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NONLINEAR, NONHOMOGENEOUS PARAMETRIC NEUMANN PROBLEMS

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(*Submitted by*)

Abstract. We consider a parametric nonlinear Neumann problem driven by a nonlinear nonhomogeneous differential operator and with a Caratheodory reaction $f(t,x)$ which is $p-$superlinear in x without satisfying the usual in such cases Ambrosetti-Rabinowitz condition. We prove a bifurcation type result describing the dependence of the positive solutions on the parameter $\lambda > 0$, we show the existence of a smallest positive solution $\overline{u}_\lambda$ and investigate the properties of the map $\lambda \to \overline{u}_\lambda$. Finally we also show the existence of nodal solutions.

1. Introduction

In this paper we study the following nonlinear parametric Neumann problem

$$(P_\lambda) \qquad \begin{cases} -div\ a\left(Du\left(z\right)\right) + \lambda \left|u\left(z\right)\right|^{p-2} u\left(z\right) = f\left(z, u\left(z\right)\right) \text{ in } \Omega, \\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega,\ \lambda > 0,\ 1 < p < \infty. \end{cases}$$

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Here $\Omega \subset \mathbb{R}^N$ is a bounded domain with a C^2-boundary $\partial\Omega$. The map $a : \mathbb{R}^N \to \mathbb{R}^N$ is continuous, strictly monotone and satisfies certain other regularity conditions which are listed in hypotheses $\mathbf{H}\left(\mathbf{a}\right)$ (see Section 2). These hypotheses are general enough to incorporate in our framework many differential operators of interest, such as the $p-$Laplacian. Also $\lambda > 0$ is a parameter and $f\left(t, z\right)$ is a Carathéodory function (i.e., for all $x \in \mathbb{R}$ $z \to f\left(z, x\right)$ is measurable and for almost all $z \in \Omega$, $x \to f\left(z, x\right)$ is continuous) which exhibits a $\left(p-1\right)-$superlinear growth in the $x-$variable, but without satisfying the usual in such case Ambrosetti-Rabinowitz condition (AR-condition for short).

Our work here was motivated by a recent paper of Motreanu-Motreanu-Papageorgiou [18], who produced constant sign and nodal solutions. Our results here complement and improve those of [18]. More precisely, the authors in [18] produced positive solutions for problem (P_λ) but did not give the precise dependence of the set of positive solutions on the parameter $\lambda > 0$. Here, we prove a bifurcation-type theorem for large values of λ, which gives a complete picture of the set of positive solutions as the parameter varies. Moreover, in [18] nodal (that is, sign-changing) solutions were produced only for the particular case of equations driven by the $p-$Laplacian. In contrast here, we generate nodal solutions for the general case. We stress that the $p-$Laplacian differential operator is homogeneous, while the differential operator in (P_λ) is not. Hence, the methods and techniques used in [18] fail in the present setting, and so a new approach is needed. Finally, we mention that a bifurcation near infinity for a different class of $p-$Laplacian Dirichlet problems was recently produced by Gasinski-Papageorgiou [12].

In the next section, we review the main mathematical tools which will be used in this paper. We also present the hypotheses on the map $y \to a\left(y\right)$ and state some useful consequences of them.

2. Mathematical background

Let $(X, \|.\|)$ be a Banach space and X^* be its topological dual. By $\langle ., . \rangle$ we denote the duality brackets for the pair (X^*, X) and $\overset{w}{\to}$ will designate weak convergence.

Let $\varphi \in C^1\left(X\right).$ We say that $x^* \in X$ is a critical point of φ if $\varphi'\left(x^*\right) = 0$. If $x^* \in X$ is a critical point of of φ then $c = \varphi\left(x^*\right)$ is a critical value of φ. We say that φ satisfies the "Palais-Smale condition" (the PS-condition for short), if the following holds:

> "every sequence $\{u_n\}_{n\geq 1} \subseteq X$ such that $\{\varphi\left(u_n\right)\}_{n\geq 1}$ is bounded in $\mathbb{R}$ and $\varphi'\left(u_n\right) \to 0$ in X^* as $n \to \infty$ admits a strongly convergent subsequence."

This compactness-type condition on the functional φ leads to a deformation theorem from which one can derive the minimax theory of the critical values of φ. One of the main results in that theory is the so called "mountain pass theorem", which we recall here:

THEOREM 2.1. *If $\varphi \in C^1(X)$ satisfies the PS-condition, u_0, $u_1 \in X$ with $\|u_1 - u_0\| > \rho > 0$ and*

$$\max\{\varphi(u_0), \varphi(u_1)\} < \inf\{\varphi(u) : \|u - u_0\| = \rho\} =: m_\rho,$$
$$\text{and } c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \varphi(\gamma(t)) \text{ where}$$
$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_0, \gamma(1) = u_1\},$$

then $c \geq m_\rho$ and c is a critical value of φ.

The main spaces that we will use in the analysis of problem (P_λ) are the Sobolev space $W^{1,p}(\Omega)$ and the Banach space $C^1(\overline{\Omega})$. The latter is an ordered Banach space with positive cone

$$\mathcal{C}_+ = \left\{u \in C^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\right\}.$$

This cone has a nonempty interior given by

$$int\ \mathcal{C}_+ = \left\{u \in \mathcal{C}_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\right\}.$$

By $\|.\|$ we denote the norm of the Sobolev space $W^{1,p}(\Omega)$, that is

$$\|u\| = \left[\|u\|_p^p + \|Du\|_p^p\right]^{\frac{1}{p}} \text{ for all } u \in W^{1,p}(\Omega),$$

where $\|.\|_p$ denotes the norm in $L^p(\Omega)$ (or $L^p(\Omega, \mathbb{R}^N)$.

Also, by $\|.\|$ we denote the $\mathbb{R}^N$-norm. However, no confusion is possible, since it will be clear from the context which norm is used. The inner product in $\mathbb{R}^N$ will be denoted by $(.,.)_{\mathbb{R}^N}$.

Let $\theta \in C^1(0,\infty)$ and assume that there exist constants $\widehat{C}$, C_0, C_1, $C_2 > 0$ such that

$$\widehat{C} \leq \frac{t\theta'(t)}{\theta(t)} \leq C_0 \text{ and } C_1 t^{p-1} \leq \theta(t) \leq C_2\left(1 + t^{p-1}\right) \text{ for all } t > 0, \tag{2.1}$$

with $1 < p < \infty$. The hypotheses on the map $a : \mathbb{R}^N \to \mathbb{R}^N$ are the following:

$\mathbf{H(a)}:$ $a(y) = a_0(\|y\|)y$ for all $y \in \mathbb{R}^N$ with $a_0(t) > 0$ for all $t > 0$ and

(*i*) $a_0 \in C^1(0,\infty)$, $t \to ta_0(t)$ is strictly increasing in $(0,\infty)$, $ta_0(t) \to 0^+$ as $t \to 0^+$ and

$$\lim_{t\to 0^+} \frac{ta_0'(t)}{a_0(t)} > -1;$$

(*ii*)

$$\|\nabla a(y)\| \leq C_3 \frac{\theta(\|y\|)}{\|y\|} \text{ for some } C_3 > 0 \text{ and all } y \in \mathbb{R}^N \backslash \{0\};$$

(iii)

$$\frac{\theta(\|y\|)}{\|y\|}\|\xi\|^2 \leq (\nabla a(y)\xi,\xi)_{\mathbb{R}^N} \text{ for all } y \in \mathbb{R}^N \setminus \{0\}, \text{ all } \xi \in \mathbb{R}^N;$$

(iv) if $G_0(t) = \int_0^t s a_0(s)\,ds$, then there exists $\xi_0 > 0$ such that

$$pG_0(t) - t^2 a_0(t) \geq -\xi_0 \text{ for all } t > 0;$$

(v) there exist $\tau, q \in (1,p)$ such that

$$t \to G_0\left(t^{\frac{1}{\tau}}\right) \text{ is convex and } \lim_{t\to 0^+}\frac{G_0(t)}{t^q} = 0.$$

Remarks: The hypotheses $\mathbf{H}(\mathbf{a})$ (i), (ii), (iii) were motivated by the regularity results of Lieberman [15] (p. 320) and the nonlinear maximum principle of Pucci-Serrin [25] (pp. 111, 120). Hypotheses $\mathbf{H}(\mathbf{a})$ (iv), (v) are particular for our problem here, but they are quite general and are satisfied by many differential operators of interest (see the example 3 below). Hypotheses $\mathbf{H}(\mathbf{a})$ imply that the primitive $G_0(.)$ is strictly convex and strictly increasing.

We set

$$G(y) = G_0(\|y\|) \text{ for all } y \in \mathbb{R}^N.$$

Evidently $G(.)$ is convex and differentiable on $\mathbb{R}^N$. We have

$$\nabla G(y) = G_0'(\|y\|)\frac{y}{\|y\|} = a_0(\|y\|)\,y \text{ for all } y \in \mathbb{R}^N \setminus \{0\},\ \nabla G(0) = 0.$$

Since $G(.)$ is convex and $G(0) = 0$, we have

$$G(y) \leq (a(y), y)_{\mathbb{R}^N} \text{ for all } y \in \mathbb{R}^N. \tag{2.2}$$

The next lemma summarizes the main properties of $G(.)$ and is an easy consequence of hypotheses $\mathbf{H}(\mathbf{a})$ (i), (ii), (iii).

Lemma 2.2. *If hypotheses* $\mathbf{H}(\mathbf{a})$ (i), (ii), (iii) *hold, then:*

(a) *the map $y \to a(y)$ is continuous and strictly monotone, hence maximal monotone too;*

(b)

$$\|a(y)\| \leq C_4\left(1 + \|y\|^{p-1}\right) \text{ for some } C_4 > 0 \text{ and all } y \in \mathbb{R}^N;$$

(c)

$$(a(y), y)_{\mathbb{R}^N} \geq \frac{C_1}{p-1}\|y\|^p \text{ for all } y \in \mathbb{R}^N.$$

This lemma together with (2.1) and (2.2) leads to the following growth estimates for $G(.)$:

COROLLARY 2.3. *If hypotheses* $\mathbf{H}(\mathbf{a})$ (i), (ii), (iii) *hold, then*

$$\frac{C_1}{p(p-1)}\|y\|^p \le G(y) \le C_5(1+\|y\|^p) \text{ for some } C_5>0, \text{ all } y\in\mathbb{R}^N.$$

Examples: The following maps $a(y)$ satisfy hypotheses $\mathbf{H}(\mathbf{a})$:

(a) $a(y)=\|y\|^{p-2}y$ with $1<p<\infty$.

This map corresponds to the $p-$Laplace differential operator defined by

$$\triangle_p u = div\left(\|Du\|^{p-2}Du\right) \text{ for all } u\in W^{1,p}(\Omega).$$

(b) $a(y)=\|y\|^{p-2}y+\|y\|^{q-2}y$ with $1<q<p<\infty$.

This map corresponds to the $(p,q)-$Laplacian defined by

$$\triangle_p u+\triangle_q u \text{ for all } u\in W^{1,p}(\Omega).$$

Such operators arise in many physical applications (see Cherfils-Ilyasov [6] and the references therein). Recently there have been existence and multiplicity results for equations driven by such operators. We mention the works of Aizicovici-Papageorgiou-Staicu [4], Cingolani-Degiovanni [7], Mugnai-Papageorgiou [22], Papa-georgiou-Radulescu [23], Sun [26].

(c) $a(y)=\left(1+\|y\|^2\right)^{\frac{p-2}{2}}y$ with $1<p<\infty$.

This map correspond to the generalized $p-$mean curvature differential operator defined by

$$div\left(\left(1+\|Du\|^2\right)^{\frac{p-2}{2}}Du\right) \text{ for all } u\in W^{1,p}(\Omega).$$

(d) $a(y)=\|y\|^{p-2}y+\frac{\|y\|^{q-2}y}{1+\|y\|^q}$ with $1<q\le p$.

$$(e)\;\; a(y)=\begin{cases}\|y\|^{p-1}y & \text{if } \|y\|<1\\ 2\|y\|^{p-2}y-\|y\|^{p-3}y & \text{if } 1<\|y\|\end{cases} \quad \text{with } 1<p<\infty.$$

Let $A: W^{1,p}(\Omega)\to W^{1,p}(\Omega)^*$ be the nonlinear map defined by

$$\langle A(u),y\rangle=\int_\Omega (a(Du),Dy)_{\mathbb{R}^N}\,dz \text{ for all } u,\, y\in W^{1,p}(\Omega). \tag{2.3}$$

From Papageorgiou-Rocha-Staicu [24] we have:

PROPOSITION 2.4. *If hypotheses* $\mathbf{H}(\mathbf{a})$ (i), (ii), (iii) *hold, then the map* $A: W^{1,p}(\Omega)\to W^{1,p}(\Omega)^*$ *defined by* (2.3) *is demicontinuous, strictly monotone (hence maximal monotone too) and of type* $(S)_+$ *(that is, if* $u_n \xrightarrow{w} u$ *in* $W^{1,p}(\Omega)$ *and*

$$\limsup_{n\to\infty}\langle A(u_n),u_n-u\rangle\le 0,$$

then $u_n\to u$ *in* $W^{1,p}(\Omega)$ *as* $n\to\infty$*).*

Let $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that

$$|f_0(z,x)| \leq a_0(z)\left(1+|x|^{r-1}\right) \text{ for a. a. } z \in \Omega, \text{ all } x \in \mathbb{R}$$

with $a_0 \in L^\infty(\Omega)_+$ and $1 < r < p^*$, where

$$p^* := \begin{cases} \frac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } p \geq N. \end{cases}$$

We set $F_0(z,x) = \int\limits_0^x f_0(z,s)\,ds$ and consider the C^1- functional $\varphi_0 : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \int\limits_\Omega G(Du(z))\,dz - \int\limits_\Omega F_0(z,u(z))\,dz \text{ for all } u \in W^{1,p}(\Omega).$$

From Motreanu-Papageorgiou [21], we have:

Proposition 2.5. *If hypotheses* $\mathbf{H(a)}$ (i), (ii), (iii) *hold,* $\varphi_0 : W^{1,p}(\Omega) \to \mathbb{R}$ *is as defined above and* $u_0 \in W^{1,p}(\Omega)$ *is a local* $C^1\left(\overline{\Omega}\right)-$ *minimizer of* φ_0, *that is, there exists* $\rho_0 > 0$ *such that*

$$\varphi_0(u_0) \leq \varphi_0(u_0+h) \text{ for all } h \in C^1\left(\overline{\Omega}\right) \text{ with } \|h\|_{C^1(\overline{\Omega})} \leq \rho_0,$$

then $u_0 \in C_0^{1,\alpha}\left(\overline{\Omega}\right)$ *for some* $\alpha \in (0,1)$ *and* u_0 *is also a* $W^{1,p}(\Omega)-$ *minimizer of* φ_0, *that is, there exists* $\rho_1 > 0$ *such that*

$$\varphi_0(u_0) \leq \varphi_0(u_0+h) \text{ for all } h \in W^{1,p}(\Omega) \text{ with } \|h\| \leq \rho_1.$$

Let X be a Banach space, $\varphi \in C^1(X)$ and $c \in \mathbb{R}$. We introduce the following sets

$$\varphi^c = \{u \in X : \varphi(u) \leq c\},$$
$$K_\varphi = \{u \in X : \varphi'(u) = 0\},$$
$$K_\varphi^c = \{u \in K_\varphi : \varphi(u) = c\}.$$

For every topological pair (Y_1, Y_2) with $Y_2 \subset Y_1 \subset X$ and every integer $k \geq 0$, by $H_k(Y_1, Y_2)$ we denote the k^{th}- relative singular homology group with integer coefficients.

Given an isolated $u \in K_\varphi^c$, the *critical groups of* φ at u are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, (\varphi^c \cap U) \setminus \{u\}), \text{ for all integers } k \geq 0,$$

where U is a neighborhood of u such that $K_\varphi \cap \varphi^c \cap U = \{u\}$.

The excision property of the singular homology implies that the above definition is independent of the particular choice of the neighborhood U.

Finally we outline some additional notations used in this paper. By $|.|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$.

Given $x \in \mathbb{R}$, we define

$$x^{\pm} = \max\{\pm x, 0\}.$$

For $u \in W^{1,p}(\Omega)$ we set $u^{\pm}(.) = u(.)^{\pm}$. We know that

$$u^{\pm} \in W^{1,p}(\Omega), \ \ |u| = u^{+} + u^{-}, \ u = u^{+} - u^{-}.$$

Given a measurable function $h : \Omega \times \mathbb{R} \to \mathbb{R}$ (for example, a Carathéodory function), we define

$$N_h(u)(.) = h(., u(.)) \text{ for all } u \in W^{1,p}(\Omega)$$

(the Nemitskii operator corresponding to h). Evidently $z \to N_h(u)(z) = h(z, u(z))$ is measurable.

3. Positive solutions

In this section, we prove a bifurcation-type theorem describing the set of positive solutions of (P_λ) as $\lambda > 0$ varies. We impose the following conditions on the reaction $f(z, x)$:

$(\mathbf{H}_1)$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ a.e. in Ω, $f(z, x) > 0$ for all $x > 0$ and

(i) there exists $a \in L^\infty(\Omega)_+$ such that

$$f(z, x) \le a(z)\left(1 + x^{r-1}\right) \text{ for a.a. } z \in \Omega, \text{all } x \ge 0, \text{ with } p < r < p^*;$$

(ii) if $F(z, x) = \int_0^x f(z, s)\, ds$ then

$$\lim_{x \to +\infty} \frac{F(z, x)}{x^p} = +\infty \text{ uniformly for a.a. } z \in \Omega;$$

(iii) there exist $\mu \in \left(\max\left\{(r-p)\frac{N}{p}, 1\right\}, p^*\right)$ and $\beta_0 > 0$ such that

$$\beta_0 \le \liminf_{x \to +\infty} \frac{f(z, x)\, x - pF(z, x)}{x^\mu} \text{ uniformly for a.a. } z \in \Omega;$$

(iv) there exist $\widehat{\delta}_0$ and $\widehat{C}_0 > 0$ such that

$$f(z, x) \ge \widehat{C}_0 x^{q-1} \text{ for a.a. } z \in \Omega, \text{all } x \in \left[0, \widehat{\delta}_0\right],$$

with $q \in (1, p)$ as in hypothesis $\mathbf{H}(\mathbf{a})(v)$.

Remarks: Since in this section we are looking for positive solutions and all the above hypotheses concern the positive half-axis $\mathbb{R}_+ = [0, \infty)$, we may assume, without any lost of generality, that $f(z, x) = 0$ for a.a. $z \in \Omega$, all $x \le 0$.

Hypotheses $(\mathbf{H}_1)(ii), (iii)$ imply that

$$\lim_{x \to \infty} \frac{f(z, x)}{x^{p-1}} = \infty \text{ uniformly for a.a. } z \in \Omega,$$

that is, for a.a. $z \in \Omega$ $f(z,.)$ is $(p-1)$-superlinear. Usually superlinear problems are treated using the so-called Ambrosetti-Rabinowitz condition (AR-condition, for short). We recall that the AR-condition (the unilateral version) says that there exist $\eta > p$ and $M > 0$ such that

(3.1)

$$0 < \eta F(z,x) \leq f(z,x)x \text{ for a.a. } z \in \Omega, \text{ all } x \geq M \text{ and } essinf\ F(.,M) > 0.$$

From (3.1) through integration, we obtain the weaker condition

$$C_6 x^\eta \leq F(z,x) \text{ for a.a. } z \in \Omega, \text{ all } x \geq M \text{ with } C_6 > 0. \tag{3.2}$$

By (3.2) and since $\eta > p$, we infer that the much weaker condition $(\mathbf{H}_1)(ii)$ holds.

Hypotheses $(\mathbf{H}_1)(ii)$, (iii) together are weaker than the AR condition and allow us to include in our framework superlinear functions with "slower" growth near $+\infty$ (see the examples bellow).

Suppose that the AR-condition holds. We may assume that $\eta > \max\left\{(r-p)\frac{N}{p}, 1\right\}$. We have

$$\begin{aligned}\frac{f(z,x)x - pF(z,x)}{x^\eta} &= \frac{f(z,x)x - \eta F(z,x)}{x^\eta} + (\eta - p)\frac{F(z,x)}{x^\eta} \\ &\geq (\eta - p)C_6 \text{ for a.a. } z \in \Omega, \text{ all } x \geq M.\end{aligned}$$

So, hypothesis $(\mathbf{H}_1)(iii)$ holds.

Hypothesis $(\mathbf{H}_1)(iv)$ implies that the reaction $f(z,.)$ exhibits a concave term near zero. Therefore our hypotheses $(\mathbf{H}_1)$ incorporate the case of equations with competing nonlinearities ("concave-convex problems").

We mention that similar or different extensions of the AR-superlinearity condition can be found in Aizicovici-Papageorgiou-Staicu [3], Costa-Magalhães [8], Li-Yang [16], and Mugnai–Papageorgiou [22].

Examples: The following functions satisfy hypotheses $(\mathbf{H}_1)$. For the sake of simplicity we drop the $z-$ dependence:

$$f_1(x) = x^{q-1} + x^{r-1} \text{ for all } x \geq 0 \text{ with } 1 < q < p < r < p^*,$$

$$f_2(x) = \begin{cases} x^{q-1} & \text{if } x \in [0,1] \\ x^{p-1}(\ln x + 1) & \text{if } 1 < x \end{cases} \quad \text{with } 1 < q < p.$$

Note that f_2 does not satisfy the AR-condition.

We introduce the following two sets:

$$\mathcal{L} = \{\lambda > 0 : (P_\lambda) \text{ admits a positive solution}\}$$

and, for $\lambda \in \mathcal{L}$,

$$\mathcal{S}(\lambda) = \text{the set of positive solutions of } (P_\lambda).$$

We start with a useful observation concerning the solution set $\mathcal{S}(\lambda)$.

PROPOSITION 3.1. *If hypotheses* $\mathbf{H}(\mathbf{a})$ (i), (ii), (iii) *and* $(\mathbf{H}_1)$ *hold, then*

$$\mathcal{S}(\lambda) \subseteq intC_+.$$

PROOF. Let $\lambda \in \mathcal{L}$ and $u \in \mathcal{S}(\lambda)$. We have

$$-div\ a(Du(z)) + \lambda u(z)^{p-1} = f(z, u(z)) \text{ a.e. in } \Omega,\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \tag{3.3}$$

(see Motreanu-Papageorgiou [20]). From Hu-Papageorgiou [14] and Winkert [27], we have that $u \in L^\infty(\Omega)$. So, we can apply the regularity result of Lieberman [15] (p. 320) and infer that $u \in C_+ \setminus \{0\}$.

Since $f \geq 0$ (see hypotheses $(\mathbf{H}_1)$), from (3.3) we have

$$div\ a(Du(z)) \leq \lambda u(z)^{p-1} \text{ for a.a. } z \in \Omega. \tag{3.4}$$

Let

$$\chi(t) = ta_0(t) \text{ for all } t > 0.$$

Then from the one-dimensional version of hypothesis $\mathbf{H}(\mathbf{a})$ (iii) we have

$$t\chi'(t) = t^2 a_0'(t) + ta_0(t) \geq C_1 t^{p-1},$$

hence

$$\int_0^t s\chi'(s)\,ds = t\chi(t) - \int_0^t \chi(s)\,ds = t^2 a_0(t) - G_0(t) \geq \frac{C_1}{p} t^p \text{ for all } t \geq 0. \tag{3.5}$$

Let

$$H(t) = t^2 a_0(t) - G_0(t) \text{ and } H_0(t) = \frac{C_1}{p} t^p \text{ for all } t \geq 0.$$

Let $s \in (0,1)$ and consider the sets

$$D_s = \{t \in (0,1) : H(t) \geq s\} \text{ and } D_s^0 = \{t \in (0,1) : H_0(t) \geq s\}.$$

From (3.5) we see that $D_s^0 \subseteq D_s$, hence we have successively: $\inf D_s^0 \leq \inf D_s$, $H^{-1}(s) \leq H_0^{-1}(s)$, and

$$\int_0^\delta \frac{1}{H^{-1}\left(\frac{\lambda}{p}s^p\right)} ds \geq \int_0^\delta \frac{1}{H_0^{-1}\left(\frac{\lambda}{p}s^p\right)} ds = C_7 \int_0^\delta \frac{ds}{s} = +\infty \text{ for some } C_7 > 0.$$

Because of (3.4) we can apply the strong maximum principle of Pucci-Serrin [25] (p. 111) and deduce that $u(z) > 0$ for all $z \in \Omega$. Then invoking the boundary point theorem of Pucci-Serrin [25] (p. 120), we conclude that $u \in int\ C_+$. Therefore $\mathcal{S}(\lambda) \subseteq int\ C_+$.

Next we show that $\mathcal{L}$ is nonempty and prove a structural property of $\mathcal{L}$, namely that $\mathcal{L}$ is a half-line.

PROPOSITION 3.2. *If hypotheses* $\mathbf{H}(\mathbf{a})$ *and* $(\mathbf{H}_1)$ *hold, then*

$$\mathcal{L} \neq \emptyset, \text{ and } \lambda \in \mathcal{L} \text{ implies that } [\lambda, +\infty) \subseteq \mathcal{L}.$$

Proof. We consider the following auxiliary Neumann

$$-div\ a\left(Du\left(z\right)\right)+u\left(z\right)^{p-1}=1 \text{ in } \Omega, \frac{\partial u}{\partial n}=0 \text{ on } \partial\Omega,\ u>0. \tag{3.6}$$

Let $K_p : L^p(\Omega) \to L^{p'}(\Omega)$ ($\frac{1}{p}+\frac{1}{p'}=1$) be the nonlinear map defined by

$$K_p\left(u\right)\left(.\right)=\left|u\left(.\right)\right|^{p-2}u\left(.\right) \text{ for all } u\in L^p\left(\Omega\right).$$

Clearly K_p is continuous and strictly monotone and so is $K_p\mid_{W^{1,p}(\Omega)}$ which implies that $K_p\mid_{W^{1,p}(\Omega)}$ is maximal monotone. Let $V : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be defined by

$$V\left(u\right)=A\left(u\right)+K_p\left(u\right) \text{ for all } u\in W^{1,p}\left(\Omega\right).$$

Using Proposition 2.4, from Gasinski-Papageorgiou [11] (p.320), we conclude that $V(.)$ is maximal monotone. Also, we have

$$\begin{aligned}\left\langle V\left(u\right),u\right\rangle &= \left\langle A\left(u\right),u\right\rangle+\left\|u\right\|_p^p \geq \frac{C_1}{p-1}\left\|Du\right\|_p^p+\left\|u\right\|_p^p \text{ (see Lemma 2.2)}\\ &\geq C_8\left\|u\right\|^p \text{ for some } C_8>0\end{aligned}$$

hence $V(.)$ is coercive. Then from in [11] (p.320), we have that $V(.)$ is surjective.

So, we can find $\overline{u}\in W^{1,p}(\Omega)$, $\overline{u}\neq 0$ such that $V(\overline{u})=0$, hence

$$A\left(\overline{u}\right)+\left|\overline{u}\right|^{p-2}\overline{u}=1. \tag{3.7}$$

On (3.7) we act with $-\overline{u}^-$ and obtain

$$\frac{C_1}{p-1}\left\|D\overline{u}^-\right\|_p^p+\left\|\overline{u}^-\right\|_p^p\leq 0 \text{ (see Lemma 2.2)},$$

hence

$$\overline{u}\geq 0, \overline{u}\neq 0.$$

Then (3.7) becomes

$$A\left(\overline{u}\right)+\overline{u}^{p-1}=1,$$

hence $\overline{u}$ is a positive solution of the auxiliary problem (3.6).

As in the proof of Proposition 3.1, using the nonlinear regularity theory (see [14], [27] and [15]) and the nonlinear maximum principle (see [25]), we have $\overline{u}\in int\ C_+$. So, we can find $C_9>0$ such that

$$\overline{u}\left(z\right)\geq C_9 \text{ for all } z\in\overline{\Omega}.$$

Let

$$\lambda_0=\frac{1+\left\|N_f\left(\overline{u}\right)\right\|_\infty}{C_9^{p-1}}$$

(see hypothesis $(\mathbf{H}_1)(i)$). Then

$$A\left(\overline{u}\right)+\lambda_0\overline{u}^{p-1}\geq N_f\left(\overline{u}\right) \text{ in } W^{1,p}\left(\Omega\right)^*. \tag{3.8}$$

Using $\overline{u} \in int\ C_+$, we introduce the following truncation of the reaction $f(z,.)$:

$$f_0(z,x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ f(z,x) & \text{if} \quad 0 \leq x \leq \overline{u}(z) \\ f(z,\overline{u}(z)) & \text{if} \quad \overline{u}(z) < x. \end{cases} \tag{3.9}$$

This is a Carathéodory function. Let

$$F_0(z,x) = \int_0^x f_0(z,s)\,ds$$

and consider the C^1-functional $\varphi_0 : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \int_\Omega G(Du(z))\,dz + \frac{\lambda_0}{p}\|u\|_p^p - \int_\Omega F_0(z,u(z))\,dz \text{ for all } u \in W^{1,p}(\Omega).$$

From (3.9) it is clear that φ_0 is coercive. Also, using the Sobolev embedding theorem, we see that φ_0 is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $u_0 \in W^{1,p}(\Omega)$ such that

$$\varphi_0(u_0) = \inf\left\{\varphi_0(u) : u \in W^{1,p}(\Omega)\right\}. \tag{3.10}$$

By virtue of $\mathbf{H}(\mathbf{a})(v)$ and $(\mathbf{H}_1)(iv)$, given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) \in \left(0, \widehat{\delta}_0\right]$ such that

$$G_0(t) \leq \frac{\varepsilon}{q}t^q \text{ for all } t \in [0,\delta],$$

hence

$$G(y) \leq \frac{\varepsilon}{q}\|y\|^q \text{ for all } y \in \mathbb{R}^N \text{ with } \|y\| \leq \delta. \tag{3.11}$$

Given $u \in int\ C_+$, we can find $t \in (0,1]$ small such that

$$tu \leq \overline{u},\ tu(z) \in (0,\delta] \text{ and } t\|Du(z)\| \in [0,\delta] \text{ for all } z \in \overline{\Omega} \tag{3.12}$$

(recall $u,\ \overline{u} \in int\ C_+$ and use Lemma 3.3 of Filippakis-Kristaly-Papageorgiou [10]). Then we have

$$\begin{aligned} \varphi_0(tu) &= \textstyle\int_\Omega G(tDu(z))\,dz + \frac{\lambda_0 t^p}{p}\|u\|_p^p - \int_\Omega F_0(z,tu(z))\,dz \\ &\leq \frac{\lambda_0 t^p}{p}\|u\|_p^p - \frac{t^q}{q}\left[\widehat{C}_0\|u\|_q^q - \varepsilon\|Du\|_q^q\right] \end{aligned} \tag{3.13}$$

(see (3.11), (3.12) and hypothesis $\mathbf{H}(\mathbf{a})(iv)$). We choose

$$\varepsilon \in \left(0, \frac{\widehat{C}_0\|u\|_q^q}{\|Du\|_q^q}\right).$$

Then from (3.13) it follows

$$\varphi_0(tu) \leq \frac{\lambda_0 t^p}{p}\|u\|_p^p - C_{10}t^q \text{ for some } C_{10} = C_{10}(u) > 0. \tag{3.14}$$

Since $q < p$ (see hypothesis $\mathbf{H}(\mathbf{a})(iv)$), choosing $t \in (0,1)$ even smaller if necessary, from (3.14) we see that

$$\varphi_0(tu) < 0.$$

which implies

$$\varphi_0(u_0) < 0 = \varphi_0(0) \ \text{(see (3.10))}$$

hence

$$u_0 \neq 0.$$

From (3.10) we have $\varphi_0'(u_0) = 0$, hence

$$A(u_0) + \lambda_0 |u_0|^{p-2} u_0 = N_{f_0}(u_0). \tag{3.15}$$

On (3.15) we act with $-u_0^- \in W^{1,p}(\Omega)$ and obtain

$$\frac{C_1}{p-1} \left\| Du_0^- \right\|_p^p + \lambda_0 \left\| u_0^- \right\|_p^p \leq 0 \ \text{(see Lemma 2.2 and (3.9))},$$

hence

$$u_0 \geq 0, \ u_0 \neq 0.$$

Also, on (3.15) we act with $(u_0 - \overline{u})^+ \in W^{1,p}(\Omega)$. We obtain

$$\begin{aligned}
&\left\langle A(u_0), (u_0-\overline{u})^+ \right\rangle + \lambda_0 \int_\Omega u_0^{p-1} (u_0-\overline{u})^+ \, dz \\
&= \int_\Omega f_0(z,u_0)(u_0-\overline{u})^+ \, dz \\
&= \int_\Omega f(z,\overline{u})(u_0-\overline{u})^+ \, dz \ \text{(see (3.9))} \\
&\leq \left\langle A(\overline{u}), (u_0-\overline{u})^+ \right\rangle + \lambda_0 \int_\Omega \overline{u}^{p-1} (u_0-\overline{u})^+ \, dz \ \text{(see (3.8))},
\end{aligned}$$

hence

$$\begin{aligned}
&\int_{\{u_0 > \overline{u}\}} \left(a(Du_0) - a(D\overline{u}), Du_0 - D\overline{u}\right)_{\mathbb{R}^N} \\
&+ \lambda_0 \int_{\{u_0 > \overline{u}\}} \left({u_0}^{p-1} - \overline{u}^{p-1}\right)(u_0 - \overline{u}) \, dz \\
&\leq 0,
\end{aligned}$$

therefore

$$\left|\{u_0 > \overline{u}\}\right|_N = 0,$$

and we conclude that

$$u_0 \leq \overline{u}.$$

So, we have proved that

$$u_0 \in [0, \overline{u}] := \left\{ u \in W^{1,p}(\Omega) : 0 \leq u(z) \leq \overline{u}(z) \ \text{for a.a. } z \in \Omega \right\}, \ u_0 \neq 0.$$

Then equation (3.15) becomes

$$A(u_0) + \lambda_0 u_0^{p-1} = N_f(u_0) \ \text{(see (3.9))}$$

therefore

$$u_0 \in \mathcal{S}(\lambda_0) \subseteq int\, C_+$$

(see Proposition 3.1) and so

$$\lambda_0 \in \mathcal{L}.$$

Now let $\lambda \in \mathcal{L}$ and $\eta > \lambda$. Then there exists $u_\lambda \in \mathcal{S}(\lambda) \subseteq int\ C_+$ (see Proposition 3.1). We have

$$A(u_\lambda) + \eta u_\lambda^{p-1} \geq A(u_\lambda) + \lambda u_\lambda^{p-1} = N_f(u_\lambda) \text{ in } W^{1,p}(\Omega)^*. \tag{3.16}$$

We truncate $f(z,.)$ at $u_\lambda(z)$ and reasoning as above with $\overline{u}$ replaced by u_λ and using (3.16) instead of (3.8) , via the direct method, we produce

$$u_\eta \in [0, u_\lambda] \cap \mathcal{S}(\eta) \subseteq [0, u_\lambda] \cap int\ C_+.$$

Therefore $\eta \in \mathcal{L}$ and so we conclude that $[\lambda, +\infty) \subseteq \mathcal{L}$.

A useful by-product of the above proof is the following corollary:

COROLLARY 3.3. *If hypotheses* $\mathbf{H}(\mathbf{a})$ *and* $(\mathbf{H}_1)$ *hold,* $\eta > \lambda \in \mathcal{L}$ *and* $u_\lambda \in \mathcal{S}(\lambda) \subseteq int\ C_+$, *then we can find* $u_\eta \in \mathcal{S}(\eta) \subseteq int\ C_+$ *such that* $u_\eta \leq u_\lambda$.

In fact, we can improve the conclusion of this corollary provided we strengthen a little the hypotheses on the reaction $f(z,x)$. The new hypotheses on the reaction $f(z,x)$ are the following:

$(\mathbf{H}_2)$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for a.a. $z \in \Omega$ $f(z,0) = 0$, $f(z,x) > 0$ for all $x > 0$, hypotheses $(\mathbf{H}_2)(i) - (iv)$ are the same as $(\mathbf{H}_1)(i) - (iv)$ and

- (v) for every $\rho > 0$, there exists $\xi_\rho > 0$ such that for a.a. $z \in \Omega$ the function $x \to f(z,x) + \xi_\rho x^{p-1}$ is nondecreasing on $[0, \rho]$.

Remarks: Note that if for a.a. $z \in \Omega$, $f(z,.) \in C^1(0,\infty)$ and $f_x(z,.)$ is $L^\infty(\Omega)$ −bounded on compact subsets of $(0,\infty)$, the hypothesis $(\mathbf{H}_2)(v)$ is automatically satisfied. So, the two examples given after hypotheses $(\mathbf{H}_1)$ satisfy $(\mathbf{H}_2)(v)$.

PROPOSITION 3.4. *If hypotheses* $\mathbf{H}(\mathbf{a})$ *and* $(\mathbf{H}_2)$ *hold,* $\eta > \lambda \in \mathcal{L}$ *and* $u_\lambda \in \mathcal{S}(\lambda) \subseteq int\ C_+$, *then we can find* $u_\eta \in \mathcal{S}(\eta) \subseteq int\ C_+$ *such that*

$$u_\lambda - u_\eta \in int\ C_+.$$

PROOF. From Corollary 3.3, we know that there exists $u_\eta \in \mathcal{S}(\eta) \subseteq int\ C_+$ such that

$$u_\eta \leq u_\lambda. \tag{3.17}$$

Let $\delta > 0$ and set $u_\eta^\delta = u_\eta + \delta \in int\ C_+$. Let $\rho = \|u_\lambda\|_\infty$ and let $\xi_\rho > 0$ be as postulated by hypothesis $(\mathbf{H}_2)(v)$. We have

$$\begin{aligned}
&- div\ a\left(Du_\eta^\delta\right) + (\lambda + \xi_\rho)\left(u_\eta^\delta\right)^{p-1} \\
&\leq -div\ a\left(Du_\eta\right) + \eta u_\eta^{p-1} - (\eta - \lambda)\, u_\eta^{p-1} + \xi_\rho u_\eta^{p-1} + \sigma(\delta) \text{ with } \sigma(\delta) \to 0^+ \text{ as } \delta \to 0^+ \\
&\leq -div\ a\left(Du_\eta\right) + (\eta + \xi_\rho)\, u_\eta^{p-1} - (\eta - \lambda)\, m_\eta^{p-1} + \sigma(\delta) \text{ with } m_\eta = \min_{\overline{\Omega}} u_\eta > 0 \\
&\leq -div\ a\left(Du_\eta\right) + (\eta + \xi_\rho)\, u_\eta^{p-1} \text{ for } \delta > 0 \text{ small} \\
&= f(z, u_\eta) + \xi_\rho u_\eta^{p-1} \text{ (since } u_\eta \in \mathcal{S}(\eta)\text{)} \\
&\leq f(z, u_\lambda) + \xi_\rho u_\lambda^{p-1} \text{ (see (3.17) and hypothesis } (\mathbf{H}_2)(v)\text{)} \\
&= -div\ a\left(Du_\lambda\right) + \xi_\rho u_\lambda^{p-1} \text{ (since } u_\lambda \in \mathcal{S}(\lambda)\text{)},
\end{aligned}$$

hence

$$u_\eta^\delta \leq u_\lambda \text{ for all } \delta > 0 \text{ small (see Damascelli [9], p.495)}$$

therefore

$$u_\lambda - u_\eta \in int\ C_+.$$

Let

$$\lambda_* = \inf \mathcal{L}.$$

In what follows, for every $\lambda > 0$, $\varphi_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ is the energy functional defined by

$$\varphi_\lambda(u) = \int_\Omega G(Du(z))\, dz + \frac{\lambda}{p}\|u\|_p^p - \int_\Omega F(z, u(z))\, dz \text{ for all } u \in W^{1,p}(\Omega).$$

Evidently $\varphi_\lambda \in C^1\left(W^{1,p}(\Omega)\right)$.

Proposition 3.5. *If hypotheses* $\mathbf{H}(\mathbf{a})$ *and* $(\mathbf{H}_1)$ *hold, then* $\lambda_* > 0$

Proof. We argue by contradiction. So, suppose that $\lambda_* = 0$ and let $\{\lambda_n\}_{n\in\mathbb{N}} \subseteq (0, \infty) \subseteq \mathcal{L}$ be such that $\lambda_n \downarrow 0$ as $n \to \infty$. We can find $u_n \in \mathcal{S}(\lambda_n)$ for $n \geq 1$, such that $\{u_n\}_{n\in\mathbb{N}}$ is nondecreasing and

$$\varphi_{\lambda_n}(u_n) < 0 \text{ for all } n \geq 1 \tag{3.18}$$

(see the last part of the proof of Proposition 3.2). From (3.18) we have

$$-\int_\Omega pF(z, u_n)\, dz \leq -\int_\Omega pG(Du_n)\, dz - \lambda_n \|u_n\|_p^p \text{ for all } n \geq 1. \tag{3.19}$$

Since $u_n \in \mathcal{S}(\lambda_n)$ for $n \geq 1$, we have

$$A(u_n) + \lambda_n u_n^{p-1} = N_f(u_n)$$

hence

$$\int_\Omega f(z, u_n)\, u_n dz = \int_\Omega (a(Du_n), Du_n)_{\mathbb{R}^N} + \lambda_n \|u_n\|_p^p \text{ for all } n \geq 1. \tag{3.20}$$

Adding (3.19) and (3.20), we obtain

$$\int_\Omega \left[f(z,u_n)u_n - pF(z,u_n)\right]dz \leq \int_\Omega \left[(a(Du_n), Du_n)_{\mathbb{R}^N} - pG(Du_n)\right]dz,$$

hence

$$\int_\Omega \left[f(z,u_n)u_n - pF(z,u_n)\right]dz \leq \xi_0 \text{ for all } n \geq 1 \text{ (see } (\mathbf{H}(\mathbf{a}))(iv)).$$

From hypotheses $(\mathbf{H}_1)(i), (iii))$, we see that we can find $\beta_1 \in (0,\beta_0)$ and $C_{11} > 0$ such that

$$\beta_1 x^\mu - C_{11} \leq f(z,x)x - pF(z,x) \text{ for a.a. } z \in \Omega, \text{ all } x \geq 0. \tag{3.21}$$

Using (3.21) and (3.20), we have

$$\|u_n\|_\mu^\mu \leq C_{12} \text{ for some } C_{12} > 0, \text{ all } n \geq 1,$$

hence

$$\{u_n\}_{n\in\mathbb{N}} \subseteq L^\mu(\Omega) \text{ is bounded.} \tag{3.22}$$

First suppose that $p < N$. It is clear from hypothesis $(\mathbf{H}_1)(iii)$ that without any lost of generality we may assume that $\mu < r < p^*$. Let $t \in (0,1)$ such that

$$\frac{1}{r} = \frac{1-t}{\mu} + \frac{t}{p^*}. \tag{3.23}$$

Invoking the interpolation inequality (see for example, Gasinski-Papageorgiou [11], p. 905), we have

$$\|u_n\|_r \leq \|u_n\|_\mu^{1-t}\|u_n\|_{p^*}^t \text{ for all } n \geq 1.$$

Then using (3.22) and the Sobolev embedding theorem we have

$$\|u_n\|_r^r \leq C_{13}\|u_n\|^{tr} \text{ for some } C_{13} > 0, \text{ all } n \geq 1. \tag{3.24}$$

Hypothesis $(\mathbf{H}_1)(i)$ implies that

$$f(z,x) \leq C_{14}(1+x^r) \text{ for a.a. } z \in \Omega, \text{ all } x \geq 0, \text{ some } C_{14} > 0. \tag{3.25}$$

From (3.20) and (3.25), we have

$$\int_\Omega (a(Du_n), Du_n)_{\mathbb{R}^N} + \lambda_n\|u_n\|_p^p \leq C_{15}(1+\|u_n\|_r^r) \text{ for some } C_{15} > 0, \text{ all } n \geq 1,$$

hence

$$\frac{C_1}{p-1}\|Du_n\|_p^p \leq C_{16}\left(1+\|u_n\|^{tr}\right) \text{ for some } C_{16} > 0, \text{ all } n \geq 1, \tag{3.26}$$

(see Lemma 2.2 and (3.24)). Recall that $u \to \|u\|_\mu + \|Du\|_p$ is an equivalent norm on the Sobolev space $W^{1,p}(\Omega)$ (see for example, Gasinski-Papageorgiou [11] (p.227)). So, from (3.22) and (3.26) we have

$$\|u_n\|^p \leq C_{17}\left(1+\|u_n\|^{tr}\right) \text{ for some } C_{17} > 0, \text{ all } n \geq 1. \tag{3.27}$$

From (3.23) and the hypothesis on μ (see $(\mathbf{H}_1)(iii)$) it follows that $tr < p$. Therefore from (3.27), we infer that

$$\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,p}(\Omega) \text{ is bounded.} \tag{3.28}$$

If $p \geq N$, then $p^* = +\infty$ and $W^{1,p}(\Omega) \hookrightarrow L^\theta(\Omega)$ for all $\theta \in [1,+\infty)$. Then the previous argument works if we replace p^* by $\eta > r > \mu$ and we choose $t \in (0,1)$ such that

$$\frac{1}{r} = \frac{1-t}{\mu} + \frac{t}{\eta},$$

that is,

$$tr = \frac{\eta(r-\mu)}{\eta-\mu}.$$

Note that $\frac{\eta(r-\mu)}{\eta-\mu} \to r-\mu$ as $\eta \to +\infty = p^*$. But by hypothesis $(\mathbf{H}_1)(iii)$, $r-\mu < p$. Therefore for $\eta > r$ large, we have $tr < p$ and so again (3.28) holds.

By virtue of (3.28) and by passing to a subsequence if necessary, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W^{1,p}(\Omega) \text{ and } u_n \to u \text{ in } L^r(\Omega) \text{ as } n\to\infty. \tag{3.29}$$

Recall that

$$A(u_n) + \lambda_n u_n^{p-1} = N_f(u_n) \text{ for all } n \geq 1. \tag{3.30}$$

On (3.30) we act with $u_n - u \in W^{1,p}(\Omega)$, pass to the limit as $n\to\infty$ and use (3.29) to obtain

$$\lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = 0,$$

hence

$$u_n \to u \text{ in } W^{1,p}(\Omega) \text{ as } n\to\infty \tag{3.31}$$

(see Proposition 2.4). So, if in (3.30) we pass to the limit as $n\to\infty$ and use (3.31) and the fact that $\lambda_n \downarrow 0$, we obtain

$$A(u) = N_f(u). \tag{3.32}$$

By the nonlinear regularity theory (see Lieberman [15]) it follows that

$$u \in C_+.$$

Claim: $u \neq 0$.

From hypotheses $(\mathbf{H}_1)(i)$, (iv), we see that we can find $C_{18} > 0$ such that

$$f(z,x) \geq \widehat{C}_0 x^{q-1} - C_{18}x^{r-1} \text{ for a.a. } z\in\Omega, \text{ all } x \geq 0.$$

Motivated by this unilateral growth estimate, we introduce the following auxiliary Neumann problem

$$\begin{aligned} &-div\, a(Du(z)) + \lambda_1 u(z)^{p-1} = \widehat{C}_0 u(z)^{q-1} - C_{18}u(z)^{r-1} \text{ in } \Omega \\ &\tfrac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega,\ u > 0. \end{aligned} \tag{3.33}$$

Let $\psi : W^{1,p}(\Omega) \to \mathbb{R}$ be the energy functional for problem (3.33) defined by

$$\psi(u) = \int_\Omega G(Du(z))\,dz + \frac{\lambda_1}{p}\|u\|_p^p - \frac{\widehat{C}_0}{q}\|u^+\|_q^q + \frac{C_{18}}{r}\|u^+\|_r^r \text{ for all } u \in W^{1,p}(\Omega).$$

Since $q < p < r$, it is clear that ψ is coercive (see Corollary 2.3). Also, by the Sobolev embedding theorem, we see that ψ is sequentially weakly lower semicontinuous. So, we can find $\widetilde{u} \in W^{1,p}(\Omega)$ such that

$$\psi(\widetilde{u}) = \inf\left\{\psi(u) : u \in W^{1,p}(\Omega)\right\}. \tag{3.34}$$

Because $q < p < r$, as in the proof of Proposition 3.2, we check that

$$\psi(\widetilde{u}) < 0 = \psi(u), \text{ hence } \widetilde{u} \neq 0.$$

From (3.34), we have

$$\psi'(\widetilde{u}) = 0$$

hence

$$A(\widetilde{u}) + \lambda_1 \|\widetilde{u}\|^{p-2}\widetilde{u} = \widehat{C}_0\left(\widetilde{u}^+\right)^{q-1} - C_{18}\left(\widetilde{u}^+\right)^{r-1}. \tag{3.35}$$

On (3.35) we act with $-\widetilde{u}^- \in W^{1,p}(\Omega)$, and using Lemma 2.2, we obtain

$$\widetilde{u} \geq 0, \widetilde{u} \neq 0.$$

Then (3.35) becomes

$$A(\widetilde{u}) + \lambda_1 \widetilde{u}^{p-1} = \widehat{C}_0 \widetilde{u}^{q-1} - C_{18}\widetilde{u}^{r-1},$$

hence $\widetilde{u}$ is a positive solution of (3.33) and

$$\widetilde{u} \in int\ C_+$$

(by nonlinear regularity [15] and the nonlinear maximum principle [25]). Moreover, as in Aizicovici-Papageorgiou-Staicu [4], we conclude that $\widetilde{u} \in int\ C_+$ is the unique positive solution of (3.33).

Let $u_1 \in \mathcal{S}(\lambda_1) \subseteq int\ C_+$ and consider the Carathéodory function

$$k(z,x) = \begin{cases} 0 & \text{if } x < 0 \\ \widehat{C}_0 x^{q-1} - C_{18}x^{r-1}, & \text{if } 0 \leq x \leq u_1(z) \\ \widehat{C}_0 u_1(z)^{q-1} - C_{18}u_1(z)^{r-1} & \text{if } u_1(z) < x. \end{cases} \tag{3.36}$$

We set

$$K(z,x) = \int_0^x k(z,s)\,ds$$

and consider the C^1-functional $\widehat{\gamma} : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\gamma}(u) = \int_\Omega G(Du(z))\,dz + \frac{\lambda_1}{p}\|u\|_p^p - \int_\Omega K(z,u(z))\,dz \text{ for all } u \in W^{1,p}(\Omega).$$

From (3.36) it is clear that $\widehat{\gamma}(.)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\widetilde{u}_0 \in W^{1,p}(\Omega)$ such that

$$\widehat{\gamma}(\widetilde{u}_0) = \inf\left\{\widehat{\gamma}(u) : u \in W^{1,p}(\Omega)\right\}. \tag{3.37}$$

As before (see the proof of Proposition 3.2), since $1 < q < p < r$, we have

$$\widehat{\gamma}(\widetilde{u}_0) < 0 = \widehat{\gamma}(0), \text{ hence } \widetilde{u}_0 \neq 0.$$

From (3.37), we have

$$\widehat{\gamma}'(\widetilde{u}_0) = 0$$

hence

$$A(\widetilde{u}_0) + \lambda_1 \|\widetilde{u}_0\|^{p-2}\widetilde{u}_0 = N_k(\widetilde{u}_0) \tag{3.38}$$

On (3.38) we first act with $-\widetilde{u}_0^- \in W^{1,p}(\Omega)$ and then with $(\widetilde{u}_0 - u_1)^+ \in W^{1,p}(\Omega)$ and obtain

$$\widetilde{u}_0 \in [0, u_1] := \left\{u \in W^{1,p}(\Omega) : 0 \leq u(z) \leq u_1(z) \text{ for a.a. } z \in \Omega\right\}$$

(see the proof of Proposition 3.2). Using (3.36) and (3.37) we obtain

$$A(\widetilde{u}_0) + \lambda_1 \widetilde{u}_0^{p-1} = \widehat{C}_0 \widetilde{u}_0^{q-1} - C_{18}\widetilde{u}_0^{r-1},$$

hence $\widetilde{u}_0$ is a positive solution of (3.33), and by the uniqueness of the positive solution of (3.33), it follows that

$$\widetilde{u}_0 = \widetilde{u} \in int\ C_+.$$

So, we can say that

$$\widetilde{u} \leq u_1 \leq u_n \text{ for all } n \geq 1$$

(recall that $\{u_n\}_{n\in\mathbb{N}}$ is nondecreasing), hence $\widetilde{u} \leq u$ (see (3.31)), therefore $u \neq 0$. This proves the Claim.

On (3.32) we act with $1 \in int\ C_+$. We obtain

$$0 = \int_\Omega f(z, u)\, dz.$$

But our hypotheses on f and the Claim, imply $\int_\Omega f(z, u)\, dz > 0$, a contradiction. This means that $\lambda_* > 0$.

If we use the stronger hypotheses $(\mathbf{H}_2)$, we can show that for $\lambda \in (\lambda_*, \infty)$, problem (P_λ) admits at least two positive solutions.

Proposition 3.6. *If hypotheses* $\mathbf{H}(\mathbf{a})$ *and* $(\mathbf{H}_2)$ *hold and* $\lambda \in (\lambda_*, \infty)$, *then* (P_λ) *admits at least two positive solutions*

$$u_\lambda,\ \widehat{u}_\lambda \in int\ C_+,\ u_\lambda \neq \widehat{u}_\lambda.$$

PROOF. Let $\eta_1, \eta_2 \in \mathcal{L}$ and assume that $\lambda_* < \eta_1 < \lambda < \eta_2$. From Proposition 3.4, we know that we can find $u_{\eta_1} \in \mathcal{S}(\eta_1) \subseteq int\ C_+$ and $u_{\eta_2} \in \mathcal{S}(\eta_2) \subseteq int\ C_+$ such that $u_{\eta_1} - u_{\eta_2} \in int\ C_+$.

We introduce the following Carathéodory function

$$w(z,x) = \begin{cases} f(z, u_{\eta_2}(z)) & \text{if} \quad x < u_{\eta_2}(z) \\ f(z,x) & \text{if} \quad u_{\eta_2}(z) \le x \le u_{\eta_1}(z) \\ f(z, u_{\eta_1}(z)) & \text{if} \quad u_{\eta_1}(z) < x. \end{cases} \tag{3.39}$$

We set

$$W(z,x) = \int_0^x w(z,s)\, ds$$

and consider the C^1-functional $\xi_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\xi_\lambda(u) = \int_\Omega G(Du(z))\, dz + \frac{\lambda}{p} \|u\|_p^p - \int_\Omega W(z, u(z))\, dz \text{ for all } u \in W^{1,p}(\Omega).$$

From (3.39) we see that $\xi_\lambda(.)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $u_\lambda \in W^{1,p}(\Omega)$ such that

$$\xi_\lambda(u_\lambda) = \inf\left\{\xi_\lambda(u) : u \in W^{1,p}(\Omega)\right\},$$

hence

$$\xi'_\lambda(u_\lambda) = 0$$

therefore

$$A(u_\lambda) + \lambda \|u_\lambda\|^{p-2} u_\lambda = N_w(u_\lambda). \tag{3.40}$$

On (3.40) we act with $(u_\lambda - u_{\eta_1})^+ \in W^{1,p}(\Omega)$ and with $(u_{\eta_2} - u_\lambda)^+ \in W^{1,p}(\Omega)$, and obtain

$$u_\lambda \in [u_{\eta_2}, u_{\eta_1}] := \left\{u \in W^{1,p}(\Omega) : u_{\eta_2}(z) \le u(z) \le u_{\eta_1}(z) \ \text{ for a.a. } z \in \Omega\right\}.$$

In fact, reasoning as in the proof of Proposition 3.4, we show that

$$u_\lambda - u_{\eta_2} \in int\ C_+ \text{ and } u_{\eta_1} - u_\lambda \in int\ C_+,$$

hence

$$u_\lambda(z) \in int_{C^1(\overline{\Omega})}[u_{\eta_2}, u_{\eta_1}]. \tag{3.41}$$

Then from (3.39) we see that $u_\lambda \in \mathcal{S}(\lambda) \subseteq int\ C_+$. So, we have produced one positive solution for (P_λ). To produce a second positive solution, we introduce the Carathéodory function $\widehat{k}(.,.)$ defined by

$$\widehat{k}(z,x) = \begin{cases} f(z, u_{\eta_2}(z)) & \text{if} \quad x < u_{\eta_2}(z) \\ f(z,x) & \text{if} \quad u_{\eta_2}(z) \le x. \end{cases} \tag{3.42}$$

Let

$$\widehat{K}(z,x) = \int_0^x k(z,s)\,ds$$

and consider the C^1−functional $\sigma_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\sigma}_\lambda(u) = \int_\Omega G(Du(z))\,dz + \frac{\lambda}{p}\|u\|_p^p - \int_\Omega \widehat{K}(z,u(z))\,dz \text{ for all } u \in W^{1,p}(\Omega).$$

As before (see the proof of Proposition 3.2) we can check that

$$K_{\widehat{\sigma}_\lambda} \subseteq [u_{\eta_2}) := \left\{u \in W^{1,p}(\Omega) : u_{\eta_2}(z) \le u(z) \text{ for a.a. } z \in \Omega\right\}. \tag{3.43}$$

From (3.39) and (3.42) we see

$$\xi_\lambda\,|_{[u_{\eta_2},u_{\eta_1}]} = \widehat{\sigma}_\lambda\,|_{[u_{\eta_2},u_{\eta_1}]}. \tag{3.44}$$

By (3.41) and (3.44) and since u_λ is a minimizer of ξ_λ, it follows that u_λ is a $C^1\left(\overline{\Omega}\right)$ −minimizer of $\widehat{\sigma}_\lambda$. Invoking Proposition 2.5, we infer that u_λ is a $W^{1,p}(\Omega)-$ minimizer of $\widehat{\sigma}_\lambda$.

We may assume that $K_{\widehat{\sigma}_\lambda}$ is finite or otherwise we have an infinity of positive solutions for problem (P_λ) (see (3.43) and (3.42)). Then, from Aizicovici-Papageorgiou-Staicu [1] (see the proof of Proposition 29), we can find $\rho \in (0,1)$ small such that

$$\widehat{\sigma}_\lambda(u_\lambda) < \inf\{\widehat{\sigma}_\lambda(u) : \|u - u_\lambda\| = \rho\} =: \widehat{m}_\lambda. \tag{3.45}$$

Hypothesis $(\mathbf{H}_2)\,(ii)$ implies

$$\widehat{\sigma}_\lambda(\xi) \to -\infty \text{ as } \xi \to +\infty,\ \xi \in \mathbb{R}. \tag{3.46}$$

In addition, minor changes in the first part of the proof of Proposition 3.5, reveal that

$$\widehat{\sigma}_\lambda \text{ satisfies the C-condition} \tag{3.47}$$

(see also Aizicovici-Papageorgiou-Staicu [3]). Then (3.45), (3.46), (3.47) permit the use of Theorem 2.1 (the mountain pass theorem). So, we can find $\widehat{u}_\lambda \in W^{1,p}(\Omega)$ such that

$$\widehat{u}_\lambda \in K_{\widehat{\sigma}_\lambda} \subseteq [u_{\eta_2}) \text{ (see (3.43)) and } \widehat{\sigma}_\lambda(u_\lambda) < \widehat{m}_\lambda \le \widehat{\sigma}_\lambda(\widehat{u}_\lambda). \tag{3.48}$$

From (3.48) we see that

$$\widehat{u}_\lambda \in \mathcal{S}(\lambda) \subseteq \mathit{int}\ C_+ \text{ (see (3.42)) and } u_\lambda \ne \widehat{u}_\lambda.$$

Next we examine what happens in the critical case $\lambda = \lambda_*$.

Proposition 3.7. *If hypotheses* $\mathbf{H(a)}$ *and* $(\mathbf{H}_1)$ *hold, then* $\lambda_* \in \mathcal{L}$ *and so,* $\mathcal{L} = [\lambda_*, +\infty)$.

Proof. Let $\{\lambda_n\}_{n\in\mathbb{N}} \subseteq \mathcal{L}$ and assume $\lambda_n \downarrow \lambda_*$. We can find $u_n \in \mathcal{S}(\lambda_n) \subseteq$ *int* C_+ such that

$$\varphi_{\lambda_n}(u_n) < 0 \text{ for all } n \geq 1.$$

Then, from the proof of Proposition 3.5, we know that

$$\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,p}(\Omega) \text{ is bounded.}$$

So, we may assume that

$$u_n \xrightarrow{w} u_* \text{ in } W^{1,p}(\Omega) \text{ and } u_n \to u_* \text{ in } L^r(\Omega) \text{ as } n \to \infty. \tag{3.49}$$

We have

$$A(u_n) + \lambda_n u_n^{p-1} = N_f(u_n) \text{ for all } n \geq 1. \tag{3.50}$$

Acting on (3.50) with $u_n - u_* \in W^{1,p}(\Omega)$, passing to the limit as $n \to \infty$ and using (3.49) we obtain

$$\lim_{n\to\infty} \langle A(u_n), u_n - u_* \rangle = 0,$$

hence

$$u_n \to u_* \text{ in } W^{1,p}(\Omega) \text{ as } n \to \infty \tag{3.51}$$

(see Proposition 2.4). Also, from the proof of Proposition 3.5), we know that

$$\widetilde{u}_0 \leq u_n \text{ for all } n \geq 1$$

(here $\widetilde{u}_0 \in$ *int* C_+ denotes the unique positive solution of the auxiliary problem (3.33)). Then from (3.51) we have

$$\widetilde{u}_0 \leq u_*, \text{ hence } u_* \neq 0.$$

If in (3.50) we pass to the limit as $n \to \infty$ and use (3.51), then

$$A(u_*) + \lambda_n u_*^{p-1} = N_f(u_*),$$

hence $u_* \in \mathcal{S}(\lambda_*) \subseteq$ *int* C_+ and so $\lambda_* \in \mathcal{L}$, hence $\mathcal{L} = [\lambda_*, +\infty)$.

In fact, we can show that for every $\lambda \in \mathcal{L} = [\lambda_*, +\infty)$ problem (P_λ) admits a smallest positive solution $\overline{u}_\lambda \in \mathcal{S}(\lambda) \subseteq$ *int* C_+. We will need this fact in the next section where we produce nodal solutions.

Proposition 3.8. *If hypotheses* $\mathbf{H}(\mathbf{a})$ *and* $(\mathbf{H}_1)$ *(resp.* $(\mathbf{H}_2)$*) hold, and* $\lambda \in \mathcal{L} = [\lambda_*, +\infty)$, *then problem* (P_λ) *admits a smallest positive solution* $\overline{u}_\lambda \in$ *int* C_+ *and the map* $\lambda \to \overline{u}_\lambda$ *is nonincreasing (resp. decreasing) and right continuous from* $\mathcal{L}$ *into* $C^1(\overline{\Omega})$.

Proof. As in Aizicovici-Papageorgiou-Staicu [2]), exploiting the monotonicity of A (see Proposition 2.4), we see that for every $\lambda \in \mathcal{L}$, $\mathcal{S}(\lambda)$ is downward directed, that is, if $u_1, u_2 \in \mathcal{S}(\lambda)$, there exists $u \in \mathcal{S}(\lambda)$ such that $u \leq u_1$, $u \leq u_2$.

Since we are looking for the smallest positive solution, and since $\mathcal{S}(\lambda)$ is downward directed, without any lost of generality, we may assume that there exists $C_{19} > 0$ such that

$$\|u\|_\infty \leq C_{19} \text{ for all } u \in W^{1,p}(\Omega). \tag{3.52}$$

From Hu-Papageorgiou [13] (p. 178), we know that we can find $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{S}(\lambda)$ such that

$$\inf \mathcal{S}(\lambda) = \inf_{n\geq 1} u_n.$$

We have

$$A(u_n) + \lambda u_n^{p-1} = N_f(u_n) \text{ for all } n \geq 1. \tag{3.53}$$

Because of (3.52) we have

$$\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,p}(\Omega) \text{ is bounded.}$$

So, we may assume that

$$u_n \xrightarrow{w} \overline{u}_\lambda \text{ in } W^{1,p}(\Omega) \text{ and } u_n \to \overline{u}_\lambda \text{ in } L^r(\Omega) \text{ as } n \to \infty. \tag{3.54}$$

Acting on (3.53) with $u_n - \overline{u}_\lambda \in W^{1,p}(\Omega)$, passing to the limit as $n \to \infty$ and using (3.54) we obtain

$$\lim_{n\to\infty} \langle A(u_n), u_n - \overline{u}_\lambda \rangle = 0,$$

hence

$$u_n \to \overline{u}_\lambda \text{ in } W^{1,p}(\Omega) \text{ as } n \to \infty \tag{3.55}$$

So, if in (3.53) we pass to the limit as $n \to \infty$ and use (3.55), then

$$A(\overline{u}_\lambda) + \lambda (\overline{u}_\lambda)^{p-1} = N_f(\overline{u}_\lambda). \tag{3.56}$$

Recall that

$$\widetilde{u} \leq u_n \text{ for all } n \geq 1,$$

where $\widetilde{u} \in int\ C_+$ is the unique positive solution of problem (3.33) with $\lambda_1 = \lambda$. Then, because of (3.55), we have

$$\widetilde{u}_0 \leq \overline{u}_\lambda,$$

hence

$$\overline{u}_\lambda \in \mathcal{S}(\lambda) \subseteq int\ C_+ \text{ and } \overline{u}_\lambda = \inf \mathcal{S}(\lambda).$$

Next, let $\eta > \lambda$ and let $\overline{u}_\lambda \in \mathcal{S}(\lambda) \subseteq int\ C_+$ be the minimal positive solution of (P_λ). If hypotheses $(\mathbf{H}_1)$ (resp. $(\mathbf{H}_2)$) hold, then from Corollary 3.3 (resp. Proposition 3.4) we know that we can find $u_\eta \in \mathcal{S}(\eta)$ such that

$$\overline{u}_\lambda \geq u_\eta \text{ (resp.} \overline{u}_\lambda - u_\eta \in int\ C_+ \text{)}$$

hence

$$\overline{u}_\lambda \geq \overline{u}_\eta \text{ (resp.} \overline{u}_\lambda - \overline{u_\eta} \in int\ C_+ \text{).}$$

This proves the desired monotonicity of the map $\lambda \to \overline{u}_\lambda$.

Finally, let $\{\lambda_n\}_{n\in\mathbb{N}} \subseteq \mathcal{L}$ such that $\lambda_n \downarrow \lambda$. We have

$$A(\overline{u}_{\lambda_n}) + \lambda_n \overline{u}_{\lambda_n}^{p-1} = N_f(\overline{u}_{\lambda_n}) \text{ for all } n \geq 1. \tag{3.57}$$

From the proof of Proposition 3.5, we know that

$$\{u_{\lambda_n}\}_{n\in\mathbb{N}} \subseteq W^{1,p}(\Omega) \text{ is bounded.} \tag{3.58}$$

Using (3.57), (3.58) and Proposition 2.4, as before, we can show that for at least a subsequence, we have

$$\overline{u}_{\lambda_n} \to \overline{u} \text{ in } W^{1,p}(\Omega) \text{ and } \overline{u} \in \mathcal{S}(\lambda) \subseteq int\ C_+. \tag{3.59}$$

We claim that $\overline{u} = \overline{u}_\lambda \in int\ C_+$. From (3.58), Hu-Papageorgiou [14] (see Proposition 5) and the regularity result of Lieberman [15] (p.320), we know that we can find $\alpha \in (0,1)$ and $C_{20} > 0$ such that

$$u_n \in C^{1,\alpha}\left(\overline{\Omega}\right) \text{ and } \|u_n\|_{C^{1,\alpha}(\overline{\Omega})} \leq C_{20} \text{ for all } n \geq 1.$$

Exploiting the compact embedding of $C^{1,\alpha}\left(\overline{\Omega}\right)$ into $C^1\left(\overline{\Omega}\right)$ and using (3.59), we infer that

$$\overline{u}_{\lambda_n} \to \overline{u} \text{ in } C^1\left(\overline{\Omega}\right). \tag{3.60}$$

If $\overline{u}_\lambda \neq \overline{u}$, then we can find $z_0 \in \Omega$ such that $\overline{u}_\lambda(z_0) \neq \overline{u}(z_0)$, hence

$$\overline{u}_\lambda(z_0) < \overline{u}_{\lambda_n} \text{ for all } n \text{ large enough (see } (3.60)). \tag{3.61}$$

But from the previous part of the proof, we have

$$\overline{u}_{\lambda_n} \leq \overline{u}_\lambda \text{ for all } n \geq 1$$

which contradicts (3.61). So, indeed $\overline{u} = \overline{u}_\lambda$ and we have proved the continuity of $\lambda \to \overline{u}_\lambda$ from $\mathcal{L}$ into $C^1\left(\overline{\Omega}\right)$.

Summarizing the situation for problem (P_λ), we can state the following bifurcation-type result.

THEOREM 3.9. *(a) If hypotheses* $\mathbf{H}(\mathbf{a})$ *and* $(\mathbf{H}_1)$ *hold, then there exists* $\lambda_* > 0$ *such that*

(i) for every $\lambda \in (0, \lambda_*)$, *problem* (P_λ) *has no positive solutions;*

(ii) for all $\lambda \geq \lambda_*$*, problem* (P_λ) *has at least one positive solution. Moreover, for every* $\lambda \geq \lambda_*$*, problem* (P_λ) *has a smallest positive solution* $\overline{u}_\lambda \in int\ C_+$ *and the map* $\lambda \to \overline{u}_\lambda$ *from* $\mathcal{L}$ *into* $C^1\left(\overline{\Omega}\right)$ *is nonincreasing and right continuous.*

(b) If hypotheses $\mathbf{H}(\mathbf{a})$ *and* $(\mathbf{H}_2)$ *hold, then there exists* $\lambda_* > 0$ *such that:*

(i) for every $\lambda \in (0, \lambda_*)$*, problem* (P_λ) *has no positive solutions;*

(ii) for $\lambda = \lambda_*$*, problem* (P_λ) *has at least one positive solution* $u_* \in int\ C_+$*;*

(iii) for every $\lambda > \lambda_*$*, problem* (P_λ) *has at least two positive solutions* u_λ*,* $\widehat{u}_\lambda \in int\ C_+$*,* $u_\lambda \neq \widehat{u}_\lambda$*. Moreover, for every* $\lambda \geq \lambda_*$*, problem* (P_λ) *has a smallest positive solution* $\overline{u}_\lambda \in int\ C_+$ *and the map* $\lambda \to \overline{u}_\lambda$ *from* $\mathcal{L}$ *into* $C^1\left(\overline{\Omega}\right)$ *is decreasing and right continuous.*

4. Nodal solutions

In this section, by imposing bilateral conditions on the reaction $f(z,.)$ we produce nodal solutions.

So, the new hypotheses on the reaction $f(z,x)$ are the following:

$(\mathbf{H}_3)$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for a.a. $z \in \Omega$ $f(z,0) = 0$, $f(z,x)x > 0$ for all $x \neq 0$ and

(i) there exists $a \in L^\infty(\Omega)_+$ such that

$$|f(z,x)| \leq a(z)\left(1 + |x|^{r-1}\right) \text{ for a.a. } z \in \Omega, \text{all } x \in \mathbb{R}, \text{ with } p < r < p^*;$$

(ii) if $F(z,x) = \int_0^x f(z,s)\,ds$ then

$$\lim_{x \to \pm\infty} \frac{F(z,x)}{|x|^p} = +\infty \text{ uniformly for a.a. } z \in \Omega$$

(iii) there exist $\mu \in \left(\max\left\{(r-p)\frac{N}{p}, 1\right\}, p^*\right)$ and $\beta_0 > 0$ such that

$$\beta_0 \leq \liminf_{x \to \pm\infty} \frac{f(z,x)x - pF(z,x)}{|x|^\mu} \text{ uniformly for a.a. } z \in \Omega;$$

(iv) there exists $\widehat{\delta}_0$ such that

$$0 < qF(z,x) \leq f(z,x)x \text{ for a.a. } z \in \Omega, \text{all } |x| \leq \widehat{\delta}_0,$$

and

$$ess\ inf_\Omega F\left(., \pm\widehat{\delta}_0\right) > 0$$

with $q \in (1,p)$ as in hypothesis $\mathbf{H}(\mathbf{a})(v)$.

Remarks: Hypothesis $(\mathbf{H}_3)(iv)$ is a dual AR-condition near zero. It implies the weak condition

$$\widehat{C}_0 |x|^q \leq F(z,x) \text{ for a.a. } z \in \Omega, \text{all } |x| \leq \widehat{\delta}_0, \text{ and some } \widehat{C}_0 > 0 \tag{4.1}$$

(see [19])). So, now we have a stronger condition near zero (see hypothesis $(\mathbf{H}_3)(iv)$). Since the conditions on $f(z,.)$ are now bilateral, reasoning as in

Section 3, we can find $\widehat{\lambda}_* > 0$ such that for all $\lambda \geq \widehat{\lambda}_*$ problem (P_λ) has a biggest negative solution $\overline{v}_\lambda \in -int\ C_+$ (in this case the set of negative solutions of (P_λ) is upward directed, that is, if v_1, v_2 are negative solutions of (P_λ), then there exists a negative solution v of (P_λ) such that $v_1 \leq v$, $v_2 \leq v$).

In what follows, we set

$$\widetilde{\lambda}_* = \max\left\{\lambda_*, \widehat{\lambda}_*\right\}$$

(see Theorem 3.9). We have the following

THEOREM 4.1. *If hypotheses* $\mathbf{H(a)}$ *and* $(\mathbf{H}_3)$ *hold and* $\lambda \geq \widetilde{\lambda}_*$ *, then problem* (P_λ) *admits a nodal solution* $y_\lambda \in C^1\left(\overline{\Omega}\right)$.

PROOF. Let $\overline{u}_\lambda \in int\ C_+$ and $\overline{v}_\lambda \in -int\ C_+$ be the two extremal constant sign solutions of (P_λ). We introduce the following Carathéodory function

$$e(z,x) = \begin{cases} f(z,\overline{v}_\lambda(z)) & \text{if} \quad x < \overline{v}_\lambda(z) \\ f(z,x) & \text{if} \quad \overline{v}_\lambda(z) \leq x \leq \overline{u}_\lambda(z) \\ f(z,\overline{u}_\lambda(z)) & \text{if} \quad \overline{u}_\lambda(z) < x. \end{cases} \tag{4.2}$$

Let

$$e_\pm(z,x) = e\left(z, \pm x^\pm\right)$$

(the positive and negative truncations of $e(z,.)$). We set

$$E(z,x) = \int_0^x e(z,s)\,ds,\ E_\pm(z,x) = \int_0^x e_\pm(z,s)\,ds,$$

and introduce the C^1-functionals $\psi_\lambda, \psi_\lambda^\pm : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_\lambda(u) = \int_\Omega G(Du(z))\,dz + \frac{\lambda}{p}\|u\|_p^p - \int_\Omega E(z,u(z))\,dz \text{ for all } u \in W^{1,p}(\Omega),$$

$$\psi_\lambda^\pm(u) = \int_\Omega G(Du(z))\,dz + \frac{\lambda}{p}\|u\|_p^p - \int_\Omega E_\pm(z,u(z))\,dz \text{ for all } u \in W^{1,p}(\Omega).$$

As before (see the proof of Proposition 3.2), we can show that

$$K_{\psi_\lambda} \subseteq [\overline{v}_\lambda, \overline{u}_\lambda],\ K_{\psi_\lambda^+} \subseteq [0, \overline{u}_\lambda],\ K_{\psi_\lambda^-} \subseteq [\overline{v}_\lambda, 0].$$

The extremality of $\overline{u}_\lambda$ and $\overline{v}_\lambda$ implies

$$K_{\psi_\lambda} \subseteq [\overline{v}_\lambda, \overline{u}_\lambda],\ K_{\psi_\lambda^+} = \{0, \overline{u}_\lambda\}\ K_{\psi_\lambda^-} = \{0, \overline{v}_\lambda\}. \tag{4.3}$$

Claim: $\overline{u}_\lambda \in int\ C_+$ and $\overline{v}_\lambda \in -int\ C_+$ are both local minimizers of ψ_λ.

It is clear from (4.2) that ψ_λ^+ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\overline{u} \in W^{1,p}(\Omega)$ such that

$$\psi_\lambda^+(\overline{u}) = \inf\left\{\psi_\lambda^+(u) : u \in W^{1,p}(\Omega)\right\}. \tag{4.4}$$

By (4.1) and since $q < p$, we see that

$$\psi_\lambda^+(\overline{u}) < 0 = \psi_\lambda^+(0),\text{ hence } \overline{u} \neq 0.$$

From (4.3) and (4.4), it follows that $\overline{u} = \overline{u}_\lambda \in int\ C_+$. Note that

$$\psi_\lambda \mid_{C_+} = \psi_\lambda^+ \mid_{C_+},$$

hence $\overline{u}_\lambda \in int\ C_+$ is a local $C^1\left(\overline{\Omega}\right)$ −minimizer of ψ_λ, therefore $\overline{u}_\lambda \in int\ C_+$ is a local $W^{1,p}(\Omega)$ −minimizer of ψ_λ (see Proposition 2.5).

Similarly for $\overline{v}_\lambda \in -int\ C_+$, using this time the functional ψ_λ^-. This proves the Claim.

Without any loss of generality, we may assume that $\psi_\lambda(\overline{v}_\lambda) \leq \psi_\lambda(\overline{u}_\lambda)$ (the reasoning is similar if the opposite inequality holds).

We assume that K_{ψ_λ} is finite (otherwise we already have infinitely many distinct nodal solutions, see (4.2) and (4.3)). By virtue of the Claim, we can find $\rho \in (0,1)$ small such that

$$\psi_\lambda(\overline{v}_\lambda) \leq \psi_\lambda(\overline{u}_\lambda) < \inf\{\psi_\lambda(u) : \|u - \overline{u}_\lambda\| = \rho\} =: \overline{m}_\lambda,\ \ \|\overline{v}_\lambda - \overline{u}_\lambda\| > \rho \tag{4.5}$$

(see Aizicovici-Papageorgiou-Staicu [1]), proof of Proposition 29). Recall that ψ_λ is coercive. Therefore

$$\psi_\lambda \text{ satisfies the } C-\text{condition.} \tag{4.6}$$

Then (4.5) and (4.6) permit the use of Theorem 2.1 (the mountain pass theorem). So, we can find $y_\lambda \in W^{1,p}(\Omega)$ such that

$$y_\lambda \in K_{\psi_\lambda} \subseteq [\overline{v}_\lambda, \overline{u}_\lambda] \text{ (see (4.3)) and } \overline{m}_\lambda \leq \psi_\lambda(y_\lambda). \tag{4.7}$$

From (4.5) and (4.7) we see that

$$y_\lambda \in [\overline{v}_\lambda, \overline{u}_\lambda],\ y_\lambda \notin \{\overline{v}_\lambda, \overline{u}_\lambda\}.$$

Hence, if we show that $y_\lambda \neq 0$, then y_λ is a nodal solution of (P_λ) (recall that $\overline{u}_\lambda$ and $\overline{v}_\lambda$ are the extremal constant sign solutions of (P_λ)), and the nonlinear regularity of Lieberman [15] will imply that $y_\lambda \in C^1\left(\overline{\Omega}\right)$.

Since y_λ is a critical point of mountain pass type for ψ_λ, we have

$$C_1(\psi_\lambda, y_\lambda) \neq 0 \tag{4.8}$$

(see Motreanu-Motreanu-Papageorgiou [19]). On the other hand, hypothesis $(\mathbf{H}_3)(iv)$ and the work of Marano-Papageorgiou [17], imply

$$C_k(\psi_\lambda, 0) = 0 \text{ for all } k \geq 0. \tag{4.9}$$

From (4.8) and (4.9) it follows that $y_\lambda \neq 0$ and so $y_\lambda \in C^1\left(\overline{\Omega}\right)$ is a nodal solution of (P_λ).

Remarks: Nodal solutions for superlinear Neumann problems driven by the $p-$ Laplacian were obtained by Aizicovici-Papageorgiou-Staicu in [2] (using the AR-condition) and in [3] (without the AR-condition). Theorem 4.1 improves substantially Theorem 3.5 in [18].

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