Minimal Time Function and Viscosity Solutions[1]

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Abstact. Two theorems in Ref. 1 are generalized. It is proved that, if $V(A,\Gamma)$ is the set of points that can be steered to the origin along a solution of the control system $x' = Ax - c$, if $c(t) \in \Gamma, \Gamma$ is a compact subset of $R^n, 0 \in \text{intrelco}\,\Gamma$, and if a rank condition holds, then the minimal time function $T(\cdot)$ is a viscosity solution of the Bellman equation

$$\max\{\langle DT(x), \gamma - Ax\rangle : \gamma \in \text{co}\,\Gamma\} - 1 = 0, \qquad x \in V(A,\Gamma)\backslash\{0\},$$

and of the Hàjek equation

$$1 - \max\{\langle DT(x), \exp[-AT(x)]\rangle : \gamma \in \text{co}\,\Gamma\} = 0, \qquad x \in V(A,\Gamma).$$

Key Words. Minimal time function, Bellman equation, Hàjek equation, viscosity solutions, linear control system.

1. Introduction

The concept of viscosity solution of first-order Hamilton-Jacobi equations was introduced by Crandall and Lions in Ref. 2. This work was reformulated and simplified by Crandall *et al.* in Ref. 3. We refer to Ref. 4 for an extensive reference on the basic aspects of this theory. Lions proved in Ref. 5 that, for a class of optimal control problems, the value function is the unique viscosity solution of the associated Hamilton-Jacobi equation. Neither this result nor the uniqueness results in Refs. 2, 3, 6, and 7 can be applied to the linear minimal time problem, which will be considered in this paper.

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We consider the linear control system

$$x' = Ax - c, \tag{1}$$

where the state vector x is a function of $t \geq 0$ with values $x(t)$ in R^n, A is a n-square matrix, and the control parameter c is a function of $t \geq 0$ with values in a subset Γ of R^n.

Let

$$C_\Gamma = \{c(\cdot): [0, \infty) \to \Gamma: c(\cdot) \text{ is locally integrable}\};$$

let

$$V(t, A, \Gamma) = \left\{ \int_0^t \exp(-sA)c(s)\, ds: c(\cdot) \in C_\Gamma \right\} \tag{2}$$

be the set of points in R^n that can be steered to the origin along a solution of (1) in time t; and let

$$V(A, \Gamma) = \bigcup_{t \geq 0} V(t, A, \Gamma). \tag{3}$$

The function $T(\cdot)$, defined by

$$T(x) = \min\{t \geq 0: x \in V(t, A, \Gamma)\}, \qquad \text{if } x \in V(A, \Gamma), \tag{4a}$$

$$T(x) = +\infty, \qquad \text{otherwise}, \tag{4b}$$

is said to be the minimal time function.

We prove that, if Γ is a compact subset of R^n, $0 \in \operatorname{intrelco} \Gamma$, and the following rank condition holds:

$$y \in C^n, \qquad A^* y = \lambda y, \qquad y^* \Gamma = \text{const} \Rightarrow y = 0, \tag{5}$$

then $T(\cdot)$ defined in (4) is a viscosity solution of the Bellman equation

$$\max\{\langle DT(x), \gamma - Ax\rangle: \gamma \in \operatorname{co} \Gamma\} - 1 = 0, \qquad x \in V(A, \Gamma)\backslash\{0\}, \tag{6}$$

and of the Hàjek equation

$$1 - \max\{\langle DT(x), \exp[-AT(x)]\rangle: \gamma \in \operatorname{co} \Gamma\} = 0,$$

$$x \in V(A, \Gamma). \tag{7}$$

This generalizes Theorem 2 and Theorem 3 in Ref. 1.

We note that, for the case when $T(\cdot)$ is locally Lipschitz, necessary and sufficient conditions of optimality were obtained by Mignanego and Pieri in Refs. 8 and 9, using the Clarke's generalized gradient.

2. Preliminaries

Let R^n be the n-dimensional Euclidian space; let $\langle\cdot, \cdot\rangle$ be the scalar product; and let $|\cdot|$ be the norm on R^n. For a subset X of R^n, we denote by co X the convex hull of X, by ∂X the boundary of X, and by intrelco X the interior of co$X$ relative to its affine hull.

For $x \in R^n$ and $r > 0$, we consider

$$B(x, r) = \{y \in R^n: |y - x| < r\}.$$

For an open subset $\Omega \subset R^n$, we denote

$$C(\Omega) = \{u(\cdot): \Omega \to R: u(\cdot) \text{ is continuous}\}.$$

Definition 2.1. Let $u(\cdot) \in C(\Omega)$ and $x \in \Omega$.

$$\partial_F^- u(x) = \left\{\xi \in R^n: \liminf_{h \to 0} \frac{u(x+h) - u(x) - \langle \xi, h\rangle}{|h|} \geq 0\right\}$$

is said to be the Fréchet subdifferential of $u(\cdot)$ in x; and

$$\partial_F^+ u(x) = \left\{\xi \in R^n: \limsup_{h \to 0} \frac{u(x+h) - u(x) - \langle \xi, h\rangle}{|h|} \leq 0\right\}$$

is said to be the Fréchet superdifferential of $u(\cdot)$ in x.

For equivalent definitions and basic properties of this Fréchet semi-differentials, we refer to Ref. 10.

In what follows, we use the following equivalent definitions:

$$\partial_F^- u(x) = \{\xi \in R^n: (\ni) r > 0, \omega(\cdot) \in C(R^+): \omega(0) = 0, u(y) \geq u(x) + \langle \xi, y - x\rangle + |y - x| \omega(|y - x|), \forall y \in B(x, r)\}, \tag{8}$$

$$\partial_F^+ u(x) = \{\xi \in R^n: (\ni) r > 0, \omega(\cdot) \in C(R^+): \omega(0) = 0, u(y) \leq u(x) + \langle \xi, y - x\rangle + |y - x| \omega(|y - x|), \forall y \in B(x, r)\}. \tag{9}$$

Let now $\Omega \subset R^n$ be open, and let $F(\cdot): \Omega \times R \times R^n \to R$ be a continuous function.

Definition 2.2. (Refs. 2 and 3). A function $u(\cdot) \in C(\Omega)$ is said to be a viscosity solution of the equation

$$F(x, u(x), Du(x)) = 0, \qquad x \in \Omega, \tag{10}$$

if the following inequalities hold:

$$F(x, u(x), \xi) \leq 0, \qquad \forall x \in \Omega, \qquad \forall \xi \in \partial_F^+ u(x), \tag{11}$$

$$F(x, u(x), \xi) \geq 0, \qquad \forall x \in \Omega, \qquad \forall \xi \in \partial_F^- u(x). \tag{12}$$

Remark 2.1. If $u(\cdot)$ is Fréchet differentiable at x, then

$$\partial_F^- u(x) = \partial_F^+ u(x) = \{Du(x)\},$$

where $Du(x)$ denote the differential of $u(x)$ at x. It follows that a classical (i.e., of class C^1) solution of (10) is also a viscosity solution.

3. Main Results

We consider the linear control system (1); and, for $x \in R^n$ and $c(\cdot) \in C_\Gamma$, we denote by $w(t, x, c)$ the solution of (1) corresponding to the control $c(\cdot)$ and the initial condition $x(0) = x$; that is,

$$w(t, x, c) = \exp(tA)x - \int_0^t \exp[(t-s)A]c(s)\, ds. \tag{13}$$

Definition 3.1. Let $x \in V(A, \Gamma)$. A control $c_x(\cdot) \in C_\Gamma$ is said to be optimal relative to x if

$$x = \int_0^{T(x)} \exp(-sA)c_x(s)\, ds.$$

Principle 3.1. (Ref. 11, p. 355). If $x \in V(A, \Gamma)$, then

(i) for every $c(\cdot) \in C_\Gamma$ and $t \in [0, T(x)]$,

$$T(w(t, x, c)) \geq T(x) - t; \tag{14}$$

(ii) if $c_x(\cdot)$ is an optimal control relative to x, then

$$T(w(t, x, c_x)) = T(x) - t. \tag{15}$$

We use this principle to prove the following result.

Theorem 3.1. If Γ is a compact subset of R^n, $0 \in \operatorname{intrelco} \Gamma$, and (5) holds, then $T(\cdot)$ defined in (4) is a viscosity solution of the Bellman equation

$$\max\{\langle DT(x), \gamma - Ax\rangle : \gamma \in \operatorname{co} \Gamma\} - 1 = 0, \qquad x \in V(A, \Gamma)\backslash\{0\}. \tag{16}$$

Proof. Since $0 \in \operatorname{intrelco} \Gamma$ and (5) holds, from (I.16.25) in Ref. 12 it follows that $V(A, \Gamma)$ is open. On the other hand, $0 \in \operatorname{intrelco} \Gamma$ implies that $0 \in \operatorname{intrel} V(t, A, \Gamma)$, for any $t > 0$; and, because Γ is compact, according to Theorem II.4.3. in Ref. 12, $T(\cdot)$ is continuous on $V(A, \Gamma)$.

Let $x \in V(A, \Gamma)\backslash\{0\}$; let $\gamma \in \Gamma$; and let

$$w(t) = \exp(tA)x - \int_0^t \exp(sA)\gamma\, ds.$$

Then, from (14), we obtain

$$T(w(t)) \geq T(x) - t, \qquad \forall t \in (0, T(x)]. \tag{17}$$

If $\xi \in \partial_F^+ T(x)$, then from (9) there exist $r_1 > 0$ and $\omega_1(\cdot): R^+ \to R^+$ such that

$$\omega_1(0) = 0,\ \omega_1(\cdot) \text{ is continuous}, \tag{18}$$

$$T(y) \leq T(x) + \langle \xi, y - x \rangle + |y - x| \omega_1(|y - x|), \qquad \forall y \in B(x, r_1). \tag{19}$$

Because

$$\lim_{t \to 0+} w(t) = x, \tag{20}$$

there exist $t_1 \in (0, T(x)]$ such that $w(t) \in B(x, r_1)$, for any $t \in (0, t_1]$; and, from (17) and (19), we obtain

$$-t \leq \langle \xi, w(t) - x \rangle + |w(t) - x| \omega_1(|w(t) - x|),$$

hence

$$-1 \leq \langle \xi, (1/t) | w(t) - x | \omega_1(|w(t) - x|). \tag{21}$$

Since

$$(1/t)[w(t) - x] = (1/t)[\exp(tA) - I]x - (1/t)\left(\int_0^t \exp(sA)\, ds\right)\gamma,$$

it follows that

$$\lim_{t \to 0+} (1/t)[w(t) - x] = Ax - \gamma. \tag{22}$$

From (18), (20), and (22), it follows that

$$\lim_{t \to 0+} (1/t)|w(t) - x| \omega_1(|w(t) - x|) = 0; \tag{23}$$

and, from (21), (22), and (23), we obtain that

$$-1 \leq \langle \xi, Ax - \gamma \rangle, \qquad \text{for any } \gamma \in \Gamma.$$

Then, also

$$\langle \xi, \gamma - Ax \rangle \leq 0, \qquad \text{for any } \gamma \in \operatorname{co} \Gamma;$$

hence

$$\max\{\langle \xi, \gamma - Ax \rangle: \gamma \in \operatorname{co} \Gamma\} \leq 1, \qquad \forall \xi \in \partial_F^+ T(x). \tag{24}$$

Let now $c_x(\cdot)$ be an optimal control relative to $x \in V(A, \Gamma) \backslash \{0\}$, and let

$$w^*(t) = \exp(tA)x - \int_0^t \exp[(t - s)A] c_x(s)\, ds.$$

Then, from (15),

$$T(w^*(t)) = T(x) - t, \qquad \text{for any } t \in (0, T(x)]. \tag{25}$$

If $\eta \in \partial_F^- T(x)$, then from (8) there exist $r_2 > 0$ and $\omega_2: R^+ \to R^+$ such that

$$\omega_2(0) = 0, \qquad \omega_2(\cdot) \text{ is continuous,} \tag{26}$$

$$T(y) \geq T(x) + \langle \eta, y - x \rangle + |y - x| \omega_2(|y - x|), \qquad \forall y \in B(x, r_2). \tag{27}$$

Because

$$\lim_{t \to 0+} w^*(t) = x, \tag{28}$$

there exists $t_2 \in (0, T(x)]$ such that

$$w^*(t) \in B(x, r_2), \qquad \text{for all } t \in (0, t_2].$$

Hence, from (25) and (27), for any $t \in (0, t_2]$, we have

$$-1 \geq \langle \eta, (1/t)[w^*(t) - x] \rangle + (1/t)|w^*(t) - x| \omega_2(|w^*(t) - x|). \tag{29}$$

Since

$$\begin{aligned} &(1/t)[w^*(t) - x] \\ &= (1/t)[\exp(tA) - I]x - (1/t) \int_0^t \exp[t - s)A] c_x(s) \, ds \\ &\in (1/t)[\exp(tA) - I]x - (1/t) W(t, A, \Gamma), \end{aligned} \tag{30}$$

where

$$W(t, A, \Gamma) = \left\{ \int_0^t \exp[(t - s)A] c(s) \, ds : c(\cdot) \in C_\Gamma \right\};$$

and since Γ is compact, from (I.15.2) in Ref. 12 it follows that

$$\lim_{t \to 0+} (1/t) W(t, A, \Gamma) = \text{co } \Gamma$$

in the Pompeiu-Hausdorff metric.

Therefore, for any sequence $t_k \to 0+$, there exist $\gamma_0 \in \text{co } \Gamma$ and a subsequence $t_{k_p} \to 0+$ such that

$$t_{k_p} \in (0, t_2], \qquad \text{for any } p \in N,$$

and

$$\lim_{p \to \infty} (1/t_{k_p}) \int_0^{t_{k_p}} \exp[(t_{k_p} - s)A] c_x(s) \, ds = \gamma_0. \tag{31}$$

From (30) and (31), it follows that

$$\lim_{p\to\infty} (1/t_{k_p})[w^*(t_{k_p})-x]=Ax-\gamma_0; \tag{32}$$

and, from (26), (28), and (32), we obtain

$$\lim_{p\to\infty} (1/t_{k_p})|w^*(t_{k_p})-x|\omega_2(|w^*(t_{k_p})-x|)=0. \tag{33}$$

From (29), (32), and (33), it follows that

$$-1\geq\langle\eta, Ax-\gamma_0\rangle,$$

hence

$$1\leq\langle\eta, \gamma_0-Ax\rangle\leq\max\{\langle\eta, \gamma-Ax\rangle: \gamma\in\operatorname{co}\Gamma\},$$

and also

$$\max\{\langle\eta, \gamma-Ax\rangle: \gamma\in\operatorname{co}\Gamma\}\geq 1, \qquad \text{for any } \eta\in\partial_F^- T(x). \tag{34}$$

From (24) and (34), it follows that $T(\cdot)$ is a viscosity solution of (16). □

Remark 3.1. Since $x=0$ is a minimum point for $T(\cdot)$, it follows that $0\in\partial_F^- T(0)$ (Proposition 4.7 in Ref. 10); and because $T(\cdot)$ is not differentiable at $x=0$ (Lemma 2 in Ref. 1), it follows that $\partial_F^+ T(0)=\varnothing$. Hence, if (24) is trivially satisfied at $x=0$, (34) can be not satisfied.

Remark 3.2. Theorem 3.1 generalizes Theorem 3 in Ref. 1, because if $T(\cdot)$ is differentiable in x, then

$$\partial_F^- T(x)=\partial_F^+ T(x)=\{DT(x)\},$$

and from (24) and (34) it follows that (16) holds.

Theorem 3.2. If Γ is a compact subset of R^n, $0\in\operatorname{intrelco}\Gamma$, and if (5) holds, then $T(\cdot)$ defined by (4) is a viscosity solution of the Hàjek equation

$$1-\max\{\langle DT(x), \exp[-AT(x)]\gamma\rangle: \gamma\in\operatorname{co}\Gamma\}=0, \qquad x\in V(A,\Gamma). \tag{35}$$

Proof. As in the proof of Theorem 3.1, $V(A,\Gamma)$ is open and $T(\cdot)$ is continuous on $V(A,\Gamma)$. Let $x\in V(A,\Gamma)$ and let $t=T(x)$.

If $\xi\in\partial_F^+ T(x)$, then there exist $r_1>0$ and $\omega_1(\cdot)$ such that (18) and (19) hold.

Let $s_k \to 0$, $s_k \neq 0$. From Proposition 2 in Ref. 1, there exist $y_k \in V(s_k, A, \Gamma)$ such that

$$y_k \to 0, \qquad T(y_k) = s_k, \tag{36a}$$

$$\mathrm{T}[\mathrm{x} + \exp(-At)y_k] = T(x) + T(y_k). \tag{36b}$$

Thus, there exists $k_0 \in N$ such that

$$x + \exp(-At)y_k \in B(x, r_1), \qquad \text{for all } k \geq k_0;$$

and from (19) and (36), we obtain, for $k \geq k_0$,

$$1 \leq \langle \xi, [1/T(y_k)] \exp(-At)y_k \rangle + |[1/T(y_k)] \exp(-At)\, y_k| \omega_1(|\exp(-At)\, y_k|). \tag{37}$$

Since Γ is compact,

$$\lim_{t \to 0} (1/t)\, V(t, A, \Gamma) = \mathrm{co}\, \Gamma$$

(Proposition 1 in Ref. 1), and there exist $\gamma_0 \in \mathrm{co}\, \Gamma$ and a subsequence $y_{k_p}/T(y_{k_p})$ such that

$$\lim_{p \to \infty} y_{k_p}/T(y_{k_p}) = \gamma_0. \tag{38}$$

From (18), (36), and (38), it follows that

$$\lim_{p \to \infty} |\exp(-At)[y_{k_p}/T(y_{k_p})]| \omega_1(|\exp(-At)\, y_{k_p}|) = 0; \tag{39}$$

and, from (37), (38), and (39), we obtain

$$1 \leq \langle \xi, \exp(-At)\, \gamma_0 \rangle.$$

Hence,

$$-1 \geq \langle -\xi, \exp(-At)\gamma_0 \rangle \geq \min\{\langle -\xi, \exp(-At)\gamma \rangle \colon \gamma \in \mathrm{co}\, \Gamma\}$$
$$= -\max\{\langle \xi, \exp(-At)\gamma \rangle \colon \gamma \in \mathrm{co}\, \Gamma\};$$

that is,

$$1 - \max\{\langle \xi, \exp(-At)\gamma \rangle \colon \gamma \in \mathrm{co}\, \Gamma\} \leq 0, \qquad \text{for any } \xi \in \partial_F^+ T(x). \tag{40}$$

If $\eta \in \partial_F^- T(x)$, then there exist $r_2 > 0$ and $\omega_2(\cdot)$ such that (26) and (27) hold.

Let $\gamma \in \partial \operatorname{co} \Gamma$. Then, from Proposition 1 in Ref. 1, there exist $y_k \in V(A, \Gamma)$ such that

$$y_k \to 0, \qquad y_k \neq 0, \qquad \lim[y_k / T(y_k)] = \gamma. \tag{41}$$

From (27), (41), and the inequality

$$T[x + \exp(-At) y_k] \leq T(x) + T(y_k)$$

(Proposition 2 in Ref. 1), it follows that there exist $k_0 \in N$ such that, for any $k \geq k_0$,

$$\begin{aligned} 1 \geq & \langle \eta, \exp(-At)[y_k / T(y_k)] \rangle \\ & + |\exp(-At)[y_k / T(y_k)]| \omega_2(|\exp(-At) y_k|); \end{aligned} \tag{42}$$

and, from (41) and (42), we obtain

$$1 \geq \langle \eta, \exp(-At) \gamma \rangle, \qquad \text{for any } \gamma \in \partial \operatorname{co} \Gamma. \tag{43}$$

From the convexity of co Γ, it follows that (43) holds for any $\gamma \in$ co Γ, hence

$$\begin{aligned} -1 & \leq \min\{\langle -\eta, \exp(-At)\gamma \rangle : \gamma \in \operatorname{co} \Gamma\} \\ & = -\max\{\langle \eta, \exp(-At)\gamma \rangle : \gamma \in \operatorname{co} \Gamma\}. \end{aligned}$$

Therefore,

$$\begin{aligned} & 1 - \max\{\langle \eta, \exp[-AT(x)]\gamma \rangle : \gamma \in \operatorname{co} \Gamma\} \geq 0, \\ & \qquad \qquad \text{for any } \eta \in \partial_F^- T(x); \end{aligned} \tag{44}$$

and, from (40) and (44), it follows that $T(\cdot)$ is a viscosity solution of (35). □

Remark 3.3. Theorem 3.2 generalizes Theorem 2 in Ref. 1, because if $T(\cdot)$ is differentiable in x, then

$$\partial_F^- T(x) = \partial_F^+ T(x) = \{DT(x)\};$$

and, from (40) and (44), it follows that the Hàjek equation

$$\max\{\langle DT(x), \exp[-AT(x)] \rangle : \gamma \in \operatorname{co} \Gamma\} = 1$$

is satisfied.

In general, $T(\cdot)$ is not the unique viscosity solution of Eqs. (16) and (35) as the following example proves.

Example 3.1. Let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \Gamma = \left\{ \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} \in R^2 : |\gamma_i| \leq 1, \, i = 1, 2 \right\}.$$

Obviously, the assumptions in Theorems 3.1 and 3.2 are satisfied; hence, $Y(\cdot)$ defined by (4) is a viscosity solution of Eqs. (16) and (35).

We prove that $u(\cdot): R^2 \to R$ defined by $u(x) = x_2$, x_2 the second component of x, is also a viscosity solution of (16) and (35). We note first that $u(\cdot)$ is not the minimal time function $T(\cdot)$:

$$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in V(A, \Gamma), \qquad u(x) = 1, \qquad \text{but } T(x) \neq 1;$$

see Ref. 9. Since $u(\cdot)$ is differentiable on R^2 and its differential is

$$Du(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

it follows that

$$\partial_F^- u(x) = \partial_F^+ u(x) = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\},$$

$$\begin{aligned} &\max\{\langle Du(x), \gamma - Ax\rangle: \gamma \in \Gamma\} \\ &= \max\left\{ \left\langle \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle: |\gamma_i| \leq 1, i = 1, 2 \right\} \\ &= \max\{\gamma_2: \gamma_2 \in [-1, 1]\} = 1, \\ &\max\{\langle Du(x), \exp[-Au(x)]\gamma: \gamma \in \Gamma\} \\ &= \max\left\{ \left\langle \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & -u(x) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} \right\rangle: |\gamma_i| \leq 1, i = 1, 2 \right\} \\ &= \max\{\gamma_2: \gamma_2 \in [-1, 1]\} = 1. \end{aligned}$$

Therefore, $u(\cdot)$ is a classical solution, hence also a viscosity solution, of Eqs. (16) and (35), which is different from $T(\cdot)$.

The uniqueness problem of the minimal time function as viscosity solution of Bellman equation was studied in Ref. 13.

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