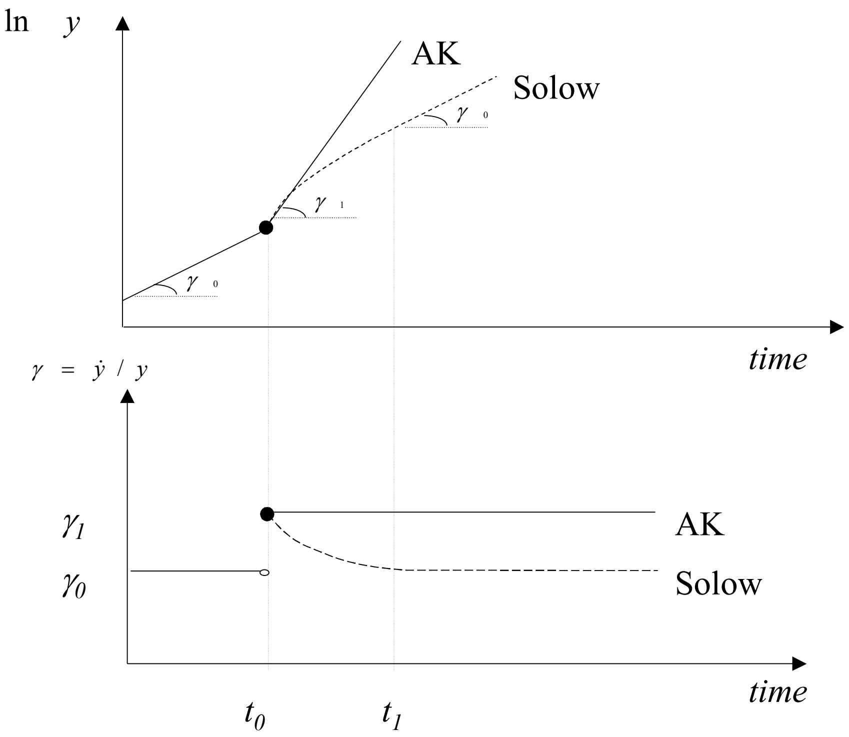
5.2.3 What happens when the saving rate increase?

The AK model differs drastically from the Solow model, in that changes in exogenous parameters alter the long run growth rate of per capita income, not only the level of per capita income.

Figure 5.2 compares the paths of per capita income in the AK model and in the Solow model following a once-and-for-all increase in the saving rate at time t_0 (the case with the

Solow model was already discussed in detail in Figure 3.3). The top part of the diagram shows levels and the bottom part shows growth rates.

Figure 5.2: The AK model and the Solow model compared for a rise in the saving rate



The figure compares the response of per capita income to an exogenous increase in the saving rate in light of the AK model versus the Solow model. While in the Solow model this gives rise to a level effect, in the AK model, there is a growth effect. The figure also suggests that in the short run the Solow model and the AK model produce similar predictions.

In the Solow model, the rate of growth of per capita income jumps initially to a higher level, but then it declines slowly over time, until returning to the previous - *exogenous* - rate of technological progress. Because of diminishing returns, the long run growth rate of per capita income is independent of the saving rate. In the AK model, the rise in the saving rate has a permanent effect on growth: there is no tendency for the growth rate of per capita income to decline as time goes by. The growth rate of per capita output is proportional to the saving rate.

5.2.4 The Harrod-Domar equation

A useful comparison between the AK model and the Solow model regards their respective behaviours in the long run. Using (5.1) and (5.4) we get:

$$\gamma = s\frac{Y}{K} - (n + \delta) \qquad (5.5)$$

This equation is known as the Harrod-Domar equation. The difference between (5.5) and (3.11) (that holds in the Solow model in the long run) refers to the variables that are exogenous and endogenous in this equation. In both models, s, n and δ are exogenous. But the two models differ in respect to the exogeneity of δ and Y/K: In the AK model, *Y/K* is exogenous and γ is endogenous. By contrast, in the Solow model, γ is exogenous and *Y/K* is endogenous.

Hence: In the Solow model, a rise in the saving rate leads to a lower average productivity of capital in the steady state. That is, Y/K declines from one steady state to the other (Figure 3.2). In the AK model, *Y/K* is constant (equal to *A)*. Hence, a rise in the saving rate can only be accommodated in the model by an increase in the growth rate of per capita income, γ.

Because the AK model predicts that changes in A or in the saving rate produce growth effects, it goes far beyond the neoclassical model in stressing the relationship between economic policies and economic growth: government policies, such as taxes and subsidies, that affect economic efficiency and consumption-saving decisions may alter the long run' rate of economic *growth*, rather than simply altering the *level* of per capita income.

5.2.5 No convergence

The AK model does not predict convergence of per capita incomes, even among similar economies. According to (5.4), two economies having the same technology and savings rates will enjoy the same growth rate of per capita income, regardless of their starting position. In that case, the respective per capita incomes will evolve in parallel without any tendency to approach each other. This contrasts to the Solow model, where countries with similar parameters should approach the same per capita income level in the steady state.

Moreover, since changes in technology (A) and in the saving rate (s) affect growth rates permanently, countries with different parameters should exhibit different growth rates of per capita income. In a world where policies differ substantially across countries, the rule should be that of divergence of per capita incomes, rather than of convergence.

5.3 Incarnations of the AK model

5.3.1 The Harrod-Domar model

The true predecessor of the AK model was developed independently by two economists, Roy Harrod, and Evsey Domar[74]. The Harrod Domar model preceded that of Solow by several years and obviously it was not motivated by any explicit intention to improve on the Solow model. The HD model was developed in the aftermath of the Great Depression, as a dynamic extension of Keynes' general theory. The aim was to discuss the business cycle in the U.S. economy. Since at that time, unemployment was very high, the focus of the model was on the relationship between investment in physical capital and output growth.

The main assumption of the Harrod-Domar model is that capital and labour are pure complements, meaning that they cannot substitute for each other in production. The underlying production function is Leontief:

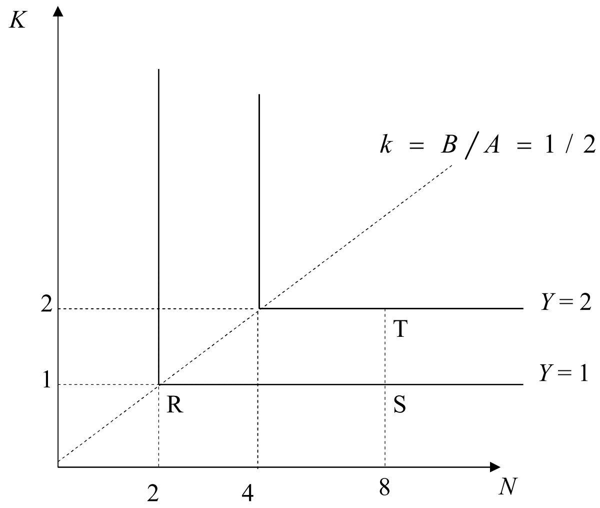
$$Y_t = \min\{AK_t, BN_t\}, \tag{5.6}$$

where A and B are positive constants. Figure 5.3 illustrates with a numerical example, considering A=1 and B=0.5. If the economy' endowments were K=1 and N=8, the economy would be producing Y=1 only, wasting 6 unit of labour (point S). From that point, expanding the quantity of labour would not deliver higher output, because labour cannot substitute for capital. The only way to expand production will be increasing the stock of physical capital. If one managed to increase the stock of capital to *K=2*, the output level would jump to *Y=2* (point T), and unemployment would be reduced to 4 units of labour. Raising production by incrementing the stock of capital (*K*) in an economy with surplus labour (*N*) is basically how the Harrod-Domar model works.

---

[74] Harrod, R. , 1939. "An essay in dynamic theory". Economic Journal 49, 14-33. Domar, E., 1946. "Capital expansion, rate of growth and unemployment", Econometrica 154, 137-47..

Figure 5.3: The Leontief production function



The figure describes two isoquants where inputs to production are complementar. The straight line B/A corresponds to the efficient combinations. If the economy' endowment point lies on the right-hand side (left hand side) of this line, there will be unemployment of labour (capital).

Mathematically, a situation of employment surplus occurs when K/N<B/A. In that case the relevant branch of the production function (5.6) is the first, implying a linear relationship between output and K, Y=AK. This branch mimics the AK model. Then, given the exogenous saving rate and the population growth rate, from (2.5)-(2.9), you'll obtain the growth rate of per capita income as described by (5.4).

The main limitation of the Harrod-Domar is that factor prices play no role in driving the economy towards full employment of labour and capital. This contrasts with the Solow model, where real wages and the real interest rate adjust to ensure full employment each moment in time. Thus, even if output is expanding over time at the rate $\dot{Y}/Y = sA - \delta$, this may not be enough for per capita income to increase: in case $sA - \delta < n$, the economy does not save the enough to keep the capital labour ratio unchanged. Population will be expanding faster than output, surplus labour will be increasing (chronic underproduction) and per capita income will decline over time. If, on the contrary, the capital stock grows faster than population ($\gamma = sA - \delta - n > 0$), then per capita income will be increasing over time, and the surplus labour will be eliminated at some point. In that stage, the relevant segment of the production function in (5.6) shifts to Y=BN, and the binding constraint in production becomes the availability of labour: beyond this point output is be bound to expand at the same rate as population, implying a constant level of per capita income thereafter[75].

Box 5.1: The ghost of financing gap

One of the reasons why the Harrod Domar equation (5.5) became so popular is that it offers a simple and appealing formula to forecast economic growth. This formula was also extensively used by international organizations, such as the World Bank, to calculate a country' financing needs.

If equation (5.5) was true, one could easily forecast a country' economic growth, using the saving rate, the depreciation rate and an estimate for the average product of capital, A. Since the later is not readily available in national accounts, a possible proxy would be the ratio of net investment to the change in real GDP over two consecutive years:

$$ICOR = \frac{\Delta K}{\Delta Y} = \frac{net \quad investment}{change \quad in \quad GDP}$$

This is the known as the "Incremental Capital-Output Ratio", ICOR.

As an example, consider a poor economy where the ICOR = 3 and the observed investment ratio (s) is 15%. Assuming a depreciation rate equal to 4%, equation (5.4) implies that output will grow at $0.15/3 - 0.04 = 0.01$. Now suppose you were a consultant in this economy, advising on poverty alleviation. You could well conclude that the saving rate in this economy was too low. If, for example, the population was expanding at 2%, this would imply a drop in per capita income…

You could, then, use the HD equation the other way around: how much should the investment rate in this country, for per capita income to increase at some desired rate? Suppose

[75] Lewis (1954) proposed an alternative path, according to which after unemployment is eliminated, wages start increasing and the economy enters in "modern economic growth". [Lewis, W., 1954. "Economic Development with Unlimited Supplies of Labour". The Manchester School of Economic and Social Studies 22, 139-191].

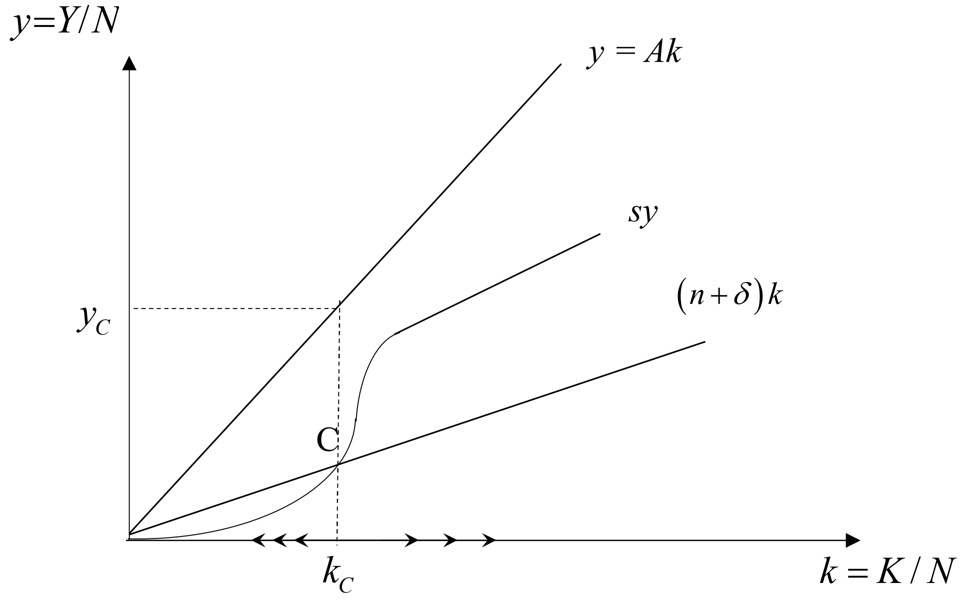
you wanted income per capita to expand at 2% per year. With the population growing at 2% and a measured ICOR equal to 3, you would need a net investment amounting to 24% of GDP. Since domestic savings were only 15%, you could request the international donors to fill the "financing gap", equal to 9% of GDP.

Economists in international institutions, such as the World Bank, the IMF, the Inter-American Development Bank, the European Bank for Reconstruction and Development used models based on the HD equation to estimate the amount of savings (and/or aid) necessary for poor countries to achieve some desired rate of economic growth. This philosophy was supported by the understanding that people living near the subsistence level cannot save the same as rich people.

Figure 5.4 revies the argument. This figure is similar to 5.1, with the difference that the saving rate starts very low and then increases with per capita income. In this case, a bifurcated growth pattern emerges, whereby per capita income continuously increases or decreases depending on the initial level of capital per worker. Now suppose a poor country was trapped in a situation with insufficient savings. Say, because a natural disaster caused the capital stock to fall below the critical level, or because a demographic explosion tilted the break-even investment line upwards, turning the existing capital stock insufficient. In this case, per capita income would begin to decline over time. Arguably, foreign aid could fix this: if foreign aid were enough to increase savings above the critical level, the country could embark on a path of self-sustained growth. In this case, external assistance would only need to be temporary.

Recent proponents of this idea include Sachs (2005) and the United Nations Millennium Development Goals Project. Sachs (2005): "(…) if the foreign assistance is substantial enough, and lasts long enough, the capital stock rises sufficiently to lift households above subsistence…growth becomes self-sustained (…)" (p. 246). United Nations (2005, p. 19): "The key to escape the trap is to raise the economy's capital stock to the point where the downwards spiral ends and self-sustained economic growth takes over" (p. 19) [76].

Figure 5.4: The AK model with a non-linear saving rate



The figure describes a case where, at low levels of income, the saving rate is an increasing function of income. In this case, the model displays one equilibrium, where per capita income is stagnant. This equilibrium is unstable: at the left, savings are insufficient, and per capita income decreases over time; at the right, per capita income increases over time.

The view that external aid would be the panacea to help poor countries to escape poverty traps was seriously criticised by William Easterly in controversial paper called "The ghost of financing gap"[77]. The author noted that, over the past four decades, large amounts of international financial assistance to the developing world did not translate into faster economic

[76] [Sachs, J., 2005. The end of poverty: economic possibilities for our times. New York: Penguin Press. United Nations, 2005. Millenium Development, Project Report, United Nations, New York].

[77] Easterly, W., 1999. "The ghost of financing gap: testing the growth model of the international financial institutions". Journal of Development Economics 60 (2), 423-438.

growth. Using a sample of 146 countries along the period from 1950 to 1992, the author failed to find a robust positive linear relationship between aid and economic growth.

Does this mean that the HD equation is wrong? Not necessarily. But probably one should not trust too much historical values of the average product of capital (the ICOR) to guess the marginal impact of new investments: if for instance, part of the external aid is diverted into unproductive uses (frivolous expenses, corruption fees), then much of the higher saving rate will be offset by a lower A. Arguably, the impact of external aid on growth shall depend on the quality of policies and institutions of the recipient country (see discussion in Box 5.3). Note however, that, until now, models have remained silent regarding the role of policies and institutions.

5.3.2 A model with Physical and Human Capital

Another way of accounting for the role of labour in production and obtain an AK type of model is by considering two types of capital, physical and human capital, and assuming that constant returns apply to the *broad* concept of capital[78].

To see this formally, consider the following production function:

$$Y = AK^{\beta}H^{1-\beta} \qquad (5.7)$$

In this production function, there are diminishing returns to physical capital and to human capital in isolation, but there are constant returns to scale in reproducible factors together. This contrasts to the Solow and the MRW models, where returns to reproducible factors are decreasing due to the presence of a non-reproducible factor, labour.

[78] Rebelo, S., 1991. Long run policy analysis and long run growth, Journal of Political Economy, 99, 500-521.

Also note that this production function does not necessarily exclude raw labour from production. Indeed, one may think human capital, *H*, as measuring *quality adjusted* labour, that is, the number of workers, *N*, multiplied by the human capital of the typical worker (*h*):

$$H = hN. \qquad (5.8)$$

The implied assumption in (5.8) is that the quantity of workers, *N*, and the quality of workers, *h*, are substitutes. With such a specification, raw labour needs no longer to be a source of diminishing returns: multiplying *h* and *K* by a given constant implies that production Y will expand proportionally, even if N remains constant. This is what we need to generate economic growth.

To see this how the model works, let's return to the MRW assumptions that people save a constant fraction of their incomes in the accumulation of human capital, just like they do for physical capital:

$$\dot{K}_t = sY_t - \delta K_t \qquad (5.9)$$

$$\dot{H}_t = s_H Y_t - \delta H_t \qquad (5.10)$$

Because of diminishing returns on each type of capital, it makes no sense for people to accumulate one type of capital faster than another. Therefore, both types of capital will be expanding at the same rate $\dot{K}/K = \dot{H}/H$. Using (5.9) and (5.10) this implies

$$\frac{H}{K} = \frac{s_H}{s} \qquad (5.11)$$

Substituting (5.11) in (5.7), we obtain a variant of the AK production function:

$$Y = A\left(\frac{s_H}{s}\right)^{1-\beta} K \qquad (5.12)$$

Comparing to (5.1) we see that now the average product of capital embodies the propensity to invest in human relative to physical capital (that is, you can look at A in equation 5.1 as including this effect).

Using this in (5.5), the growth rate of per capita income in this variant of the AK model is:

$$\gamma = s^{\beta} s_H^{1-\beta} A - n - \rho. \qquad (5.13)$$

This equation shows that it is perfectly possible to obtain sustained growth of per capita income with diminishing returns on physical capital alone. What we need is to have constant returns to all types of capital (or *reproducible* inputs) when considered together. Comparing to (5.4), what matters for growth is not the investment rate in physical capital alone, but the average investment rate in physical and human capital.

5.3.3 A two-sector model of endogenous growth

Another incarnation of the AK model was proposed by Usawa, as early as in 1965, but the contribution remained basically obscured until it was popularized by Lucas (1988)[79]. The author extended the Solow model, considering two sectors of production: one producing final goods that can be either consumed or accumulated as capital stock (productive sector) and other pooling together various activities that contribute to the efficiency of labour, such as education, and research ("educational" or "research" sector). The production sector employs labour and capital. The educational sector employs labour only.

In the model, it is assumed that workers devote a fraction $1-\mu$ of their working time to production of goods and the remaining μ to the improve labour efficiency, λ. The production function for final goods is given by:

$$Y = AK^{\beta}\left[(1-\mu)\lambda N\right]^{1-\beta} \qquad (5.14)$$

The change in technology is a positive function of the *fraction* of labour allocated to the educational sector:

$$\dot{\lambda} = b\mu\lambda \qquad (5.15)$$

The parameter b captures the productivity *in the research sector*. In this model, the linearity that is needed to generate endogenous growth (the AK feature) arises from the fact

[79] Usawa, H., 1965. Optimum technical change in an aggregative model of economic growth", International Economic Review, 6, 19-31. Lucas (1988), op. cit.

that the production function for technology (5.15) depends linearly on the level of technology, through the *standing on shoulders effect*: a constant fraction of working time devoted to research produces a constant growth rate of technology. With such an assumption, a policy change that successfully increases the proportion of time allocated to research (μ) or that improves the productivity in that sector (b) will impact positively and permanently on the growth rate of per capita income.

As for physical capital accumulation, the assumption (5.9) is retained. This model can be solved in the same manner as the Solow model. For mathematical convenience, let's rewrite the production function (5.14) as follows:

$$\tilde{y} = A(1-\mu)^{1-\beta}\tilde{k}^{\beta} \qquad (5.16)$$

Where $\tilde{y} = Y/L$, $\tilde{k} = K/L$, and $L = \lambda N$. Proceeding as usual, the fundamental dynamic equation becomes:

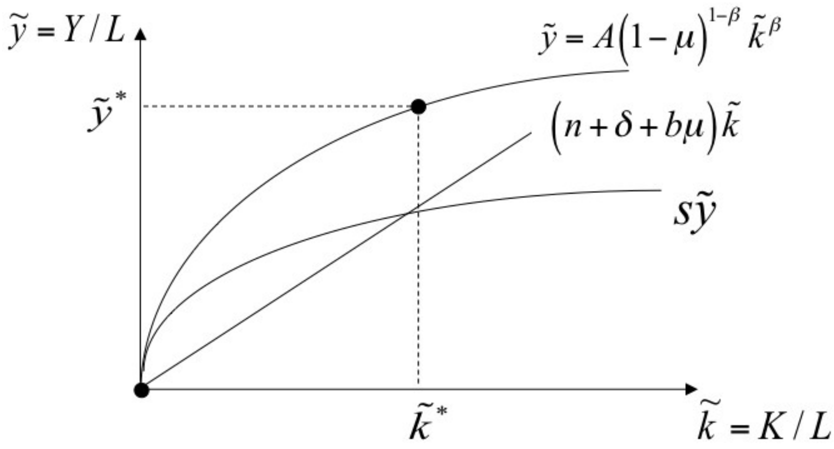
$$\dot{\tilde{k}} = sA(1-\mu)^{1-\beta}\tilde{k}^{\beta} - [n+\delta+b\mu]\tilde{k} \qquad (5.17)$$

Comparing with (3.8) you can verify how similar this model is with the Solow model. The main difference is that the parameter determining the effectiveness of labour, rather than growing exogenously, is now dependent of other parameters in the model (ie, $\gamma = b\mu$). To find the steady state, we just need to solve for $\dot{\tilde{k}} = 0$, and use the definition $\tilde{y} = y/\lambda$ to obtain:

$$y_t^* = (1-\mu)A^{\frac{1}{1-\beta}}\left(\frac{s}{n+\delta+\gamma}\right)^{\frac{\beta}{1-\beta}}\lambda_t \qquad \text{with} \qquad \lambda_t = e^{\gamma t} \qquad (5.18)$$

Figure 5.5 illustrates the steady state of the model. Because both $\tilde{y}$ and $\tilde{k}$ are constant in the steady state, the output-capital ratio is constant and so will do the interest rate.

Figure 5.5: the steady state in the two-sector model



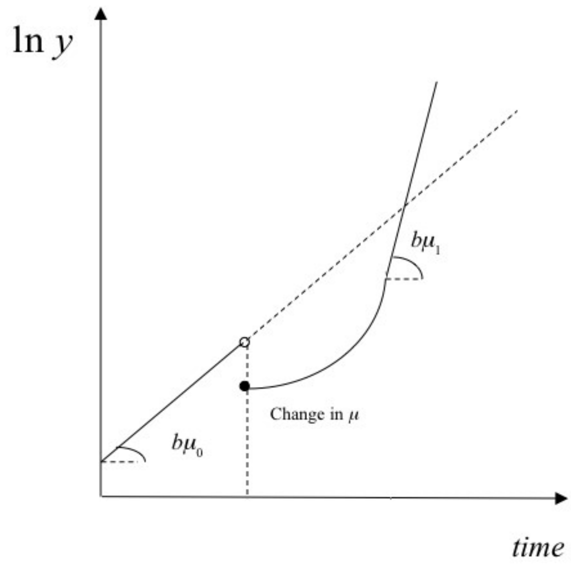
The dynamics of the Usawa model is similar to that of the Solow model. There is a stable steady state when the break-even investment line crosses the savings locus. The main difference is that both the production function and the break-even investment line are tilted when the fraction of time devoted to research changes.

This model is hybrid, in the sense that it shares characteristics with the Solow model and with the AK model. It shares with the neoclassical model the feature that it displays level effects and transition dynamics: like in the Solow model, changes in s produce "level effects", causing output per unit of efficiency to increase from one steady state to the other. However, it borrows from the AK model the linearity that is needed to obtain unceasing growth: technically, sustained growth is obtained in this model because the production function of technology is free from diminishing returns[80]. Thus, for example, a policy that induces an increase in the proportion of time that people allocate to research (μ) or in the effectiveness of that time, b, will permanently tilt the growth rate of per capita income (growth effect). The model can

[80] This assumption is not free of controversy. It could be argued that new knowledge become more difficult to achieve as the stock of knowledge increases ("fishing out effect"). In that case, the production function of knowledge would exhibit diminishing returns (ie, $\dot{\lambda} = b\mu\lambda^{\varepsilon}$ with $\varepsilon < 1$) and the growth rate of technology would fall down to zero, no matter how much effort was allocated to the educational sector. Sustained growth could not be achieved. On the other hand, the production function (5.15) is ignoring the possibility that technological progress depends on the number of researchers (rather than on the fraction of population engaged in research). Equation (5.1) is therefore just a convenient specification that prevents the growth rate of knowledge from decreasing or expanding over time.

generate sustained growth of per capita income without the need to assume exogenous shifts in the production function.

Figure 5.6 – The path of output per capita following an increase in μ



When the fraction of time employed in research increases, output in the production sector declines, causing per capita income to fall. However, there is a growth effect that, in the long run, more than offsets the negative level effect. Immediately after the shock. The growth rate of per capita income is low, reflecting the fact that the capital per unit of efficiency labour (horizontal axes in figure 5.5) is approaching a new – lower – level.

Figure 5.6 describes the path of per capita income in this economy following an increase in the proportion of time allocated to research: at the impact, there is a negative effect on per capita income, because less time is devoted to production. As the times go by, however, the growth rate of per capita output accelerates, due to the faster rate of technological expansion. Sooner or later output per capita will pass the level it would have reached had there be no change in the research effort. Referring to Figure 5.5 the production function and the break-even investment line shift in opposite directions. Hence, $\tilde{k}$ starts decreasing until meeting the new steady state. This implies that, during the transition period the growth rate of per capita output is less than the corresponding level in the steady state. As the economy approaches the new steady state, the decline in $\tilde{k}$ decelerates, implying a convergence of per capita income growth to the new steady state growth rate, $b\mu_1$.

A question that naturally arises is how people decide the optimal level of μ. Intuitively, the time allocated to the research sector versus final good production shall be determined by non-arbitrage conditions, stating that the payoff of the two activities at the margin should be the same. On the other hand, the payoff to R&D shall depend on how future production is valued today. Thus, the optimal allocation of time shall balance a variety of factors, such as the productivity of research (b), the opportunity cost of research time (the wage rate), the level of impatience of people, and so on.

5.4 The AK model with endogenous savings

Thus far, the saving rate in the AK model has been assumed exogenous. In this section we show that, when the model is extended to allow individuals to optimally trade consumption today for consumption in the future, a second channel linking efficiency to growth is opened up.

5.4.1 Adding the optimal consumption rule to the AK model

In what follows, let's recall the simplest possible optimal consumption rule, introduced in appendix 2.1:

$$\gamma = r - \rho \qquad (5.19)$$

This equation states that, as long as the interest rate is higher than the rate of time preference, there will be an incentive for households to increase consumption over time. This, in turn, is achieved through a higher saving rate. Note that in this model (because there is no transitional dynamics), consumption and income evolve in parallel each moment it time. Hence, (5.19) can be seen as describing simultaneously the growth rate of per capita consumption and the growth rate of per capita income.

To find out how the growth rate of per capita income relates with the remaining parameters of the AK model, one needs an interest rate. As before, we assume that firms are perfectly competitive and maximize profits. In this case, capital will be paid its marginal product, A. That is:

$$r + \delta = A \qquad (5.20)$$

Substituting (5.20) in (5.19) and rearranging, one obtains:

$$\gamma = A - \delta - \rho \qquad (5.21)$$

This equation describes the growth rate of per capita income in a version of the AK model where consumers are allowed to trade consumption today for consumption in the future at interest rate r. Comparing to (5.4), you see that now it is the rate of time preference, instead of the saving rate, that determines the rate of economic growth.

5.4.2 Transpiration responds to inspiration!

From the qualitative point of view, equation (5.21) brings no novelty relative to the case with exogenous savings, (5.4): a lower rate of time preference (that is, a change in consumer preferences towards more consumption in the future and less consumption today), by raising the saving rate, leads to a higher rate of capital accumulation and a higher growth rate of per capita income.

However, comparing (5.21) to (5.4), we observe that the impact of A on growth is now much larger than in the case with exogenous savings. For instance, with a saving rate equal to 20%, the impact of a unitary change in A on growth in light of (5.5) is 0.2. In light of (5.21), however, the impact of A on growth is one to one. That is: five times more.

What makes the assumption of endogenous savings so powerful that it can alter dramatically the relationship between the efficiency parameter and growth? The reason is that, when A increases, there are two effects: On one hand, when A increases for a given s, the growth rate of per capita income increases, just like in (5.4); on the other hand, when A increases, there is an additional impact mediated by the interest rate, r: a higher marginal productivity of capital translates into a higher return on capital and this, in turn, induces a higher saving rate, for each rate of time preference. Then, a higher saving rate, impacts positively on growth.

Formally, the double impact of A on growth under endogenous savings can be assessed substituting (5.21) in (5.4) and solving for s, to obtain the (endogenous) saving rate:

$s=1-(\rho-n)/A$. For the problem to be well behaved, $\rho>n$ must hold[81]. Taking the partial derivative in respect to A, we verify that the impact of a change in A on the saving rate is $\partial s/\partial A=(\rho-n)/A^2$. The total impact of a change in A is the sum of the direct impact of A on γ with the indirect impact, through s: $d\gamma/dA=\partial\gamma/\partial A+(\partial\gamma/\partial s)(\partial s/\partial A)=s+(\rho-n)/A=1$.

This finding is of the upmost importance to understand the mechanics of many endogenous growth models. A typical assessment based on equation (5.4) is that a country may either grow through "inspiration" (A) or through "transpiration" (s). But, we just saw that "transpiration" responds to "inspiration": a more efficient resource allocation, leading to a higher marginal product of capital, implies a higher return on investment. Thus, agents will be willing to forego a higher proportion of their consumption to save more.

With this finding, one may rewrite equation (5.4) in the following form:

$$\gamma = s\left(\overset{-}{\rho},\overset{+}{A}\right)A-(n+\delta) \qquad (5.22)$$

The fact that capital accumulation is itself endogenous in respect to the level of productivity has important policy implications: how can governments improve the level of efficiency in an economy? Is just a question of implementing the right policies and imitate best practices? Or there are institutional and cultural elements that prevent governments from adopting the best policies? Or is it shall be the case that, irrespectively the good intentions of governments, that are natural factors such as climate and location that prevent productivity in different geographical locations to approach each other, dooming countries facing adverse geographical conditions to permanent lower growth?

5.4.3 Proximate versus fundamental causes of economic growth

[81] We skip the proof of this.

According to equation (5.4), a low rate of economic growth can be explained either because a country does not invest enough (s) or because it has low productivity of capital (A). Dealing with the development question at a deeper level, however, one may ask why some countries save and invest more than others and why some countries reach higher levels of efficiency than others. In other words, one would like to consider as endogenous some of the parameters that the model takes as exogenous.

To some extent, equation (5.22) is a step in that direction: according to this equation, individuals will save more in countries where the productivity of capital is higher[82]. This gives parameter A a key role in this model.

The following chapters will be devoted to a better understanding of what is behind parameter A. In this quest, we will relate the level of A to the quality of economic policies, among other factors. We will argue that countries with sound economic policies are expected to achieve higher efficiency levels than countries with poor economic policies. But another question will immediately arise: why do some countries implement better policies than others? To answer this question, we need to address the incentives of policymakers. These, in turn depend on the quality of political institutions. These, in turn, are embedded in even deeper factors underlying human societies, such as social norms, culture and geography.

In a word, as one deepens the analysis, we move from the *proximate* causes of economic growth (the exogenous parameters in equation 5.4), to the *fundamental causes* of economic growth, which ultimately determine why in a given country the parameter values are what they are. These fundamental causes are essential to understand why some societies make choices that translate into the adoption of better policies and more modern technologies.

This is not to say that simple models like (5.4) are useless. On the contrary, they are essential to understand *the mechanics of economic growth*. In particular, the role of investment

[82] Equation (5.22) stresses the causality from “inspiration” to “transpiration”. However, the reversal may also be true. In Chapter 7 we’ll address precisely some theories according to which the level of A is enhanced by capital accumulation. The possibility of mutual causation implies that savings and efficiency may reinforce each other, both positively and negatively, raising the possibility of multiple equilibria and poverty traps.

and technology as *mediator*s between country characteristics and economic performances. But dealing with the growth question at a deeper level, one may want to understand what is behind the parameters that this model takes as exogenous.

Box 5.2. Developing accounting by Hall and Jones

In a famous article, Hall and Jones (1999) prosed an exercise of development accounting based on a production function of the form[83]:

$$Y = K^{1/3}(\lambda H)^{2/3} \text{ with } H = hN. \qquad (5.22)$$

In (5.22), K denotes for physical capital, H for human capital, h for human capital per worker, N the number of workers and λ is a measure of productivity.

The authors proposed the following re-specification:

$$y = \left[\left(\frac{K}{Y}\right)^{\frac{1}{2}} h\right]\lambda = X\lambda \qquad (5.23)$$

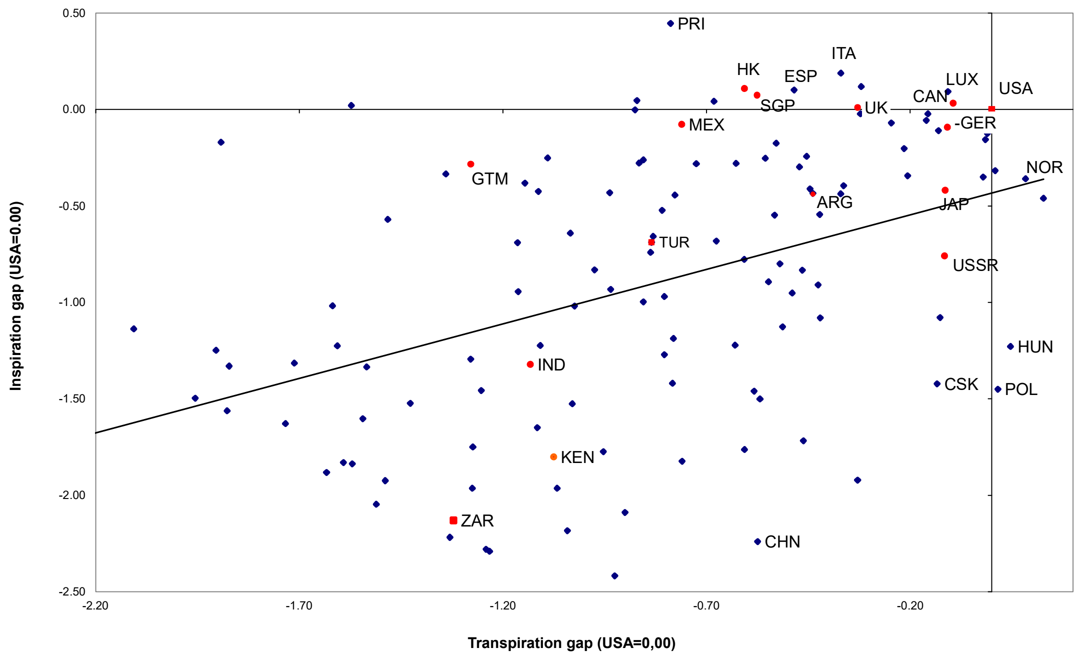
This equation breaks down differences in output per capita into differences in the capital-output ratio, differences in human capital per worker and differences in productivity. The first two terms (in the square brackets) account for the contribution of inputs (*transpiration, X*) and the last term measures TFP (*inspiration, λ*). As in standard growth accounting, the later is obtained as a residual. Hall and Jones implemented this decomposition for the year of 1988. In their calculations, all variables are measured relative to the United States. Human capital per worker *h* was computed as a function of the (observed) average years of schooling.

[83] Hall, R., and C. Jones, 1999. "Why do some countries produce so much more output per worker than others?", The Quarterly Journal of Economics 114 (1), 83-116.

Figure 13.7 summarises the authors calculations of X and λ. For each country, the vertical axes measures the estimated value of λ relative to the corresponding value in the US, in logs. For instance, according to the figure, Italy has a positive *inspiration gap* vis-à-vis the world leader, while USSR has negative *inspiration gap*. The horizontal axes measures the extent to which a country accumulated more or less human and physical capital (as captured by X) than the US. For instance, according to the figure, Norway has positive *transpiration gap* while Argentina has a negative *transpiration gap*.

In the figure, the positively sloped line is a 45º line, describing the points for which the two gaps are of the same size. For example, Argentina and Turkey exhibit patterns that are "balanced" in terms of inspiration gap and transpiration gap. On the contrary, the USSR had a large inspiration gap and very low transpiration gap, reflecting a massive investment in physical and human capital in a poor innovative environment and low ability to implement foreign technology. An interesting pattern in Figure 5.7 is that countries with large inspiration gaps also tend to exhibit large transpiration gaps. In other words, the two variables are largely correlated across countries.

Figure 5.7: Inspiration gap versus transpiration gap for 127 countries (US=0.00)



Source: Hall and Jones (1999).

As we know, correlation is not causality. Hall and Jones (1999) argued that this correlation was capturing a causality from λ to X. The authors conjectured that cross-country differences in physical and human capital accumulation and productivity are fundamentally related to cross-country differences in "social infrastructure", that is, "the institutions and government policies that determine the economic environment within which individuals accumulate skills, and firms accumulate capital and produce output" (p.84). A social infrastructure will be favourable to economic development if it provides an environment that is supportive of investment and innovation, rather than promoting theft, corruption or confiscatory taxation.

To test this proposition, the authors constructed a measure of "social infrastructure", as a combination of two indexes. The first is a measure of government anti-diversion policies (defined as an average of five indexes: law and order, bureaucratic quality, corruption, risk of expropriation and government repudiation of contracts). The second is a measure of openness to trade. Controlling for feed-back effects, the authors found that countries with a good social infrastructure tend to have high capital intensities, high human capital per worker and high output per worker.

These results point to a distinction between the proximate causes of growth (as captured by capital accumulation) and the deep causes, which ultimately determine the individual's willingness to produce and invest. According to this interpretation, policy choices such as the size of the government, the rate of inflation and innovation are all best thought as outcomes (that is, proximate causes) rather than as determinants. At the deeper level, the quality of institution plays the key role in determining the quality of policies that in turn shape the incentives to produce and invest.

5.5 Empirical controversies

5.5.1 Level effects or growth effects?

The AK model differs dramatically from the exogenous growth model, in terms of the relationship it establishes between the investment rate and economic growth. In the Solow model, a change in investment rate alters the steady state level of per capita income; in the AK model, a change in investment rate alters the growth rate of per capita income. This prediction suggests a natural way of assessing what is the model that better complies with the real world facts: if, on average, countries that invest more achieve higher rates of per capita income, eventually that will be supportive of the AK model.

Table 5.1 displays some challenging evidence provided by Klenow and Rodriguez-Clare (2005)[84]. The table shows the estimates of nine regression equations, implemented for three samples: a large sample with 96 countries, a subsample consisting in OECD economies, and a subsample consisting in non- OECD economies. In each panel, two dependent variables are experimented: the growth rate of per capita income along 1960-2000 ("growth"), and the log of per capita in 2000 ("level"). In the left panel, the explanatory variable is the investment rate in physical capital; in the right-panel, the independent variable is the investment rate in human capital.

Table 5.1. Investment rates corelate more with levels than with growth rates

	Independent variable: S			**Independent variable: S_H**		
	Dependent variable: Y/N		number of	Dependent variable: Y/N		number of
	Growth	**Level**	countries	**Growth**	**Level**	countries
All countries						
beta	0.111	1.25	96	0.21	0.313	74
Std.dev	(0.017)	(0.13)		(0.060)	(0.026)	
R-sq	0.32	0.48		0.15	0.67	
OECD						
beta	0.02	0.76	23	-0.259	0.119	21
Std.dev	(0.047)	(0.358)		(0.078)	(0.024)	
R-sq	0.01	0.18		0.37	0.56	
Non-OECD						
beta	0.124	0.842	73	0.367	0.314	53
Std.dev	(0.023)	(0.162)		(0.095)	(0.043)	
R-sq	0.29	0.28		0.22	0.51	

[84] Klenow, P. and Rodriguez-Clare, A., 2005. "Externalities and Growth". In Aghion, P., and Durlauf, S. (eds), Handbook of Economic Growth, North Holland, Amsterdam, Chapter 11, 817-866.

Source: Klenow and Rodriguez-Clare (2005). Notes: Y/N is GDP pr worker; S is the physical capital investment rate; SH is years of schooling (for the 25+ population) divided by 60 years (working live). These variables are averages over 1960-2000.

As shown in the table, investment rates in physical capital and in human capital are both positively correlated with growth in the sample of 96 countries and in the non-OECD sub-sample. In the OECD sample, investment in physical is not significant, and investment in human capital gets the wrong sign in explaining economic growth. In contrast, we observe that both investment variables are significant and with the right sign when predicting per capita income levels. In general, the correlation coefficient is higher when the dependent variable is per capita income in levels than when the dependent variable is the growth rate of per capita income. Broadly speaking, this evidence is more supportive of the new-classical growth model than the AK model.

5.5.2 Tests on conditional convergence

Another key difference between the neo-classical model and the AK model refers to convergence. The new-classical model predicts conditional convergence. This hypothesis states that countries converge to the respective steady states, and that the speed of convergence varies in direct proportion to the distance to the steady state. This property of the neoclassical model contrasts to the AK model, where changes in parameters impact once-and-for all on the growth rate of per capita income, without any tendency for per capita income to return to a previous path. Hence, another possible avenue to disentangle what is the "true" model is to test whether conditional convergence holds in reality.

The workhorse of empirical research in tests for conditional convergence are the so-called *cross-country growth regressions*[85]. Basically, the method consists in estimating the growth rate of per capita income as a function of a range of variables that control for differences in steady states, and also for the initial level of per capita GDP, that controls for transition dynamics. Conditional convergence is assessed by the significance of the coefficient on the initial level of per capita income.

An example of cross-country growth regression was already introduced in Box 4.2. In the MRW formulation, differences in steady states are accounted for by differences in investment rates in human and in physical capital. As for parameter A, the authors simply assumed it to vary randomly across economies, reflecting differences in climate, natural resources, institutions and other, without disentangling their differential contributions.

More generally, one may extend the model to include a range of possible determinants of A. In general, the equation to be estimated in an extended neoclassical framework will be:

$$\ln y_t - \ln y_0 = a - b \ln y_0 + \psi X + \chi Z + u_t , \qquad (5.24)$$

where X is a vector of variables capturing factor accumulation that are present in the MRW model (propensities to invest in physical and human capital, and the population growth rates) and Z is a vector of other variables determining the level of A. The conditional convergence hypothesis is assessed investigating the significance of *b*. In case this parameter is positive and significant this means that growth rates are proportional to the distance to steady states, which accords to the idea of conditional convergence. In case the assumption *b=0* is not rejected, then changes in the explanatory variables impact on the growth rate permanently, supporting the endogenous growth model (5.5).

---

[85] Cross country growth regressions were popularized by Barro (1991). The pioneers of this approach were however Robison (1971) and Kormendi and Meguire (1985). [Barro, R. J., 1991. "Economic growth in a cross-section of countries", *Quarterly Journal of Economics* 106:2, 407-43. Robinson, S., (1971). Source of growth in Less Developed Countries: a cross-section study, *Quarterly Journal of Economics* 85 (3), 391-408. Kormendi, R., Meguire, P., 1985. Macroeconomis determinants of growth: cross country evidence, *Journal of Monetary Economics*, 16 (2), 141-63].

The advantage of cross-country growth regressions relative to simple growth accounting is that, rather than estimating A as a residual, they try to identify the policies and other factors that underlie the cross-country differences in A. Variables that have been found significant in empirical work include proxies for the quality of policies and institutions (such as trade openness, the rule law, political risk, inflation, financial depth) and of geographical conditions. The discussion in Box 5.3 illustrates this.

In general, the evidence with cross-country growth regressions using large samples of countries has been favourable to the conditional convergence hypothesis (an illustration in table 5.2). That is, the coefficient on the initial level of per capita income has been found to be, in general, negative and significant in cross-country growth regressions. This fact inspired Robert Barro to state: "*It is surely an irony that one of the lasting contributions of endogenous growth theory is that is stimulated empirical work that demonstrated the explanatory power of the neoclassical growth model*" (Barro (1997), p. x).

Box 5.3. The external aid controversy

The question as to whether external aid helps or not a country achieve faster economic growth is obviously very important from the policy point of view. With no surprise, this question has been subject to empirical scrutiny.

A branch in the literature has investigated the possibility of the impact of aid being conditional on the recipient country' characteristics. A particularly influent study was a background paper to the 1998 World Bank *Assessing Aid* report, by Burnside and Dollar[86]. The authors run some regressions trying to explain the growth rates of per capita income along the period from 1970 to 1993, using a sample of 56 developing countries. The original results of Burnside and Dollar are reproduced in columns (1) and (2) of Table 5.2. In equation (1), the growth rate of per capita GDP is correlated with: the logarithm of initial per capita GDP

[86] Burnside, C. and Dollar, D., 2000. "Aid, policies and growth", American Economic Review 90 (4), 847-868.

(capturing conditional convergence); the degree of ethnic fractionalisation, the rate of political assassinations and the product of these two variables (to capture political instability); an index of institutional quality; the ratio of money to GDP (to capture financial development); two regional dummies, for sub-Saharan Africa and East Asia; a "policy index" (compounding the government budget surplus, inflation and openness to international trade, to capture the quality of domestic policies); and external aid as a percentage of GDP.

In column (1), we see that the t-ratio on AID/GDP is too low (0.28 in column 1). The authors then concluded that aid, by itself, does not explain growth. Column (2) differs from column (1) by adding an interaction term, given by the product of the variables AID/GDP and the Policy Index. Because this last variable was found to be significant while AID/GDP alone was not, the authors concluded that aid only leads to more growth in a sound policy framework[87].

Table 5.2. Growth regressions explaining the growth rate of per capita GDP in 56 developing countries

[87] The authors also tested the possibility of aid to be detrimental to policies. However, no significant relationship was found between the amount of aid received and the quality of the domestic policies.

	(1)	(2)	(3)
Estimation method	**OLS**	**OLS**	**OLS**
Initial GDP	-0.61	-0.60	-0.54
	(1.09)	(1.05)	(0.96)
Ethnic fractionalization	-0.54	-0.42	0.12
	(0.75)	(0.58)	(0.16)
Assassinations	-0.44*	-0.45*	-0.38
	(1.69)	(1.73)	(1.55)
Ethnic fractionalization * Assassinations	0.82*	0.79*	0.70
	(1.86)	(1.80)	(1.63)
Institutional quality	0.64**	0.69**	0.69**
	(3.76)	(4.06)	(4.02)
M2/GDP (lagged)	0.014	0.012	-0.02
	(1.08)	(0.86)	(1.54)
Sub-Saharan Africa	-1.60**	-1.87**	-1.58**
	(2.19)	(2.49)	(2.04)
East Asia	0.91*	1.31**	1.57**
	(1.69)	(2.26)	(2.63)
Burnside-Dollar policy index	1.00**	0.71**	0.78**
	(7.14)	(3.74)	(4.05)
Aid/GDP	0.034	-0.021	1.49**
	(0.28)	(0.13)	(3.92)
(Aid/GDP) * policy index		0.19**	0.09
		(2.71)	(1.34)
Fraction of land in tropics			-0.70
			(1.32)
(Aid/GDP) * fract. of land in tropics			-1.52**
			(4.02)
Observations	275	270	270
Countries	56	56	56
R^2	0.36	0.36	

Sources: Burnside and Dollar (2002) for columns (1) and (2) (regressions (3) and (4) in the original paper; Dalgaard et al., (2004) for column (3) (column 5 in the original) 'Notes: The dependent variable is real per capita GDP growth. All regressions include time dummies. Robust t-statistics in parentheses.'* significant at 10%; ** significant at 5%.

These results caused a significant reaction in the economic profession, as it implied that foreign aid is useless in countries pursuing bad policies. Not surprisingly, they were subject to an intense scrutiny by other researchers. In general, this further investigation revealed sensitivity of the Burnside and Dollar results in respect to alternative specifications of the regression model or of the sample period[88]. Burnside and Dollar then shifted their focus from the quality of policies to the quality of institutions. Using a cross section of 124 countries over the 1990s, they found that, while aid alone is not significant related to growth, aid interacted with the degree of institutional quality is significant[89].

This was not however a final word. In column 3 of table 5.1 we reproduce the results of another study, by Daalgard et al. (2004)[90]. The authors stressed the role of geography in explaining growth and accordingly they included the fraction of a country's land located in the tropics as explanatory variable. In column (3), we see that the policy-aid interaction becomes insignificant, while aid and aid interacted with the climate became significant. These results suggest that aid has a positive impact on growth, but the impact decreases for countries located in the tropics. This last result is, of course, disappointing because it points to a critical role of geography - which cannot be changed by human actions - rather than of policy, which people can change.

In the last few years, many other studies have investigated the extent to which the impact of aid on growth is conditional on third variables. The main conclusion is that there is no definitive answer regarding the variable that better interacts with aid: some studies suggests it is policy, others point to the critical role of institutions, and some others to geography. This disparity of results suggests that the inter-play between aid, local circumstances and growth is eventually too complex to be captured by a simple regression equation. It also points to the difficulty in establishing empirically the right determinants of economic growth, when the candidate variables are so many, as compared to the limited number of observations (see discussion in Box 5.4).

---

[88] Easterly, W., Levine, R. Roodman, D. 2004. "Aid, Policies, and Growth: Comment," American Economic Review, vol. 94(3), pages 774-780, June. Islam, N., 2005. Regime changes, economic policies and the effects of aid on growth. The Journal of Development Studies 41 (8), 1467-1492.

[89] Burnside, C. and Dollar, D., 2004b. "Aid, Policies, and Growth: Revisiting the evidence", World Bank Policy Research Working Paper 3251, March.

[90] Daalgard, C., Hansen, H., Tarp, F., 2004. On the empirics of foreign aid and growth. Economic Journal 114(496): F191-F216.

Box 5.4. Pitfalls of cross-country growth regressions

The Empirical analysis using cross country growth regressions face a number of limitations[91].

First, because the theory does not provide an unambiguous guide to the choice of elements of Z, there is a lot of uncertainty regarding the right model specification. In practice researchers have proposed more and more variables to complement the baseline specification, each one stressing a causal relationship between a particular variable and growth. This, in turn, brings a familiar econometric problem: because explanatory variables tend to be correlated to each other (countries performing badly in a given indicator also tend to perform badly in other indicators), there is a large scope for multicolinerity: the significance of each variable in the equation is influenced by the particular combination of variables included in the regression. This problem makes very difficult to assess empirically which variable is more correlated to growth and how much (e.g, if inflation rates, exchange rate volatility and political instability go wrong together, how one can disentangle the various contributions to growth?).

Second, there is a problem of endogeneity: although it may appear natural that the parameter estimates (ψ and χ in equation 5.25) contain information of causal effects on economic growth, this is not necessarily true. Some right-hand-side variables may be econometrically endogenous in the sense that they are jointly determined with the rate of economic growth: for instance, the same factors that make a country invest more in physical capital may also have a direct effect on a country growth rate. In that case, the estimated parameter will be biased and will provide little information regarding the direction of causality.

Third, even if all variables on the right-hand side were exogenous, many of them could be "symptoms", rather than "syndromes". For instance, consider the measurement of human capital. Shall we choose the secondary school enrolment or the primary school enrolment?

[91] For a discussion, see Durlauf, S., Johnson, P., 1995. Multiple regimes and cross-country growth behaviour. Journal of Applied Econometrics, 10, 365-384.

Since these tend to be correlated to each other, they render one another insignificant when both are included in the regression equation. So which one should we choose? Moreover, a given symptom may be interpreted as capturing different syndromes. For example, a negative correlation between inflation and growth means bad macroeconomic management or a large tax evasion that forces the government to rely on revenues from money creation?

Fourth, there is a problem of parameter heterogeneity: parameter values estimated with cross section exercises that pool together very different countries may fail to accurately capture any of each. As once stated by Arnold Harberger: "What do Thailand, the Dominican Republic, Zimbabwe, Greece and Bolivia have in common that merits their being put in the same regression analysis?.

Fifth, the lack of a structural model stating how much the parameter A depends on each policy variables makes it difficult to go beyond general statements on observed correlations and to provide a convincing interpretation of the results.

Other problems of cross-country-growth regressions include: the presence of outliers, measurement errors, and model linearity. Despite the extensive econometric improvements that have been adopted to overcome these limitations, the results of cross-country growth regressions have still to be taken with caution.

5.6 Discussion

The AK model stresses the relationship between policies and economic growth. This contrasts with the Solow model, whereby changes in the key parameters only produce level effects. The empirical evidence has not been, however, very favourable to the simpler version of the AK model. In general, country characteristics, such as the saving rate and aggregate efficiency are found to influence the *levels* of per capita income, rather than *growth rates*. This view is supported by an extensive empirical literature favourable to the conditional convergence hypothesis.

Does this mean that we shall abandon the AK model? The answer is no.

First, remember that the important link between efficiency and growth is also present in the neoclassical model: the difference is that in the later the growth effect will be transitory. That is, you may interpret the AK model as a *short-run* version of the neoclassical growth model. With half of the transition period between steady states in the neoclassical model taking as long as 35 years, whatever the true model is, we are doomed to accept that policy actions may influence economic growth for a *considerable* period of time[92].

Second, the AK model is much easier to solve than the Solow model. Because of this, from the expositional point of view, it is often more convenient to study the impact of particular policies in the context of the AK model than in the context of the Solow model, especially when the math becomes too complex. Of course, in doing so, one shall take into account that any conclusion regarding the impact of the policy on growth would be spelled out in term of level effects, if adapted to the context of the neo-classical model. In some of the upcoming chapters, we will follow this approach.

Last, but not at all the least, the basic AK model illustrates how linearity avoids the basic problem of diminishing returns. Linearity is a basic feature of most endogenous growth models focusing on technological change. The Usawa model provides an illustration of this: in that model, linearity in production of technology is the key to generate continuous growth. The AK model can therefore be interpreted as a toy version of more complex endogenous growth models, whereby knowledge expands through investments in R&D. Knowledge shares with capital the characteristic that it can be built over time by sacrificing some of today's consumptions. Interpreting investment as foregone consumption in a broad sense (that is, including R&D), is the what we need to rescue the AK model as a very valid framework to think the mechanics of economic growth.

[92] Easterly (2005) calibrated a simple neoclassical growth model with a share of total capital equal to 2/3 (which accords to MRW) and with other reasonable values for the remaining parameters. He found that a tax decrease from 30% to zero raises per capita income by a factor of 2.25 times. The author also showed that immediately after the change in policy, the growth rate of the economy jumps up by almost 8 percentage points relative to its steady state. Only in the very long run (more than 5 decades after), the growth effect wears off and the growth rate returns to its long run level. The author concluded that policies have significant effects in the neoclassical model, too (pp. 1024-1026).

5.7 Key ideas of Chapter 5

- The AK model reveals in a simple manner that getting rid of diminishing returns, factor accumulating alone can generate continuous growth of per capita income. In the context of the AK model, changes in the saving rate produce "growth effects" rather than "level effects".
- Extending the AK model to the case with endogenous savings, the direct effect of aggregate efficiency on growth is reinforced by an indirect effect via a higher return on savings. The implication is that, wherever financial markets are more developed, the impact of policy changes on growth is more dramatic.
- The model with endogenous savings appeals to the distinction between proximate causes of growth, like the savings rate and aggregate efficiency, and fundamental causes of growth, that determine why some countries have higher investment rates and better efficiency than others.
- Extending the model in a variety of ways, we found that one can interpret K in a broad sense, including other reproducible inputs to production and technology.
- In its simpler formulation, the AK models displays no transitional dynamics. There are some hybrid models, however, like the Usawa model, that at the same time display growth effects and transition dynamics. In light of this extension, a reallocation of resources away from production towards the research sector implies a temporary output loss but it delivers a faster rate of technological progress.
- The Harrod Domar equation inspired the idea that complementing low domestic savings in poor countries by foreign aid would be a key to generate economic growth. In practice, however, the impact of external aid on the growth varied significantly across countries, depending on the quality of domestic policies, institutions, and geography.
- The empirical evidence of conditional convergence has been more favourable to the neo-classical model than to the simpler versions of the AK model whereby higher saving rates generate faster economic growth.
- The AK model can be seen as a toy version of more complex models of endogenous growth based on technological change.

Problems and Exercises

Key concepts

- *Harrod Domar equation. Level effect versus growth effect. Proximate versus fundamental causes of economic growth. Cross-country growth regressions.*

Essay questions:

- Referring to the Harrod-Domar equation, compare the AK model and the Solow model in respect to the variables that are exogenous and endogenous. In particular, examine the impact of an increase in the saving rate in light of the two models.

- Comment: "Poor countries, with underdeveloped financial markets, are more likely to tolerate bad policies than rich countries with developed capital markets".
- Explain why the Usawa model is hybrid. In the context of this model, which policies could influence the rate of economic growth?

Exercises

5.1.[**AK Simple**] Consider an economy where the production function is given by Y=AK.. In this economy, the saving rate is *s*, the population grows at rate *n* and the capital depreciation rate is δ. (a) Does this production function satisfy the usual neoclassical properties? Why? (b) Describe analytically and graphically the dynamics of per capita income in this economy. Is there any stable equilibrium? (c) Does this model predict convergence of per capita incomes across economies? (d) Describe, comparing with the Solow model, the impact of: (i) a fall in the population growth rate; (ii) An increase in A.

5.2.[**AK endogenous savings**] Consider an economy, where the production function is given by Y=0.2K, the population grows at 2% per year, the capital depreciates at 3% and the saving rate is 25%. (a) Find out the growth rate of per capita income in this economy. (b) What will be the effect of A increasing to 0.25? (c) What would be the effect of a simultaneous increase of A to 0.25 and s to 0.4? (d) Now assume that the saving rate was endogenous, as implied by the following optimal consumption rule: $\gamma_t = r_t - 0.17$. Analyse in this case the implications of an increase in efficiency from 0.2 to 0.25. (e) Comparing the two models, find out the expression that relates the saving rate to efficiency (A). Explain why a change in the efficiency parameter (A) impacts more on growth when savings are endogenous.

5.3.[**Harrod-Domar**] Consider a closed economy without government, where population is equal to one thousand inhabitants, and constant over time. In this economy, the relevant production function is given by $Y = 0.5K$, capital deteriorates at the rate of $\delta = 0.04$ per year, and consumption is a linear function of income, according to $C = 8000 + 0.84Y$. (a) Suppose the initial capital stock in this economy was exatcly $K = 200.000$. In this case, per capital income: (i) will be growing over time; (ii) will be decreasing over time; (iii) will be stagnant. (b) Suppose that, due to some external support, the capital stock in this economy jumped temporarily to $K = 250.000$. In this case, the long run growth rate of per capita income will approach: (i) 0; (ii) 0.008; (iii) 0.04. (c) Suppose this economy started out with a capital stock equal to $K = 250.000$, but the production function was actually given by $Y = \min[0.5K, 150N]$. In this case, the long run growth rate of per capita income will approach: (i) 0; (ii) 0.008; (iii) 0.04.(d) In the conditions set out in (c), the long run value of the capital-labour ratio will be: (i) decreasing over time; (ii) k=300; (iii) k=400; (iv) increasing over time.

5.4.[Rebelo] Consider an economy where the aggregate production function is given by $Y = K^{0.5}H^{0.5}$, where *H=hN, N* is the number of workers and *h* measures the quality of labour. We also know that in this country 20% of income is saved and invested in physical capital, the population is constant, and the physical capital stock depreciates at an annual rate of 4%. (a) Assume for the moment that the quality of labour evolves exogenously according to $\dot{h}/h = 0.01$. (a1) Discuss that assumption and its implications for the model; ((a2) Determine the equilibrium values of *k = K/N* and *y = Y/N*. Describe the equilibrium of the model on a graph and discuss its stability; (a3) How well does this model explain the stylized facts of economic growth? (b) Now assume that $\dot{H} = s_H Y - \delta H = 0.0125Y - 0.04H$, where sh refers to the investment rate in human capital. (b1) Does this version of the model still verify the neo-classical properties? Why? (b2) Compute the growth rate of per capita income in this economy and explain its dynamics with the help of a graph. Describe the income per capita dynamics of this model. Represent it on a graph. (c) Compare, in the light of both models: (c1) the short run and the long run effects of a rise in the saving rate. (c2) the convergence hypothesis.

5.5. [**Usawa**] Consider an economy where $Y_t = K_t^{1/2}\left[(1-\mu)(N_t\lambda_t)\right]^{1/2}$, the saving rate is *20%*, population is constant, and capital depreciates at the rate $\delta = 0.04$. Technology evolves according to $\lambda_t = e^{\gamma t}$, $\gamma = b\mu$, and b=0.05. (a) Assume for a moment that $\mu = 0$. (a1) Find out the steady state level of per capita income. (a2) Represent in a graph. (a3) In the steady state, per capita income and the interest rate evolve according to the Kaldor stylized facts? (b) (b1) Explain the equation describing the technological change. Examine the implications of an increase in μ to $\mu = 0.2$. In particular: (b2) Describe the change in equilibrium with the help of a graph. (b3) Compute the new equilibrium values of $\tilde{y}$ and $\tilde{k}$. (b4) Describe the time path of per capita income before and after the change. (b5) In the steady state, per capita income and the interest rate evolve according to the Kaldor stylized facts? (b6) was the change in μ welfare improving? (c) Compare the impacts of an increase in the saving rate in cases (a) and (b). (d) Does this model display conditional convergence?

5.6.[**Usawa**] Consider an economy where the production function is given by $Y_t = (1-\mu)^{1/2} K_t^{1/2} (N\lambda_t)^{1/2}$, were $\mu = 0.75$ is the fraction of time devoted to production. In this economy, the saving rate is 15%, the population is constant and capital does not depreciate. The productivity of labour accumulates at the rate $\gamma = \dot{\lambda}/\lambda = b\mu$, where b=0.02. (a) Explain the equation describing technological progress. (b) Using the equation describing the change in $\tilde{k} = K/L$ (the fundamental dynamic equation), find out the steady state values of $\tilde{y}$ and $\tilde{k}$. (c) Examine the implications of an increase in the saving rate from s=15% to s=18%. In particular, compute the new equilibrium values of $\tilde{y}$ and $\tilde{k}$. Describe the change in a graph and explain what will happen to the interest rate. (d) Returning to the initial figures, examine the implication of an increase in b to b=0.04. In particular, compute the new equilibrium values of $\tilde{y}$ and $\tilde{k}$. Describe the change in a graph and explain what will happen to the interest rate. (e) Compare the effects on the path of per capita income, y=Y/N, of the changes described in c) and in (d).

5.7. Consider the following production function and law of motion of per capita consumption: $Y_t = K_t^{\beta} N_t^{\alpha} H_t^{1-\beta-\alpha}$, with $\alpha, \beta \leq 1$, and $\gamma = r - \rho$. Assume that the depreciation rate is identical for the two capital types and that population does not grow over time. (a) Suppose that $\alpha=0, \beta<1$. Explain if it is possible to obtain sustained grow in the long-run through factor accumulation. Would conditional convergence hold in that case? What would be the impact of an increase in ρ on the interest rate and on per capita income? (b) Suppose that $\alpha+\beta<1$, $\alpha, \beta \neq 0$. Discuss the advantages of this parameterization comparing them to the results obtain in (a). Would conditional convergence hold in this case? And sustained growth of per capita income? (c) Suppose that $\alpha+\beta=1$, $\alpha, \beta \neq 0$. Would conditional convergence hold in that case? And sustained growth of per capita income? What would be the impact of an increase in ρ on the interest rate and on per capita income?