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Continuous Selections of Solution Sets of Lipschitzean Differential Inclusions

By

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1. Introduction

Recently, Cellina and Ornelas studied in [C], [O], [P-S] and [C-O] the existence of a continuous map $\xi \to x_\xi$ such that x_ξ is a Caratheodory solution of the differential inclusion

$$\dot{x} \in F(t, x), \quad x(0) = \xi.$$

They assumed that the right-hand side is Lipschitz continuous with respect to x, with values in $\boldsymbol{R}^n$ and ξ belongs to a compact set.

The purpose of the present paper is to give a generalization of those results. Namely, we consider the Cauchy problems

$$(P_s) \qquad \dot{x} \in F(t, x, s), \quad x(0) = \xi(s)$$

where the right-hand side is Lipschitz continuous in x and lower semicontinuous in s with values in a separable Banach space. Assuming that the initial data depends continuously on s, we show the existence of a continuous map $s \to x_s$, where the x_s are solutions of (P_s). The proof is based on an argument different from the one used by Cellina and Ornelas; it relies on a selection theorem of Bressan and Colombo [B-C].

Our result contains as a special case the selection theorems due to Antosiewicz and Cellina [An-C], Bressan and Colombo [B-C] and Fryszkowski [F_1].

2. Preliminaries

Denote by I the interval $[0, 1]$ and by $\mathscr{L}$ the σ-field of Lebesgue measurable subsets of I. Let S be a separable metric space and X a separable Banach space with the norm $|\cdot|$. $\mathscr{P}(X)$ will stand for the family of all nonempty closed subsets of X with the Hausdorff distance d_H and $\mathscr{B}(S)$ for the family of Borel

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subsets of S.

Denote by $L^1(I, X)$ the Banach space of Bochner integrable functions $u: I \to X$, with the norm $\|u\| = \int_I |u(t)|dt$ and by AC(I, X) the Banach space of absolutely continuous functions $u: I \to X$ with the norm $\|u\|_{AC} = |u(0)| + \|\dot{u}\|$.

A subset $K \subseteq L^1(I, X)$ is called decomposable if for every $u, v \in K$ and any $A \in \mathscr{L}$,

$$u\chi_A + v\chi_{I \setminus A} \in K, \tag{2.1}$$

where χ_A stands for the characteristic function of A. The family of all nonempty closed and decomposable subsets of $L^1(I, X)$ is denoted by $\mathscr{D}$.

It is known that $K \in \mathscr{D}$ iff there exists a measurable map $F: I \to \mathscr{P}(X)$ such that

$$K = \{u \in L^1(I, X): u(t) \in F(t) \text{ a.e. in } I\},$$

and that K is nonempty iff the function $t \to d(0, F(t))$ is integrable, where d denotes the usual point-to-set distance. For more details on decomposable sets and set-valued maps we refer to [H-U].

A multivalued map $G: S \to \mathscr{P}(X)$ is called lower semicontinuous (l.s.c.) if the set $\{s \in S: G(s) \subseteq C\}$ is closed in S for any closed $C \subseteq X$.

According to [B-C] (Theorem 3) a l.s.c. map $G: S \to \mathscr{D}$ admits a continuous selection, i.e. there exists a continuous map $g: S \to L^1(I, X)$ such that $g(s) \in G(s)$ for all $s \in S$ (see also [F_1]).

Consider a map $F: I \times S \to \mathscr{P}(X)$ and set

$$G_F(s) = \{v \in L^1(I, X): v(t) \in F(t, s) \text{ a.e. in } I\}. \tag{2.2}$$

The following proposition is a combined version of Proposition 2 and Theorem 3 from [B-C] and Proposition 2 from [F_2].

Proposition 2.1. *Assume $F: I \times S \to \mathscr{P}(X)$ is $\mathscr{L} \otimes \mathscr{B}(S)$ measurable and l.s.c. in s. Then the map $s \to G_F(s)$ given by* (2.2) *is l.s.c. from S into $\mathscr{D}$ iff there exists a continuous $\beta: S \to L^1(I, \boldsymbol{R})$ such that for every $s \in S$*

$$\beta(s)(t) \geq d(0, F(t, s)) \text{ a.e. in } I. \tag{2.3}$$

Proof. The necessity is obvious since if $g(\cdot)$ is a continuous selection of $G_F(\cdot)$ then $\beta(s)(t) = |g(s)(t)|$ satisfies (2.3).

In order to prove that (2.3) is also sufficient, let $C \subseteq L^1(I, X)$ be an arbitrary closed set and let $s_n \to s_0$ be such that $G_F(s_n) \subseteq C$. Take any $v_0 \in G_F(s_0)$ and consider measurable selections $v_n(t)$ of $t \to F(t, s_n)$ such that

$$|v_n(t) - v_0(t)| < d(v_0(t), F(t, s_n)) + \frac{1}{n} \text{ a.e. in } I. \tag{2.4}$$

The existence of such v_n follows from Proposition 2 in [B-C]. Let us notice that since for every t the map $s \to F(t, s)$ is l.s.c. then for every $x \in X$

$$(2.5) \qquad s \longrightarrow d(x, F(t, s)) \text{ is u.s.c.}$$

Therefore from (2.4) we obtain that

$$(2.6) \qquad u_n(t) \longrightarrow v_0(t) \text{ a.e. in } I.$$

We show that $v_n \to v_0$ in $L^1(I, X)$. From (2.4) we have

$$(2.7) \qquad |v_n(t) - v_0(t)| < |v_0(t)| + \beta(s_n)(t) + \frac{1}{n} \text{ a.e. in } I.$$

Denote by $a_n(t)$ the right-hand side of (2.7) and observe that the sequence $a_n(\cdot)$ is strongly convergent in $L^1(I, \boldsymbol{R})$. Thus it is bounded in $L^1(I, \boldsymbol{R})$ and uniformly integrable, so the same holds for the sequence of functions $t \to |v_n(t) - v_0(t)|$. Therefore, $v_n \to v_0$ in $L^1(I, X)$, because of (2.6).

Since C is closed and $v_n \in C$, $v_0 \in C$ as well. But v_0 is an arbitrary point of $G_F(s_0)$, hence $G_F(s_0) \subseteq C$, that was to be proved.

Theorem 3 and Proposition 4 in [B-C] or Proposition 2.2 and Theorem 1 in [F_1] imply:

Proposition 2.2. *Consider a l.s.c. multivalued map* $G: S \to \mathscr{D}$ *and assume that* $\varphi: S \to L^1(I, X)$ *and* $\psi: S \to L^1(I, R)$ *are continuous maps and for every* $s \in S$ *the set*

$$H(s) = \text{cl}\{u \in G(s): |u(t) - \varphi(s)(t)| < \psi(s)(t) \text{ a.e. in } I\}$$

is nonempty. Then the map $H: S \to \mathscr{D}$ *is l.s.c., so it admits a continuous selection.* (cl *stands for the closure*).

Consider a map $F: I \times X \times S \to \mathscr{P}(X)$. We shall assume the following hypotheses on F:

(H1): F is $\mathscr{L} \otimes \mathscr{B}(X \times S)$ measurable.

(H2): There exists a map $s \to k(\cdot, s)$ continuous from S into $L^1(I, \boldsymbol{R})$ such that $k(t, s) > 0$ and for any $s \in S$ and $x, y \in X$,

$$d_H(F(t, x, s), F(t, y, s)) \le k(t, s)|x - y| \text{ a.e. in } I.$$

(H3): For any (t, x) the map $s \to F(t, x, s)$ is l.s.c..

(H4): For any continuous map $s \to y(\cdot, s)$ from S into AC(I, X), there exists a continuous map $\beta_y: S \to L^1(I, \boldsymbol{R})$ such that for every $s \in S$

$$(2.8) \qquad \beta_y(s)(t) \ge d(\dot{y}(t, s), F(t, y(t, s), s)) \text{ a.e. in } I.$$

Notice that due to (H2) and Proposition 2.1 the assumption (H4) may be

replaced by the equivalent condition:
($H4_0$): There exists a continuous map $\beta_0: S \to L^1(I, \boldsymbol{R})$ such that for any $s \in S$

$$\beta_0(s)(t) \geq d(0, F(t, 0, s)) \text{ a.e. in } I.$$

Indeed, it easily follows from the inequality

(2.9) $d(\dot{y}(t, s), F(t, y(t, s), s)) \leq |\dot{y}(t, s)| + d(0, F(t, 0, s)) + k(t, s)|y(t, s)|$ a.e. in I.

Let us remark that from Proposition 2.1 and (H3) it follows that (H4) is also equivalent to the condition:
(H4′): For any continuous map $s \to y(\cdot, s)$ from S into AC(I, X), the map $G_y(\cdot)$ defined by

$$G_y(s) = \{v \in L^1(I, X): v(t) \in F(t, y(t, s), s) \text{ a.e. in } I\}$$

is l.s.c. from S into $\mathscr{D}$.
Similarly ($H4'_0$) holds iff
($H4'_0$): The map $G_0(\cdot)$ defined by

(2.10) $$G_0(s) = \{v \in L^1(I, X): v(t) \in F(t, 0, s) \text{ a.e. in } I\}$$

is l.s.c. from S into $\mathscr{D}$.

Indeed, if $s \to y(\cdot, s)$ is continuous from S into AC(I, X), then the map $s \to F(t, y(t, s), s) - \dot{y}(t, s)$ is l.s.c. and $\mathscr{L} \otimes \mathscr{B}(S)$ measurable, so we can apply Proposition 2.1.

3. Main result

Let $F: I \times X \times S \to \mathscr{P}(X)$ and consider the following Cauchy problems

(P_s) $$\dot{x} \in F(t, x, s), \quad x(0) = \xi(s),$$

where $\xi: S \to X$ is a continuous function.

For given s, by a solution of (P_s) we mean a function $x \in \text{AC}(I, X)$ with $x(0) = \xi(s)$ such that

$$\dot{x}(t) \in F(t, x(t), s) \quad \text{a.e.}.$$

The main result of this paper is the following

Theorem 3.1. *Suppose F satisfies* (H1), ..., (H4). *Then for any continuous map $s \to y(\cdot, s)$ from S into* AC(I, X) *and $s \to \beta(s) = \beta_y(s)$ from S into $L^1(I, \boldsymbol{R})$ satisfying* (2.8) *and for every $\varepsilon > 0$, there exists a function $x: I \times X$ such that*
(a) *For every s the function $t \to x(t, s)$ is a solution of* (P_s).
(b) *The map $s \to x(\cdot, s)$ is continuous from S into* AC(I, X).
(c) *For every $s \in S$*

$$|\dot{y}(t, s) - \dot{x}(t, s)| \leq \varepsilon + \varepsilon k(t, s)e^{m(t,s)} + k(t, s)|y(0, s) - \xi(s)|e^{m(t,s)}$$
$$+ k(t, s)\int_0^t \beta(s)(\tau)e^{m(t,s)-m(\tau,s)}d\tau + \beta(s)(t) \text{ a.e. in } I.$$

(d) *For all* $(t, s) \in I \times S$

$$|[y(t, s) - x(t, s)] - [y(0, s) - \xi(s)]| \leq \varepsilon e^{m(t,s)} + |y(0, s) - \xi(s)|(e^{m(t,s)} - 1)$$
$$+ \int_0^t \beta(s)(\tau)e^{m(t,s)-m(\tau,s)}d\tau,$$

where $m(t, s) = \int_0^t k(\tau, s)d\tau$.

Remark. denote by $\mathscr{R}(s)$ the closed subset of $AC(I, X)$ consisting of all solutions of (P_s). Theorem 3.1 provides the existence of a continuous selection of the map $\mathscr{R}$. This implies the selection theorems due to Antosiewicz and Cellina [An-C], Bressan and Colombo [B-C] and Fryszkowski [F_1].

Proof of Theorem 3.1. We may assume that for any $(t, s) \in I \times S$

$$y(t, s) = 0 \quad \text{and} \quad \xi(s) = 0.$$

In fact, denote by

$$\tilde{F}(t, z, s) = F(t, z + y(t, s) - y(0, s) + \xi(s), s) - \dot{y}(t, s)$$

and consider the problem

$$(\tilde{P}_s) \qquad \dot{z} \in \tilde{F}(t, z, s), \quad z(0) = 0.$$

Now the function

$$x(t, s) = z(t, s) + y(t, s) - y(0, s) + \xi(s)$$

is a desired solution of (P_s), whenever z satisfies (a), ..., (d) for $(\tilde{P}_s)$ with

$$\tilde{\beta}(s)(t) = \beta(s)(t) + |\dot{y}(t, s)| + k(t, s)|\xi(s) - y(0, s)| \geq d(0, \tilde{F}(t, 0, s)) \text{ a.e. in } I.$$

Fix $\varepsilon > 0$, set $\varepsilon_n = ((n + 1)/(n + 2))\varepsilon$ and put

$$\beta_n(s)(t) = \int_0^t \beta(s)(u)\frac{(m(t, s) - m(u, s))^{n-1}}{(n-1)!}du + \frac{m(t, s)^{n-1}}{(n-1)!}\varepsilon_n.$$

We shall construct a Cauchy sequence of successive approximations $x_n(t, s)$, $x_n(\cdot, s) \in AC(I, X)$, such that for all $n \geq 0$, $x_n(0, s) = 0$ and

(i) $\qquad s \longrightarrow x_n(\cdot, s)$ are continuous,

(ii) $\qquad \dot{x}_{n+1}(t, s) \in F(t, x_n(t, s), s)$ a.e. in I,

(iii) $\qquad |\dot{x}_{n+1}(t, s) - \dot{x}_n(t, s)| \leq k(t, s)\beta_n(s)(t)$ a.e. in I,

where, for simplicity, $k(t, s)\beta_0(s)(t)$ is understood as $\beta(s)(t) + \varepsilon_0$.

Remark that repeating for any s the calculations provided in [Au-C], (formula (14), page 122) we can conclude that

$$\text{(3.1)} \quad \int_0^t k(u, s)\beta_n(s)(u)du$$

$$= \int_0^t \beta(s)(u)\frac{(m(t, s) - m(u, s))^n}{n!}du + \frac{m(t, s)^n}{n!}\varepsilon_n < \beta_{n+1}(s)(t) \text{ a.e. in } I.$$

Therefore, from (iii), we also have

$$\text{(3.2)} \quad |x_{n+1}(t, s) - x_n(t, s)| < \beta_{n+1}(s)(t) \text{ a.e. in } I.$$

Set $x_0(t, s) = 0$ and denote by

$$G_0(s) = \{v \in L^1(I, X) \colon v(t) \in F(t, x_0(t, s), s) \text{ a.e. in } I\}.$$

Consider the map H_0 defined by

$$H_0(s) = \mathrm{cl}\{v \in G_0(s) \colon |v(t)| < \beta(s)(t) + \varepsilon_0\}.$$

Propositin 2.2 applied to H_0 implies the existence of a continuous map $h_0 \colon S \to L^1(I, X)$ such that

$$h_0(s)(t) \in F(t, x_0(t, s), s) \text{ a.e. in } I$$

and

$$|h_0(s)(t)| \leq \beta(s)(t) + \varepsilon_0.$$

Define

$$x_1(t, s) = \int_0^t h_0(s)(\tau)d\tau$$

and notice that

$$|x_1(t, s) - x_0(t, s)| \leq \int_0^t |h_0(s)(\tau)d\tau| < \int_0^t \beta(s)(\tau)d\tau + \varepsilon_0 < \beta_1(s)(t) \text{ a.e. in } I.$$

Suppose we have defined the functions $x_0, \ldots, x_n$ satisfying (i), (ii) and (iii). Observe that

$$d(\dot{x}_n(t, s), F(t, x_n(t, s), s)) \leq d_H(F(t, x_{n-1}(t, s), s), F(t, x_n(t, s), s))$$

$$\leq k(t, s)|x_n(t, s) - x_{n-1}(t, s)|.$$

The latter and (3.2) yield

$$\text{(3.3)} \quad d(\dot{x}_n(t, s), F(t, x_n(t, s), s)) < k(t, s)\beta_n(s)(t) \text{ a.e. in } I.$$

Denote

$$G_n(s) = \{v \in L^1(I, X) : v(t) \in F(t, x_n(t, s), s) \text{ a.e. in } I\}$$

and consider the map

$$H_n(s) = \mathrm{cl}\{v \in G_n(s) : |v(t) - \dot{x}_n(t, s)| < k(t, s)\beta_n(s)(t) \text{ a.e. in } I\}. \tag{3.4}$$

$H_n(s)$ is nonempty because of (3.3). By Proposition 2.2 there exists a continuous map $h_n : S \to L^1(I, X)$ such that

$$h_n(s)(t) \in F(t, x_n(t, s), s) \text{ a.e. in } I$$

and

$$|h_n(s)(t) - \dot{x}_n(t, s)| \leq k(t, s)\beta_n(s)(t) \text{ a.e. in } I.$$

Define

$$x_{n+1}(t, s) = \int_0^t h_n(s)(\tau)\, d\tau.$$

Clearly, x_{n+1} satisfies (i), (ii) and (iii).

From (iii) and (3.1) we obtain that

$$\| x_{n+1}(\cdot, s) - x_n(\cdot, s) \|_{AC} \leq \beta_{n+1}(s)(1). \tag{3.5}$$

The right-hand side of (3.4) can be estimated by

$$\beta_{n+1}(s)(1) \leq \int_0^1 \beta(s)(t) \frac{\| k(\cdot, s) \|^n}{n!}\, dt + \frac{m(1, s)^n}{n!} \varepsilon_{n+1}.$$

Therefore

$$\beta_{n+1}(s)(1) \leq \frac{\| k(\cdot, s) \|^n}{n!} (\| \beta(s) \| + \varepsilon),$$

since

$$m(t, s) - m(u, s) = \int_u^t k(\tau, s)\, d\tau \leq \| k(\cdot, s) \|,$$

and

$$m(1, s) = \| k(\cdot, s) \|.$$

Hence we have

$$\| x_{n+1}(\cdot, s) - x_n(\cdot, s) \|_{AC} \leq \frac{\| k(\cdot, s) \|^n}{n!} (\| \beta(s) \| + \varepsilon). \tag{3.6}$$

The functions $s \to \|\beta(s)\|_{AC}$ and $s \to \|k(\cdot, s)\|_{AC}$ are continuous. Therefore, (3.6) implies that for every s, the sequence $\{x_n(\cdot, s')\}$ satisfies the Cauchy condition uniformly in s' on some neighbourhood of s. Hence $s \to x(\cdot, s)$ where $x(t, s) = \lim x_n(t, s)$ is continuous from S into $\mathrm{AC}(I, X)$. To see that the function $t \to x(t, s)$ is a solution of (P_s) it is enough to notice that

$$d(\dot{x}_{n+1}(t, s), F(t, x(t, s), s)) \le k(t, s)|x_n(t, s) - x(t, s)|.$$

We shall now prove (c) and (d).

By adding the inequalities (iii) for all n, we obtain that

$$\begin{aligned}|\dot{x}_{n+1}(t, s)| &\le \beta(s)(t) + \sum_{i=1}^{n} |\dot{x}_{i+1}(t, s) - \dot{x}_i(t, s)| + \varepsilon_0 \\ &\le \beta(s)(t) + k(t, s)\int_0^t \beta(s)(u)\left[\sum_{i=1}^{n} \frac{(m(t, s) - m(u, s))^{i-1}}{(i-1)!}\right] du \\ &\quad + \varepsilon k(t, s)\left[\sum_{i=1}^{n} \frac{m(t, s)^{i-1}}{(i-1)!}\right] + \varepsilon.\end{aligned}$$

Similarly, by adding (3.2) we get

$$\begin{aligned}|x_{n+1}(t, s)| &\le \sum_{i=0}^{n} |x_{i+1}(t, s) - x_i(t, s)| \\ &\le \int_0^t \beta(s)(u)\left[\sum_{i=0}^{n} \frac{(m(t, s) - m(u, s))^{i}}{i!}\right] du + \varepsilon\left[\sum_{i=0}^{n} \frac{m(t, s)^{i}}{i!}\right].\end{aligned}$$

So, by passing to the limit and using the identity $e^{-m(t,s)} + \int_0^t k(u, s)e^{-m(u,s)}\,du = 1$ we obtain (c) and (d). This ends the proof.

4. Properties of the solution sets

In what follows we assume that $F: I \times X \times S \to \mathscr{P}(X)$ satisfies (H1), ..., (H4). Denote by $\mathscr{R}(s)$ the closed subset of $\mathrm{AC}(I, X)$ consisting of all solutions of (P_s). From Theorem 3.1 we already know that $s \to \mathscr{R}(s)$ admits a continuous selection. Now we shall provide some other properties of this map.

Theorem 4.1. *Fix $s_0 \in S$ and $x_0 \in \mathscr{R}(s_0)$. Then there exists a continuous selection $r: S \to \mathrm{AC}(I, X)$ of $\mathscr{R}$ such that $r(s_0) = x_0$.*

Proof. Using the same argument as at the beginning of the proof of Theorem 3.1 we may assume that $x_0 = 0$ so we have

$$0 \in F(t, 0, s_0).$$

Consider the map $F_*: I \times X \times S \to \mathscr{P}(X)$ defined by

$$F_*(t, x, s) = \begin{cases} F(t, x, s) & \text{if } s \neq s_0 \\ \{0\} & \text{if } s = s_0 \end{cases}.$$

Clearly, F_* satisfies (H1), (H2) and (H3). We claim that also (H4) holds and moreover one can choose a continuous $\beta_*: S \to L^1(I, \boldsymbol{R})$ with

$$\beta_*(s_0) = 0.$$

From the definition of F_*, we see that

$$d(0, F_*(t, 0, s)) = d(0, F(t, 0, s)).$$

Consider

$$P(s) = \mathrm{cl}\{v \in L^1(I, \boldsymbol{R}): v(t) > d(0, F_*(t, 0, s)) \text{ a.e. in } I\}$$

and notice that

$$0 \in P(s_0).$$

By Proposition 2.1, $P(\cdot)$ is l.s.c. from S into $\mathscr{D}$. Therefore it admits a continuous selection $\beta_*(\cdot)$ such that $\beta_*(s_0) = 0$. This proves the claim.

Repeating the same construction as in the proof of Theorem 3.1, we see that $0 \in H_n(s_0)$ for any n, $H_n(s)$ as in (3.4). So, we can always choose a continuous selection h_n of H_n such that $h_n(s_0) = 0$. Hence, the sequence of approximate solutions $x_n(t, s)$ is such that for all n, $x_n(\cdot, s_0) = 0$ and the same holds for the limit. This completes the proof.

Denote by r_{s_0,x_0} a selection of $\mathscr{R}$ such that

$$r_{s_0,x_0}(s_0) = x_0.$$

Clearly, for every $s \in S$

$$\mathscr{R}(s) = \{r_{s_0,x_0}(s): s_0 \in S, \quad x_0 \in \mathscr{R}(s_0)\}.$$

Theorem 4.2. *The map $\mathscr{R}: S \to \mathscr{P}(\mathrm{AC}(I, X))$ is l.s.c. and admits a continuous selection. Moreover, if S is compact, then there exists a countable family $\{r_n\}$ of selections of $\mathscr{R}$ such that*

$$\mathscr{R}(s) = \mathrm{cl}\{r_n(s): n \in \boldsymbol{N}\}.$$

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