Local growth envelopes of spaces of generalized smoothness: the critical case

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Abstract

The concept of local growth envelope of a quasi-normed function space is applied to the spaces of Besov and Triebel-Lizorkin type of generalized smoothness (s, Ψ) in the critical case $s = n/p$, where s stands for the main smoothness, Ψ is a perturbation and p stands for integrability. The expression obtained for the behaviour of the local growth envelope functions (which, as expected, depends on Ψ) shows the ability to be generalized to a form unifying both critical ($s = n/p$) and subcritical ($s < n/p$) cases.

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1 Introduction

In [2] we have studied the local growth envelopes of the spaces $B^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ and $F^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ of generalized smoothness in the sub-critical case $n(1/p - 1)_+ =: \sigma_p < s < n/p$, extending previous results of Haroske [7] and Triebel [17] dealing with the same type of issues for the more classical Besov and Triebel-Lizorkin spaces.

We have postponed to the present work the extension of the results of [7] and [17] dealing with the critical case $\sigma_p < s = n/p$. One of the reasons had to do with the fact that the technique of interpolation used in the study of the sub-critical case was not powerful enough in order to deal with the critical case (this was already clear in the more classical setting studied by Haroske and Triebel). A second reason was that the behaviour observed for the local growth envelope function in the classical critical case did not give

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a clue for what it should be in the generalized setting. A third reason was the difficulty in overcoming some technical details.

However, in the end the results obtained were much rewarding, because we not only solved the problem of characterizing the local growth envelopes of the spaces $B_{pq}^{(s,\Psi)}(\mathbb{R}^n)$ and $F_{pq}^{(s,\Psi)}(\mathbb{R}^n)$ in the critical case $\sigma_p < s = n/p$, as we did it in a way that unifies with the sub-critical case $\sigma_p < s < n/p$ and – we hope – which may open the door to extensions to even more general settings (so, extension to the spaces studied, for example, in [5] would be worth considering).

Functions spaces of generalized smoothness have been considered since the middle of the seventies of the last century, in particular by the Russian school, and have been again in the center of interest in recent times. In particular because they are relevant in recent investigations in the theory of stochastic processes, where they appear in a natural way. For a short description of this and some historical remarks, please check [5], where other relevant references can also be found.

We give now a more concrete flavour of the results which are proved in this paper.

Denoting by A either B or F, the main objective is to characterize the ability of local growth for functions of the spaces $A_{pq}^{(s,\Psi)}(\mathbb{R}^n)$, when these spaces are not continuously embedded in $L_\infty(\mathbb{R}^n)$ (thus the situation $s > n/p$ has no interest). This is in part done by studying the behaviour of the local growth envelope function

$$\mathcal{E}_{\mathrm{LG}}|A_{pq}^{(s,\Psi)}(t) := \sup\{f^*(t) : \|f|A_{pq}^{(s,\Psi)}(\mathbb{R}^n)\| \leq 1\}$$

near 0, where f^* stands for the decreasing rearrangement of f (so we need f to be in L_1^{loc}, which in part explains the restriction $s > \sigma_p$).

When $\Psi \equiv 1$ and, consequently, we are dealing with the Besov and Triebel-Lizorkin spaces $A_{pq}^s(\mathbb{R}^n)$, Haroske [7] and Triebel [17] have proved that $\mathcal{E}_{\mathrm{LG}}|A_{pq}^s(t)$ behaves like $t^{s/n-1/p}$ near 0 in the sub-critical case and like $|\log t|^{1/u'}$ near 0, with $u = q$ when $A = B$ and $u = p$ when $A = F$, in the critical case (with u' standing for the conjugate exponent of u and where u is here assumed to be greater than 1, as otherwise the question is of no interest).

In [2] we have shown that $\mathcal{E}_{\mathrm{LG}}|A_{pq}^{(s,\Psi)}(t)$ behaves like $t^{s/n-1/p}\Psi(t)^{-1}$ near 0 in the sub-critical case. In the present work we prove that it behaves like $\left(\int_{t^{1/n}}^1 \Psi(y)^{-u'} \frac{dy}{y}\right)^{1/u'}$ near 0, with u as above, in the critical case (where here Ψ is assumed to be such that $(\Psi(2^{-j})^{-1})_{j\in\mathbb{N}} \notin \ell_{u'}$, as otherwise the question is of no interest), this giving a clue for describing the behaviour of the local growth envelope function near 0 of either critical and sub-critical cases, both for the classical or generalized settings, in the following unified form (which we actually prove in the course of the paper that reduces to the previously given expressions in each specific situation considered before):

$$\Phi_{r,u'}(t) := \left(\int_{t^{1/n}}^{1} y^{-\frac{n}{r}u'} \Psi(y)^{-u'} \, \frac{dy}{y} \right)^{1/u'} ,$$

where $-n/r = s - n/p$.

Following the idea of Haroske [7] and Triebel [17] when introducing the concept of local growth envelope, we also study the behaviour of an individual f^* against $\Phi_{r,u'}$ and (assuming we have picked up a continuous Ψ) the Borel measure $\mu_{r,u'}$ associated with $-\log \Phi_{r,u'}$ in some interval $(0, \varepsilon]$ (for some small positive ε), thus proving that the best exponent v such that

$$\left(\int_0^{\varepsilon} \left(\frac{f^*(t)}{\Phi_{r,u'}(t)} \right)^{v} \mu_{r,u'}(dt) \right)^{1/v} \leq c \, \| f | A_{pq}^{(s,\Psi)}(\mathbb{R}^n) \| \, , \qquad (1)$$

for some constant $c = c(v)$ and all $f \in A_{pq}^{(s,\Psi)}(\mathbb{R}^n)$, is q when $A = B$ and p when $A = F$, just like in the classical setting.

Notice that, when $\Psi \equiv 1$, $r = \infty$ and $v = u$, (1) can be written as

$$\left(\int_0^{\varepsilon} \left(\frac{f^*(t)}{|\log t|} \right)^{u} \frac{dt}{t} \right)^{1/u} \leq c \, \| f | A_{pq}^{s}(\mathbb{R}^n) \| \, ,$$

which may become apparent that we are pushing forward in a direction already followed by many others. We refer the reader to [17, 11.8(v), 13.5] for historical references to the subject and related more recent developments, where the names of Adams, Brézis, Brudnyi, Cwikel, Edmunds, Gold'man, Hansson, Kaljabin, Kerman, Krbec, Maz'ya, Moser, Netrusov, Peetre, Pick, Pohozaev, Pustylnik, Schmeisser, Strichartz, Triebel, Trudinger, Wainger, Yudovich and Ziemer are cited. See also related recent results of Opic and Trebels [12] dealing with some Bessel potential spaces of generalized smoothness.

We finish this introduction by collecting some general notation used throughout the paper.

As usual, $\mathbb{R}^n$ denotes the n-dimensional real Euclidean space, $\mathbb{N}$ the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We use the equivalence "$\sim$" in

$$a_k \sim b_k \quad \text{or} \quad \varphi(x) \sim \psi(x)$$

always to mean that there are two positive numbers c_1 and c_2 such that

$$c_1 \, a_k \leq b_k \leq c_2 \, a_k \quad \text{or} \quad c_1 \, \varphi(x) \leq \psi(x) \leq c_2 \, \varphi(x)$$

for all admitted values of the discrete variable k or the continuous variable x, where $(a_k)_k$, $(b_k)_k$ are non-negative sequences and φ, ψ are non-negative functions. Given two quasi-Banach spaces X and Y, we write $X \hookrightarrow Y$ if

$X \subset Y$ and the natural embedding of X into Y is continuous. All unimportant positive constants will be denoted by c, occasionally with additional subscripts within the same formula. If not otherwise indicated, log is always taken with respect to base 2. Since we will not deal with function spaces defined on domains different from $\mathbb{R}^n$, in most cases we shall omit the "$\mathbb{R}^n$" from their notation.

2 Functions of interest

We shall be concerned with function spaces of generalized smoothness of Besov and Triebel-Lizorkin type, where the usual main smoothness parameter s is replaced by a couple (s, Ψ), where Ψ is an admissible function according to the following definition.

Definition 2.1 *A positive monotone function Ψ on the interval $(0,1]$ is called admissible if*

$$\Psi(2^{-j}) \sim \Psi(2^{-2j}), \quad j \in \mathbb{N}_0. \tag{2}$$

Example 2.2 If $b \in \mathbb{R}$ then

$$\Psi_b(x) = (1 + |\log x|)^b, \quad x \in (0,1], \tag{3}$$

is an admissible function; we return to this particular choice in the sequel for illustration.

The proposition below gives some properties of admissible functions that will be useful in the sequel. We refer to Lemma 2.3 of [2], where a simple proof can be found. The reader may also want to refer to the beginning of [10], where some other useful properties of admissible functions are stated and proved.

Proposition 2.3 *Let Ψ be an admissible function.*

(i) *There exist constants $b \geq 0$, $c_1, c_2 > 0$ such that*

$$c_1 \, (1 + |\log t|)^{-b} \leq \inf_{0<s\leq 1} \frac{\Psi(ts)}{\Psi(s)} \leq \sup_{0<s\leq 1} \frac{\Psi(ts)}{\Psi(s)} \leq c_2 \, (1 + |\log t|)^{b},$$

for any $t \in (0,1]$.

(ii) *For any $a, d > 0$, there is $\delta > 0$ such that*

$$\Psi(at^d) \sim \Psi(t), \quad t \in (0, \delta).$$

The functions we are going to introduce now will be central in the estimates to be presented later.

Definition 2.4 *Let $r, u \in (0,\infty]$ and Ψ be a continuous admissible function. Define $\Phi_{r,u} : (0, 2^{-n}] \to \mathbb{R}$ by*

$$\Phi_{r,u}(t) := \left(\int_{t^{1/n}}^{1} y^{-\frac{n}{r}u} \Psi(y)^{-u} \, \frac{dy}{y} \right)^{1/u}$$

(modified to $\sup_{t^{1/n} \le y \le 1} y^{-\frac{n}{r}} \Psi(y)^{-1}$ if $u = \infty$).

Proposition 2.5 *$\Phi_{r,u}$ given as above is a positive, monotonically decreasing and continuous function. In the case $u \neq \infty$ we can even say that $\Phi_{r,u}$ is differentiable, its derivative being given by*

$$\Phi'_{r,u}(t) = -\frac{1}{un} \, t^{-\frac{u}{r}-1} \Psi(t^{1/n})^{-u} \Phi_{r,u}(t)^{1-u} \,, \qquad t \in (0, 2^{-n}]. \tag{4}$$

Proof. That $\Phi_{r,u}$ is positive and monotonically decreasing is obvious. In the case $u \neq \infty$ it is also clear that $\Phi_{r,u}$ is continuous, as it is a positive power of a (Riemann) indefinite integral. Since the integrand is even continuous, then $\Phi_{r,u}$ is also differentiable and elementary calculations lead to the expression (4).

It only remains to show that $\Phi_{r,\infty}$ is also continuous.

Let $t \in (0, 2^{-n}]$ and $t_k \uparrow t$, $k \in \mathbb{N}$.

If there is $k_1 \in \mathbb{N}$ such that $\Phi_{r,\infty}(t_{k_1})$ is attained in $[t^{1/n}, 1]$, then for $k \ge k_1$ it always holds $\Phi_{r,\infty}(t_k) = \Phi_{r,\infty}(t)$ and therefore $\lim_{k\to\infty} \Phi_{r,\infty}(t_k) = \Phi_{r,\infty}(t)$.

If $\Phi_{r,\infty}(t_k)$ is always attained in $[t_k^{1/n}, t^{1/n})$, say in y_k, then $\Phi_{r,\infty}(t_k) = y_k^{-\frac{n}{r}} \Psi(y_k)^{-1}$, so that, by the continuity of Ψ, $\lim_{k\to\infty} \Phi_{r,\infty}(t_k) = t^{-\frac{1}{r}} \Psi(t^{1/n})^{-1}$. If $\Phi_{r,\infty}(t) = t^{-\frac{1}{r}} \Psi(t^{1/n})^{-1}$, we already got $\lim_{k\to\infty} \Phi_{r,\infty}(t_k) = \Phi_{r,\infty}(t)$. Otherwise, it must be $t^{-\frac{1}{r}} \Psi(t^{1/n})^{-1} < \Phi_{r,\infty}(t) < y_k^{-\frac{n}{r}} \Psi(y_k)^{-1}$ and, by the continuity of Ψ, there exists $z_k \in (y_k, t^{1/n})$ such that $\Phi_{r,\infty}(t) = z_k^{-\frac{n}{r}} \Psi(z_k)^{-1}$. Given any $\varepsilon > 0$ it is thus possible, again using the continuity of Ψ, to choose $k_0 \in \mathbb{N}$ such that $|y^{-\frac{n}{r}} \Psi(y)^{-1} - t^{-\frac{1}{r}} \Psi(t^{1/n})^{-1}| < \varepsilon$ whenever $y \in [t_{k_0}^{1/n}, t^{1/n}]$, from which follows that

$$|y^{-\frac{n}{r}} \Psi(y)^{-1} - z^{-\frac{n}{r}} \Psi(z)^{-1}| < 2\varepsilon \tag{5}$$

whenever $y, z \in [t_{k_0}^{1/n}, t^{1/n}]$. In particular (5) is true when $y = y_k$ and $z = z_k$, for any $k \ge k_0$, that is,

$$|\Phi_{r,\infty}(t_k) - \Phi_{r,\infty}(t)| < 2\varepsilon \;\text{ whenever }\; k \ge k_0 \,.$$

Let now $t \in (0, 2^{-n})$ and $t_k \downarrow t$, $k \in \mathbb{N}$.

If $\Phi_{r,\infty}(t)$ is attained in $(t^{1/n}, 1]$, say in y_0, then for $k \in \mathbb{N}$ such that $t_k^{1/n} \le y_0$ we have $\Phi_{r,\infty}(t_k) = \Phi_{r,\infty}(t)$, and therefore $\lim_{k\to\infty} \Phi_{r,\infty}(t_k) =$

$\Phi_{r,\infty}(t)$. Otherwise, $\Phi_{r,\infty}(t) = t^{-\frac{1}{r}}\Psi(t^{1/n})^{-1}$ and, by the continuity of Ψ,

$$\Phi_{r,\infty}(t) \geq \Phi_{r,\infty}(t_k) \geq t_k^{-\frac{1}{r}}\Psi(t_k^{1/n})^{-1} \underset{k\to\infty}{\longrightarrow} t^{-\frac{1}{r}}\Psi(t^{1/n})^{-1} = \Phi_{r,\infty}(t)\,,$$

so that we also have $\lim_{k\to\infty}\Phi_{r,\infty}(t_k) = \Phi_{r,\infty}(t)$. □

Proposition 2.6 *Let $\Phi_{r,u}$ be given as in Definition 2.4.*

(i) *If $r \neq \infty$, then $\Phi_{r,u}(t) \sim t^{-\frac{1}{r}}\Psi(t)^{-1}$ in $(0, 2^{-n}]$. The same happens if $r, u = \infty$ and Ψ is not bounded away from zero.*

(ii) *If Ψ is bounded away from 0, then $\Phi_{\infty,\infty}(t) \sim 1$ in $(0, 2^{-n}]$.*

(iii) *If $u \neq \infty$ and $\Psi \equiv 1$, then $\Phi_{\infty,u}(t) \sim |\log t|^{1/u}$ in $(0, 2^{-n}]$.*

Proof. Note that we are dealing with positive continuous functions in $(0, 2^{-n}]$, so that it suffices to show that the equivalences hold near 0.

(i) First consider the case $r, u \neq \infty$.

Observe that

$$\Phi_{r,u}(t) = A(t)\, t^{-\frac{1}{r}}\Psi(t^{1/n})^{-1}\,, \tag{6}$$

with

$$\begin{aligned} A(t) &= \left(\int_{t^{1/n}}^{1}\left(\frac{y}{t^{1/n}}\right)^{-\frac{n}{r}u}\left(\frac{\Psi(y)}{\Psi(t^{1/n})}\right)^{-u}\frac{dy}{y}\right)^{1/u} \\ &= \left(\frac{r}{un}\right)^{1/u}\left(\int_0^{-\frac{u}{r}\ln t} e^{-z}\left(\frac{\Psi(t^{1/n})}{\Psi(e^{zr/(un)}t^{1/n})}\right)^{u} dz\right)^{1/u}, \end{aligned} \tag{7}$$

where we have changed variables according to the rule $y = t^{1/n}e^{\frac{r}{un}z}$.

The fraction inside the integral in (7) is, by Proposition 2.3(i), bounded above by $c_1(1+z)^b$, for some constants $c_1 > 0$ and $b \geq 0$, so that

$$\begin{aligned} A(t) &\leq c_2\left(\int_0^1 e^{-z}\,dz + \int_1^{-\frac{u}{r}\ln t} e^{-z}z^{bu}\,dz\right)^{1/u} \\ &\leq c_2(1 - e^{-1} + \Gamma(bu+1))^{1/u} \;\leq\; c_3\,, \end{aligned} \tag{8}$$

where $c_2, c_3 > 0$ are constants, Γ stands for the Euler function and the second integral only shows up when $1 < -\frac{u}{r}\ln t$.

On the other hand, again by Proposition 2.3(i), the fraction inside the integral in (7) is bounded below by $c_4(1+z)^{-b}$, for some constant $c_4 > 0$, so that for $t \in (0, e^{-r/u})$ we have

$$A(t) \geq c_5\left(\int_0^1 e^{-z}\,dz\right)^{1/u} \geq c_6\,, \tag{9}$$

where $c_5, c_6 > 0$ are constants.

Since, from Proposition 2.3(ii), there exists $\delta > 0$ such that $\Psi(t^{1/n}) \sim \Psi(t)$ in $(0, \delta)$, (6), (8) and (9) together prove part (i) when $r, u \neq \infty$.

Consider now $u = \infty$ (again with $r \neq \infty$).

When Ψ is monotonically increasing, the result is obvious.

When Ψ is monotonically decreasing, on the one hand one has, by definition of supremum, $\Phi_{r,\infty}(t) \geq t^{-\frac{1}{r}} \Psi(t^{1/n})^{-1}$; on the other hand, $\Phi_{r,\infty}(t) \leq B(t)\, t^{-\frac{1}{r}} \Psi(t^{1/n})^{-1}$, with

$$\begin{aligned} B(t) &= \sup_{t^{1/n} \leq y \leq 1} \left(\frac{t^{1/n}}{y} \right)^{\frac{n}{r}} \frac{\Psi(t^{1/n})}{\Psi(y)} \\ &\leq c_7 \sup_{t^{1/n} \leq y \leq 1} \left(\frac{t^{1/n}}{y} \right)^{\frac{n}{r}} \left(1 + |\log \frac{t^{1/n}}{y}|\right)^b \\ &= c_7 \sup_{t^{1/n} \leq z \leq 1} z^{\frac{n}{r}} (1 - \log z)^b \leq c_8 \,, \end{aligned}$$

where $c_7, c_8 > 0$ and $b \geq 0$ are constants and we have taken advantage of Proposition 2.3(i). The result again follows with the help of Proposition 2.3(ii).

Consider, finally, the case when $r, u = \infty$ and Ψ is not bounded away from 0.

Since Ψ is monotonic, then it must be increasing, so that $\Phi_{\infty,\infty}(t) = \Psi(t^{1/n})^{-1} \sim \Psi(t)^{-1}$ for small $t > 0$.

(ii) If Ψ is increasing, $\Phi_{\infty,\infty}(t) = \Psi(t^{1/n})^{-1} \geq \Psi(1)^{-1}$, a positive constant. On the other hand, under the hypothesis that Ψ is bounded away from 0, there exists $c_9 > 0$ such that $\Psi(t^{1/n})^{-1} \leq c_9$.

If Ψ is decreasing, $\Phi_{\infty,\infty}(t) = \Psi(1)^{-1}$, a positive constant.

(iii) It follows from straightforward calculations that, in the case $u \neq \infty$ and $\Psi \equiv 1$, $\Phi_{\infty,u}(t) = (-\frac{1}{n \log e} \log t)^{1/u}$. □

Proposition 2.7 *Let $\Phi_{r,u}$ be as in Definition 2.4. Then*

$$\Phi_{r,u}(t) \sim \left(\sum_{j=1}^{[|\log t|/n]} 2^{j\frac{n}{r}u} \Psi(2^{-j})^{-u} \right)^{1/u} \quad \text{in } (0, 2^{-n}]$$

(with the right-hand side modified to $\sup_{j=1,\dots,[|\log t|/n]} 2^{j\frac{n}{r}} \Psi(2^{-j})^{-1}$ if $u = \infty$).

Proof. Taking advantage of the definition of admissible function, we have, for all $t \in (0, 2^{-n}]$,

$$\begin{aligned}
\left(\sum_{j=1}^{[|\log t|/n]} 2^{j\frac{n}{r}u} \Psi(2^{-j})^{-u} \right)^{1/u} &\le c_1 \left(\sum_{j=1}^{[|\log t|/n]} \int_{2^{-j}}^{2^{-(j-1)}} y^{-\frac{n}{r}u} \Psi(y)^{-u} \, \frac{dy}{y} \right)^{1/u} \\
&\le c_1 \left(\int_{t^{1/n}}^{1} y^{-\frac{n}{r}u} \Psi(y)^{-u} \, \frac{dy}{y} \right)^{1/u} \\
&\le c_1 \left(\sum_{j=1}^{[|\log t|/n]+1} \int_{2^{-j}}^{2^{-(j-1)}} y^{-\frac{n}{r}u} \Psi(y)^{-u} \, \frac{dy}{y} \right)^{1/u} \\
&\le c_2 \left(\sum_{j=1}^{[|\log t|/n]+1} 2^{j\frac{n}{r}u} \Psi(2^{-j})^{-u} \right)^{1/u} \\
&\le c_3 \left(\sum_{j=1}^{[|\log t|/n]} 2^{j\frac{n}{r}u} \Psi(2^{-j})^{-u} \right)^{1/u}
\end{aligned}$$

and

$$\begin{aligned}
\sup_{j=1,\dots,[|\log t|/n]} 2^{j\frac{n}{r}} \Psi(2^{-j})^{-1} &\le c_1 \sup_{j=1,\dots,[|\log t|/n]} \left(\sup_{2^{-j} \le y \le 2^{-(j-1)}} y^{-\frac{n}{r}} \Psi(y)^{-1} \right) \\
&\le c_1 \sup_{t^{1/n} \le y \le 1} y^{-\frac{n}{r}} \Psi(y)^{-1} \\
&\le c_1 \sup_{j=1,\dots,[|\log t|/n]+1} \left(\sup_{2^{-j} \le y \le 2^{-(j-1)}} y^{-\frac{n}{r}} \Psi(y)^{-1} \right) \\
&\le c_2 \sup_{j=1,\dots,[|\log t|/n]+1} 2^{j\frac{n}{r}} \Psi(2^{-j})^{-1} \\
&\le c_3 \sup_{j=1,\dots,[|\log t|/n]} 2^{j\frac{n}{r}} \Psi(2^{-j})^{-1} \,,
\end{aligned}$$

where, in both cases, c_1, c_2 and c_3 are suitable positive constants. □

3 Function spaces of generalized smoothness

3.1 Introduction

Before introducing the function spaces under consideration we need to recall some notation. By $\mathcal{S}$ we denote the Schwartz space of all complex-valued, infinitely differentiable and rapidly decreasing functions on $\mathbb{R}^n$ and by $\mathcal{S}'$ the dual space of all tempered distributions on $\mathbb{R}^n$. Furthermore, L_1^{loc} stands for the collection of all complex-valued locally Lebesgue-integrable functions on

$\mathbb{R}^n$ and L_p, with $0 < p \le \infty$, is the usual quasi-Banach space with respect to the Lebesgue measure, quasi-normed by

$$\|f \mid L_p\| := \Big(\int_{\mathbb{R}^n} |f(x)|^p \, dx\Big)^{1/p},$$

with the usual modification if $p = \infty$. Let $\varphi_0 \in \mathcal{S}$ with

$$\varphi_0(x) = 1 \quad \text{if} \quad |x| \le 1 \quad \text{and} \quad \operatorname{supp} \varphi_0 \subset \{x \in \mathbb{R}^n : |x| \le 2\}, \tag{10}$$

and for each $j \in \mathbb{N}$ let $\varphi_j(x) := \varphi_0(2^{-j}x) - \varphi_0(2^{-j+1}x)$, $x \in \mathbb{R}^n$. Then $(\varphi_j)_{j \in \mathbb{N}_0}$ form a smooth dyadic resolution of unity. Given any $f \in \mathcal{S}'$, we denote by $\mathcal{F}f$ and $\mathcal{F}^{-1}f$ its Fourier transform and its inverse Fourier transform, respectively.

Definition 3.1 *Let $0 < p, q \le \infty$, $s \in \mathbb{R}$ and Ψ be an admissible function.*
(i) *Then $B_{pq}^{(s,\Psi)}$ is the collection of all $f \in \mathcal{S}'$ such that*

$$\|f \mid B_{pq}^{(s,\Psi)}\| := \Big(\sum_{j=0}^{\infty} 2^{jsq} \, \Psi(2^{-j})^q \, \|\mathcal{F}^{-1}[\varphi_j \mathcal{F} f] \mid L_p\|^q\Big)^{1/q} \tag{11}$$

(with the usual modification if $q = \infty$) is finite.
(ii) *Let $0 < p < \infty$. Then $F_{pq}^{(s,\Psi)}$ is the collection of all $f \in \mathcal{S}'$ such that*

$$\|f \mid F_{pq}^{(s,\Psi)}\| := \Big\|\Big(\sum_{j=0}^{\infty} 2^{jsq} \, \Psi(2^{-j})^q \, |\mathcal{F}^{-1}[\varphi_j \mathcal{F} f](\cdot)|^q\Big)^{1/q} \Big| \, L_p\Big\| \tag{12}$$

(with the usual modification if $q = \infty$) is finite.

Remark 3.2 The above spaces, quasi-normed by (11) and (12), were introduced by Edmunds and Triebel in [3, 4] and also considered by Moura in [10, 11], where they have remarked that such spaces are independent of the resolution of unity taken, in the sense of equivalent quasi-norms. If $\Psi \equiv 1$ then the spaces $B_{pq}^{(s,\Psi)}$ and $F_{pq}^{(s,\Psi)}$ coincide with the usual Besov and Triebel-Lizorkin spaces, B_{pq}^s and F_{pq}^s, respectively, and the following elementary embeddings hold:

$$A_{pq}^{s+\varepsilon} \hookrightarrow A_{pq}^{(s,\Psi)} \hookrightarrow A_{pq}^{s-\varepsilon}, \tag{13}$$

for all $\varepsilon > 0$ and $A \in \{B, F\}$.

Example 3.3 With the particular choice of Ψ_b given by (3) we obtain spaces $B_{pq}^{s,b}$ consisting of those $f \in \mathcal{S}'$ for which

$$\|f \mid B_{pq}^{s,b}\| = \Big(\sum_{j=0}^{\infty} 2^{jsq} \, (1+j)^{bq} \, \|(\varphi_j \widehat{f})^\vee \mid L_p\|^q\Big)^{1/q}$$

is finite (usual modification for $q = \infty$); similarly for $F_{pq}^{s,b}$. These spaces were studied by Leopold in [9].

An important tool is the characterization of the spaces of generalized smoothness by means of atomic decompositions. We state this here only for the B-spaces. We refer to [10] or [11] for a complete description. We need some preparation.

As for $\mathbb{Z}^n$, it stands for the lattice of all points in $\mathbb{R}^n$ with integer-valued components, $Q_{\nu m}$ denotes a cube in $\mathbb{R}^n$ with sides parallel to the axes of coordinates, centred at $2^{-\nu}m = (2^{-\nu}m_1 \ldots, 2^{-\nu}m_n)$, and with side length $2^{-\nu}$, where $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ and $\nu \in \mathbb{N}_0$. If Q is a cube in $\mathbb{R}^n$ and $r > 0$ then rQ is the cube in $\mathbb{R}^n$ concentric with Q and with side length r times the side length of Q.

Definition 3.4 (i) *Let $K \in \mathbb{N}_0$ and $c > 1$. A K times differentiable complex-valued function a in $\mathbb{R}^n$ (continuous if $K = 0$) is called an 1_K-atom if*

$$\operatorname{supp} a \subset c\,Q_{0m}, \quad \textit{for some} \quad m \in \mathbb{Z}^n$$

and

$$|D^\alpha a(x)| \leq 1, \quad \textit{for} \quad |\alpha| \leq K.$$

(ii) *Let $K \in \mathbb{N}_0$, $L + 1 \in \mathbb{N}_0$ and $c > 1$. A K times differentiable complex-valued function a in $\mathbb{R}^n$ (continuous if $K = 0$) is called an $(s, p, \Psi)_{K,L}$-atom if for some $\nu \in \mathbb{N}_0$,*

$$\operatorname{supp} a \subset c\,Q_{\nu m}, \quad \textit{for some} \quad m \in \mathbb{Z}^n,$$

$$|D^\alpha a(x)| \leq 2^{-\nu(s-\frac{n}{p})+|\alpha|\nu}\,\Psi(2^{-\nu})^{-1}, \quad \textit{for} \quad |\alpha| \leq K,$$

and

$$\int_{\mathbb{R}^n} x^\beta a(x)\,dx = 0, \quad \textit{if} \quad |\beta| \leq L.$$

If the atom a is located at $Q_{\nu m}$, that means

$$\operatorname{supp} a \subset c\,Q_{\nu m}, \quad \text{with } \nu \in \mathbb{N}_0,\ m \in \mathbb{Z}^n,$$

then we write it as $a_{\nu m}$. The sequence spaces b_{pq} are defined as follows:

Definition 3.5 *Let $\lambda = (\lambda_{\nu m})_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n} \subset \mathbb{C}$. Then*

$$b_{pq} = \Big\{ \lambda : \|\lambda \mid b_{pq}\| = \Big(\sum_{\nu=0}^{\infty} \Big(\sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}|^p \Big)^{q/p} \Big)^{1/q} < \infty \Big\}$$

(with the usual modification if $p = \infty$ or/and $q = \infty$).

If $0 < p \leq \infty$ then $\sigma_p := n(1/p - 1)_+ = \max\{0, n(1/p - 1)\}$.

Theorem 3.6 *Let $c > 1$, $K \in \mathbb{N}_0$ and $L + 1 \in \mathbb{N}_0$ with*

$$K \geq (1 + [s])_+ \quad \text{and} \quad L \geq \max(-1, [\sigma_p - s]) \tag{14}$$

be fixed. Then $f \in \mathcal{S}'$ belongs to $B_{pq}^{(s,\Psi)}$ if, and only if, it can be represented as

$$f = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \, a_{\nu m}, \quad \text{convergence being in } \mathcal{S}', \tag{15}$$

where $a_{\nu m}$ are 1_K-atoms ($\nu = 0$) or $(s, p, \Psi)_{K,L}$-atoms ($\nu \in \mathbb{N}$) according to Definition 3.4 and $\lambda \in b_{pq}$. Furthermore

$$\inf \|\lambda \mid b_{pq}\|, \tag{16}$$

where the infimum is taken over all admissible representations (15), is an equivalent quasi-norm in $B_{pq}^{(s,\Psi)}$.

3.2 Rearrangement properties

If f is an extended complex-valued measurable function on $\mathbb{R}^n$ which is finite a.e., then the decreasing rearrangement of f is the function defined on $[0, \infty)$ by

$$f^*(t) := \inf\{\lambda \geq 0 : m_f(\lambda) \leq t\}, \quad t \geq 0, \tag{17}$$

with m_f being the distribution function given by

$$m_f(\lambda) := \big|\{x \in \mathbb{R}^n : |f(x)| > \lambda\}\big|, \quad \lambda \geq 0.$$

As usual, the convention $\inf \emptyset = \infty$ is assumed and $|\cdot|$ denotes Lebesgue measure when applied to measurable subsets of $\mathbb{R}^n$. Moreover, the maximal function of f^* is the function

$$f^{**}(t) := \frac{1}{t} \int_0^t f^*(\tau) \, d\tau, \quad t > 0.$$

We assume that the reader is familiar with basic facts concerning rearrangements: these may be found in [1]. In particular we shall need the sub-additivity property

$$(f + g)^{**}(t) \leq f^{**}(t) + g^{**}(t), \quad t > 0. \tag{18}$$

By analogy, in the case of a (multiple) sequence $(\alpha_m)_{m \in \mathbb{Z}^n} \subset \mathbb{C}$, its decreasing rearrangement is defined as the sequence $(\alpha_l^*)_{l \in \mathbb{N}}$, where

$$\alpha_l^* := \inf\{\lambda \geq 0 : \#\{m \in \mathbb{Z}^n : |\alpha_m| > \lambda\} < l\}, \quad l \in \mathbb{N}. \tag{19}$$

We also define

$$\alpha_l^{**} := \frac{1}{l} \sum_{k=1}^{l} \alpha_k^*, \quad l \in \mathbb{N}.$$

Proposition 3.7 *Let $p \in (1, \infty]$. Let $(\alpha_m)_{m \in \mathbb{Z}^n}$, $(\alpha_l^*)_{l \in \mathbb{N}}$ and $(\alpha_l^{**})_{l \in \mathbb{N}}$ be as above. Then*

$$\|(\alpha_m)_{m\in\mathbb{Z}^n}|\ell_p\| = \|(\alpha_l^*)_{l\in\mathbb{N}}|\ell_p\| \leq \|(\alpha_l^{**})_{l\in\mathbb{N}}|\ell_p\| \leq \frac{p}{p-1}\,\|(\alpha_l^*)_{l\in\mathbb{N}}|\ell_p\|\,,$$

where $\frac{p}{p-1}$ should be interpreted as 1 when $p = \infty$.

Proof. The result is obvious for $p = \infty$. As to the case $p \in (1, \infty)$, the equality (which, actually, holds also for $0 < p \leq 1$) follows from [1, Prop. 1.8 in Ch. 2, p. 43] applied to the counting measure in $\mathbb{Z}^n$ and the last inequality is due to Hardy and Landau [6, pp. 239-240]. □

Proposition 3.8 *Let $d > 1/2$ and $(d_j)_{j\in\mathbb{N}_0}$ be a sequence of positive numbers. Let $(a_{jm})_{j\in\mathbb{N}_0, m\in\mathbb{Z}^n}$ be a sequence of complex-valued measurable functions on $\mathbb{R}^n$ such that, for each j and m, $\operatorname{supp} a_{jm} \subset 2dQ_{jm}$ and $|a_{jm}(x)| \leq d_j$, $\forall x \in \mathbb{R}^n$, where Q_{jm} is a dyadic cube as defined previously, in subsection 3.1. Let $(\lambda_{jm})_{j\in\mathbb{N}_0, m\in\mathbb{Z}^n}$ be a sequence of complex numbers and define, for each $j \in \mathbb{N}_0$,*

$$f_j(x) := \sum_{m\in\mathbb{Z}^n} \lambda_{jm} a_{jm}(x)\,, \quad x \in \mathbb{R}^n\,. \tag{20}$$

(i) *There are positive constants C and D, depending only on n and d, such that*

$$f_j^*(t) \leq D d_j \sum_{l=1}^{\infty} \lambda_{jl}^* \chi_{jl}(t)\,, \quad t \geq 0\,,\ j \in \mathbb{N}_0\,,$$

and

$$f_j^{**}(t) \leq D d_j \sum_{l=1}^{\infty} \lambda_{jl}^{**} \chi_{jl}(t)\,, \quad t > 0\,,\ j \in \mathbb{N}_0\,, \tag{21}$$

where χ_{jl} stands for the characteristic function of the set $[C2^{-jn}(l-1), C2^{-jn}l)$, $l \in \mathbb{N}$, $(\lambda_{jl}^*)_{l\in\mathbb{N}}$ is the decreasing rearrangement of $(\lambda_{jm})_{m\in\mathbb{Z}^n}$, $j \in \mathbb{N}_0$, and $\lambda_{jl}^{**} := \frac{1}{l}\sum_{k=1}^{l} \lambda_{jk}^*$, $l \in \mathbb{N}$.

(ii) *If, for some $j \in \mathbb{N}_0$ and $p \in (0, \infty]$, $(\lambda_{jm})_{m\in\mathbb{Z}^n} \in \ell_p(\mathbb{Z}^n)$, then $f_j \in L_p$, for the same j and p.*

(iii) *Let $p \in [1, \infty]$, $q \in (0, \infty]$ and assume $(\lambda_{jm})_{j\in\mathbb{N}_0, m\in\mathbb{Z}^n} \in b_{pq}$, where b_{pq} was introduced in Definition 3.5. If, moreover, $(d_j 2^{-j\frac{n}{p}})_{j\in\mathbb{N}_0} \in \ell_{q'}$, where q' is conjugate to q (with $q' = \infty$ when $0 < q \leq 1$), then the series*

$$\sum_{j=0}^{\infty} f_j$$

converges in L_p to a function f satisfying

$$f^{**}(t) \le \sum_{j=0}^{\infty} f_j^{**}(t)\,, \quad t > 0\,.$$

Proof. **(i)** Note that (20) makes sense pointwise, the sum being finite in, say, each set of the form $2dQ_{jm}$, and therefore each f_j is a complex-valued measurable function.

Let $D/2$ be an upper bound for the maximum number of different sets $2dQ_{jm}$ (for the same $j \in \mathbb{N}_0$) with non-empty intersection.

Observe that, for each $j \in \mathbb{N}_0$ and $x \in \mathbb{R}^n$, there is an $m \in \mathbb{Z}^n$ such that $x \in 2dQ_{jm}$ and

$$|f_j(x)| \le \frac{D}{2} d_j |\lambda_{jm}|\,.$$

Therefore, for each $j \in \mathbb{N}_0$ and $l \in \mathbb{N}$, and if λ^*_{jl} is finite,

$$\begin{aligned}
&|\{x \in \mathbb{R}^n : |\tfrac{2}{D} d_j^{-1} f_j(x)| > \lambda^*_{jl}\}| \\
&\quad \le \quad |\{x \in \mathbb{R}^n : \exists m \in \mathbb{Z}^n \text{ s.t. } x \in 2dQ_{jm} \text{ and } |\lambda_{jm}| > \lambda^*_{jl}\}| \\
&\quad \le \quad |\bigcup_{m\in\mathbb{Z}^n \text{s.t.} |\lambda_{jm}|>\lambda^*_{jl}} 2dQ_{jm}| \\
&\quad \le \quad \sum_{m\in\mathbb{Z}^n \text{s.t.} |\lambda_{jm}|>\lambda^*_{jl}} |2dQ_{jm}| \\
&\quad \le \quad (2d)^n 2^{-jn}(l-1)\,.
\end{aligned}$$

Choose $C := (2d)^n$.

We then have, for $t \ge C2^{-jn}(l-1)$, that $(\frac{2}{D} d_j^{-1} f_j)^*(t) \le \lambda^*_{jl}$ and, consequently,

$$f_j^*(t) \le \frac{D}{2} d_j \sum_{l=1}^{\infty} \lambda^*_{jl} \chi_{jl}(t)\,, \quad t \ge 0\,,\ j \in \mathbb{N}_0\,.$$

Let now $t > 0$. Therefore, for each $j \in \mathbb{N}_0$, $l \in \mathbb{N}$ and $C2^{-jn}(l-1) \le t < C2^{-jn}l$,

$$f_j^{**}(t) \le D d_j \lambda^{**}_{jl}\,,$$

from which (21) follows easily.

(ii) Assume first that $0 < p < \infty$. Then we have

$$\begin{aligned}
\int_0^\infty f_j^*(t)^p\, dt \quad &\le \quad \sum_{l=1}^{\infty} D^p d_j^p \int_{C2^{-jn}(l-1)}^{C2^{-jn}l} \lambda^{*p}_{jl}\, dt \\
&= \quad D^p d_j^p C 2^{-jn} \sum_{l=1}^{\infty} \lambda^{*p}_{jl} \\
&= \quad D^p d_j^p C 2^{-jn} \sum_{m\in\mathbb{Z}^n} |\lambda_{jm}|^p \ < \ \infty\,,
\end{aligned}$$

hence $f_j \in L_p$.

In the case $p = \infty$, observe that $f_j^*(0) \le Dd_j \lambda_{j1}^* = Dd_j \sup_{m\in\mathbb{Z}^n} |\lambda_{jm}| < \infty$, hence $f_j \in L_\infty$.

(iii) Given $M, L \in \mathbb{N}$ with $M > L$, in the case $1 \le p < \infty$ we can write

$$\begin{aligned}
\Big\| \sum_{j=L}^{M} f_j \,\Big|\, L_p \Big\| &\le \sum_{j=L}^{M} \left(\int_0^\infty f_j^*(t)^p \, dt \right)^{1/p} \\
&\le \sum_{j=L}^{M} DC^{1/p} d_j 2^{-j\frac{n}{p}} \left(\sum_{m\in\mathbb{Z}^n} |\lambda_{jm}|^p \right)^{1/p} \\
&\le DC^{1/p} \left(\sum_{j=L}^{M} \left(\sum_{m\in\mathbb{Z}^n} |\lambda_{jm}|^p \right)^{q/p} \right)^{1/q} \left(\sum_{j=L}^{M} d_j^{q'} 2^{-j\frac{n}{p}q'} \right)^{1/q'}
\end{aligned}$$

(with the usual modifications if $q = \infty$ or $0 < q \le 1$).

From the hypothesis it follows that $\left(\sum_{j=0}^{M} f_j \right)_{M\in\mathbb{N}}$ is fundamental in the complete space L_p, hence it converges in this space to, say, f.

When $p = \infty$ we have $\| \sum_{j=L}^{M} f_j | L_\infty \| \le \sum_{j=L}^{M} f_j^*(0) \le \sum_{j=L}^{M} Dd_j \sup_{m\in\mathbb{Z}^n} |\lambda_{jm}|$, and the proof of the convergence of $\sum_{j=0}^{\infty} f_j$ to some function f in L_p follows as before.

From $f = \sum_{j=0}^{\infty} f_j$ in L_p it follows that $|f| \le \sum_{j=0}^{\infty} |f_j|$ pointwise a.e., where the last sum might possibly be infinity at some points, and from here we can apply the subadditivity property (18) as well as other properties of maximal functions (cf. [1, Prop. 3.2 of Ch. 2, pp. 52-53] for the case when the functions are finite a.e.) to get

$$f^{**}(t) \le \sum_{j=0}^{\infty} f_j^{**}(t) \,, \quad t > 0 \,.$$

□

Corollary 3.9 *Given $p \in [1,\infty]$, $q \in (0,\infty]$, $s \in \mathbb{R}$ and Ψ an admissible function, $B_{pq}^{(s,\Psi)} \subset L_1^{\mathrm{loc}}$ either if $s > 0$ or if $s = 0$ and $(\Psi(2^{-j})^{-1})_{j\in\mathbb{N}_0} \in \ell_{q'}$.*

Proof. We first remark that the result for $s > 0$ can also be obtained by comparing with Besov spaces with $\Psi \equiv 1$, via (13). But here we have a direct proof, even for the case when $\Psi \equiv 1$.

From Theorem 3.6 we know that any $f \in B_{pq}^{(s,\Psi)}$ is the limit, in $\mathcal{S}'$, of f_j's as in Proposition 3.8, where $(\lambda_{jm})_{j\in\mathbb{N}_0, m\in\mathbb{Z}^n} \in b_{pq}$, $d_0 = 1$ and $d_j = 2^{-j(s-\frac{n}{p})}\Psi(2^{-j})^{-1}$ for $j \in \mathbb{N}$. Since $(d_j 2^{-j\frac{n}{p}})_{j\in\mathbb{N}} = (2^{-js}\Psi(2^{-j})^{-1})_{j\in\mathbb{N}}$ is in $\ell_{q'}$ either when $s > 0$ or when $s = 0$ and $(\Psi(2^{-j})^{-1})_{j\in\mathbb{N}_0} \in \ell_{q'}$, then part (iii) of Proposition 3.8 guarantees that $\sum_{j=0}^{\infty} f_j$ also converges in L_p. Since $L_p \hookrightarrow \mathcal{S}'$ (recall $p \ge 1$), then, under the given conditions, $f \in L_p$ and, therefore, is locally integrable. □

Example 3.10 Recall our example Ψ_b given by (3) and the corresponding spaces in Example 3.3. Assume $p \in [1,\infty]$, $q \in (0,\infty]$ and $s, b \in \mathbb{R}$. Then Corollary 3.9 yields that $B^{s,b}_{pq} \subset L_1^{\mathrm{loc}}$ either if $s > 0$ or

$$s = 0 \qquad \text{and} \qquad \begin{cases} b > 1/q', & \text{if } q > 1; \\ b \geq 0, & \text{if } 0 < q \leq 1. \end{cases}$$

3.3 Extremal functions

Let φ be the compactly supported C^∞ function on $\mathbb{R}^n$ defined by

$$\varphi(x) := e^{-1/(1-|x|^2)} \quad \text{if} \quad |x| < 1 \quad \text{and} \quad \varphi(x) := 0 \quad \text{if} \quad |x| \geq 1. \tag{22}$$

Proposition 3.11 *Let $0 < p < \infty$, $0 < q \leq \infty$ and Ψ be an admissible function. Let $b = (b_j)_{j\in\mathbb{N}}$ be a non-negative sequence in ℓ_q and put*

$$f(x) := \sum_{j=1}^{\infty} b_j \, \Psi(2^{-j})^{-1} \, \varphi(2^{j-1}x), \quad x \in \mathbb{R}^n, \tag{23}$$

where φ is the function given by (22). Then $f \in B^{(n/p,\Psi)}_{pq}$ and, moreover,

$$\|f \mid B^{(n/p,\Psi)}_{pq}\| \leq c \, \|b \mid \ell_q\| \tag{24}$$

for some $c > 0$ which does not dependent on b. If, in addition, there exist a strictly increasing sequence $(j_k)_{k\in\mathbb{N}}$ of natural numbers and a positive constant d such that

$$b_{j_k} \geq d \, b_{j_{k+1}}, \quad k \in \mathbb{N}, \qquad b_j = 0 \quad \textit{for} \quad j \neq j_k,\ k \in \mathbb{N}, \tag{25}$$

and

$$\Psi(2^{-j_k}) \sim \Psi(2^{-j_{k+1}}), \qquad k \in \mathbb{N}, \tag{26}$$

then, for the decreasing rearrangement f^* of f the following inequalities hold:

$$f^*(t) \leq c_1 \sum_{\ell=1}^{k+1} b_{j_\ell} \, \Psi(2^{-j_\ell})^{-1} \quad \textit{for} \quad t \geq |\omega_n| \, 2^{-j_k n}, \qquad \textit{and} \tag{27}$$

$$f^*(t) \geq c_2 \sum_{\ell=1}^{k+1} b_{j_\ell} \, \Psi(2^{-j_\ell})^{-1} \quad \textit{for} \quad 0 < t < |\omega_n| \, (1 - 2^{-n}) \, 2^{-j_k n}, \tag{28}$$

with $k \in \mathbb{N}$, c_1, c_2 positive constants which depend only on φ, Ψ and d, and $|\omega_n|$ standing for the Lebesgue measure of the unit ball in $\mathbb{R}^n$.

Proof. Since the functions

$$a_j(x) := \Psi(2^{-j})^{-1}\,\varphi(2^{j-1}x), \quad x \in \mathbb{R}^n,\ j \in \mathbb{N}, \tag{29}$$

are (up to constants, independently of j) $(n/p, p, \Psi)_{K,-1}$-atoms, for some fixed $K \in \mathbb{N}$ with $K > n/p$, and $b \in \ell_q$, then (24) is an immediate consequence of Theorem 3.6.

Assume now that (25) and (26) hold true. Let $2^{-(j_k+1)} \le |x| \le 2^{-j_k}$, for some $k \in \mathbb{N}$. We then have

$$f(x) = \sum_{\ell=1}^{k+1} b_{j_\ell}\,\Psi(2^{-j_\ell})^{-1}\,\varphi(2^{j_\ell-1}x) \le e^{-1}\sum_{\ell=1}^{k+1} b_{j_\ell}\,\Psi(2^{-j_\ell})^{-1} \tag{30}$$

and, on the other hand,

$$\begin{aligned} f(x) &\ge e^{-\frac{4}{3}}\sum_{\ell=1}^{k} b_{j_\ell}\,\Psi(2^{-j_\ell})^{-1} \ge \frac{e^{-\frac{4}{3}}}{2}\Big(\sum_{\ell=1}^{k} b_{j_\ell}\,\Psi(2^{-j_\ell})^{-1} + b_{j_k}\,\Psi(2^{-j_k})^{-1}\Big) \\ &\ge \frac{e^{-\frac{4}{3}}}{2}\Big(\sum_{\ell=1}^{k} b_{j_\ell}\,\Psi(2^{-j_\ell})^{-1} + c\,b_{j_{k+1}}\,\Psi(2^{-j_{k+1}})^{-1}\Big) \\ &\ge c'\sum_{\ell=1}^{k+1} b_{j_\ell}\,\Psi(2^{-j_\ell})^{-1}, \end{aligned} \tag{31}$$

where c' is a constant depending only on φ, Ψ and on the constant d (according to (25)). Thus, if λ is such that $0 < \lambda < c'\sum_{\ell=1}^{k+1} b_{j_\ell}\,\Psi(2^{-j_\ell})^{-1}$, for some $k \in \mathbb{N}$ and with c' as above, then

$$\begin{aligned} m_f(\lambda) &\ge \Big|\Big\{x \in \mathbb{R}^n : f(x) \ge c'\sum_{\ell=1}^{k+1} b_{j_\ell}\,\Psi(2^{-j_\ell})^{-1}\Big\}\Big| \\ &\ge \Big|\Big\{x \in \mathbb{R}^n : 2^{-(j_k+1)} \le |x| \le 2^{-j_k}\Big\}\Big| = |\omega_n|\,(1-2^{-n})\,2^{-j_k n}, \end{aligned}$$

and, if $\lambda \ge e^{-1}\sum_{\ell=1}^{k+1} b_{j_\ell}\,\Psi(2^{-j_\ell})^{-1}$ then

$$\begin{aligned} m_f(\lambda) &\le \Big|\Big\{x \in \mathbb{R}^n : f(x) > e^{-1}\sum_{\ell=1}^{k+1} b_{j_\ell}\,\Psi(2^{-j_\ell})^{-1}\Big\}\Big| \\ &\le \Big|\Big\{x \in \mathbb{R}^n : |x| \le 2^{-j_k}\Big\}\Big| = |\omega_n|\,2^{-j_k n}, \end{aligned}$$

where $|\omega_n|$ denotes the Lebesgue measure of the unit ball in $\mathbb{R}^n$. The above estimates yield (27) and (28) with $c_1 = e^{-1}$ and $c_2 = c'$. □

Remark 3.12 In the sequel and for technical reasons, we will consider the function g given by

$$g(x) := f(R^{1/n}x), \qquad x \in \mathbb{R}^n, \tag{32}$$

where f is the function in (23) and $R := |\omega_n|(1-2^{-n})2^{-n}$ (so, a constant depending only on n). Analogously to f, it turns out that

$$g \in B_{pq}^{(n/p,\Psi)} \quad \text{and} \quad \|g \mid B_{pq}^{(n/p,\Psi)}\| \leq c\,\|b \mid \ell_q\|, \tag{33}$$

for some constant c, independent of b. Concerning the decreasing rearrangement of g, it holds

$$g^*(t) = f^*(Rt), \qquad t > 0, \tag{34}$$

and, in particular,

$$g^*(2^{-j_k n}) \geq c \sum_{\ell=1}^{k+1} b_{j_\ell}\,\Psi(2^{-j_\ell})^{-1}, \qquad k \in \mathbb{N}, \tag{35}$$

for some positive constant c, independent of b.

3.4 Local growth envelopes

As we briefly mentioned in the Introduction – and explained in some detail in [2] –, regarding the study of local growth envelopes in the context of the spaces $A_{pq}^{(s,\Psi)}$, of interest are the spaces so that

$$A_{pq}^{(s,\Psi)} \subset L_1^{\mathrm{loc}} \quad \text{but} \quad A_{pq}^{(s,\Psi)} \not\hookrightarrow L_\infty.$$

As for the inclusion $A_{pq}^{(s,\Psi)} \subset L_1^{\mathrm{loc}}$, this is the case if $s > \sigma_p$ and impossible if $s < \sigma_p$. The borderline $s = \sigma_p$ deserves a careful attention and we transfer this topic to a later occasion, although something in this direction is already contained in Corollary 3.9. A complete characterization for the usual Besov and Triebel-Lizorkin spaces is known, cf. [13].
When $\sigma_p < s < n/p$, independently of Ψ and q we never have embeddings in L_∞. This corresponds to the so-called sub-critical case and in [2] we have achieved final answers for the correspondent local growth envelopes. When $s > n/p$ we always have embeddings in L_∞, so that the remaining case is then $s = n/p$, the so-called critical case, which we shall consider in this paper.

In the following we present the complete description for the embeddings in L_∞ in the critical case. In the context of $1 < p, q < \infty$ the result is due to Kalyabin [8]. We recall that for $0 < r \leq \infty$ the number r' is given by $1/r' = (1-1/r)_+$.

Proposition 3.13 *Let $0 < p, q \leq \infty$ and Ψ be an admissible function.*

(i) *Then*

$$B_{pq}^{(n/p,\Psi)} \hookrightarrow L_\infty \quad \textit{if, and only if,} \quad \left(\Psi(2^{-j})^{-1}\right)_{j\in\mathbb{N}} \in \ell_{q'}.$$

(ii) *Let* $0 < p < \infty$. *Then*

$$F_{pq}^{(n/p,\Psi)} \hookrightarrow L_\infty \quad \textit{if, and only if,} \quad \big(\Psi(2^{-j})^{-1}\big)_{j\in\mathbb{N}} \in \ell_{p'}.$$

In both cases L_∞ *can be replaced by* C, *the space of all complex-valued bounded and uniformly continuous functions on* $\mathbb{R}^n$.

Proof. We will prove here only the sufficiency of the conditions since the necessity will follow as a by-product of later considerations (see Remark 4.3 below).

Let $(\varphi_j)_{j\in\mathbb{N}_0}$ be the usual resolution of unity and let $f \in B_{pq}^{(n/p,\Psi)}$. By (1.3.2/5) and Remark 1.4.1/4 in [15], we have

$$\|\mathcal{F}^{-1}[\varphi_j\mathcal{F}f] \mid L_\infty\| \le c\,2^{jn/p}\,\|\mathcal{F}^{-1}[\varphi_j\mathcal{F}f] \mid L_p\|, \qquad j\in\mathbb{N}_0. \tag{36}$$

Let first $0 < q \le 1$ and suppose that $\big(\Psi(2^{-j})^{-1}\big)_{j\in\mathbb{N}} \in \ell_\infty$. Since then $\ell_q \hookrightarrow \ell_1$, using (36) we obtain

$$\begin{aligned}\sum_{j=0}^{\infty} \|\mathcal{F}^{-1}[\varphi_j\mathcal{F}f] \mid L_\infty\| &\le c\sum_{j=0}^{\infty} 2^{jn/p}\,\|\mathcal{F}^{-1}[\varphi_j\mathcal{F}f] \mid L_p\| \\ &\le c\Big(\sum_{j=0}^{\infty} 2^{jnq/p}\,\|\mathcal{F}^{-1}[\varphi_j\mathcal{F}f] \mid L_p\|^q\Big)^{1/q} \\ &\le c\sup_{j\in\mathbb{N}_0}\Psi(2^{-j})^{-1}\Big(\sum_{j=0}^{\infty} 2^{jnq/p}\,\Psi(2^{-j})^q\,\|\mathcal{F}^{-1}[\varphi_j\mathcal{F}f] \mid L_p\|^q\Big)^{1/q} \\ &= c\,\|\Psi(2^{-j})^{-1} \mid \ell_\infty\|\,\|f \mid B_{pq}^{(n/p,\Psi)}\|.\end{aligned}$$

Now let $1 < q \le \infty$. Assuming that $\big(\Psi(2^{-j})^{-1}\big)_{j\in\mathbb{N}} \in \ell_{q'}$, using (36) and applying Hölder's inequality we get

$$\begin{aligned}\sum_{j=0}^{\infty} \|\mathcal{F}^{-1}[\varphi_j\mathcal{F}f] \mid L_\infty\| &\le c\,\|\Psi(2^{-j})^{-1} \mid \ell_{q'}\|\times \\ &\quad\times\Big(\sum_{j=0}^{\infty} 2^{jnq/p}\,\Psi(2^{-j})^q\|\mathcal{F}^{-1}[\varphi_j\mathcal{F}f] \mid L_p\|^q\Big)^{1/q} \\ &= c\,\|\Psi(2^{-j})^{-1} \mid \ell_{q'}\|\,\|f \mid B_{pq}^{(n/p,\Psi)}\|\end{aligned}$$

(with the usual modification if $q = \infty$). Hence, in both cases of q and under the corresponding assumption on the sequence $\big(\Psi(2^{-j})^{-1}\big)_{j\in\mathbb{N}}$, we have shown that

$$B_{pq}^{(n/p,\Psi)} \hookrightarrow B^0_{\infty 1}.$$

This leads to $B_{pq}^{(n/p,\Psi)} \hookrightarrow C$ as $B^0_{\infty 1} \hookrightarrow C$, cf. e.g. [14, 2.2.9/(1), p. 68].

Since $F^{n/p}_{pq} \hookrightarrow B^0_{\infty p}$ – cf. e.g. [16, 11.4(iii)] – then Proposition 3.4 of [2] yields

$$F^{(n/p,\Psi)}_{pq} \hookrightarrow B^{(0,\Psi)}_{\infty p}.$$

Therefore, by what has been proved above, $\left(\Psi(2^{-j})^{-1}\right)_{j\in\mathbb{N}_0} \in \ell_{p'}$ implies $F^{(n/p,\Psi)}_{pq} \hookrightarrow C$. □

Example 3.14 In case of our particular example Ψ_b given by (3) and the corresponding spaces in Example 3.3, Proposition 3.13 reads as

$$B^{n/p,b}_{pq} \hookrightarrow L_\infty \quad \text{if, and only if,} \quad \begin{cases} b > 1/q', & \text{if } \ q > 1; \\ b \geq 0, & \text{if } \ 0 < q \leq 1, \end{cases}$$

where $p, q \in (0, \infty]$ and $b \in \mathbb{R}$.

According to what was pointed out before, regarding the local growth envelope of interest are the spaces $A^{(s,\Psi)}_{pq}$ so that

$$\sigma_p < s < \frac{n}{p} \quad \text{or} \quad \sigma_p < s = \frac{n}{p} \quad \text{and} \quad \begin{cases} \left(\Psi(2^{-j})^{-1}\right)_{j\in\mathbb{N}} \notin \ell_{q'} & \text{if} \quad A = B, \\ \left(\Psi(2^{-j})^{-1}\right)_{j\in\mathbb{N}} \notin \ell_{p'} & \text{if} \quad A = F. \end{cases} \tag{37}$$

We can even say, in such a case, that

$$\mathcal{E}_{\mathrm{LG}}|A^{(s,\Psi)}_{pq}(t) := \sup\{f^*(t) : \|f|A^{(s,\Psi)}_{pq}\| \leq 1\}$$

(which is finite for $t > 0$, in view of Proposition 4.1 and Corollary 4.5) defines a decreasing function which is positive in $(0, \varepsilon]$, for some $\varepsilon \in (0, 1)$, and which tends to ∞ as t goes to 0 (cf. Theorem 4.4 of [2] in case of $s < n/p$ and Proposition 4.2 below in case of $s = n/p$). Therefore, it makes sense to ask for the behaviour of $\mathcal{E}_{\mathrm{LG}}|A^{(s,\Psi)}_{pq}(t)$ near zero, which gives an indication of the ability of local growth for functions in $A^{(s,\Psi)}_{pq}$.

Let $\mathcal{E}_{\mathrm{LG}}$ be the set of all functions $f : (0, \varepsilon] \to \mathbb{R}^+$, for any $\varepsilon \in (0, 1]$, which are decreasing and consider the following equivalence relation in $\mathcal{E}_{\mathrm{LG}}$: given $f, g \in \mathcal{E}_{\mathrm{LG}}$, one says that f and g are equivalent (and write $f \sim_{\mathrm{LG}} g$) if

$$\exists\, c_1, c_2 > 0 : \forall\, t \in (0, \varepsilon], \ c_1\, g(t) \leq f(t) \leq c_2\, g(t),$$

where $(0, \varepsilon]$ is the smallest of the domains of f and g.

Definition 3.15 *The local growth envelope function of $A^{(s,\Psi)}_{pq}$, for s, p, q and Ψ satisfying (37), is the equivalence class $[\mathcal{E}_{\mathrm{LG}}|A^{(s,\Psi)}_{pq}]$. We shall also call local growth envelope function of $A^{(s,\Psi)}_{pq}$ any representative in such a class. We even call local growth envelope function of $A^{(s,\Psi)}_{pq}$ any function $f : (0, \varepsilon] \to \mathbb{R}^+$, for some $\varepsilon \in (0, 1]$, – even if not decreasing – such that $f \sim \mathcal{E}_{\mathrm{LG}}|A^{(s,\Psi)}_{pq}$ in $(0, \varepsilon]$, and use it to represent the equivalence class $[\mathcal{E}_{\mathrm{LG}}|A^{(s,\Psi)}_{pq}]$.*

Remark 3.16 Note that different equivalent quasi-norms taken in the same space $A_{pq}^{(s,\Psi)}$ give rise to the same equivalence class $[\mathcal{E}_{\mathrm{LG}}|A_{pq}^{(s,\Psi)}]$.

Let again s, p, q and Ψ be so that (37) holds true.

Assume there exists a continuous representative $\mathcal{E}_{\mathrm{LG}}A_{pq}^{(s,\Psi)} \in [\mathcal{E}_{\mathrm{LG}}|A_{pq}^{(s,\Psi)}]$ (we shall later see that this is indeed the case). Let $(0,\varepsilon]$, $0<\varepsilon<1$, be its domain.

Define $H(t) := -\log \mathcal{E}_{\mathrm{LG}}A_{pq}^{(s,\Psi)}(t)$ and note that H is a (finite) real increasing function on $(0,\varepsilon]$ which tends to $-\infty$ when t goes to 0. There is only a Borel measure (i.e., a measure defined on the Borel sets) μ_H in $(0,\varepsilon]$ such that $\mu_H\big([a,b]\big) = H(b) - H(a)$, $\forall\,[a,b] \subset (0,\varepsilon]$. Its restriction to each such $[a,b]$ is the Stieltjes-Borel measure associated with $H|_{[a,b]}$.

We recall here Proposition 12.2 of [17], which will be useful in the sequel.

Proposition 3.17 (i) *Let $0<\varepsilon<1$ and $h:(0,\varepsilon]\to\mathbb{R}^+$ be a continuous, decreasing function such that $\lim_{t\to 0^+} h(t) = \infty$. Let $H(t) := -\log h(t)$, $t\in(0,\varepsilon]$, and μ_H be the associated Borel measure in $(0,\varepsilon]$, as above. Let $0<u_1<u_2<\infty$. There are $c_1, c_2 > 0$ such that*

$$\begin{aligned}\sup_{t\in(0,\varepsilon]} \frac{\gamma(t)}{h(t)} &\le c_2 \left(\int_{(0,\varepsilon]} \left(\frac{\gamma(t)}{h(t)}\right)^{u_2} \mu_H(dt)\right)^{1/u_2} \\ &\le c_1 \left(\int_{(0,\varepsilon]} \left(\frac{\gamma(t)}{h(t)}\right)^{u_1} \mu_H(dt)\right)^{1/u_1}\end{aligned}$$

for all non-negative decreasing functions γ on $(0,\varepsilon]$.
(ii) *Let $0<\varepsilon<1$ and h_1 and h_2 be functions as the h above and satisfying $h_1 \sim h_2$ in $(0,\varepsilon]$. Let $H_i := -\log h_i$ and μ_{H_i} be the associated Borel measure in $(0,\varepsilon]$, $i=1,2$, as before. Let $0<u\le\infty$. Then*

$$\left(\int_{(0,\varepsilon]} \left(\frac{\gamma(t)}{h_1(t)}\right)^{u} \mu_{H_1}(dt)\right)^{1/u} \sim \left(\int_{(0,\varepsilon]} \left(\frac{\gamma(t)}{h_2(t)}\right)^{u} \mu_{H_2}(dt)\right)^{1/u}$$

(with the sup-norm if $u=\infty$) for all non-negative decreasing functions γ on $(0,\varepsilon]$, where the equivalence constants are independent of γ.

Remark 3.18 Note that this proposition makes clear that an expression like

$$\left(\int_{(0,\varepsilon]} \left(\frac{\gamma(t)}{h(t)}\right)^{u} \mu_H(dt)\right)^{1/u} \tag{38}$$

must be interpreted as $\sup_{t\in(0,\varepsilon]}(\gamma(t)/h(t))$ when $u=\infty$.

In the important case when H happens to be continuously differentiable in $(0,\varepsilon]$, we have $\mu_H(dt) = H'\,dt$, and for the functions we want to integrate in (38) we can calculate the integral as the improper Riemann integral

$$\int_0^\varepsilon \left(\frac{\gamma(t)}{h(t)}\right)^u H'(t)\,dt.$$

Definition 3.19 *Let s, p, q, Ψ be according to* (37) *and $0 < u \le \infty$. Then*

$$\mathfrak{E}_{\mathrm{LG}} A_{pq}^{(s,\Psi)} := \big([\mathcal{E}_{\mathrm{LG}}|A_{pq}^{(s,\Psi)}], u\big)$$

is called the local growth envelope of $A_{pq}^{(s,\Psi)}$ if u is the minimum (assuming that it exists) of all $v > 0$ such that

$$\exists\, c(v) > 0 : \forall\, f \in A_{pq}^{(s,\Psi)},$$
$$\left(\int_{(0,\varepsilon]} \left(\frac{f^*(t)}{h(t)}\right)^v \mu_H(dt)\right)^{1/v} \le c(v)\,\|f|A_{pq}^{(s,\Psi)}\|, \qquad (39)$$

where $h(t)$ is a continuous representative in $[\mathcal{E}_{\mathrm{LG}}|A_{pq}^{(s,\Psi)}]$ with domain $(0,\varepsilon]$, $0 < \varepsilon < 1$.

We must remark that this definition makes sense, namely that the infimum of all such v's is independent of the chosen continuous representative $h(t)$ in $[\mathcal{E}_{\mathrm{LG}}|A_{pq}^{(s,\Psi)}]$, as follows by using some standard arguments of measure and integration theory, the definition of $\mathcal{E}_{\mathrm{LG}}|A_{pq}^{(s,\Psi)}$ and Proposition 3.17(ii). Recall, on the other hand, that we are assuming that there exists at least one such representative – and we have already mentioned that this is indeed the case, as will be apparent later. Recall also that the definition of $\mathcal{E}_{\mathrm{LG}}|A_{pq}^{(s,\Psi)}$ guarantees that (39) holds at least for $v = \infty$. Remark also that the definition does not discard the possibility that there is no such thing called the local growth envelope of $A_{pq}^{(s,\Psi)}$: this would be the case if the infimum of the mentioned v's were not a minimum. We shall, however, see (in Section 4) that the minimum is really attained, and therefore all mentioned spaces have local growth envelopes.

Instead of $\big([\mathcal{E}_{\mathrm{LG}}|A_{pq}^{(s,\Psi)}], u\big)$, we shall usually write $\big(h(t), u\big)$ for the local growth envelope of $A_{pq}^{(s,\Psi)}$ with (37), where $h(t)$ is any continuous representative in $[\mathcal{E}_{\mathrm{LG}}|A_{pq}^{(s,\Psi)}]$. Instead of $h(t)$, we can also use in the couple any local growth envelope function as considered in Definition 3.15, though it must be borne in mind that for the construction of the measure μ_H we shall only use continuous representatives in $[\mathcal{E}_{\mathrm{LG}}|A_{pq}^{(s,\Psi)}]$.

4 Local growth envelopes for $B_{pq}^{(s,\Psi)}$ and $F_{pq}^{(s,\Psi)}$

We start by getting the upper estimates needed in order to determine the growth envelopes and, as we shall see, in this part we deal both with the critical and the sub-critical case, though the latter was already studied in [2]. The reason is that the technique used here is completely different from the technique of interpolation with a function parameter used in [2] for the corresponding upper estimates. Here we use a more direct approach, as Haroske and Triebel did for the critical case in the classical setting, but we show that the same approach can also be used in the sub-critical case. Though not explicitly mentioned in the assertion that follows, when convenient we assume in its proof that the admissible function Ψ satisfies the condition $\Psi(1) = 1$. There is no loss of generality in doing this.

Recall that the notation σ_p stands for $n(\frac{1}{p} - 1)_+$.

Proposition 4.1 *Let $0 < p, q \leq \infty$ and $s \in \mathbb{R}$ be such that $\sigma_p < s \leq n/p$ and Ψ be a continuous admissible function. Define $r \in (1, \infty]$ by the equation $s - n/p = -n/r$ and let $\Phi_{r,q'}$ be as in Definition 2.4 (now with q' in the place of u). In the case $s = n/p$ assume further that $(\Psi(2^{-j})^{-1})_{j\in\mathbb{N}} \notin \ell_{q'}$. Then there exists $\varepsilon \in (0, 1)$ and $c > 0$ such that*

$$\mathcal{E}_{\mathrm{LG}}|B_{pq}^{(s,\Psi)}(t) \leq c\, \Phi_{r,q'}(t)\,, \qquad \forall t \in (0, \varepsilon], \tag{40}$$

and, for each $v \in [q, \infty]$, there exists $c(v) > 0$ such that

$$\left(\int_0^{\varepsilon} \left(\frac{f^*(t)}{\Phi_{r,q'}(t)} \right)^v \mu_{r,q'}(dt) \right)^{1/v} \leq c(v) \|f|B_{pq}^{(s,\Psi)}\|\,, \qquad \forall f \in B_{pq}^{(s,\Psi)} \tag{41}$$

(with the modification (49) if $v = \infty$), where $\mu_{r,q'}$ denotes the Borel measure associated with $-\log \Phi_{r,q'}$ in $(0, \varepsilon]$ (in accordance with subsection 3.4).

Proof. First note that the hypotheses imply that $p \neq \infty$ and $B_{pq}^{(s,\Psi)} \subset L_1^{\mathrm{loc}}$.

Fix $d > 1/2$ as in Proposition 3.8 and consider the corresponding constants C and D. Define $\varepsilon := C 2^{-k_0 n} \leq 2^{-n}$, for a suitable chosen $k_0 \in \mathbb{N}$.

Step 1. First we assume $p > 1$ and $q = \infty$ and prove (40) and the modified version (45) of (41) (which is the correct interpretation of the latter in the case $v = \infty$).

Given $f \in B_{p\infty}^{(s,\Psi)}$ consider a corresponding atomic decomposition $\sum_{j=0}^{\infty} f_j$ (convergence in $\mathcal{S}'$), where f_j have the same meaning as in (20), for given atoms a_{jm} in $B_{p\infty}^{(s,\Psi)}$ and complex numbers λ_{jm} satisfying $(\lambda_{jm})_{j\in\mathbb{N}_0, m\in\mathbb{Z}^n} \in b_{p\infty}$. Together with our hypotheses, this guarantees that Proposition 3.8 can be applied with $d_j := 2^{-j(s-\frac{n}{p})}\Psi(2^{-j})^{-1}$, $j \in \mathbb{N}_0$. In particular, f is also the limit, in L_p, of the series $\sum_{j=0}^{\infty} f_j$. This justifies de following inequalities:

$$\begin{aligned} \sup_{0<t\le\varepsilon} \frac{f^*(t)}{\Phi_{r,1}(t)} &= \sup_{k\ge k_0} \sup_{C2^{-(k+1)n}<t\le C2^{-kn}} \frac{f^*(t)}{\Phi_{r,1}(t)} \\ &\le \sup_{k\ge k_0} \frac{f^{**}(C2^{-(k+1)n})}{\Phi_{r,1}(C2^{-kn})} \\ &\le \sup_{k\ge k_0} \left(\frac{\sum_{j=0}^{k} f_j^{**}(C2^{-(k+1)n})}{\Phi_{r,1}(C2^{-kn})} + \frac{\sum_{j=k+1}^{\infty} f_j^{**}(C2^{-(k+1)n})}{\Phi_{r,1}(C2^{-kn})} \right). \end{aligned} \tag{42}$$

Since, for $0 \le j \le k$, $C2^{-(k+1)n} \in (0, C2^{-jn})$, then (21), Proposition 2.7, Proposition 3.7 and the admissibility of Ψ allow us to write

$$\begin{aligned} \frac{\sum_{j=0}^{k} f_j^{**}(C2^{-(k+1)n})}{\Phi_{r,1}(C2^{-kn})} &\le c_1 \frac{\sum_{j=0}^{k} 2^{j\frac{n}{r}} \Psi(2^{-j})^{-1} \lambda_{j1}^*}{\sum_{j=0}^{k} 2^{j\frac{n}{r}} \Psi(2^{-j})^{-1}} \\ &\le c_1 \sup_{j=0,\dots,k} \lambda_{j1}^* \\ &\le c_1 \sup_{j\in\mathbb{N}_0} \|(\lambda_{jl}^*)_{l\in\mathbb{N}} | \ell_p\| \\ &= c_1 \|(\lambda_{jm})_{j\in\mathbb{N}_0, m\in\mathbb{Z}^n} | b_{p\infty}\|, \end{aligned} \tag{43}$$

where $c_1 > 0$ depends only on n, d, s, p and Ψ.

Since, for $j \ge k+1$, $C2^{-(k+1)n} \in [C2^{-jn}2^{(j-k-1)n}, C2^{-jn}(2^{(j-k-1)n}+1))$ and $\lambda_{j,2^{(j-k-1)n}+1}^{**} \le 2^{2n/p}(2^n-1)^{-1/p} 2^{-(j-k)\frac{n}{p}} \left(\sum_{l=1}^{\infty} \lambda_{jl}^{**p} \right)^{1/p}$, then (21), Proposition 2.7, Proposition 3.7 and the admissibility of Ψ allow us to write

$$\begin{aligned} &\frac{\sum_{j=k+1}^{\infty} f_j^{**}(C2^{-(k+1)n})}{\Phi_{r,1}(C2^{-kn})} \\ &\le c_1 \frac{\sum_{j=k+1}^{\infty} 2^{j\frac{n}{r}} \Psi(2^{-j})^{-1} \lambda_{j,2^{(j-k-1)n}+1}^{**}}{\sum_{j=0}^{k} 2^{j\frac{n}{r}} \Psi(2^{-j})^{-1}} \\ &\le c_2 \frac{\sum_{j=k+1}^{\infty} 2^{k\frac{n}{r}} \Psi(2^{-j})^{-1} 2^{-(j-k)s} \|(\lambda_{jl}^{**})_{l\in\mathbb{N}} | \ell_p\|}{\sum_{j=0}^{k} 2^{j\frac{n}{r}} \Psi(2^{-j})^{-1}} \\ &\le c_3 \left(\sum_{j=k+1}^{\infty} 2^{-(j-k)s} \frac{\Psi(2^{-j})^{-1}}{\Psi(2^{-k})^{-1}} \right) \sup_{j\ge k+1} \|(\lambda_{jm})_{m\in\mathbb{Z}^n} | \ell_p\| \\ &\le c_4 \left(\sum_{j=1}^{\infty} 2^{-js}(1+j)^b \right) \|(\lambda_{jm})_{j\in\mathbb{N}_0, m\in\mathbb{Z}^n} | b_{p\infty}\|, \end{aligned} \tag{44}$$

where $c_2, c_3, c_4 > 0$ depend only on n, d, s, p and Ψ and $b \ge 0$ is determined by Ψ.

Putting (42), (43) and (44) together, we get

$$\sup_{0<t\le\varepsilon} \frac{f^*(t)}{\Phi_{r,1}(t)} \le c_5 \|(\lambda_{jm})_{j\in\mathbb{N}_0, m\in\mathbb{Z}^n}|b_{p\infty}\| < \infty\,,$$

for some $c_5 > 0$ depending only on n, d, s, p and Ψ, and, with the help of Theorem 3.6,

$$\sup_{0<t\le\varepsilon} \frac{f^*(t)}{\Phi_{r,1}(t)} \le c_5 \|f|B^{(s,\Psi)}_{p\infty}\|\,. \tag{45}$$

From this it easily follows, in the case $p>1$ and $q=\infty$, that (40) holds and, in particular, that $\mathcal{E}_{\mathrm{LG}}|B^{(s,\Psi)}_{p\infty}(t)$ is finite for each $t\in(0,\varepsilon]$.

Step 2. Now we prove (40) and (41) in the case $p>1$ and $1<q<\infty$.

We start with the proof of (41) when $v=q$.

Given $f\in B^{(s,\Psi)}_{pq}$ consider a corresponding atomic decomposition $\sum_{j=0}^\infty f_j$ (convergence in $\mathcal{S}'$), where f_j have the same meaning as in (20), for given atoms a_{jm} in $B^{(s,\Psi)}_{pq}$ and complex numbers λ_{jm} satisfying $(\lambda_{jm})_{j\in\mathbb{N}_0,m\in\mathbb{Z}^n}\in b_{pq}$. Together with our hypotheses, this guarantees that Proposition 3.8 can be applied with $d_j := 2^{-j(s-\frac{n}{p})}\Psi(2^{-j})^{-1}$, $j\in\mathbb{N}_0$. In particular, f is also the limit, in L_p, of the series $\sum_{j=0}^\infty f_j$. This together with Proposition 2.5, the discussion in subsection 3.4 and the admissibility of Ψ allow us to write that

$$\begin{aligned}
&\left(\int_0^\varepsilon \left(\frac{f^*(t)}{\Phi_{r,q'}(t)}\right)^q \mu_{r,q'}(dt)\right)^{1/q}\\
&= \left(\frac{1}{q'n}\right)^{1/q}\left(\sum_{k=k_0}^\infty \int_{C2^{-(k+1)n}}^{C2^{-kn}} \left(\frac{f^*(t)}{\Phi_{r,q'}(t)^{q'}}\right)^q t^{-\frac{q'}{r}-1}\Psi(t^{1/n})^{-q'}\,dt\right)^{1/q}\\
&\le c_6\left(\sum_{k=k_0}^\infty \left(\frac{f^{**}(C2^{-(k+1)n})}{\Phi_{r,q'}(C2^{-kn})^{q'}}\right)^q 2^{k\frac{n}{r}q'}\Psi(2^{-k})^{-q'}\right)^{1/q} \qquad (46)\\
&\le c_6\left(\sum_{k=k_0}^\infty 2^{k\frac{n}{r}q'}\Psi(2^{-k})^{-q'}\left(\frac{\sum_{j=0}^k f_j^{**}(C2^{-(k+1)n})}{\Phi_{r,q'}(C2^{-kn})^{q'}}\right)^q\right)^{1/q}\\
&\quad + c_6\left(\sum_{k=k_0}^\infty 2^{k\frac{n}{r}q'}\Psi(2^{-k})^{-q'}\left(\frac{\sum_{j=k+1}^\infty f_j^{**}(C2^{-(k+1)n})}{\Phi_{r,q'}(C2^{-kn})^{q'}}\right)^q\right)^{1/q}.
\end{aligned}$$

Since, for $0\le j\le k$, $C2^{-(k+1)n}\in(0,C2^{-jn})$, then (21), Proposition 2.7, Proposition 3.7 and the admissibility of Ψ allow us to write

$$\begin{aligned}
&\left(\sum_{k=k_0}^{\infty} 2^{k\frac{n}{r}q'}\Psi(2^{-k})^{-q'}\left(\frac{\sum_{j=0}^{k} f_j^{**}(C2^{-(k+1)n})}{\Phi_{r,q'}(C2^{-kn})^{q'}}\right)^{q}\right)^{1/q}\\
&\leq c_7\left(\sum_{k=k_0}^{\infty} 2^{k\frac{n}{r}q'}\Psi(2^{-k})^{-q'}\left(\frac{\sum_{j=0}^{k} 2^{j\frac{n}{r}}\Psi(2^{-j})^{-1}\lambda_{j1}^{*}}{\sum_{j=0}^{k} 2^{j\frac{n}{r}q'}\Psi(2^{-j})^{-q'}}\right)^{q}\right)^{1/q}\\
&\leq c_8\left(\sum_{k=0}^{\infty}\lambda_{k1}^{*q}\right)^{1/q} \qquad (47)\\
&\leq c_8\left(\sum_{k=0}^{\infty}\|(\lambda_{kl}^{*})_{l\in\mathbb{N}}|\ell_p\|^{q}\right)^{1/q}\\
&= c_8\|(\lambda_{jm})_{j\in\mathbb{N}_0,m\in\mathbb{Z}^n}|b_{pq}\|\,,
\end{aligned}$$

where in the second inequality we have used a generalization of Hardy's inequality (cf. [6, p. 247]).

Since, for $j\geq k+1$, $C2^{-(k+1)n}\in[C2^{-jn}2^{(j-k-1)n}, C2^{-jn}(2^{(j-k-1)n}+1))$ and $\lambda_{j,2^{(j-k-1)n}+1}^{**}\leq 2^{2n/p}(2^n-1)^{-1/p}2^{-(j-k)\frac{n}{p}}\left(\sum_{l=1}^{\infty}\lambda_{jl}^{**p}\right)^{1/p}$, then (21), Proposition 2.7, Proposition 3.7 and the admissibility of Ψ allow us to write

$$\begin{aligned}
&\left(\sum_{k=k_0}^{\infty} 2^{k\frac{n}{r}q'}\Psi(2^{-k})^{-q'}\left(\frac{\sum_{j=k+1}^{\infty} f_j^{**}(C2^{-(k+1)n})}{\Phi_{r,q'}(C2^{-kn})^{q'}}\right)^{q}\right)^{1/q}\\
&\leq c_7\left(\sum_{k=k_0}^{\infty} 2^{k\frac{n}{r}q'}\Psi(2^{-k})^{-q'}\left(\frac{\sum_{j=k+1}^{\infty} 2^{j\frac{n}{r}}\Psi(2^{-j})^{-1}\lambda_{j,2^{(j-k-1)n}+1}^{**}}{\sum_{j=0}^{k} 2^{j\frac{n}{r}q'}\Psi(2^{-j})^{-q'}}\right)^{q}\right)^{1/q}\\
&\leq c_9\left(\sum_{k=k_0}^{\infty} 2^{k\frac{n}{r}q'}\Psi(2^{-k})^{-q'}\left(\frac{\sum_{j=k+1}^{\infty} 2^{k\frac{n}{r}}\Psi(2^{-j})^{-1}2^{-(j-k)s}\|(\lambda_{jl}^{**})_{l\in\mathbb{N}}|\ell_p\|}{\sum_{j=0}^{k} 2^{j\frac{n}{r}q'}\Psi(2^{-j})^{-q'}}\right)^{q}\right)^{1/q}\\
&\leq c_{10}\left(\sum_{k=k_0}^{\infty}\left(\sum_{j=k+1}^{\infty} 2^{-(j-k)s}\frac{\Psi(2^{-j})^{-1}}{\Psi(2^{-k})^{-1}}\|(\lambda_{jm})_{m\in\mathbb{Z}^n}|\ell_p\|\right)^{q}\right)^{1/q}\\
&\leq c_{11}\left(\sum_{k=k_0}^{\infty}\left(\sum_{l=1}^{\infty} 2^{-ls}(1+l)^{b}\|(\lambda_{k+l,m})_{m\in\mathbb{Z}^n}|\ell_p\|\right)^{q}\right)^{1/q} \qquad (48)\\
&\leq c_{11}\left(\sum_{l=1}^{\infty} 2^{-ls}(1+l)^{b}\right)\|(\lambda_{jm})_{j\in\mathbb{N}_0,m\in\mathbb{Z}^n}|b_{pq}\|\,,
\end{aligned}$$

where in the last part we have used a generalized Minkowski inequality.

As in Step 1, $b \geq 0$ is determined by Ψ, and the positive constants c_6 to c_{11} depend only on n, d, s, p, q and Ψ.

Putting (46), (47) and (48) together, we get

$$\left(\int_0^\varepsilon \left(\frac{f^*(t)}{\Phi_{r,q'}(t)}\right)^q \mu_{r,q'}(dt)\right)^{1/q} \leq c_{12}\|(\lambda_{jm})_{j\in\mathbb{N}_0,m\in\mathbb{Z}^n}|b_{pq}\|\,,$$

for some $c_{12} > 0$ depending only on n, d, s, p, q and Ψ, and, with the help of Theorem 3.6, the case $1 < v = q < \infty$ (and with $p > 1$) of (41) follows easily. To prove (41) for any $v \geq q$ (even for $v = \infty$, in which case it should be interpreted as

$$\sup_{0<t\leq\varepsilon} \frac{f^*(t)}{\Phi_{r,q'}(t)} \leq c(\infty)\|f|B^{(s,\Psi)}_{pq}\|\,, \quad \forall f \in B^{(s,\Psi)}_{pq}\,), \tag{49}$$

one just has to use Proposition 3.17.

From (49) it also easily follows, still in the case $p > 1$ and $1 < q < \infty$, that (40) holds and, in particular, that $\mathcal{E}_{\mathrm{LG}}|B^{(s,\Psi)}_{pq}(t)$ is finite for each $t \in (0, \varepsilon]$.

Step 3. Still with $p > 1$, we deal now with the proof of (40) and (41) for $0 < q \leq 1$.

Again, we start by proving (41) when $v = q$.

The sub-critical case $s < n/p$ can be dealt with the same type of discretization of the integral as in Step 2, taking advantage of the rough estimate $\mu_{r,q'}[C2^{-(k+1)n}, C2^{-kn}] \leq$ constant. Since the result for this case is already known (cf. [2]) and we would run into problems if we applied the same type of discretization when dealing with the critical case $s = n/p$, we shall omit the details for the case $s < n/p$ and deal now only with what is our main concern in this paper, namely the case $s = n/p$.

We recall that when considering $s = n/p$ we are also assuming that $(\Psi(2^{-j})^{-1})_{j\in\mathbb{N}} \notin \ell_{q'}$. Since we are now dealing only with $0 < q \leq 1$, this means that $(\Psi(2^{-j})^{-1})_{j\in\mathbb{N}}$ is unbounded, that is, $\lim_{t\to 0+}\Psi(t) = 0$. Recall also that in this case Ψ must be increasing (and that, in any case, Ψ is positive in (0,1]).

As a consequence, we can build a sequence $(\alpha_k)_{k\in\mathbb{N}_0}$ in the following way: $\alpha_0 = C2^{-t_0 n}$, where $t_0 = k_0$; for every $k \in \mathbb{N}$, $\alpha_k = C2^{-t_k n}$, where $t_k \geq t_{k-1} + 1$ is such that $c' \leq \Psi(C2^{-t_k n})\Psi(C2^{-t_{k-1} n})^{-1} \leq 1/2$, for some positive constant c' depending only on n and Ψ.

We then start as in Step 2: given $f \in B^{(s,\Psi)}_{pq}$ consider a corresponding atomic decomposition $\sum_{j=0}^\infty f_j$ (convergence in $\mathcal{S}'$), where f_j have the same meaning as in (20), for given atoms a_{jm} in $B^{(s,\Psi)}_{pq}$ and complex numbers λ_{jm} satisfying $(\lambda_{jm})_{j\in\mathbb{N}_0,m\in\mathbb{Z}^n} \in b_{pq}$. Together with our hypotheses, this guarantees that Proposition 3.8 can be applied with $d_j := 2^{-j(s-\frac{n}{p})}\Psi(2^{-j})^{-1}$, $j \in \mathbb{N}_0$. In particular, f is also the limit, in L_p, of the series $\sum_{j=0}^\infty f_j$.

We can then write, recalling also the definitions of $\Phi_{r,u}$ and $\mu_{r,u}$ (now for $r, u = \infty$), the discussion in subsection 3.4 and the admissibility of Ψ,

$$\begin{aligned}
&\left(\int_0^\varepsilon \left(\frac{f^*(t)}{\Phi_{\infty,\infty}(t)}\right)^q \mu_{\infty,\infty}(dt)\right)^{1/q} \\
&\leq \; c_{13}\left(\sum_{k=0}^{\infty}\int_{\alpha_{k+1}}^{\alpha_k}\left(\frac{f^*(t)}{\Psi(t)^{-1}}\right)^q \mu_{\infty,\infty}(dt)\right)^{1/q} \\
&\leq \; c_{13}\left(\sum_{k=0}^{\infty}\left(\frac{f^{**}(\alpha_{k+1})}{\Psi(\alpha_k)^{-1}}\right)^q \mu_{\infty,\infty}([\alpha_{k+1},\alpha_k])\right)^{1/q} \qquad (50) \\
&\leq \; c_{14}\left(\sum_{k=0}^{\infty}\Psi(\alpha_k)^q \sum_{j=0}^{j<t_{k+1}} f_j^{**}(\alpha_{k+1})^q + \sum_{k=0}^{\infty}\Psi(\alpha_k)^q \sum_{j\geq t_{k+1}}^{\infty} f_j^{**}(\alpha_{k+1})^q\right)^{1/q},
\end{aligned}$$

where, for example, $\sum_{j=0}^{j<t_{k+1}}$ means that the sum is made on integers j from 0 to the nearest integer less than t_{k+1}.

Since, for $0 \leq j < t_{k+1}$, $\alpha_{k+1} \in (0, C2^{-jn})$, then (21) and the admissibility of Ψ allow us to write (with the understanding that $t_l := 0$ when the index l is a negative number)

$$\begin{aligned}
\sum_{k=0}^{\infty}\Psi(\alpha_k)^q \sum_{j=0}^{j<t_{k+1}} f_j^{**}(\alpha_{k+1})^q \;&\leq\; D^q \sum_{k=0}^{\infty}\Psi(\alpha_k)^q \sum_{h=0}^{k+1}\sum_{j\geq t_{h-1}}^{j<t_h}\Psi(2^{-j})^{-q}\lambda_{j1}^{*q} \\
&\leq\; c_{15}\sum_{k=0}^{\infty}\sum_{h=0}^{k+1}\left(\frac{\Psi(\alpha_k)}{\Psi(\alpha_h)}\right)^q \sum_{j\geq t_{h-1}}^{j<t_h}\lambda_{j1}^{*q} \\
&\leq\; c_{16}\sum_{k=0}^{\infty}\sum_{h=0}^{k+1}2^{-(k-h)q}\sum_{j\geq t_{h-1}}^{j<t_h}\lambda_{j1}^{*q} \\
&\leq\; c_{16}\sum_{k=0}^{\infty}\sum_{l=-1}^{\infty}2^{-lq}\sum_{j\geq t_{k-l-1}}^{j<t_{k-l}}\lambda_{j1}^{*q} \\
&=\; c_{16}\sum_{l=-1}^{\infty}2^{-lq}\sum_{k=0}^{\infty}\sum_{j\geq t_{k-l-1}}^{j<t_{k-l}}\lambda_{j1}^{*q} \\
&\leq\; c_{16}\left(\sum_{l=-1}^{\infty}2^{-lq}\right)\left(\sum_{j=0}^{\infty}\lambda_{j1}^{*q}\right). \qquad (51)
\end{aligned}$$

Since, for $j \geq [t_{k+1}]$, $\alpha_{k+1} \in [C2^{-jn}(l_{kj}-1), C2^{-jn}l_{kj})$, where l_{kj} is the only natural number satisfying the inequalities $l_{kj} - 1 \leq 2^{(j-t_{k+1})n} < l_{kj}$, and $\lambda_{j,l_{kj}}^{**} \leq c_{17}2^{-(j-[t_{k+1}])\frac{n}{p}}\left(\sum_{l=1}^{\infty}\lambda_{jl}^{**p}\right)^{1/p}$, we have, again with the help of

(21), Proposition 3.7 and the admissibility of Ψ,

$$
\begin{aligned}
&\sum_{k=0}^{\infty}\Psi(\alpha_k)^q \sum_{j\geq t_{k+1}}^{\infty} f_j^{**}(\alpha_{k+1})^q \\
&\quad\leq\; D^q\sum_{k=0}^{\infty}\Psi(\alpha_k)^q \sum_{j=[t_{k+1}]}^{\infty}\Psi(2^{-j})^{-q}\lambda_{j,l_{kj}}^{**q} \\
&\quad\leq\; c_{18}\sum_{k=0}^{\infty}\sum_{j=[t_{k+1}]}^{\infty}\Psi(\alpha_k)^q\Psi(2^{-j})^{-q}2^{-(j-[t_{k+1}])sq}\|(\lambda_{jm})_{m\in\mathbb{Z}^n}|\ell_p\|^q \\
&\quad=\; c_{18}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\Psi(2^{-[t_{k+1}]-l})^{-q}}{\Psi(C2^{-t_kn})^{-q}}2^{-lsq}\|(\lambda_{[t_{k+1}]+l,m})_{m\in\mathbb{Z}^n}|\ell_p\|^q \\
&\quad\leq\; c_{19}\sum_{l=0}^{\infty}2^{-lsq}(1+l)^{bq}\sum_{k=0}^{\infty}\|(\lambda_{[t_{k+1}]+l,m})_{m\in\mathbb{Z}^n}|\ell_p\|^q \\
&\quad\leq\; c_{19}\left(\sum_{l=0}^{\infty}2^{-lsq}(1+l)^{bq}\right)\|(\lambda_{jm})_{j\in\mathbb{N}_0,m\in\mathbb{Z}^n}|b_{pq}\|^q\,. \qquad (52)
\end{aligned}
$$

As in the previous Steps, $b\geq 0$ is determined by Ψ, and the positive constants c_{13} to c_{19} depend only on n,d,s,p,q and Ψ.

Putting (50), (51) and (52) together, we get

$$\left(\int_0^{\varepsilon}\left(\frac{f^*(t)}{\Phi_{\infty,\infty}(t)}\right)^q\mu_{\infty,\infty}(dt)\right)^{1/q}\leq c_{20}\|(\lambda_{jm})_{j\in\mathbb{N}_0,m\in\mathbb{Z}^n}|b_{pq}\|\,,$$

for some $c_{20}>0$ depending only on n,d,s,p,q and Ψ, and, with the help of Theorem 3.6, the case $0<v=q\leq 1$, $s=n/p$ (and with $p>1$) of (41) follows easily. As mentioned before, the situation when $s<n/p$ can be dealt with in a similar – though easier – way.

To prove (41) for any $v\geq q$ (even for $v=\infty$, with the interpretation (49)) one just has to use Proposition 3.17.

From (49) it also easily follows, still in the case $p>1$ and $0<q\leq 1$, that (40) holds and, in particular, that $\mathcal{E}_{\mathrm{LG}}|B_{pq}^{(s,\Psi)}(t)$ is finite for each $t\in(0,\varepsilon]$.

Step 4. We extend now the validity of (40) and (41) to $0<p\leq 1$.

Note that, given $0<p\leq 1$ and $\sigma_p<s\leq n/p$, we actually have $s>n(\frac{1}{p}-1)$, that is, $s-\frac{n}{p}>-n$, so that there are $\gamma,\delta>0$ such that $s-\frac{n}{p}=\delta-\frac{n}{1+\gamma}$, and therefore $B_{pq}^{(s,\Psi)}\hookrightarrow B_{1+\gamma,q}^{(\delta,\Psi)}$ (cf. [10, Prop. 1.9(iv)]). As we have already proved (41) for the space on the right-hand side, we can thus write

$$\left(\int_0^{\varepsilon}\left(\frac{f^*(t)}{\Phi_{r,q'}(t)}\right)^v\mu_{r,q'}(dt)\right)^{1/v}\leq c(v)\|f|B_{1+\gamma,q}^{(\delta,\Psi)}\|\leq c_{21}\|f|B_{pq}^{(s,\Psi)}\|,\quad f\in B_{pq}^{(s,\Psi)},$$

with modification if $v=\infty$. Actually, it is from this modification that, as usual, (40) is also obtained for the extended range of the parameter p, not to mention the finiteness of $\mathcal{E}_{\mathrm{LG}}|B_{pq}^{(s,\Psi)}(t)$ for each $t\in(0,\varepsilon]$. □

In the next result we consider only the critical case, as the technique of proof is essentially the same as the one used in [2] for the sub-critical case.

Proposition 4.2 *Let* $0 < p < \infty$, $0 < q \leq \infty$ *and* Ψ *be a continuous admissible function. Then there exists* $c > 0$ *such that*

$$\mathcal{E}_{\mathrm{LG}}|A^{(n/p,\Psi)}_{pq}(t) \geq c\,\Phi_{\infty,u'}(t), \quad t \in (0, 2^{-n}], \tag{53}$$

where

$$u = \begin{cases} q & \text{if} \quad A = B, \\ p & \text{if} \quad A = F. \end{cases}$$

Proof. *Step 1.* In this step we deal with the case $A = B$. Suppose first that $1 < q \leq \infty$. For each $J \in \mathbb{N}$ we denote by g_J the function g in (32) with $b = (b_j)_{j\in\mathbb{N}}$ being the sequence defined by

$$b_j := \begin{cases} \Psi(2^{-j})^{1-q'} \Big(\sum_{k=1}^{J} \Psi(2^{-k})^{-q'}\Big)^{-1/q} & \text{for} \quad j = 1, \cdots, J, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $b_j \geq c\, b_{j+1}$, $j \in \mathbb{N}$, for some positive constant c, as Ψ is an admissible function. Moreover, $\|b \mid \ell_q\| = 1$. By (35),

$$g^*_J(2^{-Jn}) \geq c \sum_{j=1}^{J+1} b_j\, \Psi(2^{-j})^{-1} \geq c \Big(\sum_{j=1}^{J} \Psi(2^{-j})^{-q'}\Big)^{1/q'}, \qquad J \in \mathbb{N}.$$

Therefore, having into consideration (33) and the property $(\lambda f)^* = |\lambda|\, f^*$, we obtain

$$\mathcal{E}_{\mathrm{LG}}|B^{(n/p,\Psi)}_{pq}(2^{-Jn}) \geq c_1\, g^*_J(2^{-Jn}) \geq c_2 \Big(\sum_{j=1}^{J} \Psi(2^{-j})^{-q'}\Big)^{1/q'}, \qquad J \in \mathbb{N}, \tag{54}$$

where the constants are independent of J. Now let $2^{-(J+1)n} \leq t \leq 2^{-Jn}$, for some $J \in \mathbb{N}$. In virtue of (54), using the monotonicity of $\mathcal{E}_{\mathrm{LG}}|B^{(n/p,\Psi)}_{pq}$, the admissibility of Ψ and Proposition 2.7, we get

$$\begin{aligned} \mathcal{E}_{\mathrm{LG}}|B^{(n/p,\Psi)}_{pq}(t) &\geq \mathcal{E}_{\mathrm{LG}}|B^{(n/p,\Psi)}_{pq}(2^{-Jn}) \geq c_2 \Big(\sum_{j=1}^{J} \Psi(2^{-j})^{-q'}\Big)^{1/q'} \\ &\geq c_3 \Big(\sum_{j=1}^{J+1} \Psi(2^{-j})^{-q'}\Big)^{1/q'} \geq c_3 \Big(\sum_{j=1}^{[|\log t|/n]} \Psi(2^{-j})^{-q'}\Big)^{1/q'} \\ &\geq c_4\, \Phi_{\infty,q'}(t), \end{aligned}$$

where the constants involved do not depend on J, and the proof of (53) for $A = B$ and $q > 1$ is then complete.

Now let $0 < q \leq 1$. For each $j \in \mathbb{N}$ let a_j be as in (29) and let δ be a positive number so that $\varphi^*(\delta) > 0$. Then the functions

$$A_j(x) := a_j(\delta^{1/n}x), \quad x \in \mathbb{R}^n,\ j \in \mathbb{N},$$

are (up to constants, independently of j) also $(n/p, p, \Psi)_{K,-1}$-atoms for some fixed $K \in \mathbb{N}$ with $K > n/p$. In particular, it holds

$$\|A_j \mid B_{pq}^{(n/p,\Psi)}\| \sim 1, \quad j \in \mathbb{N}.$$

For fixed $J \in \mathbb{N}$ and $j \in \{1, \cdots, J+1\}$, we have

$$\begin{aligned} A_j^*(2^{-Jn}) = a_j^*(\delta\, 2^{-Jn}) &= \inf\{\lambda \geq 0 : m_{a_j}(\lambda) \leq \delta\, 2^{-Jn}\} \\ &= \inf\{\lambda \geq 0 : 2^{-(j-1)n} m_\varphi(\lambda\, \Psi(2^{-j})) \leq \delta\, 2^{-Jn}\} \\ &= \Psi(2^{-j})^{-1} \inf\{\lambda \geq 0 : m_\varphi(\lambda) \leq \delta\, 2^{(j-J-1)n}\} \\ &\geq \Psi(2^{-j})^{-1}\, \varphi^*(\delta), \end{aligned}$$

leading to

$$\begin{aligned} \mathcal{E}_{\mathrm{LG}}|B_{pq}^{(n/p,\Psi)}(2^{-Jn}) &\geq c_1 \sup\{A_j^*(2^{-Jn}) : j = 1, \cdots, J\} \\ &\geq c_2 \sup_{j=1,\cdots,J+1} \Psi(2^{-j})^{-1}, \qquad J \in \mathbb{N}. \end{aligned}$$

Now let $2^{-(J+1)n} \leq t \leq 2^{-Jn}$, for some $J \in \mathbb{N}$. Using the monotonicity of $\mathcal{E}_{\mathrm{LG}}|B_{pq}^{(n/p,\Psi)}$ and Proposition 2.7, we get

$$\begin{aligned} \mathcal{E}_{\mathrm{LG}}|B_{pq}^{(n/p,\Psi)}(t) \geq \mathcal{E}_{\mathrm{LG}}|B_{pq}^{(n/p,\Psi)}(2^{-Jn}) &\geq c \sup_{j=1,\cdots,J+1} \Psi(2^{-j})^{-1} \\ &\geq c \sup_{j=1,\cdots,[|\log t|/n]} \Psi(2^{-j})^{-1} \geq c'\, \Phi_{\infty,\infty}(t), \end{aligned}$$

where the constants involved do not depend on J, concluding the proof for the B-spaces.

Step 2. Notice that

$$B_{rp}^{(n/r,\Psi)} \hookrightarrow F_{pq}^{(n/p,\Psi)} \quad \text{for} \quad 0 < r < p < \infty. \tag{55}$$

We refer to Example 3.5 of [2]. Then the assertion (53) for the F-spaces follows from the corresponding assertion for the B-spaces, proved in Step 1.
□

Remark 4.3 The necessity of the conditions in Proposition 3.13 can be inferred from Proposition 4.2. In view of Proposition 2.7, this is immediately the case if $p < \infty$, since the unboundedness of $\mathcal{E}_{\mathrm{LG}}|A_{pq}^{(n/p,\Psi)}$ implies

$A^{(n/p,\Psi)}_{pq} \not\hookrightarrow L_\infty$. In the case of $p=\infty$ (hence for the B-spaces) the conclusion follows from the previous one due to the embedding

$$B^{(n/p_0,\Psi)}_{p_0 q} \hookrightarrow B^{(0,\Psi)}_{\infty q} \quad \text{for} \quad 0<p_0<\infty.$$

We just point out that in Proposition 3.13 no continuity assumption on Ψ is required, in contrast with Proposition 4.2; but this is immaterial in the reasoning above. Indeed, given an arbitrary admissible function there is always a continuous equivalent admissible function giving rise to an equivalent quasi-norm in $A^{(s,\Psi)}_{pq}$.

Theorem 4.4 *Let $0<p,q\le\infty$ and $s\in\mathbb{R}$ be such that $\sigma_p<s\le n/p$ and Ψ be a continuous admissible function. Define $r\in(1,\infty]$ by the equation $s-n/p=-n/r$. In the case $s=n/p$ assume further that $(\Psi(2^{-j})^{-1})_{j\in\mathbb{N}}\notin \ell_{q'}$. Then*

$$\mathfrak{E}_{\mathrm{LG}} B^{(s,\Psi)}_{pq} = \big(\Phi_{r,q'}, q\big),$$

with $\Phi_{r,q'}$ as in Definition 2.4.

Proof. Having into consideration Proposition 2.6, the case $\sigma_p<s<n/p$ was already proved in [2, Thm. 4.4], though, as mentioned previously, we have given in Proposition 4.1 a different approach for some parts of its proof; so that from now on we shall deal with the case $s=n/p$, for what we assume

$$(\Psi(2^{-j})^{-1})_{j\in\mathbb{N}} \notin \ell_{q'}. \tag{56}$$

We remark that, in view of propositions 4.1 and 4.2, we have just to prove the optimality of the exponent q.

Step 1. Let first $1<q\le\infty$. Assume that for some $v\in(0,q)$ it was possible to find $c(v)>0$ such that

$$\left(\int_0^\varepsilon \left(\frac{f^*(t)}{\Phi_{\infty,q'}(t)}\right)^v \mu_{\infty,q'}(dt)\right)^{1/v} \le c(v)\, \|f|B^{(n/p,\Psi)}_{pq}\|, \quad \forall\, f\in B^{(n/p,\Psi)}_{pq}, \tag{57}$$

where $\mu_{\infty,q'}$ denotes the Borel measure associated with $-\log\Phi_{\infty,q'}$ in $(0,\varepsilon]$ and ε is as in Proposition 4.1. Notice that, by Proposition 2.7, there are positive constants c_1, c_2 such that

$$c_1\Big(\sum_{j=1}^{[|\log t|/n]} \Psi(2^{-j})^{-q'}\Big)^{1/q'} \le \Phi_{\infty,q'}(t) \le c_2\Big(\sum_{j=1}^{[|\log t|/n]} \Psi(2^{-j})^{-q'}\Big)^{1/q'}, \quad t\in(0,\varepsilon]. \tag{58}$$

Due to (56) we can construct a strictly increasing sequence $(t_k)_{k\in\mathbb{N}_0}$ of natural numbers in the following way:

(i) t_0 is such that $2^{-t_0 n}\le\varepsilon$;

(ii) t_{k+1}, $k \in \mathbb{N}_0$, is the smallest integer satisfying

$$\frac{\sum_{j=1}^{t_{k+1}} \Psi(2^{-j})^{-q'}}{\sum_{j=1}^{t_k} \Psi(2^{-j})^{-q'}} \geq \Big(\frac{2c_2}{c_1}\Big)^{q'}, \tag{59}$$

with c_1, c_2 as in (58).

We remark that in such a case

$$\frac{\sum_{j=1}^{t_{k+1}-1} \Psi(2^{-j})^{-q'}}{\sum_{j=1}^{t_k} \Psi(2^{-j})^{-q'}} < \Big(\frac{2c_2}{c_1}\Big)^{q'};$$

so that, using the admissibility of Ψ,

$$\frac{\sum_{j=1}^{t_{k+1}} \Psi(2^{-j})^{-q'}}{\sum_{j=1}^{t_k} \Psi(2^{-j})^{-q'}} \leq \frac{\sum_{j=1}^{t_{k+1}-1} \Psi(2^{-j})^{-q'} + c\,\Psi(2^{-(t_{k+1}-1)})^{-q'}}{\sum_{j=1}^{t_k} \Psi(2^{-j})^{-q'}} \leq c', \tag{60}$$

for all $k \in \mathbb{N}$.
For each $J \in \mathbb{N}$, let $b = (b_j)_{j\in\mathbb{N}}$ be defined by

$$b_j := \begin{cases} \Psi(2^{-j})^{1-q'}\Big(\sum_{\ell=1}^{t_k} \Psi(2^{-\ell})^{-q'}\Big)^{-1/q} & \text{for} \quad \begin{array}{l} j = t_{k-1}+1, \cdots, t_k, \\ k = 1, \cdots, J \end{array} \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$\|b \mid \ell_q\| = \Big(\sum_{k=1}^{J} \sum_{j=t_{k-1}+1}^{t_k} \Psi(2^{-j})^{-q'}\Big(\sum_{\ell=1}^{t_k} \Psi(2^{-\ell})^{-q'}\Big)^{-1}\Big)^{1/q} \leq J^{1/q}$$

(with the usual modification if $q = \infty$). Let $k \in \{1, \cdots, J\}$. For $j \in \{t_{k-1}+1, \cdots, t_k - 1\}$ we have

$$\begin{aligned} b_{j+1} =& \Psi(2^{-(j+1)})^{1-q'}\Big(\sum_{\ell=1}^{t_k} \Psi(2^{-\ell})^{-q'}\Big)^{-1/q} \\ \leq& c\,\Psi(2^{-j})^{1-q'}\Big(\sum_{\ell=1}^{t_k} \Psi(2^{-\ell})^{-q'}\Big)^{-1/q} = c\,b_j \end{aligned}$$

and, for $k \neq J$,

$$\begin{aligned} b_{t_k+1} =& \Psi(2^{-(t_k+1)})^{1-q'}\Big(\sum_{\ell=1}^{t_{k+1}} \Psi(2^{-\ell})^{-q'}\Big)^{-1/q} \\ \leq& c\,\Psi(2^{-t_k})^{1-q'}\Big(\sum_{\ell=1}^{t_k} \Psi(2^{-\ell})^{-q'}\Big)^{-1/q} = c\,b_{t_k}. \end{aligned}$$

Thus $b_j \geq c\, b_{j+1}$, $j \in \{t_0+1, \cdots t_J\}$, for some constant $c > 0$ independent of j and J. Denote by g_J the function given by (32) with the above-described sequence. According to Remark 3.12 we have

$$\|g_J \mid B_{pq}^{(n/p,\Psi)}\| \leq c\, J^{1/q} \tag{61}$$

and

$$\begin{aligned} g_J^*(2^{-t_k n}) &\geq c \sum_{\ell=1}^{k} \Big(\sum_{i=1}^{t_\ell} \Psi(2^{-i})^{-q'} \Big)^{-1/q} \sum_{j=t_{\ell-1}+1}^{t_\ell} \Psi(2^{-j})^{-q'} \\ &\geq c \Big(\sum_{i=1}^{t_k} \Psi(2^{-i})^{-q'} \Big)^{-1/q} \sum_{j=t_{k-1}+1}^{t_k} \Psi(2^{-j})^{-q'}, \quad k \in \{1, \cdots, J\}, \end{aligned} \tag{62}$$

where $c > 0$ is independent of J. From (57), using (61), the monotonicity of g_J^* and $\Phi_{\infty,q'}$, (62), (58), (59) and (60), we obtain for any $J \in \mathbb{N}$,

$$\begin{aligned} J^{1/q} &\geq c \Big(\sum_{k=1}^{J} \int_{2^{-t_{k+1}n}}^{2^{-t_k n}} \Big(\frac{g_J^*(t)}{\Phi_{\infty,q'}(t)} \Big)^v \mu_{\infty,q'}(dt) \Big)^{1/v} \\ &\geq c \Big(\sum_{k=1}^{J} \Big(\frac{g_J^*(2^{-t_k n})}{\Phi_{\infty,q'}(2^{-t_{k+1}n})} \Big)^v \mu_{\infty,q'}\big([2^{-t_{k+1}n}, 2^{-t_k n}]\big) \Big)^{1/v} \\ &\geq c' \Big\{ \sum_{k=1}^{J} \Big(\sum_{i=1}^{t_k} \Psi(2^{-i})^{-q'} \Big)^{-v/q} \Big(\sum_{j=t_{k-1}+1}^{t_k} \Psi(2^{-j})^{-q'} \Big)^v \Big(\sum_{j=1}^{t_{k+1}} \Psi(2^{-j})^{-q'} \Big)^{-v/q'} \Big\}^{1/v} \\ &\geq c'' \Big\{ \sum_{k=1}^{J} \Big(\sum_{j=t_{k-1}+1}^{t_k} \Psi(2^{-j})^{-q'} \Big)^v \Big(\sum_{j=1}^{t_k} \Psi(2^{-j})^{-q'} \Big)^{-v} \Big\}^{1/v} \geq c''' J^{1/v}, \end{aligned}$$

which is impossible for $v < q$. We have also used above the fact that

$$\Big(\sum_{j=t_{k-1}+1}^{t_k} \Psi(2^{-j})^{-q'} \Big) \Big(\sum_{i=1}^{t_k} \Psi(2^{-i})^{-q'} \Big)^{-1} \geq c,$$

for some positive constant c and for all $k \in \mathbb{N}$, consequence of (59).

Step 2. Now let $0 < q \leq 1$. Then $q' = \infty$. We modify appropriately Step 1. Assume that (57) holds true for some $v \in (0, q)$. The counterpart of (58) reads as follows:

$$c_1 \sup_{j=1,\cdots,[|\log t|/n]} \Psi(2^{-j})^{-1} \leq \Phi_{\infty,\infty}(t) \leq c_2 \sup_{j=1,\cdots,[|\log t|/n]} \Psi(2^{-j})^{-1},\ t \in (0, \varepsilon]. \tag{63}$$

We remark that (56) implies that the admissible function Ψ has to be monotone increasing and we construct a strictly increasing sequence $(t_k)_{k\in\mathbb{N}_0}$ of natural numbers in the following way:

(i) t_0 is such that $2^{-t_0 n} \le \varepsilon$;

(ii) t_{k+1}, $k \in \mathbb{N}_0$, is the smallest integer satisfying

$$\frac{\Psi(2^{-t_{k+1}})^{-1}}{\Psi(2^{-t_k})^{-1}} \ge \frac{2c_2}{c_1}, \tag{64}$$

with c_1, c_2 as in (63).

Notice that then

$$\frac{\Psi(2^{-t_k})^{-1}}{\Psi(2^{-t_{k+1}})^{-1}} \sim \frac{\Psi(2^{-t_k})^{-1}}{\Psi(2^{-(t_{k+1}-1)})^{-1}} > \frac{c_1}{2c_2}, \quad k \in \mathbb{N}. \tag{65}$$

For each $J \in \mathbb{N}$, let $b = (b_j)_{j\in\mathbb{N}}$ be defined by

$$b_j := \begin{cases} 1 & \text{if} \quad j = t_k,\ k \in \{1, \cdots, J\} \\ 0 & \text{otherwise.} \end{cases}$$

We have $\|b \mid \ell_q\| = J^{1/q}$. Then the corresponding function g_J, as in step 1, satisfies

$$\|g_J \mid B^{(n/p,\Psi)}_{pq}\| \le c_3\, J^{1/q} \quad \text{and} \quad g^*_J(2^{-t_k n}) \ge c_4\, \Psi(2^{-t_k})^{-1},\ k \in \{1, \cdots J\}, \tag{66}$$

where $c_3, c_4 > 0$ are independent of J. From (57), using the monotonicity of g^*_J and $\Phi_{\infty,\infty}$, (63), (64), (65) and (66), we obtain for any $J \in \mathbb{N}$,

$$\begin{aligned} J^{1/q} &\ge c \left(\sum_{k=1}^{J} \int_{2^{-t_{k+1}n}}^{2^{-t_k n}} \Big(\frac{g^*_J(t)}{\Phi_{\infty,\infty}(t)} \Big)^v \mu_{\infty,\infty}(dt) \right)^{1/v} \\ &\ge c \left(\sum_{k=1}^{J} \Big(\frac{g^*_J(2^{-t_k n})}{\Phi_{\infty,\infty}(2^{-t_{k+1}n})} \Big)^v \mu_{\infty,\infty}\big([2^{-t_{k+1}n}, 2^{-t_k n}]\big) \right)^{1/v} \\ &\ge c' \left\{ \sum_{k=1}^{J} \Big(\frac{\Psi(2^{-t_k})^{-1}}{\sup_{j=1,\cdots,t_{k+1}} \Psi(2^{-j})^{-1}} \Big)^v \right\}^{1/v} \\ &\ge c'' J^{1/v}, \end{aligned}$$

which is impossible for $v < q$. □

Corollary 4.5 *Let $0 < p, q \le \infty$ and $s \in \mathbb{R}$ be such that $\sigma_p < s \le n/p$ and Ψ be a continuous admissible function. Define $r \in (1, \infty]$ by the equation $s - n/p = -n/r$. In the case $s = n/p$ assume further that $(\Psi(2^{-j})^{-1})_{j\in\mathbb{N}} \notin \ell_{p'}$. Then*

$$\mathfrak{E}_{\mathrm{LG}} F^{(s,\Psi)}_{pq} = \big(\Phi_{r,p'}, p\big),$$

with $\Phi_{r,p'}$ according to Definition 2.4.

Proof. The assertion for $\sigma_p < s < n/p$ is covered by [2, Thm. 4.4], having into consideration Proposition 2.6(i), though it can now also be easily deduced by the same technique that follows, *mutatis mutandis*. The remaining case, i.e. the case $s = n/p$, it is a consequence of Theorem 4.4 and the embeddings

$$B^{(n/p_1,\Psi)}_{p_1 p} \hookrightarrow F^{(n/p,\Psi)}_{pq} \hookrightarrow B^{(n/p_2,\Psi)}_{p_2 p} \quad \text{for} \quad 0 < p_1 < p < p_2 < \infty$$

(see Example 3.5 of [2]). □

Example 4.6 We return to our example Ψ_b given by (3). Assume $p, q \in (0, \infty]$ and $b < 1/q'$. Then Theorem 4.4 yields that

$$\mathfrak{E}_{\mathrm{LG}} B^{n/p,b}_{pq} = \big(|\log t|^{-(b-1/q')}, q \big).$$

When $q \in (1, \infty]$ it also makes sense to consider $b = 1/q'$ (cf. Example 3.14), and in this case we get that

$$\mathfrak{E}_{\mathrm{LG}} B^{n/p,b}_{pq} = \big((\log |\log t|)^{1/q'}, q \big).$$

Remark 4.7 Analogously to what has been observed in Remark 4.3, we point out that there is no loss of generality in assuming the continuity of the admissible function Ψ in Theorem 4.4 and Corollary 4.5. Actually we just have to keep in mind that for an arbitrary admissible function, the function $\Phi_{r,u'}$ to appear in the local growth envelope – and, specially, the corresponding measure $\mu_{r,u'}$ – should be built by means of an equivalent continuous admissible function.

Remark 4.8 Theorem 4.4 and Corollary 4.5 describe in a rather condensed way some sharp inequalities. It is not difficult to see that, together with Proposition 3.17(i), they even imply, under the hypotheses assumed, that, given a positive monotonically decreasing function κ on $(0, \varepsilon]$, for some small enough $\varepsilon \in (0, 1)$, and $0 < v \le \infty$,

$$\left(\int_0^\varepsilon \left(\kappa(t) \frac{f^*(t)}{\Phi_{r,q'}(t)} \right)^v \mu_{r,q'}(dt) \right)^{1/v} \le c \, \| f | B^{(s,\Psi)}_{pq} \| \tag{67}$$

(with the appropriate modification if $v = \infty$) holds for some $c > 0$ and all $f \in B^{(s,\Psi)}_{pq}$ if, and only if, κ is bounded and $v \ge q$;

$$\left(\int_0^\varepsilon \left(\kappa(t) \frac{f^*(t)}{\Phi_{r,p'}(t)} \right)^v \mu_{r,p'}(dt) \right)^{1/v} \le c \, \| f | F^{(s,\Psi)}_{pq} \|$$

(with the appropriate modification if $v = \infty$) holds for some $c > 0$ and all $f \in F^{(s,\Psi)}_{pq}$ if, and only if, κ is bounded and $v \ge p$.

Notice also that it is possible to give explicit expressions for $\mu_{r,q'}(dt)$ and $\mu_{r,p'}(dt)$ (even for general Ψ) in some cases: for example, if $u \neq \infty$,

$$\mu_{r,u}(dt) \sim \frac{dt}{\Phi_{r,u}(t)^u \Psi(t)^u \, t^{u/r+1}} \tag{68}$$

(cf. Proposition 2.5); since $\Phi_{r,u}(t) \sim t^{-1/r}\Psi(t)^{-1}$ when $r \neq \infty$ (cf. Proposition 2.6(i)), then in the case $r, u \neq \infty$ the measure $\mu_{r,u}(dt)$ can be further simplified to $\frac{dt}{t}$.

And either by using (68) or by calculating directly from the local growth envelope function given in Example 4.6, in the interesting case of this example when q is assumed in $(1, \infty]$ and b equals $1/q'$, one has

$$\mu_{\infty,q'}(dt) \sim \frac{dt}{(\log|\log t|)|\log t|\, t}$$

and the inequality (67) above then reads

$$\left(\int_0^{\varepsilon} \left(\kappa(t) \frac{f^*(t)}{(\log|\log t|)^{\frac{1}{q'}+\frac{1}{v}}} \right)^v \frac{dt}{|\log t|\, t} \right)^{1/v} \leq c\, \|f|B_{pq}^{n/p,b}\|.$$

Remark 4.9 Our main results stated in Theorem 4.4 and Corollary 4.5 recover the results of Haroske [7] and Triebel [17] for the usual Besov and Triebel-Lizorkin spaces in the critical case. Indeed, having into account Proposition 2.6(iii), when $\Psi \equiv 1$ what we have obtained is:

$$\mathfrak{E}_{\mathrm{LG}} B_{pq}^{(n/p,\Psi)} = (|\log t|^{1/q'}, q), \qquad 0 < p < \infty,\ 1 < q \leq \infty;$$
$$\mathfrak{E}_{\mathrm{LG}} F_{pq}^{(n/p,\Psi)} = (|\log t|^{1/p'}, p), \qquad 1 < p < \infty,\ 0 < q \leq \infty.$$

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