ON A p-SUPERLINEAR NEUMANN p-LAPLACIAN EQUATION

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Abstract. We consider a nonlinear Neumann problem, driven by the $p-$ Laplacian, and with a nonlinearity which exhibits a $p-$ superlinear growth near infinity, but does not necessarily satisfy the Ambrosetti-Rabinowitz condition. Using variational methods based on critical point theory, together with suitable truncation techniques and Morse theory, we show that the problem has at least three nontrivial solutions, of which two have a fixed sign (one positive and the other negative

1. Introduction

Let $Z \subseteq \mathbb{R}^N$ be a bounded domain with a C^2 boundary ∂Z. In this paper we study the following nonlinear elliptic problem:

$$(1.1)\qquad \begin{cases} -\triangle_p x(z) = f(z, x(z)) \text{ a.e. on } Z, \\ \frac{\partial x}{\partial n} = 0, \text{ on } \partial Z. \end{cases}$$

Here $\triangle_p$ denotes the $p-$Laplacian differential operator defined by

$$\triangle_p u(z) = div\left(\|Du(z)\|_{\mathbb{R}^N}^{p-2} Du(z)\right), 1 < p < \infty,$$

$n(.)$ stands for the outward unit normal on ∂Z, and $f(z,x)$ is a nonlinear Caratheodory function.

Our aim is to prove a multiplicity theorem for problem (1.1), when the nonlinearity $f(z,.)$ exhibits a p-superlinear growth near infinity. To deal with such a problem, in most papers, it is assumed that the nonlinearity $x \to f(z,x)$ satisfies the so called *Ambrosetti-Rabinowitz condition* (*AR-condition* for short).

We recall that this condition says that there exist $q > p$ and $M > 0$ such that, for almost all $z \in Z$ and all $|x| \geq M$, we have

$$0 < qF(z,x) \leq f(z,x)x,$$

with $F(z,x) = \int_0^x f(z,s)\,ds$ (the primitive of $f(z,.)$). A direct integration of this inequality, implies that for almost all $z \in Z$ and all $|x| \geq M$, we have $F(z,x) \geq \eta |x|^q$ for some $\eta > 0$, which implies the strict p-superlinear growth near infinity of the potential function $F(z,.)$. This condition is employed in the works of Bartsch-Liu [10], Degiovanni-Lancelotti [14], Liu [22], and Perera [26], where the authors sudy the corresponding Dirichlet problem. We should also mention that in the above

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papers, with the exception of Bartsch-Liu [10], we find existence but no multiplicity results. Multiplicity results, but for thesemilinear (i.e., $p = 2$) equation with a superlinear nonlinearity and Dirichlet boundary conditions, can be found in the works of Struwe [29] and Wang [31]. The study of the corresponding problem for the Neumann $p-$Laplacian, in some sense, is lagging behind. Recently there have been some multiplicity results for Neumann $p-$Laplacian problems, but under different conditions which do not cover the case of $p-$superlinear perturbations. We mention the works of Anello [6], Bonanno-Candito [11], Faraci [16], Filippakis-Gasinski-Papageorgiou [17], Motreanu-Motreanu-Papageorgiou [24], Motreanu-Papageorgiou [25], Ricceri [28] and Wu-Tan [32]. In the papers of Anello [6], Bonanno-Candito [11], Faraci [16] and Ricceri [28], it is assumed that $p > N$ (low dimensional problems) and this allows the authors to exploit the fact that the Sobolev space $W^{1,p}(Z)$ is embedded compactly in $C\left(\overline{Z}\right)$. In these works the approach is essentially similar and is based on an abstract multiplicity result of Ricceri [27] or variants of it. Wu-Tan [32] also assume $p > N$, but they use variational methods based on critical point theory. Filippakis-Gasinski-Papageorgiou [17] and Motreanu-Papageorgiou [25] assume bounded and symmetric nonlinearities and use minimax techniques based on the second deformation theorem and the symmetric mountain pass theorem. Motreanu-Motreanu-Papageorgiou [24] consider eigenvalue problems with a parameter λ near resonance, and they allow nonlinearities $f(z, x)$ which are $p-$linear and $p-$superlinear.

In our recent work [2], we consider problems with a $p-$superlinear nonlinearity satisfying the AR condition, and prove multiplicity results with precise sign information for the solutions. Our approach in [2] is purely variational and a crucial role is played by a new variational characterization of $\lambda_1 > 0$ (the first nonzero eigenvalue of the negative Neumann $p-$Laplacian) that was earlier proved by us in [3]. Finally, we also mention the recent papers [4], [7], where related $p-$Laplacian Neumann problems are discussed under different assumptions, by using variational techniques in combination with the method of upper-lower solutions and Morse theory [4], and respectively variational and degree theoretic arguments [7].

In this paper, we prove a multiplicity theorem (three nontrivial solutions) for problems with $p-$ superlinear nonlinearities, which need not satisfy the AR-condition. Our approach combines minimax arguments based on critical point theory with suitable truncation techniques and methods from Morse theory.

2. Preliminaries

In this section, for the convenience of the reader, we recall some basic definitions and facts from critical point theory and from Morse theory, which we will need in the sequel. The reader is referred to [12], [18], [23] for more details.

Let $(X, \|.\|)$ be a Banach space, X^* its topological dual, and let $\langle ., . \rangle$ denote the duality brackets for the pair (X^*, X). Let $\varphi \in C^1(X)$. We say that φ satisfies the *Cerami condition* at the level $c \in \mathbb{R}$ (the C_c-condition, for short), if every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that

$$\varphi(x_n) \to c \text{ and } (1 + \|x_n\|)\varphi'(x_n) \to 0 \text{ in } X^* \text{ as } n \to \infty$$

has a strongly convergent subsequence. If this condition holds at every level $c \in \mathbb{R}$, then we say that φ satisfies the C-condition.

This compactness notion plays a key role in the following minimax theorem for the critical values of a C^1-functional, known in the literature as the mountain pass theorem; see, e.g. [8], [18].

Theorem 1. *If* $(X, \|.\|)$ *is a Banach space,* $\varphi \in C^1(X)$, $x_0,\ x_1 \in X$, $\rho > 0$,

$$\max\{\varphi(x_0), \varphi(x_1)\} < \inf\{\varphi(x) : \|x - x_0\| = \rho\} = \eta, \ \ \|x_1 - x_0\| > \rho,$$

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = x_0, \ \gamma(1) = x_1\}, \ c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \varphi(\gamma(t))$$

and φ *satisfies the* C_c-*condition then* $c \geq \eta$ *and* c *is a critical value of* φ, *i.e., there exists* $x^* \in X$ *such that* $\varphi'(x^*) = 0$ *and* $\varphi(x^*) = c$.

Given $\varphi \in C^1(X)$ we introduce the following notation:

$$\varphi^c = \{x \in X : \varphi(x) \leq c\} \ \text{(the sublevel set of } \varphi \textit{ at } c \in \mathbb{R}),$$
$$K = \{x \in X : \varphi'(x) = 0\} \ \text{(the critical set of } \varphi),$$

and

$$K_c = \{x \in K : \varphi(x) = c\} \ \text{(the critical set of } \varphi \textit{ at the level } c \in \mathbb{R}).$$

Let $Y_2 \subseteq Y_1 \subseteq X$ and let $k \geq 0$ be an integer. By $H_k(Y_1, Y_2)$ we denote the k^{th}-relative singular homology group of the pair (Y_1, Y_2) with integer coefficients. Let $x_0 \in X$ be an isolated critical point of $\varphi \in C^1(X)$ and $c = \varphi(x_0)$. The critical groups of φ at x_0 are defined by

$$C_k(\varphi, x_0) = H_k(\varphi^c \cap U, (\varphi^c \cap U) \setminus \{x_0\}) \ \text{ for all } k \geq 0,$$

where U is a neighborhood of x_0 such that $K \cap \varphi^c \cap U = \{x_0\}$ (see Chang [12] and Mawhin-Willem [23]). By the excision property of the singular homology theory, we see that the above definition of critical groups is independent of the particular neighborhood U we use.

Now suppose that $\varphi \in C^1(X)$ satisfies the $C-$condition and $-\infty < \inf \varphi(K)$. Let $c < \inf \varphi(K)$. The critical groups of φ at infinity, are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \ \text{ for all } k \geq 0$$

(see Bartsch-Li [9]). The deformation lemma (which is valid since φ satisfies the $C-$condition, see Bartolo-Benci-Fortunato [8] and Gasinski-Papageorgiou [18], p.636) implies that the above definition of critical groups of φ at infinity is independent of the choice of $c < \inf \varphi(K)$. If $K = \{x_0\}$, then

$$C_k(\varphi, \infty) = C_k(\varphi, x_0) \ \text{ for all } k \geq 0.$$

Suppose K is finite. The Morse-type numbers of φ are defined by

$$M_k = \sum_{x\in K} rank \ C_k(\varphi, x) \ \text{ for all } k \geq 0.$$

The Betti-type numbers of φ, are defined by

$$\beta_k = rank \ C_k(\varphi, \infty) \ \text{ for all } k \geq 0.$$

According to Morse theory (see [9], [12] and [23]) the *Poincare-Hopf formula*

$$\sum_{k\geq 0} (-1)^k M_k = \sum_{k\geq 0} (-1)^k \beta_k, \tag{2.1}$$

holds if all M_k, β_k are finite and the series converge.

In the analysis of problem (1.1) we will use of the following two spaces:

$$C_n^1\left(\overline{Z}\right)=\left\{x\in C^1\left(\overline{Z}\right):\frac{\partial x}{\partial n}=0\text{ on }\partial Z\right\}$$

and

$$W_n^{1,p}\left(Z\right)=\overline{C_n^1\left(\overline{Z}\right)}^{\|.\|}$$

where $\|.\|$ denotes the $W^{1,p}\left(Z\right)-$ norm. Both spaces are ordered Banach spaces, with order cones given by

$$C_+=\left\{x\in C_n^1\left(\overline{Z}\right):x\left(z\right)\geq 0\text{ for all }z\in\overline{Z}\right\}.$$

and respectively

$$W_+=\left\{x\in W_n^{1,p}\left(Z\right):x\left(z\right)\geq 0\text{ a.e. on }Z\right\}.$$

Moreover, we know that C_+ has nonempty interior, given by

$$int\ C_+=\left\{x\in C_+:x\left(z\right)>0\text{ for all }z\in\overline{Z}\right\}.$$

3. Hypotheses and Auxiliary Results

Throughout this section and the remainder of the paper, $\|.\|_p$ denotes the norm of $L^p\left(\mathbb{R}\right)$ or $L^p\left(\mathbb{R}^N\right)$, while $\|.\|$ denotes the norm of $W_n^{1,p}\left(Z\right)$. Also, for any $x\in\mathbb{R}$, we set $x^{\pm}=\max\left\{\pm x,0\right\}$. Finally, we use $\xrightarrow{w}$ to denote the weak convergence, $|.|_N$ to designate the Lebesgue measure on $\mathbb{R}^N$, and χ_E to indicate the characteristic function of a subset E of Z.

The hypotheses on the nonlinearity $f\left(z,x\right)$ are the following:

(H) $f:Z\times\mathbb{R}\rightarrow\mathbb{R}$ is a function such that $f\left(z,0\right)=0$ a.e. on Z and

(i) for all $x\in\mathbb{R}$, $z\rightarrow f\left(z,x\right)$ is measurable;

(ii) for almost all $z\in Z$, $x\rightarrow f\left(z,x\right)$ is continuous;

(iii) for almost all $z\in Z$ and all $x\in\mathbb{R}$

$$\left|f\left(z,x\right)\right|\leq a\left(z\right)+c\left|x\right|^{r-1},$$

with $a\in L^\infty\left(Z\right)_+$, $c>0$ and $p<r<p^*$, where

$$p^*=\begin{cases}\frac{Np}{N-p} & \text{if } p<N\\ +\infty & \text{if } p\geq N.\end{cases}$$

(iv) if $F\left(z,x\right)=\int\limits_0^x f\left(z,s\right)ds$, then $\lim\limits_{|x|\to\infty}\frac{F(z,x)}{|x|^p}$ uniformly for a.a. $z\in Z$, and there exists $\mu\in\left(\left(r-p\right)\max\left\{1,\frac{N}{p}\right\},r\right]$ such that

$$\liminf_{|x|\to\infty}\frac{f\left(z,x\right)x-pF\left(z,x\right)}{\left|x\right|^{\mu}}>0\text{ uniformly for a.a. }z\in Z;$$

(v) there exists $\delta>0$ such that $F\left(z,x\right)\leq 0$ for a.a. $z\in Z$, and all $|x|\leq\delta$, and

$$\lim_{x\to 0}\frac{f\left(z,x\right)}{\left|x\right|^{p-2}x}=0\text{ uniformly for a.a. }z\in Z;$$

(vi) there exists $c_0>0$ such that for a.a. $z\in Z$

$$f\left(z,x\right)\geq -c_0x^{p-1}\text{ for all }x\geq 0\text{ and }f\left(z,x\right)\leq c_0\left|x\right|^{p-1}\text{ for all }x\leq 0.$$

Remark: Hypothesis $(H)(v)$ implies that the nonlinerity $f(z,.)$ is p-superlinear near infinity for a.a. $z \in Z$ (see Costa-Magalhães [13]). However, it does not need to satisfy the AR-condition, as the example that follows illustrates.

Example: Consider the following function $F(x)$ $(x \in \mathbb{R})$ (for the sake of simplicity we drop the $z-$dependence):

$$F(x) = \begin{cases} \frac{1}{q}|x|^q - \frac{1}{p}|x|^p & \text{if } |x| \leq 1 \\ \frac{\xi}{p}|x|^p \ln|x| + c|x| & \text{if } |x| > 1 \end{cases},$$

with

$$1 < p < q < \infty,\ c = \frac{p-q}{pq} < 0 \text{ and } \xi = \frac{q-p}{pq} > 0.$$

Evidently, $F \in C^1(\mathbb{R})$ and $f(x) = F'(x)$ satisfies hypotheses $(\mathbf{H})$. Indeed, take $r = p + \varepsilon$, with $\varepsilon > 0$ such that $N\varepsilon < p^2$ and $\mu = p$. Note however that $f(x)$ does not satisfy the AR-condition.

Let $\varphi : W_n^{1,p}(Z) \to \mathbb{R}$ be the Euler functional for problem (1.1), defined by

$$\varphi(x) = \frac{1}{p}\|Dx\|_p^p - \int_Z F(z, x(z))\,dz \text{ for all } x \in W_n^{1,p}(Z).$$

Clearly $\varphi \in C^1\left(W_n^{1,p}(Z)\right)$.

As we already mentioned in the Introduction, we will also use truncation techniques. For this reason, we introduce the following truncations of the nonlinearity $f(z,.)$:

$$f_+(z,x) = \begin{cases} 0 & \text{if } x \leq 0 \\ f(z,x) & \text{if } x \geq 0 \end{cases} \text{ and } f_-(z,x) = \begin{cases} f(z,x) & \text{if } x \leq 0 \\ 0 & \text{if } x \geq 0 \end{cases}.$$

We set $F_\pm(z,x) = \int\limits_0^x f_\pm(z,s)\,ds$. Let $0 < \varepsilon < 1$ and introduce the functionals $\varphi_\pm^\varepsilon : W_n^{1,p}(Z) \to \mathbb{R}$ defined by

$$\varphi_\pm^\varepsilon(x) = \frac{1}{p}\|Dx\|_p^p + \frac{\varepsilon}{p}\|x\|_p^p - \int\limits_Z F_\pm(z,x(z))\,dz \mp \frac{\varepsilon}{p}\left\|x^\pm\right\|_p^p \text{ for all } x \in W_n^{1,p}(Z).$$

Evidently $\varphi_\pm^\varepsilon \in C^1\left(W_n^{1,p}(Z)\right)$. In what follows, we denote by $\langle .,. \rangle$ the duality brackets for the pair $\left(W_n^{1,p}(Z)^*, W_n^{1,p}(Z)\right)$. Let $A : W_n^{1,p}(Z) \to W_n^{1,p}(Z)^*$ be the nonlinear map, defined by

$$\langle A(x), u \rangle = \int_Z \|Dx\|_{\mathbb{R}^N}^{p-2} (Dx, Du)_{\mathbb{R}^N}\,dz \text{ for all } x,\, u \in W_n^{1,p}(Z).$$

It is easy to verify (see, e.g., [2]) that A is of type $(S)_+$, i.e., if $x_n \xrightarrow{w} x$ in $W_n^{1,p}(Z)$ and

$$\limsup_{n\to\infty} \langle A(x_n), x_n - x \rangle \leq 0$$

then $x_n \to x$ in $W_n^{1,p}(Z)$. Also, we introduce the maps $N,\ N_\pm : W_n^{1,p}(Z) \to L^{r'}(Z) \subseteq W_n^{1,p}(Z)^*$ $(\frac{1}{r} + \frac{1}{r'} = 1)$, defined by

$$N(u)(.) = f(., u(.)) \text{ and } N_\pm(u)(.) = f_\pm(., u(.)) \text{ for all } u \in W_n^{1,p}(Z).$$

Note that

$$\varphi'(x) = A(x) - N(x) \text{ and } \left(\varphi_\pm^\varepsilon\right)'(x) = A(x) + \varepsilon|x|^{p-2}x - N_\pm(x) \mp \varepsilon\left(x^\pm\right)^{p-1}.$$

Proposition 1. *If hypotheses* $(\mathbf{H})$ *hold, then the functionals* $\varphi_{\pm}^{\varepsilon}$ *and* φ *satisfy the* $C-$*condition.*

Proof. We complete the proof for $\varphi_{+}^{\varepsilon}$, the proofs for $\varphi_{-}^{\varepsilon}$ and φ being similar. So, let $\{x_n\}_{n\geq 1} \subseteq W_n^{1,p}(Z)$ be a sequence such that

$$\left|\varphi_{+}^{\varepsilon}(x_n)\right| \leq M_1, \text{ for some } M_1 > 0, \text{ all } n \geq 1 \tag{3.1}$$

and

$$(1+\|x_n\|)\left(\varphi_{+}^{\varepsilon}\right)'(x_n) \to 0 \text{ in } W_n^{1,p}(Z)^* \text{ as } n \to \infty. \tag{3.2}$$

Claim: The sequence $\{x_n\}_{n\geq 1} \subseteq W_n^{1,p}(Z)$ is bounded.

From (3.2), we have

$$\left|\left\langle \left(\varphi_{+}^{\varepsilon}\right)'(x_n), u\right\rangle\right| \leq \frac{\varepsilon_n}{1+\|x_n\|}\|u\| \text{ for all } u \in W_n^{1,p}(Z), \text{ with } \varepsilon_n \downarrow 0,$$

hence

$$\left|\langle A(x_n), u\rangle + \varepsilon\int_Z |x_n|^{p-2} x_n u dz - \int_Z f_+(z, x_n) u dz - \varepsilon\int_Z \left(x_n^+\right)^{p-1} u dz\right|$$
$$\leq \frac{\varepsilon_n}{1+\|x_n\|}\|u\|. \tag{3.3}$$

First choose $u = -x_n^- \in W_n^{1,p}(Z)$ in (3.3). Then

$$\left\|Dx_n^-\right\|_p^p + \varepsilon\left\|x_n^-\right\|_p^p \leq \varepsilon_n,$$

therefore

$$x_n^- \to 0 \text{ in } W_n^{1,p}(Z) \text{ as } n \to \infty. \tag{3.4}$$

Next, we choose $u = x_n^+ \in W_n^{1,p}(Z)$ in (3.3). Then

$$-\left\|Dx_n^+\right\|_p^p + \int_Z f_+(z, x_n) x_n^+ dz \leq \varepsilon_n, \text{ for all } n \geq 1. \tag{3.5}$$

Also, from (3.1) and (3.4), we have

$$\left\|Dx_n^+\right\|_p^p - \int_Z pF_+(z, x_n) dz \leq M_2 \text{ for some } M_2 > 0, \text{ all } n \geq 1. \tag{3.6}$$

Adding (3.5) and (3.6) we obtain

$$\int_Z \left(f_+(z, x_n) x_n^+ - pF_+(z, x_n)\right) dz \ \leq M_3 \text{ for some } M_3 > 0, \text{ all } n \geq 1. \tag{3.7}$$

By virtue of hypothesis $(\mathbf{H})(iv)$, we can find $\beta > 0$ and $M_4 = M_4(\beta) > 0$ such that

$$0 < \beta|x|^{\mu} \leq f(z,x)x - pF(z,x) \text{ for a.a. } z \in Z, \text{ all } |x| \geq M_4,$$

hence

$$0 < \beta x^{\mu} \leq f_+(z,x)x - pF_+(z,x) \text{ for a.a. } z \in Z, \text{ all } x \geq M_4. \tag{3.8}$$

On the other hand, hypothesis $(\mathbf{H})(iii)$ implies that for some $M_5 > 0$

$$|f_+(z,x)x - pF_+(z,x)| \leq M_5, \text{ for a.a. } z \in Z \text{ and all } x < M_4. \tag{3.9}$$

Combining (3.8) and (3.9) we see that

$$\beta\left(x^+\right)^{\mu} - M_6 \leq f_+(z,x)x^+ - pF_+(z,x) \text{ for a.a. } z \in Z, \text{ all } x \in \mathbb{R} \tag{3.10}$$

with $M_6 = M_5 + \beta M_4^{\mu}$. We use (3.10) in (3.7) and obtain

$$\beta \left\| x_n^+ \right\|_{\mu}^{\mu} \leq M_7 \text{ for some } M_7 > 0, \text{ all } n \geq 1,$$

hence

$$\left\{ x_n^+ \right\}_{n\geq 1} \subseteq L^{\mu}(Z) \text{ is bounded.} \tag{3.11}$$

By hypothesis $(\mathbf{H})\,(iv)$, we have $\mu \leq r < p^*$. Hence, we can find $t \in [0,1)$ such that

$$\frac{1}{r} = \frac{1-t}{\mu} + \frac{t}{p^*}.$$

Imvoking a classical interpolation inequality (see, for instance, Gasinski-Papageorgiou [18], p.905), we have

$$\left\| x_n^+ \right\|_r \leq \left\| x_n^+ \right\|_{\mu}^{1-t} \left\| x_n^+ \right\|_{p^*}^{t}$$

whence (see(3.11))

$$\left\| x_n^+ \right\|_r^r \leq M_8 \left\| x_n^+ \right\|_{p^*}^{tr} \text{ for some } M_8 > 0, \text{ all } n \geq 1. \tag{3.12}$$

Recall that (cf.(3.3))

$$\left| \left\| D x_n^+ \right\|_p^p - \int_Z f_+ (z, x_n)\, x_n^+ dz \right| \leq \varepsilon_n, \text{ for all } n \geq 1. \tag{3.13}$$

Note that hypotheses $(\mathbf{H})\,(iii)$ and (iv) imply that given $\varepsilon > 0$, we can find $c_\varepsilon > 0$, such that

$$|f(z,x)\,x| \leq \varepsilon\,|x|^p + c_\varepsilon\,|x|^r \text{ for a.a. } z \in Z, \text{ all } x \in \mathbb{R},$$

hence

$$f_+(z,x)\,x^+ \leq \varepsilon \left(x^+\right)^p + c_\varepsilon \left(x^+\right)^r \text{ for a.a. } z \in Z, \text{ all } x \in \mathbb{R}.$$

Using this inequality and (3.12) in (3.13), we obtain

$$\left\| D x_n^+ \right\|_p^p \leq \varepsilon_n + \varepsilon \left\| x_n^+ \right\|_p^p + \widehat{c}_\varepsilon \left\| x_n^+ \right\|_{p^*}^{tr} \text{ for some } \widehat{c}_\varepsilon > 0, \text{ all } n \geq 1. \tag{3.14}$$

Arguing by contradiction, suppose that $\|x_n^+\| \to \infty$. Set

$$y_n = \frac{x_n^+}{\left\| x_n^+ \right\|}, n \geq 1.$$

Then $\|y_n\| = 1$ for all $n \geq 1$, and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_n^{1,p}(Z) \text{ and } y_n \to y \text{ in } L^p(Z),\ y \geq 0. \tag{3.15}$$

We write $x_n^+ = y_n \|x_n^+\|$ in (3.14) and divide by $\|x_n^+\|^p$. Then

$$\|D y_n\|_p^p \leq \frac{\varepsilon_n}{\left\| x_n^+ \right\|^p} + \varepsilon \|y_n\|_p^p + \frac{\widehat{c}_\varepsilon}{\left\| x_n^+ \right\|^{p-tr}} \|y_n\|_{p^*}^{tr} \text{ for all } n \geq 1. \tag{3.16}$$

The hypothesis $\mu > (r-p) \max\left\{1, \frac{N}{p}\right\}$ (see $(\mathbf{H})\,(iv)$) is equivalent to saying that $tr < p$. So, if in (3.16), we pass to the limit as $n \to \infty$ and we use (3.15), we obtain

$$\|Dy\|_p^p \leq \varepsilon \|y\|_p^p \leq \varepsilon \|y\|^p \leq \varepsilon \text{ (since } \|y\| = 1).$$

But recall that $\varepsilon > 0$ was arbitrary. So we let $\varepsilon \downarrow 0$ and obtain $\|Dy\|_p = 0$, therefore

$$y \equiv \xi \in \mathbb{R}.$$

If $\xi = 0$, then from (3.16) we have $Dy_n \to 0$ in $L^p(Z, \mathbb{R}^n)$ hence (see (3.15)),

$$y_n \to 0 \text{ in } W_n^{1,p}(Z),$$

a contradiction since $\|y_n\| = 1$ for all $n \geq 1$.

If $\xi > 0$ (recall that $y \geq 0$, see (3.15)), then $x_n^+(z) \to \infty$ for a.a. $z \in Z$. From (3.1) we have

$$\int_Z \frac{F_+(z, x_n(z))}{\|x_n^+\|^p} dz \leq \frac{M_1}{\|x_n^+\|^p} + 1 \text{ (recall that } \|y_n\| = 1 \text{ for all } n \geq 1). \tag{3.17}$$

On the other hand, by virtue of hypothesis $(\mathbf{H})(iv)$, given $\theta > 0$, we can find $M_9 = M_9(\theta)$ such that

$$\frac{F(z,x)}{|x|^p} \geq \theta > 0 \text{ for a.a. } z \in Z, \text{ all } x \geq M_9,$$

hence

$$\frac{F_+(z,x)}{x^p} \geq \theta > 0 \text{ for a.a. } z \in Z, \text{ all } |x| \geq M_9, \tag{3.18}$$

Using (3.18) and hypothesis $(\mathbf{H})(iii)$, we have

$$\begin{aligned}
&\int_Z \frac{F_+(z, x_n(z))}{\|x_n^+\|^p} dz \\
&= \int_{\{x_n^+ \geq M_9\}} \frac{F_+(z, x_n(z))}{x_n^+(z)^p} y_n(z)^p \, dz + \int_{\{x_n^+ < M_9\}} \frac{F_+(z, x_n(z))}{\|x_n^+\|^p} dz \\
&\geq \int_{\{x_n^+ \geq M_9\}} \theta y_n(z)^p \, dz + \int_{\{x_n^+ < M_9\}} \frac{F_+(z, x_n(z))}{\|x_n^+\|^p} dz \\
&\geq \theta \int_{\{x_n^+ \geq M_9\}} y_n(z)^p \, dz - \frac{M_{10}}{\|x_n^+\|^p} \text{ for some } M_{10} > 0, \text{ all } n \geq 1.
\end{aligned} \tag{3.19}$$

Since $x_n^+(z) \to \infty$ for a.a. $z \in Z$, we have

$$\chi_{\{x_n^+ \geq M_9\}}(z) \to \chi_Z(z) = 1 \text{ for a.a. } z \in Z.$$

Hence it follows that

$$\theta \int_{\{x_n^+ \geq M_9\}} y_n(z)^p \, dz \to \theta \xi^p |Z|_N.$$

So, if in (3.19) we pass to the limit as $n \to \infty$, we obtain

$$\liminf_{n \to \infty} \int_Z \frac{F_+(z, x_n(z))}{\|x_n^+\|^p} \geq \theta \xi^p |Z|_N.$$

Recall that $\theta > 0$ was arbitrary. So, we let $\theta \to \infty$ to conclude that

$$\lim_{n \to \infty} \int_Z \frac{F_+(z, x_n(z))}{\|x_n^+\|^p} = \infty. \tag{3.20}$$

Comparing (3.17) and (3.20), we reach a contradiction. This proves the **Claim**.

By virtue of the **Claim**, we may assume that

$$x_n \xrightarrow{w} x \text{ in } W_n^{1,p}(Z) \text{ and } x_n \to x \text{ in } L^r(Z). \tag{3.21}$$

If in (3.3) we choose $u = x_n - x \in W_n^{1,p}(Z)$ and pass to the limit as $n \to \infty$, using (3.21), we obtain

$$\lim_{n \to \infty} \langle A(x_n), x_n - x \rangle = 0.$$

But A is of type $(S)_+$. So, it follows that $x_n \to x$ in $W_n^{1,p}(Z)$. This proves that φ_+^ε satisfies the C-condition.

Similarly, we show that $\varphi_{-}^{\varepsilon}$ and φ satisfy the C-condition. $\square$

Proposition 2. *If hypotheses* $(\mathbf{H})$ *hold, then* $x=0$ *is a local minimizer for the functionals* $\varphi_{+}^{\varepsilon}$, $\varphi_{-}^{\varepsilon}$ *and* φ.

Proof. Again we carry out the proof for $\varphi_{+}^{\varepsilon}$, the proofs for $\varphi_{-}^{\varepsilon}$ and φ being similar.

Let $x \in C_n^1\left(\overline{Z}\right)$ be such that $\|x\|_{C_n^1(\overline{Z})} \leq \delta$, with $\delta>0$ as in hypothesis $(\mathbf{H})\,(v)$. Then

$$F\left(z, x\left(z\right)\right) \leq 0 \text{ a.e. on } Z \text{ (see } (\mathbf{H})\,(v)\text{)}. \tag{3.22}$$

Consequently, for $x \in C_n^1\left(\overline{Z}\right)$ with $\|x\|_{C_n^1(\overline{Z})} \leq \delta$, we have

$$\varphi_{+}^{\varepsilon}\left(x\right) \geq \frac{1}{p}\left\|Dx\right\|_p^p - \int_Z F_{+}\left(z, x\left(z\right)\right) dz \text{ (since } \left\|x^{+}\right\|_p \leq \left\|x\right\|_p) \geq 0$$

(see (3.22)). Therefore, $x=0$ is a local $C_n^1\left(\overline{Z}\right)-$ minimizer of $\varphi_{+}^{\varepsilon}$. Invoking Proposition 2.5 of Motreanu-Motreanu-Papageorgiou [24] (see also [2], Proposition 3 when $p \geq 2$) we infer that $x=0$ is also a local $W_n^{1,p}\left(Z\right)-$ minimizer of $\varphi_{+}^{\varepsilon}$.

Similarly, we show that $x=0$ is a local minimizer of $\varphi_{-}^{\varepsilon}$ and φ, too. $\square$

Proposition 3. *If hypotheses* $(\mathbf{H})$ *hold, then problem* (1.1) *has two solutions of constant sign* $x_0 \in int\ C_{+}$ *and* $v_0 \in -int\ C_{+}$.

Proof. By virtue of Proposition 3 and arguing as in Aizicovici-Papageorgiou-Staicu [5] (see the proof of Proposition 10), we can find $\rho>0$ small such that

$$0=\varphi_{+}^{\varepsilon}\left(0\right) < \inf\left\{\varphi_{+}^{\varepsilon}\left(x\right) : \|x\|=\rho\right\} =: \gamma_\rho^{+} \tag{3.23}$$

On account of hypothesis $(\mathbf{H})\,(iv)$, it is clear that

$$\varphi_{+}^{\varepsilon}\left(\sigma\right) = -\int_Z F_{+}\left(z,\sigma\right) dz \to -\infty \text{ as } \sigma \to \infty. \tag{3.24}$$

Then Proposition 2 together with (3.23) and (3.24) enables us to use the mountain pass theorem (see Theorem 1). So, we obtain $x_0 \in W_n^{1,p}\left(Z\right)$ such that

$$0=\varphi_{+}^{\varepsilon}\left(0\right) < \gamma_\rho^{+} \leq \varphi_{+}^{\varepsilon}\left(x_0\right) \tag{3.25}$$

and

$$\left(\varphi_{+}^{\varepsilon}\right)'\left(x_0\right) = 0. \tag{3.26}$$

By (3.25) it is clear that $x_0 \neq 0$. From (3.26), we have

$$A\left(x_0\right) + \varepsilon\left|x_0\right|^{p-2} x_0 = N_{+}\left(x_0\right) + \varepsilon\left(x_0^{+}\right)^{p-1}.$$

We act with $-x_0^{-} \in W_n^{1,p}\left(Z\right)$ on the above relation and obtain

$$\varepsilon\left\|x_0^{-}\right\| = 0,$$

hence

$$x_0 \in W_{+},\ x_0 \neq 0.$$

Moreover, using the nonlinear Green identity, as in Motreanu-Papageorgiou [25], we show that $x_0 \in W_{+}$ is a solution of problem (1.1). In addition, the nonlinear regularity theory (see, for example, Gasinski-Papageorgiou [18]) implies that $x_0 \in C_{+}$. By virtue of hypothesis $(\mathbf{H})\,(vi)$, we have

$$-\triangle_p x_0\left(z\right) = f\left(z, x_0\left(z\right)\right) \geq -c_0 x_0\left(z\right)^{p-1} \text{ a.e. on } Z,$$

hence

$$\triangle_p x_0(z) \le c_0 x_0(z)^{p-1} \text{ a.e. on } Z.$$

Invoking the nonlinear strong maximum principle of Vazquez [30], we conclude that $x_0 \in int\, C_+$. Similarly, working this time with the finctional φ_-^{ε}, we obtain a second constant sign solution $v_0 \in -int\, C_+$. □

Proposition 4. *If hypotheses* $(\mathbf{H})$ *hold, then* $C_k(\varphi,\infty) = 0$ *for all* $k \ge 0$.

Proof. Hypotheses $(\mathbf{H})(iii), (iv)$ imply that given any $\theta > 0$, we can find $M_{11} = M_{11}(\theta) > 0$ such that

$$F(z,x) \ge \frac{\theta}{p}|x|^p - M_{11} \text{ for a.a. } z \in Z, \text{ all } x \in \mathbb{R}. \tag{3.27}$$

Hence, if $u \in \partial B_1 = \left\{u \in W_n^{1,p}(Z) : \|u\| = 1\right\}$ and $t > 0$, then

$$\begin{aligned} \varphi(tu) &= \frac{t^p}{p}\|Du\|_p^p - \int_Z F(z,tu)\,dz \le \frac{t^p}{p} - \frac{\theta t^p}{p}\|u\|_p^p + M_{11}|Z|_N \\ &= \frac{t^p}{p}\left(1 - \theta\|u\|_p^p\right) + M_{11}|Z|_N. \end{aligned} \tag{3.28}$$

So, if we choose $\theta > \frac{1}{\|u\|_p^p}$, then from (3.28) it is clear that

$$\varphi(tu) \to -\infty \text{ as } t \to +\infty. \tag{3.29}$$

Also, because of hypothesis $(\mathbf{H})(iv)$, we can find $\beta > 0$ and $M_{12} > 0$, such that

$$f(z,x)x - pF(z,x) \ge \beta|x|^\mu \text{ for a.a. } z \in Z, \text{ all } |x| \ge M_{12}.$$

Hence, for any $v \in W_n^{1,p}(Z)$, we have

$$\begin{aligned} &\int_Z pF(z,v)\,dz - \int_Z f(z,v)v\,dz \\ &= \int_{\{|v| \ge M_{12}\}} pF(z,v)\,dz + \int_{\{|v| < M_{12}\}} pF(z,v)\,dz \\ &- \int_{\{|v| \ge M_{12}\}} f(z,v)v\,dz - \int_{\{|v| < M_{12}\}} f(z,v)v\,dz \\ &\le - \int_{\{|v| \ge M_{12}\}} \beta|x|^\mu\,dz + c_1, \end{aligned} \tag{3.30}$$

where

$$c_1 = \xi M_{12}|Z|_N(p+1) \text{ with } \xi = \operatorname{ess\,sup}\{|f(z,x)| : z \in Z, |x| < M_{12}\}.$$

Let $c_2 = c_1 + 1$ and pick $\lambda < -c_2$. Then by virtue of (3.29), for $t > 0$ large and $u \in \partial B_1$, we have

$$\varphi(tu) = \frac{1}{p}\left(t^p\|Du\|_p^p - \int_Z pF(z,tu)\,dz\right) \le \lambda. \tag{3.31}$$

In view of (3.29) and of the fact that $\varphi(0) = 0$, we infer that there exists $t^* > 0$ such that

$$\varphi(t^* u) = \lambda.$$

Then

$$\begin{aligned}\frac{d}{dt}\varphi(tu) &= \langle \varphi'(tu), u\rangle = t^{p-1}\|Du\|_p^p - \int_Z f(z,tu)\,udz \\ &= \frac{1}{t}\left(t^p\|Du\|_p^p - \int_Z f(z,tu)\,tudz\right) \\ &\le \frac{1}{t}\left(t^p\|Du\|_p^p - \int_Z pF(z,tu)\,dz + c_1\right) \text{ (see (3.30))} \\ &\le \frac{1}{t}(\lambda + c_1) < 0 \text{ (recall } \lambda < -c_2 = -(c_1+1)\text{), for all } t \ge t^*.\end{aligned}$$

It follows that there exists a unique $\tau(u) > 0$, such that

$$\varphi(\tau(u)u) = \lambda \text{ for all } u \in \partial B_1.$$

Moreover, the implicit function theorem guarantees that $\tau \in C(\partial B_1)$.

For $u \neq 0$, we set $\widehat{\tau}(u) = \frac{1}{\|u\|}\tau\left(\frac{u}{\|u\|}\right)$. Evidently $\widehat{\tau} \in C\left(W_n^{1,p}(Z)\setminus\{0\}\right)$ and

$$\varphi(\widehat{\tau}(u)u) = \lambda \text{ for all } u \in W_n^{1,p}(Z)\setminus\{0\}.$$

Moreover, if $\varphi(u) = \lambda$, then $\widehat{\tau}(u) = 1$. We define

$$\widehat{\tau}_0(u) = \begin{cases} 1 & \text{if } \varphi(u) \le \lambda \\ \widehat{\tau}(u) & \text{if } \varphi(u) \ge \lambda \end{cases}.$$

From the above, it is clear that $\widehat{\tau}_0 \in C\left(W_n^{1,p}(Z)\setminus\{0\}\right)$. We introduce the map $h : [0,1] \times \left(W_n^{1,p}(Z)\setminus\{0\}\right) \to W_n^{1,p}(Z)\setminus\{0\}$, defined by

$$h(t,u) = (1-t)u + t\widehat{\tau}_0(u)u.$$

The continuity of $\widehat{\tau}_0$ implies that h is continuous, too. Also, we have

$$h(0,u) = u,\ h(1,u) \in \varphi^\lambda \text{ for all } u \in W_n^{1,p}(Z)\setminus\{0\}$$

and if $u \in \varphi^\lambda$, then $h(t,u) = u$ for all $t \in [0,1]$. These properties imply that φ^λ is a strong deformation retract of $W_n^{1,p}(Z)\setminus\{0\}$. Clearly, ∂B_1 is a retract of $W_n^{1,p}(Z)\setminus\{0\}$ (consider the radial retraction). Therefore, choosing $\lambda < \inf \varphi(K)$, we conclude that

$$\varphi^\lambda \text{ is homotopy equivalent to } \partial B_1$$

(see Dugundji [15], pp. 325, 365). Hence

$$H_k\left(W_n^{1,p}(Z), \varphi^\lambda\right) = H_k\left(W_n^{1,p}(Z), \partial B_1\right) \text{ for all } k \ge 0$$

(see Granas-Dugundji [19], p.387), therefore

$$C_k(\varphi, \infty) = 0 \text{ for all } k \ge 0$$

(see Granas-Dugundji [19] and recall that ∂B_1 is contractible in itself). □

Proposition 5. *If hypotheses* $(\mathbf{H})$ *hold, then* $C_k\left(\varphi_\pm^\varepsilon, \infty\right) = 0$ *for all* $k \ge 0$.

Proof. We present the proof for φ_+^ε, the proof for φ_-^ε being similar.

By virtue of hypotheses $(\mathbf{H})\,(iii),(iv)$, given $\theta > 0$, we can find $M_{13} = M_{13}(\theta) > 0$ such that

$$F_+(z,x) \ge \frac{\theta}{p}\left(x^+\right)^p - M_{13} \text{ for a.a. } z \in Z, \text{ all } x \in \mathbb{R}. \tag{3.32}$$

We consider the set $S_+ = \{u \in \partial B_1 : u^+ \neq 0\}$. Then, for $u \in S_+$ and $t > 0$, we have

$$\begin{aligned} \varphi_+^\varepsilon (tu) &\leq \frac{t^p}{p}(1+\varepsilon) - \frac{t^p}{p}(\theta+\varepsilon)\left\|u^+\right\|_p^p + M_{13}|Z|_N \\ &= \frac{t^p}{p}\left(1+\varepsilon-(\theta+\varepsilon)\left\|u^+\right\|_p^p\right) + M_{13}|Z|_N \end{aligned} \tag{3.33}$$

(see (3.32) and recall that $\|u\| = 1$). Since $\theta > 0$ is arbitrary, from (3.33), we infer that

$$\varphi_+^\varepsilon (tu) \to -\infty \text{ as } t \to +\infty. \tag{3.34}$$

Also, by hypothesis $(\mathbf{H})(iv)$, we can find $\beta > 0$ and $M_{14} > 0$, such that

$$f_+(z,x)x - pF_+(z,x) \geq \beta x^\mu \text{ for a.a. } z \in Z, \text{ all } x \geq M_{14}. \tag{3.35}$$

Recalling that $f_+(z,x) = F_+(z,x) = 0$ for a.a. $z \in Z$ and all $x \leq 0$, for any $v \in W_n^{1,p}(Z)$, we have

$$\int_Z pF_+(z,v)\,dz - \int_Z f_+(z,v)\,v dz \leq -\int_{\{v \geq M_{14}\}} \beta v^\mu dz + c_3, \tag{3.36}$$

where

$$c_3 = \widehat{\xi} M_{14}|Z|_N (p+1) \text{ with } \widehat{\xi} = \operatorname{ess\,sup}\{|f(z,x)| : z \in Z, 0 \leq x < M_{14}\}$$

(see (3.35) and the proof of Proposition 5).

Let $c_4 = c_3 + 1$ and pick $\lambda < -c_4$. Because of (3.34), for $t > 0$ large and $u \in S_+$, we have

$$\varphi_+^\varepsilon (tu) = \frac{1}{p}\left(t^p\|Du\|_p^p + \varepsilon\left\|u^-\right\|_p^p - \int_Z pF_+(z,tu)\,dz\right) \leq \lambda. \tag{3.37}$$

Recalling (3.34) and the fact that $\varphi_+^\varepsilon(0) = 0$, we infer that there exists $\widehat{t} > 0$ such that $\varphi_+^\varepsilon\left(\widehat{t}u\right) = \lambda$.

Then, as before (see the proof of Proposition 5)

$$\begin{aligned} \frac{d}{dt}\varphi_+^\varepsilon (tu) &= \left\langle \left(\varphi_+^\varepsilon\right)'(tu), u \right\rangle \\ &= \frac{1}{t}\left(t^p\|Du\|_p^p + \varepsilon t^p\|u\|_p^p - \int_Z f_+(z,tu)\,tudz - \varepsilon t^p\left\|u^+\right\|_p^p\right) \\ &\leq \frac{1}{t}\left(t^p\|Du\|_p^p + \varepsilon t^p\left\|u^-\right\|_p^p - \int_Z pF_+(z,tu)\,dz + c_3\right) \text{ (see (3.36))} \\ &\leq \frac{1}{t}(\lambda + c_3) \text{ (see (3.37))} \\ &< 0 \text{ (recall } \lambda < -(c_3+1)), \text{ for all } t \geq \widehat{t}. \end{aligned}$$

Thus, by the implicit function theorem, we can find a unique $\tau^+ \in C(S_+)$ such that

$$\varphi\left(\tau^+(u)\,u\right) = \lambda \text{ for all } u \in S_+.$$

Let $D_+ = \left\{x \in W_n^{1,p}(Z) : x^+ \neq 0\right\}$. We define

$$\widehat{\tau^+}(u) = \frac{1}{\|u\|}\tau^+\left(\frac{u}{\|u\|}\right) \text{ for all } u \in D_+.$$

Clearly $\widehat{\tau^+} \in C(D_+)$ and $\varphi_+^\varepsilon\left(\widehat{\tau^+}(u)\,u\right) = \lambda$ for all $u \in D_+$. Moreover, if $\varphi_+^\varepsilon(u) = \lambda$, then $\widehat{\tau^+}(u) = 1$. We define

$$\widehat{\tau}_0^+(u) = \begin{cases} 1 & \text{if } \varphi_+^\varepsilon(u) \le \lambda \\ \widehat{\tau^+}(u) & \text{if } \varphi_+^\varepsilon(u) \ge \lambda \end{cases}. \tag{3.38}$$

Then $\widehat{\tau}_0^+ \in C(D_+)$. We introduce the map $h_+ : [0,1] \times D_+ \to D_+$, defined by

$$h_+(t,u) = (1-t)\,u + t\widehat{\tau}_0^+(u)\,u.$$

The continuity of $\widehat{\tau}_0^+$ implies the continuity of h_+. In addition, we have

$$h_+(0,u) = u,\ h_+(1,u) \in \left(\varphi_+^\varepsilon\right)^\lambda \text{ for all } u \in D_+$$

and if $u \in \left(\varphi_+^\varepsilon\right)^\lambda$ then $h_+(t,u) = u$ for all $t \in [0,1]$ (see (3.38). Hence, we infer that $\left(\varphi_+^\varepsilon\right)^\lambda$ is a strong deformation retract of D_+.

Next, we show that D_+ is contractible in itself. To this end, let $u_0 \in int\ C_+$ be the L^p- normalized principal eigenvalue of $\left(-\triangle_p, W_n^{1,p}(Z)\right)$. Consider the map $\widehat{h} : [0,1] \times D_+ \to D_+$, defined by

$$\widehat{h}(t,u) = \frac{(1-t)\,u + u_0}{\|(1-t)\,u + u_0\|}.$$

Note that $[(1-t)\,u + u_0]^+ \neq 0$ and so $\widehat{h}$ is well defined. Evidently $\widehat{h}$ is continuous and $\widehat{h}(1,u) = \frac{u_0}{\|u_0\|} \in D_+$. Therefore D_+ is contractible in itself. Then, reasoning as in the proof of Proposition 5, we conclude that

$$C_k\left(\varphi_+^\varepsilon, \infty\right) = 0 \text{ for all } k \ge 0$$

In a similar way, we show that

$$C_k\left(\varphi_-^\varepsilon, \infty\right) = 0 \text{ for all } k \ge 0,$$

using this time the set $D_- = \left\{x \in W_n^{1,p}(Z) : x^- \neq 0\right\}$. □

4. Main Result

In this section, we prove a multiplicity (three solutions) theorem for problem (1.1).

Theorem 2. *If hypotheses* (**H**) *hold, then problem* (1.1) *has at least three nontrivial solutions* $x_0 \in int\ C_+$, $v_0 \in -int\ C_+$ *and* $u_0 \in int\ C_n^1\left(\overline{Z}\right)$.

Proof. From Proposition 4, we already have two solutions $x_0 \in int\ C_+$ and $v_0 \in -int\ C_+$.

Suppose that $\{0, x_0, v_0\}$ are the only critical points of φ, or otherwise we have a third nontrivial critical point , hence a third nontrivial solution of (1.1) (belonging to $C_n^1\left(\overline{Z}\right)$ by nonlinear regularity theory), and we are done.

Claim 1: $C_k\left(\varphi_+^\varepsilon, x_0\right) = C_k\left(\varphi_-^\varepsilon, v_0\right) = \delta_{k,1}\mathbb{Z}$ for all $k \ge 0$.

We complete the proof for the pair $\left(\varphi_+^\varepsilon, x_0\right)$, the proof for the pair $\left(\varphi_-^\varepsilon, v_0\right)$ being similar. Note that the critical points of φ_+^ε belong to C_+, and so are the critical points of φ, too. Since we have assumed that $\{0, x_0, v_0\}$ are the only critical points of φ, we see that $\{0, x_0\}$ are the only critical points of φ_+^ε. We choose λ, $\xi \in \mathbb{R}$ such that

$$\lambda < 0 = \varphi_+^\varepsilon(0) < \xi < \varphi_+^\varepsilon(x_0).$$

Then, we consider the following triple of sets

$$\left(\varphi_+^\varepsilon\right)^\lambda \subseteq \left(\varphi_+^\varepsilon\right)^\xi \subseteq W_n^{1,p}(Z) =: W.$$

We have the following long exact sequence

$$...H_k\left(W, \left(\varphi_+^\varepsilon\right)^\lambda\right) \xrightarrow{j_*} H_k\left(W, \left(\varphi_+^\varepsilon\right)^\xi\right) \xrightarrow{\partial} H_{k-1}\left(\left(\varphi_+^\varepsilon\right)^\xi, \left(\varphi_+^\varepsilon\right)^\lambda\right)... \tag{4.1}$$

where j_* is the homomorphism induced by the inclusion

$$\left(W, \left(\varphi_+^\varepsilon\right)^\lambda\right) \xrightarrow{j} \left(W, \left(\varphi_+^\varepsilon\right)^\xi\right)$$

and ∂ is the boundary homomorphism. From the choice of the levels λ and ξ and since $\{0, x_0\}$ are the only critical points of φ_+^ε, we have

$$H_k\left(W, \left(\varphi_+^\varepsilon\right)^\lambda\right) = C_k\left(\varphi_+^\varepsilon, \infty\right) = 0 \text{ for all } k \geq 0 \tag{4.2}$$

(see Proposition 6),

$$H_k\left(W, \left(\varphi_+^\varepsilon\right)^\xi\right) = C_k\left(\varphi_+^\varepsilon, x_0\right) \text{ for all } k \geq 0 \tag{4.3}$$

and

$$H_{k-1}\left(\left(\varphi_+^\varepsilon\right)^\xi, \left(\varphi_+^\varepsilon\right)^\lambda\right) = C_{k-1}\left(\varphi_+^\varepsilon, 0\right) = \delta_{k-1,0}\mathbb{Z} = \delta_{k,1}\mathbb{Z} \text{ for all } k \geq 0 \tag{4.4}$$

(see Proposition 3). The exactness of (4.1) together with (4.2) imply that the boundary homomorphism is an isomorphism between the groups in (4.3) and (4.4). This proves Claim 1.

Claim 2: $C_k(\varphi, x_0) = C_k\left(\varphi_+^\varepsilon, x_0\right)$ and $C_k(\varphi, v_0) = C_k\left(\varphi_-^\varepsilon, v_0\right)$ for all $k \geq 0$.

We do the proof for the pair $\left(\varphi_+^\varepsilon, x_0\right)$, since the proof for the pair $\left(\varphi_-^\varepsilon, v_0\right)$ is similar.

Consider the homotopy

$$\psi(t, x) = t\varphi_+^\varepsilon(x) + (1-t)\varphi(x) \text{ for all } (t, x) \in [0,1] \times W_n^{1,p}(Z).$$

We show that, without loss of generality, we can say that for some $r > 0$, x_0 is the only critical point of $\psi(t, .)$ in $B_r(x_0) = \left\{x \in W_n^{1,p}(Z) : \|x - x_0\| < r\right\}$, for all $t \in [0,1]$. Indeed, if this is not the case, we can find $\{t_n\}_{n\geq 1} \subseteq [0,1]$ and $\{x_n\}_{n\geq 1} \subseteq W_n^{1,p}(Z)$ such that $t_n \to t$, $x_n \to x$ in $W_n^{1,p}(Z)$ and $\psi_x'(t_n, x_n) = 0$. Then

$$A(x_n) + t_n\varepsilon |x_n|^{p-2} x_n = t_n N_+(x_n) + (1-t_n) N(x_n) + (1-t_n)\varepsilon \left(x_n^+\right)^{p-1},$$

therefore

$$\begin{cases} -\triangle_p x_n(z) = t_n f_+(z, x_n(z)) + (1-t_n) f(z, x_n(z)) + (1-t_n)\varepsilon \left(x_n^+(z)\right)^{p-1} \\ \qquad\qquad +\varepsilon t_n \left[\left(x_n^-(z)\right)^{p-1} - \left(x_n^+(z)\right)^{p-1}\right] \text{ a.e. on } Z, \\ \frac{\partial x_n}{\partial n} = 0 \text{ on } \partial Z. \end{cases}$$

The nonlinearity on the right hand side of the above Neumann problem has subcritical growth. Hence by $L^\infty(Z)-$ regularity (see Ladyzhenskaya-Uraltseva [20]) we can find an $M_{15} > 0$ such that $\|x_n\|_\infty \leq M_{15}$ for all $n \geq 1$. Then (see [21]) we can find $\alpha \in [0,1]$ and $M_{16} > 0$, both independent of $n \geq 1$, such that

$$x_n \in C_n^{1,\alpha}\left(\overline{Z}\right) \text{ and } \|x_n\|_{C_n^{1,\alpha}(\overline{Z})} \leq M_{16} \text{ for all } n \geq 1. \tag{4.5}$$

From (4.5) and the compact embedding of $C_n^{1,\alpha}\left(\overline{Z}\right)$ into $C_n^1\left(\overline{Z}\right)$ (see, for example Adams [1], p.11), we have that $\{x_n\}_{n\geq 1}\subseteq C_n^{1,\alpha}\left(\overline{Z}\right)$ is relatively compact, hence we may assume that

$$x_n \to x_0 \text{ in } C_n^1\left(\overline{Z}\right).$$

Since $x_0 \in int\ C_+$, it folows that we can find an integer $n_0 \geq 1$ such that $x_n \in int\ C_+$ for all $n \geq n_0$. Then

$$f_+\left(z, x_n\left(z\right)\right) = f\left(z, x_n\left(z\right)\right) \text{ for all } n \geq n_0,$$

and so, all x_n with $n \geq n_0$ are solutions of (1.1) and we are done.

Therefore, we may assume that, for some $r > 0$, x_0 is the only critical point in $B_r\left(x_0\right)$ of $\psi\left(t, .\right)$, for all $t \in [0,1]$. Invoking the homotopy invariance property of singular homology, we have

$$C_k\left(\psi\left(0,.\right), x_0\right) = C_k\left(\psi\left(1,.\right), x_0\right) \text{ for all } k \geq 0,$$

hence

$$C_k\left(\varphi, x_0\right) = C_k\left(\varphi_+^{\varepsilon}, x_0\right) \text{ for all } k \geq 0.$$

Similarly, we show that

$$C_k\left(\varphi, v_0\right) = C_k\left(\varphi_-^{\varepsilon}, v_0\right) \text{ for all } k \geq 0.$$

This proves Claim 2.

Recall that, because of Proposition 3, we have

$$C_k\left(\varphi, 0\right) = \delta_{k,0}\mathbb{Z} \text{ for all } k \geq 0 \tag{4.6}$$

(see Chang [12], p.33 and Mawhin-Willem [23], p.175).

Then from Propositions 5, 6, Claims 1, 2, (4.6) and the Poincare-Hopf formula (see (2.1)) we have

$$\left(-1\right)^0 + 2\left(-1\right)^1 = 0,$$

a contradiction. This means that φ has a third nontrivial critical point $u_0 \in W_n^{1,p}\left(Z\right)$, which solves (1.1) and belongs to $C_n^1\left(\overline{Z}\right)$ (by the nonlinearity regularity theory). $\square$

Remark: When $p = 2$ (semilinear problems), Theorem 7 extends to Neumann problems the work of Wang [31], where the boundary condition is of Dirichlet type. However, note that in contrast with Wang [31], where $f\left(z, x\right) = f\left(x\right)$, we do note assume that f is of class C^1 and we do not use the AR-condition.

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