Set-Valued Anal
DOI 10.1007/s11228-008-0100-1

Eigenvalue Problems for Hemivariational Inequalities

Nikolaos S. Papageorgiou ·
Sandrina Rafaela Andrade Santos · **Vasile Staicu**

Received: 13 March 2008 / Accepted: 24 June 2008

Abstract We consider a semilinear eigenvalue problem with a nonsmooth potential (hemivariational inequality). Using a nonsmooth analog of the local Ambrosetti–Rabinowitz condition (AR-condition), we show that the problem has a nontrivial smooth solution. In the scalar case, we show that we can relax the local AR-condition. Finally, for the resonant $\lambda = \lambda_1$ problem, using the nonsmooth version of the local linking theorem, we show that the problem has at least two nontrivial solutions. Our approach is variational, using minimax methods from the nonsmooth critical point theory.

Keywords Locally Lipschitz function · Generalized subdifferential · Linking set · AR-condition · Multiple solutions

Mathematics Subject Classifications (2000) 35J20 · 35J60

1 Introduction

Let $Z \subseteq \mathbb{R}^n$ be a bounded domain with a C^2-boundary ∂Z. The goal of this paper is to study the following semilinear eigenvalue problem with a nonsmooth potential (hemivariational inequality)

$$\begin{cases} -\Delta x(z) - \lambda a(z)\, x(z) \in \partial j(z, x(z)) \text{ a.e. on } Z, \\ x|_{\partial Z} = 0. \end{cases} \tag{1.1}$$

N. S. Papageorgiou
Department of Mathematics, National Technical University,
Zografou Campus, Athens 15780, Greece
e-mail: npapg@math.ntua.gr

S. R. Andrade Santos · V. Staicu (✉)
Department of Mathematics, Aveiro University, 3810-193 Aveiro, Portugal
e-mail: vasile@ua.pt

S. R. Andrade Santos
e-mail: sandrinasant@gmail.com

Here $a \in L^{\infty}(Z)_{+}$ is a function with strictly positive essential infimum, and $j(z,x)$ is a measurable potential function which is locally Lipschitz and, in general, nonsmooth in the $x \in \mathbb{R}$ variable. By $\partial j(z,x)$, we denote the generalized (Clarke) subdifferential of $j(z,\cdot)$ (see Section 2). Such problems are known as *hemivariational inequalities*. They arise in mechanics and engineering, if one wants to consider more realistic laws of set-valued and nonmonotone nature, which correspond to nonsmooth and nonconvex energy functionals and their study requires tools and techniques from nonsmooth and multivalued analysis. For concrete applications, we refer to the book of Naniewicz–Panagiotopoulos [20].

Eigenvalue problems for hemivariational inequalities provide information about the stability and bifurcation properties of the solutions and have attracted the interest of many authors. We mention the works of Barletta–Marano [1], Cirstea–Radulescu [4], Goeleven–Motreanu–Panagiotopoulos [11], Marano-Molica Bisci–Motreanu [14], Motreanu–Panagiotopoulos [16, 17] (for semilinear problems) and by Gasinski–Papageorgiou [7, 8], Motreanu–Motreanu–Papageorgiou [15], Motreanu–Radulescu [19] and Papageorgiou–Papageorgiou [21] (for quasilinear problems).

Here, we prove the existence of a nontrivial solution for every value of the parameter $\lambda \in \mathbb{R}$. An analogous *smooth* result was proved by Rabinowitz [22], p.30, who assumed that $j \in C^1\left(\overline{Z}, \mathbb{R}\right)$ and that $\partial j(z,x) = \{f(z,x)\}$, where $f(.,.)$ satisfies the sign condition

$$f(z,x)\,x \geqslant 0 \text{ for all } (z,x) \in \overline{Z} \times \mathbb{R}.$$

The work of Rabinowitz was extended to hemivariational inequalities by Barletta–Marano [1]. Both works assume the so-called Ambrosetti–Rabinowitz condition (AR-condition, for short), which dictates a superquadratic behavior for the potential $x \to j(z,x)$. Rabinowitz uses a local version of the AR-condition (i.e., it is valid only for $|x|$ large). Barletta–Marano [1], use a kind of global AR-condition and also employ an additional condition (see (j_5) in [1]).

Here, we use only a local AR-condition (see Hypothesis $H(j)_1$ (v)) and this makes our existence theorem (see Theorem 4), a more genuine nonsmooth generalization of Theorem 5.16, p.30, of Rabinowitz [22]. In fact, even when restricted to the smooth case, our result improves and refines the aforementioned theorem of Rabinowitz [22]. Moreover, for the scalar problem (i.e., $N = 1$, hence Eq. 1.1 becomes an ordinary differential inclusion), we are able to replace the AR-condition by a weaker one (see hypothesis $H(j)_2$ (iv)). Finally, when $\lambda = \lambda_1$, with $\lambda_1 > 0$ being the principal (first) eigenvalue of $\left(-\Delta, H_0^1(Z), a\right)$ ($a \in L^{\infty}(Z)_{+}$ being a weight function), under conditions on the potential function $j(z,\cdot)$, which permit resonance both at zero and at infinity, we prove a multiplicity theorem (see Theorem 6).

Our approach is variational, using minimax methods from nonsmooth critical point theory.

2 Mathematical Background

The nonsmooth critical point theory which we employ in the variational arguments, is based mainly on the subdifferential theory for locally Lipschitz functions. So, we start by recalling some basic notions and facts from this theory. Our main reference in this direction is the book of Clarke [5].

Let X be a Banach space. By X^* we denote its topological dual and by $\langle \cdot, \cdot \rangle$ the duality brackets for the pair (X, X^*). Let $\varphi\colon X \to \mathbb{R}$ be a locally Lipschitz functional. The generalized directional derivative $\varphi^0(x; h)$ of φ at $x \in X$ in the direction $h \in X$, is defined by

$$\varphi^0(x; h) = \limsup_{\substack{x' \to x \\ \lambda \downarrow 0}} \frac{\varphi(x' + \lambda h) - \varphi(x')}{\lambda}.$$

It is straightforward to check that $h \to \varphi^0(x; h)$ is continuous, sublinear and so it is the support function of a nonempty, convex and w*-compact set $\partial\varphi(x) \subseteq X^*$ defined by

$$\partial\varphi(x) = \left\{x^* \in X^* : \left\langle x^*, h\right\rangle \leqslant \varphi^0(x; h) \text{ for all } h \in X\right\}.$$

The multifunction $x \to \partial\varphi(x)$ is called the generalized (or Clarke) subdifferential of φ. If $\varphi : X \to \mathbb{R}$ is continuous convex, then the generalized subdifferential of φ coincides with the subdifferential in the sense of convex analysis, given by

$$\partial_c\varphi(x) = \left\{x^* \in X^* : \left\langle x^*, h\right\rangle \leqslant \varphi(x + h) - \varphi(x) \text{ for all } h \in X\right\}.$$

If $\varphi \in C^1(X)$, then $\partial\varphi(x) = \left\{\varphi'(x)\right\}$. We say that $x \in X$ is a critical point of φ, if $0 \in \partial\varphi(x)$ and in this case $c = \varphi(x)$ is a critical value of φ. We can easily check that, if $x \in X$ is a local extremum of φ (i.e., $x \in X$ is a local minimum or a local maximum of φ), then x is a critical point of φ.

Given a locally Lipschitz function $\varphi\colon X \to \mathbb{R}$, we say that φ satisfies the Palais–Smale condition at level $c \in \mathbb{R}$ (PS_c-condition, for short), if every sequence $\{x_n\}_{n\geqslant 1} \subseteq X$ such that

$$\varphi(x_n) \to c \text{ and } m(x_n) = \inf\left[\left\|x^*\right\| : x^* \in \partial\varphi(x_n)\right] \to 0, \qquad \text{as } n \to \infty,$$

has a strongly convergent subsequence.

We say that φ satisfies the PS-condition, if it satisfies the PS_c-condition for every $c \in \mathbb{R}$. Sometimes, it is more appropriate to use a slightly more general compactness notion, the so-called C-condition. So, we say that φ satisfies the Cerami condition at the level $c \in \mathbb{R}$ (the C_c-condition, for short), if every sequence $\{x_n\}_{n\geqslant 1} \subseteq X$ such that

$$\varphi(x_n) \to c \text{ and } (1 + \|x_n\|)\, m(x_n) \to 0, \qquad \text{as } n \to \infty,$$

has a strongly convergent subsequence.

We say that φ satisfies the C-condition, if the C_c-condition holds for all $c \in \mathbb{R}$. Evidently, the C-condition is weaker than the PS-condition. The two conditions coincide, if φ is bounded below.

The topological notion of linking sets is crucial in the minimax characterization of the critical values of a locally Lipschitz function.

Definition 1 Let Y be a Hausdorff topological space, E_0, E and D are nonempty closed subsets of Y, with $E_0 \subseteq E$. We say that the pair $\{E, E_0\}$ is linking with D in Y, if

(a) $E_0 \cap D = \varnothing$;
(b) for any $\gamma \in C(E, Y)$, with $\gamma|_{E_0} = id|_{E_0}$, we have $\gamma(E) \cap D \neq \varnothing$.

Using this notion, we have the following general minimax principle for the critical values of a locally Lipschitz function (see Kourogenis–Papageorgiou [13]).

Theorem 1 *If E_0, E and D are nonempty, closed subsets of X, $\{E, E_0\}$ is linking with D in X, $\varphi\colon X \to \mathbb{R}$ is locally Lipschitz, $\sup_{E_0}\varphi \leqslant \inf_D \varphi$, φ satisfies the C_c-condition, where*

$$c = \inf_{\gamma\in\Gamma}\sup_{x\in E}\varphi(\gamma(x)) \ \ \text{and}\ \Gamma = \left\{\gamma \in C(E, X) : \gamma|_{E_0} = id|_{E_0}\right\},$$

then $c \geqslant \inf_D \varphi$ and c is a critical value of φ. Moreover, if $c = \inf_D \varphi$, then there exists a critical point x of φ, such that $\varphi(x) = c$ and $x \in E$.

With suitable choices of linking sets, we produce nonsmooth versions of well-known minimax theorems. We mention the *nonsmooth mountain pass theorem*, which we shall need in the sequel.

Theorem 2 *If $x_0, x_1 \in X$ with $\|x_1 - x_0\| > r > 0$,*

$$\max\{\varphi(x_0), \varphi(x_1)\} \leqslant \inf[\varphi(x) : \|x\| = r]$$

and φ satisfies the PS_c-condition, where

$$c = \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]}\varphi(\gamma(t)), \ \text{with}\ \Gamma = \left\{\gamma \in C([0,1], X) : \gamma(0) = x_0, \gamma(1) = x_1\right\},$$

then $c \geqslant \inf_{\|x\|=r}\varphi(x)$ and c is a critical value of φ. Moreover, if $c = \inf_{\|x\|=r}\varphi(x)$, then there exists a critical point x of φ, with $\varphi(x) = c$ and $\|x\| = r$.

More about the nonsmooth critical point theory can be found in the books of Carl–Le–Motreanu [3], Gasinski–Papageorgiou [8] and Motreanu–Radulescu [18].

Recently, Kandilakis–Kourogenis–Papageorgiou [12] (see also Gasinski–Papageorgiou [8], p.178), extended to a nonsmooth setting the local linking theorem of Brezis–Nirenberg [2].

Theorem 3 *If $X = Y \oplus V$, with $\dim Y < +\infty$, φ is bounded below, satisfies the C-condition, $\varphi(0) = 0$, $\inf_X \varphi < 0$ and there exists $r > 0$, such that*

$$\begin{cases} \varphi(x) \leqslant 0 \ \text{if}\ x \in Y, & \|x\| \leqslant r \\ \varphi(x) \geqslant 0 \ \text{if}\ x \in V, & \|x\| \leqslant r \end{cases}$$

(*local linking at* 0)*, then φ has, at least, two nontrivial critical points.*

Let us recall some basic facts about the spectrum of $\left(-\Delta, H_0^1(Z), a\right)$. For details we refer, for example, to Gasinski–Papageorgiou [9]. So, we consider the following linear eigenvalue problem with weight $a \in L^\infty(Z)_+$, with *ess* inf $a > 0$:

$$\begin{cases} -\Delta x(z) = \lambda a(z)\, x(z), \ \text{a.e. on}\ Z \\ x|_{\partial Z} = 0, \end{cases} \tag{2.1}$$

The problem has a sequence of distinct eigenvalues $\{\lambda_m\}_{m\geqslant 1}$, which are all positive, $\lambda_m < \lambda_{m+1}$, for all $m \geqslant 1$, $\lambda_m \to +\infty$ as $m \to \infty$ and $\lambda_1 > 0$ is simple (i.e., the corresponding eigenspace, $E(\lambda_1)$, is one-dimensional). If $\{u_n\}_{n\geqslant 1} \subseteq H_0^1(Z)$ are the eigenfunctions corresponding to these eigenvalues, then $u_n \in H_0^1(Z) \cap C^\infty(Z)$ and

$$\int_Z (Du_n, Du_k)_{\mathbb{R}^N}\, dz = 0, \qquad \int_Z a(z)\, u_n(z)\, u_k(z)\, dz = 0, \qquad \text{for all } n \neq k.$$

Moreover, if ∂Z is a C^k-manifold ($1 \leqslant k \leqslant \infty$), then $u_n \in C^k(\overline{Z})$, for all $n \geqslant 1$. For every integer $m \geqslant 1$, by $E(\lambda_m)$ we denote the eigenspace corresponding to the eigenvalue λ_m. This space has the unique continuation property, namely if $u \in E(\lambda_m)$ and u vanishes on a set of positive measure, then $u \equiv 0$. We set

$$Y_m = \overset{m}{\underset{i=1}{\oplus}} E(\lambda_i) \text{ and } V_m = \overline{\underset{i\geqslant m}{\oplus} E(\lambda_i)}, \qquad \text{for all } m \geqslant 1.$$

We have the following variational characterizations of the eigenvalues (using the so-called Rayleigh quotient):

$$\lambda_1 = \min\left[\frac{\|Dx\|_2^2}{\int_Z ax^2 dz} : x \in H_0^1(Z), \qquad x \neq 0\right] \tag{2.2}$$

and for $m \geqslant 2$,

$$\lambda_m = \max_{x\in Y_m,\, x\neq 0} \frac{\|Dx\|_2^2}{\int_Z ax^2 dz} = \min_{x\in V_m,\, x\neq 0} \frac{\|Dx\|_2^2}{\int_Z ax^2 dz}$$

$$= \min\left\{\max_{x\in Y,\, x\neq 0} \frac{\|Dx\|_2^2}{\int_Z ax^2 dz} : Y \subseteq H_0^1(Z), \qquad \dim Y = m.\right\} \tag{2.3}$$

We shall need the following simple facts about the component spaces Y_m and V_m.

Lemma 1

(a) *If $\theta \in L^\infty(Z)_+$, $\theta(z) \leqslant \lambda_m$ a.e. on Z and $\theta \neq \lambda_m$, then there exists $\xi_0 > 0$, such that*

$$\psi_0(x) = \|Dx\|_2^2 - \int_Z \theta(z)\, a(z)\, x(z)^2\, dz \geqslant \xi_0 \|Dx\|_2^2, \qquad \textit{for all } x \in V_m.$$

(b) *If $\gamma \in L^\infty(Z)_+$, $\gamma(z) \geqslant \lambda_m$ a.e. on Z and $\gamma \neq \lambda_m$, then there exists $\xi_1 > 0$, such that*

$$\psi_1(x) = \int_Z \gamma(z)\, a(z)\, x(z)^2\, dz - \|Dx\|_2^2 \geqslant \xi_1 \|Dx\|_2^2, \qquad \textit{for all } x \in Y_m.$$

Proof

(a) We proceed by contradiction. So suppose that the result is not true. Exploiting the 2-homogeneity of ψ_0, we can find $\{x_n\}_{n\geqslant 1} \subseteq V_m$, such that $\|Dx_n\|_2 = 1$, for all $n \geqslant 1$ and $\psi_0(x_n) \downarrow 0$. We may assume that

$$x_n \xrightarrow{w} x \text{ in } H_0^1(Z) \text{ and } x_n \to x \text{ in } L^2(Z).$$

Hence in the limit as $n \to \infty$, we obtain

$$\|Dx\|_2^2 \leqslant \int_Z \theta a x^2 dz \leqslant \lambda_m \int_Z a x^2 dz.$$

If $x = 0$, then $Dx_n \to 0$ in $L^2\left(Z, \mathbb{R}^N\right)$, a contradiction, since $\|Dx_n\|_2 = 1$, for all $n \geqslant 1$. So, $x \neq 0$ and $x \in V_m$. Hence, by virtue of Eq. 2.3 , we have $\|Dx\|_2^2 = \lambda_m \|x\|_2^2$ and so $x \in E(\lambda_m)$. Because $E(\lambda_m)$ has the unique continuation property, it follows that $x(z) \neq 0$ a.e. on Z. Therefore

$$\|Dx\|_2^2 < \lambda_m \int_Z a x^2 dz,$$

a contradiction to Eq. 2.3.

(b) The proof of this part is done similarly as (a) and so it is omitted. □

3 Existence of Solutions

The hypotheses on the nonsmooth potential $j(z, x)$ are the following:

H(j)$_1$: j: $Z \times \mathbb{R} \to \mathbb{R}_+$ is a function such that $j(z, 0) = 0$ a.e. on Z and

- (i) for all $x \in \mathbb{R}$, $z \to j(z, x)$ is measurable;
- (ii) for almost all $z \in Z$, $x \to j(z, x)$ is locally Lipschitz;
- (iii) for almost all $z \in Z$, all $x \in \mathbb{R}$ and all $u \in \partial j(z, x)$, we have

$$|u| \leqslant a_0(z) + c_0 |x|^{r-1},$$

  with $a_0 \in L^\infty(Z)_+$, $c_0 > 0$ and $2 < r < 2^*$;
- (iv) $\lim\limits_{x \to 0} \frac{j(z,x)}{x^2} = 0$ uniformly for a.a. $z \in Z$;
- (v) there exist $\mu > 2$ and $M > 0$, such that for almost all $z \in Z$ and all $|x| \geqslant M$

$$0 < \widehat{c} \leqslant \mu j(z, x) \leqslant -j^0(z, x; -x).$$

Remark 1 Hypothesis $H(j)_1$ (v) is a nonsmooth analog of the local AR-condition. In Lemma 2 below, we show that this hypothesis implies the superquadratic growth $x \to j(z, x)$.

Example 1 The following functions satisfy hypotheses $H(j)_1$. In what follows $a_0 \in L^\infty(Z)_+$.

$$j_1(z, x) = \begin{cases} \dfrac{a_0(z)}{3} |x|^3 & \text{if } |x| \leqslant 1 \\ 1/\mu\, |x|^\mu - x^2 \ln |x| - 1/\mu + \dfrac{a_0(z)}{3} & \text{if } |x| > 1 \end{cases}, \qquad \mu > 2,$$

$$j_2(z, x) = \begin{cases} \dfrac{a_0(z)}{2} x^2 \ln(|x| + 1) & \text{if } |x| \leqslant 1 \\ 1/\mu\, |x|^\mu - 1/\mu + \dfrac{a_0(z)}{2} \ln 2 & \text{if } |x| > 1 \end{cases}, \qquad \mu > 2,$$

and

$$j_3(z,x) = \frac{a_0(z)}{\mu}|x|^{\mu} + \frac{1}{2}x^2 \ln(|x|+1), \qquad \mu > 2.$$

Note that $j_2(z,x)$ does not satisfy the hypotheses of Barletta–Marano [1], who assume a global AR-condition. Also note that $j_3(z,x)$ is a C^1 function with respect to x.

We start with a simple lemma, which highlights the consequences of the nonsmooth AR-condition (see hypothesis $H(j)_1$ (v)).

Lemma 2 *If* $j_0 : \mathbb{R} \to \mathbb{R}$ *is locally Lipschitz and we can find* $\mu > 2$ *and* $M > 0$*, such that*

$$0 < \mu j_0(x) \leqslant -(j_0)^0(x; -x), \qquad \textit{for all } |x| \geqslant M,$$

then there exist $c_1, c_2 > 0$*, such that*

$$c_1\|x\|^{\mu} - c_2 \leqslant j_0(x), \qquad \textit{for all } x \in \mathbb{R}.$$

Proof Let $x \in \mathbb{R}$, with $|x| \geqslant M$ and consider the function $\xi_0 : [1,\infty) \to \mathbb{R}_+$ defined by

$$\xi_0(r) = j_0(rx), \qquad r \geqslant 1.$$

Evidently ξ_0 is locally Lipschitz. From the nonsmooth chain rule (see Clarke [5], p. 45), we have

$$\partial_r(\xi_0(r)) \subseteq x\partial j_0(rx). \tag{3.1}$$

The function $r \to \xi_0(r)$ is differentiable almost everywhere and if $r \in [1,\infty)$ is such a point of differentiability of $\xi_0(\cdot)$, we have $\frac{d}{dr}\xi_0(r) \in \partial\xi_0(r)$ (see Clarke [5], p. 32). So, from Eq. 3.1 and the hypothesis of the Lemma, we have

$$\mu\xi_0(r) = \mu j_0(rx) \leqslant -(j_0)^0(rx; -rx) \leqslant r\frac{d}{dr}\xi_0(r), \qquad \text{for a.a. } r \geqslant 1,$$

hence

$$\mu/r \leqslant \frac{\frac{d}{dr}\xi_0(r)}{\xi_0(r)}, \qquad \text{for a.a. r} \geqslant 1.$$

Integrating this inequality from 1 to $r \geqslant 1$, we obtain

$$\ln r^{\mu} \leqslant \ln\frac{\xi_0(r)}{\xi_0(1)},$$

so

$$r^{\mu}\xi_0(1) \leqslant \xi_0(r),$$

therefore

$$r^{\mu} j_0(x) \leqslant j_0(rx), \qquad \text{for all } r \geqslant 1 \text{ and all } |x| \geqslant M. \tag{3.2}$$

Then, for all $|x| \geqslant M$, we have

$$j_0(x) = j_0\left(\frac{|x|}{M} M \frac{x}{|x|}\right) \geqslant \frac{|x|^\mu}{M^\mu} j_0\left(\frac{Mx}{|x|}\right) \text{ (see (3.2))}$$
$$\geqslant \frac{|x|^\mu}{M^\mu} \min\{j_0(M), j_0(-M)\} = c_1 |x|^\mu \tag{3.3}$$

for some $c_1 > 0$. On the other hand, if $|x| < M$, then we can find $\widehat{c}_1 > 0$, such that

$$|j_0(x)| \leqslant \widehat{c}_1. \tag{3.4}$$

From Eqs. 3.3 and 3.4 it follows that

$$j_0(x) \geqslant c_1 |x|^\mu - c_2,$$

for all $x \in \mathbb{R}$, with $c_2 = \widehat{c}_1 + c_1 M^\mu > 0$. □

Let $\varphi_\lambda : H_0^1(Z) \to \mathbb{R}$ be the Euler functional for problem Eq. 1.1, defined by

$$\varphi_\lambda(x) = 1/2 \|Dx\|_2^2 - \lambda/2 \int_Z a(z) x(z)^2 dz - \int_Z j(z, x(z)) dz$$

for all $x \in H_0^1(Z)$. We know that φ_λ is Lipchitz continuous on bounded sets, hence it is locally Lipschitz (see Clarke [5], p. 83).

Proposition 1 *If hypotheses $H(j)_1$ hold and $\lambda_k \leqslant \lambda < \lambda_{k+1}$ for some $k \geqslant 1$, then φ_λ satisifies the PS-condition.*

Proof Let $\{x_n\}_{n\geqslant 1} \subseteq H_0^1(Z)$ be a sequence such that

$$|\varphi_\lambda(x_n)| \leqslant M_1, \tag{3.5}$$

for some $M_1 > 0$, all $n \geqslant 1$ and

$$m_\lambda(x_n) = \inf\left[\|x^*\| : x^* \in \partial\varphi_\lambda(x_n)\right] \to 0 \text{ as } n \to \infty. \tag{3.6}$$

Since $\partial\varphi_\lambda(x_n) \subseteq H^{-1}(Z) = H_0^1(Z)^*$ is nonempty and weakly compact and in any Banach space, the norm functional is weakly lower semicontinuous, by the Weierstrass theorem, we can find $x_n^* \in \partial\varphi_\lambda(x_n)$ such that $m_\lambda(x_n) = \|x_n^*\|$ for all $n \geqslant 1$. Let $\langle\cdot,\cdot\rangle$ denote the duality brackets for the pair $\left(H^{-1}(Z), H_0^1(Z)\right)$ and let $A \in \mathcal{L}\left(H_0^1(Z), H^{-1}(Z)\right)$ be defined by

$$\langle A(x), y\rangle = \int_Z (Dx, Dy)_{\mathbb{R}^N} dz, \qquad \text{for all } x, \qquad y \in H_0^1(Z).$$

We have

$$x_n^* = A(x_n) - \lambda a x_n - u_n, \qquad \text{with } u_n \in N(x_n), \tag{3.7}$$

where

$$N(v) = \left\{u \in L^{r'}(Z) : u(z) \in \partial j(z, v(z)) \text{ a.e. on } Z\right\}$$

for all $v \in H_0^1(Z)$ and with $1/r + 1/r' = 1$. Let $\eta \in (2, \mu)$. We have

$$\eta/2 \|Dx_n\|_2^2 - \lambda\eta/2 \int_Z a x_n^2 dz - \int_Z \eta j(z, x_n) dz \leqslant \eta M_1 \text{ (see (3.5))} \tag{3.8}$$

and

$$\left| \langle A(x_n), x_n \rangle - \lambda \int_Z a x_n^2 dz - \int_Z u_n x_n dz \right| \leqslant \varepsilon_n \|x_n\|,$$

with $\varepsilon_n \downarrow 0$ (see Eqs. 3.6, 3.7), therefore

$$-\|Dx_n\|_2^2 + \lambda \int_Z a x_n^2 dz + \int_Z u_n x_n dz \leqslant \varepsilon_n \|x_n\|,$$

so

$$-\|Dx_n\|_2^2 + \lambda \int_Z a x_n^2 dz - \int_Z j^0(z, x_n; -x_n) dz \leqslant \varepsilon_n \|x_n\|. \tag{3.9}$$

Adding Eqs. 3.8 and 3.9, we obtain

$$(\eta/2 - 1)\|Dx_n\|_2^2 - \lambda(\eta/2 - 1)\int_Z a x_n^2 dz - \int_Z \left[\eta j(z, x_n) + j^0(z, x_n; -x_n)\right] dz \\ \leqslant \varepsilon_n \|x_n\| + \eta M_1, \tag{3.10}$$

hence

$$(\eta/2 - 1)\|Dx_n\|_2^2 - c_3\|x_n\|_2^2 - \int_Z \left[\eta j(z, x_n) + j^0(z, x_n; -x_n)\right] dz \\ \leqslant \varepsilon_n \|x_n\| + \eta M_{1,} \qquad \text{with } c_3 = c_3(\lambda) = \lambda(\eta/2 - 1)\|a\|_\infty > 0. \tag{3.11}$$

Note that

$$\begin{aligned} &\int_Z \left[\eta j(z, x_n) + j^0(z, x_n; -x_n)\right] dz \\ &= \int_Z \left[\mu j(z, x_n) + j^0(z, x_n; -x_n)\right] dz - (\mu - \eta)\int_Z j(z, x_n) dz \\ &= \int_{\{|x_n| < M\}} \left[\mu j(z, x_n) + j^0(z, x_n; -x_n)\right] dz \\ &\quad + \int_{\{|x_n| \geqslant M\}} \left[\mu j(z, x_n) + j^0(z, x_n; -x_n)\right] dz - (\mu - \eta)\int_Z j(z, x_n) dz \\ &\leqslant c_4 - (\mu - \eta)\int_Z j(z, x_n) dz, \qquad \text{for some } c_4 > 0 \end{aligned} \tag{3.12}$$

(see hypotheses $H(j)_1$ (iii),(v)). Returning to Eq. 3.11 and using 3.12, we have

$$(\eta/2 - 1)\|Dx_n\|_2^2 - c_3\|x_n\|_2^2 - c_4 + (\mu - \eta)\int_Z j(z, x_n) dz \leqslant \varepsilon_n \|x_n\| + \eta M_{1,}$$

so

$$(\eta/2 - 1)\|Dx_n\|_2^2 - c_3\|x_n\|_2^2 + (\mu - \eta) c_1 \|x_n\|_\mu^\mu \leqslant \varepsilon_n \|x_n\| + c_5, \tag{3.13}$$

for some $c_5 > 0$ (see Lemma 2). Since $\mu > 2$ and using Young's inequality with $\varepsilon > 0$, we obtain

$$\|x_n\|_2^2 \leqslant c_6 \|x_n\|_\mu^2 \leqslant \varepsilon \|x_n\|_\mu^\mu + c_7(\varepsilon),$$

for some $c_6, c_7(\varepsilon) > 0$. So Eq. 3.13 becomes

$$(\eta/2 - 1)\|Dx_n\|_2^2 + ((\mu - \eta) c_1 - \varepsilon)\|x_n\|_\mu^\mu \leqslant \varepsilon_n \|x_n\| + c_8(\varepsilon), \tag{3.14}$$

with $c_8(\varepsilon) = c_5 + c_7(\varepsilon) > 0$. Choosing $0 < \varepsilon \leqslant (\mu - \eta) c_1$, from Eq. 3.14 and Poincaré's inequality, we conclude that $\{x_n\}_{n\geqslant 1} \subseteq H_0^1(Z)$ is bounded. Hence by passing to a suitable subsequence, if necessary, we may assume that

$$x_n \xrightarrow{w} x \text{ in } H_0^1(Z)$$

and

$$x_n \to x \text{ in } L^2(Z) \text{ and in } L^r(Z) \text{ (recall } r < 2^*).$$

From Eq. 3.6, we know that

$$\left|\langle A(x_n), x_n - x\rangle - \lambda \int_Z a x_n (x_n - x)\, dz - \int_Z u_n (x_n - x)\, dz\right|$$
$$\leqslant \varepsilon_n \|x_n - x\|. \tag{3.15}$$

Evidently, we have

$$\int_Z a x_n (x_n - x)\, dz \to 0 \text{ and } \int_Z u_n (x_n - x)\, dz \to 0 \text{ as } n \to \infty.$$

Hence, from Eq. 3.15, we obtain

$$\langle A(x_n), x_n - x\rangle \to 0 \text{ as } n \to \infty.$$

Note that $A(x_n) \xrightarrow{w} A(x)$ in $H^{-1}(Z)$. Hence

$$\|Dx_n\|_2^2 = \langle A(x_n), x_n\rangle \to \langle A(x), x\rangle = \|Dx\|_2^2.$$

But $Dx_n \xrightarrow{w} Dx$ in $L^2(Z, \mathbb{R}^N)$. So, from the Kadec–Klee property of Hilbert spaces, it follows that $Dx_n \to Dx$ in $L^2(Z, \mathbb{R}^N)$, therefore $x_n \to x$ in $H_0^1(Z)$. This proves that for $\lambda \in [\lambda_k, \lambda_{k+1})$, the functional φ satisfies the PS-condition. □

Proposition 2 *If hypotheses $H(j)_1$ hold and $\lambda < \lambda_1$, then φ_λ satisfies the PS-condition.*

Proof Since by hypothesis $\lambda < \lambda_1$, we have that

$$|x|^2 = \|Dx\|_2^2 - \lambda \int_Z a x^2 dz,$$

is an equivalent norm for $H_0^1(Z)$ (see Lemma 1(a)). Let $\{x_n\}_{n\geqslant 1} \subseteq H_0^1(Z)$ be a PS-sequence (i.e., it satisfies Eqs. 3.5 and 3.6). From Eqs. 3.10 and 3.12 we have

$$(\eta/2 - 1)|x_n|^2 + (\mu - \eta)\int_Z j(z, x_n)\, dz \leqslant c_9,$$

for some $c_9 > 0$, all $n \geqslant 1$, therefore

$$(\eta/2 - 1)\,|x_n|^2 + c_{10}\,\|x_n\|_\mu^\mu \leqslant c_{11}, \tag{3.16}$$

for some $c_{10}, c_{11} > 0$, all $n \geqslant 1$, (see Lemma 2). Since $\eta \in (2, \mu)$, from Eq. 3.16 we infer that $\{x_n\}_{n\geqslant 1} \subseteq H_0^1(Z)$ is bounded and so we finish this proof as that of Proposition 1. □

In the sequel, we set

$$H_- = \bigoplus_{i=1}^{k-1} E(\lambda_i) \text{ and } H_+ = \overline{\bigoplus_{i\geqslant k+1} E(\lambda_i)}.$$

Proposition 3 *If hypotheses $H(j)_1$ hold and $\lambda_k \leqslant \lambda < \lambda_{k+1}$, for some $k \geqslant 2$, then we can find $\rho > 0$ and $\beta > 0$, such that*

$$\varphi_\lambda(x) \geqslant \beta > 0$$

for all $x \in H_+$, with $\|x\| = \rho$.

Proof By virtue of hypothesis $H(j)_1$ (iv), given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$, such that

$$j(z, x) \leqslant \varepsilon/2x^2 \tag{3.17}$$

for a.a. $z \in Z$ and all $|x| \leqslant \delta$. On the other hand, hypothesis $H(j)_1$ (iii) and the mean value theorem for locally Lipschitz functions (see Clarke [5], p. 41) imply that

$$j(z, x) \leqslant c_{12}\,|x|^\tau \tag{3.18}$$

for a.a. $z \in Z$, all $|x| > \delta$ and some $c_{12} > 0$, $\tau > 2$. From Eqs. 3.17 and 3.18 it follows that

$$j(z, x) \leqslant \varepsilon/2\,|x|^2 + c_{12}\,|x|^\tau \tag{3.19}$$

for a.a. $z \in Z$, all $x \in \mathbb{R}$. Let $x \in H_+$. Then

$$\begin{aligned} \varphi_\lambda(x) &= 1/2\,\|Dx\|_2^2 - \lambda/2\int_Z ax^2 dz - \int_Z j(z, x(z))\,dz \\ &\geqslant 1/2\,\|Dx\|_2^2 - \lambda/2\int_Z ax^2 dz - \varepsilon/2\,\|x\|_2^2 - c_{13}\,\|x\|^\tau \text{ for some } c_{13} > 0 \\ &\geqslant \frac{c_{14}}{2}\,\|Dx\|_2^2 - \frac{\varepsilon c_{15}}{2\lambda_{k+1}}\,\|Dx\|_2^2 - c_{16}\,\|Dx\|_2^\tau \text{ for some } c_{14}, c_{15}, c_{16} > 0 \end{aligned} \tag{3.20}$$

(see Eq. 3.19 and Lemma 1(a)). From Eq. 3.20 we see that, if $\varepsilon < \frac{c_{14}}{c_{15}}$, then

$$\varphi_\lambda(x) \geqslant c_{17}\,\|Dx\|_2^2 - c_{16}\,\|Dx\|_2^\tau \tag{3.21}$$

for all $x \in H_+$ and some $c_{17} > 0$. Because $\tau > 2$, from Eq. 3.21 and Poincaré's inequality, we see that if $\rho \in (0, 1)$ is small, then

$$\varphi_\lambda(x) \geqslant \beta > 0$$

for all $x \in H_+$, with $\|x\| = \rho$. □

We continue to assume that $\lambda_k \leqslant \lambda < \lambda_{k+1}$, for some $k \geqslant 1$. Let $e \in E(\lambda_{k+1})$, with $\|De\|_2 = \rho$ and consider the following half-ball

$$E = \{w = v + re : v \in V = H_- \oplus E(\lambda_k),\ \|w\| \leqslant R, r \geqslant 0\},$$

with $R > \rho$ to be fixed in the process of the proof. We have

$$\partial E = E_0 = E_1 \cup E_2,$$

where

$$E_1 = \{w = v \in V : \|w\| \leqslant R\} \text{ (the basis of the half-ball)}$$

and

$$E_2 = \{w = v + re : v \in V,\ \|w\| = R, r \geqslant 0\} \text{ (the hemisphere).}$$

Proposition 4 *If hypotheses $H(j)_1$ hold and $\lambda_k \leqslant \lambda < \lambda_{k+1}$ for some $k \geqslant 1$, then $\varphi|_{E_0} \leqslant 0$.*

Proof First we examine what happens on E_1. If $w \in E_1$, then $w = v \in V$ and $\|v\| \leqslant R$. So

$$\begin{aligned}\varphi(v) &= 1/2\,\|Dv\|_2^2 - \lambda/2\int_Z av^2dz - \int_Z j(z,v)\,dz\\ &\leqslant 1/2\,\|Dv\|_2^2 - \lambda/2\int_Z av^2dz \text{ (recall } j \geqslant 0)\\ &\leqslant 1/2\,(\lambda_k - \lambda)\int_Z av^2dz \text{ (recall ((2.3)))}\\ &\leqslant 0 \text{ (since } \lambda_k \leqslant \lambda). \end{aligned} \tag{3.22}$$

Next we examine what happens on E_2. So, let $w \in E_2$. Exploiting the orthogonality of the component spaces, we have

$$\begin{aligned}\varphi(w) &= 1/2\,\|Dw\|_2^2 - \lambda/2\int_Z aw^2dz - \int_Z j(z,w)\,dz\\ &\leqslant 1/2\,\|Dv\|_2^2 - \lambda/2\int_Z av^2dz + r^2/2\,\|De\|_2^2 - \lambda r^2/2\int_Z ae^2dz - c_{18}\,\|w\|_\mu^\mu + c_{19}\end{aligned} \tag{3.23}$$

for some $c_{18}, c_{19} > 0$ (see Lemma 2). We have

$$1/2\,\|Dv\|_2^2 - \lambda/2\int_Z av^2dz \leqslant 0 \text{ (see Lemma 1 } (b)) \tag{3.24}$$

and

$$r^2/2\left(\|De\|_2^2 - \lambda\int_Z ae^2dz\right) = r^2/2\left(1 - \frac{\lambda}{\lambda_{k+1}}\right)\|De\|_2^2 \text{ (since } e \in E(\lambda_{k+1})). \tag{3.25}$$

Returning to Eq. 3.23 and using Eqs. 3.24 and 3.25, we obtain

$$\varphi(w) \leqslant r^2/2\left(1 - \frac{\lambda}{\lambda_{k+1}}\right)\rho^2 - c_{18}\,\|w\|_\mu^\mu + c_{19} \tag{3.26}$$

for some $c_{18}, c_{19} > 0$. The space $W = V \oplus \mathbb{R}e = H_- \oplus E(\lambda_k) \oplus \mathbb{R}e$ is finite dimensional. So, all norms are equivalent. Hence we can find $c_{20} > 0$, such that $\|w\| \leqslant c_{20}\|w\|_\mu$, for all $w \in W$. Using this fact in Eq. 3.26, we obtain

$$\begin{aligned}\varphi(w) &\leqslant r^2/2\left(1 - \frac{\lambda}{\lambda_{k+1}}\right)\rho^2 - c_{21}\|w\|^\mu + c_{19}, \qquad \text{with } c_{21} = c_{18}/c_{19} > 0 \\ &= r^2/2\left(1 - \frac{\lambda}{\lambda_{k+1}}\right)\rho^2 - c_{21}R^\mu + c_{19}\ (\text{since } w \in E_2). \end{aligned} \tag{3.27}$$

From Eq. 3.27 it is clear that, if we choose $R > \rho$ large enough, we shall have

$$\varphi|_{E_2} \leqslant 0. \tag{3.28}$$

From Eq. 3.22 and 3.28, it follows that $\varphi|_{E_0} \leqslant 0$. □

Now we are ready for the existence theorem.

Theorem 4 *If hypotheses $H(j)_1$hold, then for every $\lambda \in \mathbb{R}$, problem Eq. 1.1 has a nontrivial solution $x \in C^1\left(\overline{Z}\right)$.*

Proof First we assume that $\lambda_k \leqslant \lambda < \lambda_{k+1}$, for some $k \geqslant 1$. Let E be the half-ball introduced earlier,

$$E_0 = \partial E = E_1 \cup E_2 \text{ and } D = H_+ \cap \partial B_\rho.$$

Claim $\{E, E_0\}$ is linking with D in $H_0^1(Z)$.

Clearly $D \cap E_0 = \varnothing$. Also, let $\gamma \in \Gamma = \left\{\gamma \in C\left(E, H_0^1(Z)\right) : \gamma|_{E_0} = id|_{E_0}\right\}$. We need to show that $\gamma(E) \cap D \neq \varnothing$. To this end, let p_V be the orthogonal projection onto V and let $\widehat{r}_{E_0} : W\backslash\{e\} \to E_0$ be a retraction map. Suppose that $\gamma(E) \cap D = \varnothing$. Then the map

$$x \to \widehat{r}_{E_0}\left(p_V(\gamma(x)) + 1/\rho\,\|(I - p_V)\gamma(x)\|\,e\right)$$

is a retraction of E onto $\partial E = E_0$. But E is homeomorphic to a finite dimensional ball and as it is well-known, in a finite dimensional space no such retraction is possible (see, for example, Denkowski–Migorski–Papageorgiou [6], p. 196). Therefore $\gamma(E) \cap D \neq \varnothing$ and so $\{E, E_0\}$ and D link in $H_0^1(Z)$. Because of Propositions 1, 3 and 4, we can apply Theorem 1 and obtain $x \in H_0^1(Z)$ such that

$$\varphi(x) \geqslant \beta > 0 = \varphi(0) \tag{3.29}$$

and

$$0 \in \partial\varphi(x)\ \ (\text{i.e., } x \text{ is a critical point of } \varphi). \tag{3.30}$$

From Eq. 3.29, it is clear that $x \neq 0$, while from Eq. 3.30 we have

$$A(x) - \lambda a x = u,$$

with $u \in N(x)$, therefore

$$-\Delta x(z) - \lambda a(z) x(z) = u(z) \in \partial j(z, x(z)) \text{ a.e. on } Z, x|_{\partial Z} = 0.$$

Moreover, standard regularity theory implies $x \in C^1\left(\overline{Z}\right)$. Next, we assume that $\lambda < \lambda_1$. As we already pointed earlier, in this case,

$$|x|^2 = \|Dx\|_2^2 - \lambda \int_Z ax^2 dz, \qquad x \in H_0^1(Z),$$

is an equivalent norm on the Sobolev space $H_0^1(Z)$. If $y \in C_0^1\left(\overline{Z}\right)$, $y(z) > 0$ for all $z \in Z$, then

$$\begin{aligned} \varphi(ty) &\leqslant c_{22} t^2 \|Dy\|_2^2 - \int_Z j(z, ty(z))\, dz, && \text{for some } c_{22} > 0 \text{ and all } t > 0 \\ &\leqslant c_{22} t^2 \|Dy\|_2^2 - c_{23} t^{\mu} \|y\|_{\mu}^{\mu} + c_{24}, && \text{for some } c_{23}, c_{24} > 0 \text{ (see Lemma 2)} \end{aligned} \tag{3.31}$$

Since $\mu > 2$,from Eq. 3.31, it follows that

$$\varphi(ty) \to -\infty, \qquad \text{as } t \to +\infty. \tag{3.32}$$

On the other hand, for every $x \in H_0^1(Z)$, we have

$$\begin{aligned} \varphi(x) &= 1/2\, |x|^2 - \int_Z j(z, x(z))\, dz \\ &\geqslant 1/2\, |x|^2 - \varepsilon/2\, \|x\|_2^2 - c_{12} \|x\|^{\tau} \ \text{(see (3.19))} \\ &\geqslant 1/2\,(c_{25} - \varepsilon) \|x\|^2 - c_{12} \|x\|^{\tau}, \qquad \text{for some } c_{25} > 0. \end{aligned}$$

Choosing $\varepsilon < c_{25}$, we infer that

$$\varphi(x) \geqslant c_{26} \|x\|^2 - c_{12} \|x\|^{\tau}, \tag{3.33}$$

for some $c_{26} > 0$ and for all $x \in H_0^1(Z)$. Because $\tau > 2$, from Eq. 3.33, we infer that, if we choose $r \in (0, 1)$ small, then

$$\varphi(x) \geqslant \widehat{\beta} > 0 = \varphi(0) \geqslant \varphi(ty), \tag{3.34}$$

for all $x \in H_0^1(Z)$, with $\|x\| = \rho$, and $t > \rho$, such that $\|ty\| > \rho$ (see Eq. 3.32). Because of Eq. 3.34 and Proposition 2, we can apply Theorem 2 and obtain $x \in H_0^1(Z)$, such that

$$\varphi(x) \geqslant \widehat{\beta} > 0 = \varphi(0) \tag{3.35}$$

and

$$0 \in \partial\varphi(x). \tag{3.36}$$

As before, from Eq. 3.35, we have that $x \neq 0$, while from Eq. 3.36 it follows that $x \in C^1\left(\overline{Z}\right)$ and solves Eq. 1.1. □

In the scalar case (i.e. $N = 1$, ordinary differential inclusion), we can weaken the hypotheses. So, we consider the following scalar Dirichlet problem

$$\begin{cases} -x''(t) - \lambda a(t)\, x(t) \in \partial j(t, x(t)) \text{ a.e. on } T = \left[0, b\right], \\ x(0) = x(b) = 0. \end{cases} \tag{3.37}$$

The hypotheses on the nonsmooth potential are the following:

H(j)$_2$: $j\colon T \times \mathbb{R} \to \mathbb{R}_+$ is a function, such that $j(t, 0) = 0$ a.e. on T and

(i) for all $x \in \mathbb{R}$, $t \to j(t, x)$ is measurable;
(ii) for almost all $t \in T$, $x \to j(t, x)$ is locally Lipschitz;
(iii) for almost all $t \in T$, all $x \in \mathbb{R}$ and all $u \in \partial j(t, x)$

$$|u| \leqslant a_0(t) + c_0 |x|^r,$$

with $a_0 \in L^1(T)_+$, $c_0 > 0$ and $1 < r$;
(iv) $\lim\limits_{x\to 0} \frac{j(t,x)}{x^2} = 0$, uniformly for a.a. $t \in T$;
(v) $\lim\limits_{|x|\to\infty} \frac{j(t,x)}{x^2} = +\infty$, uniformly for a.a. $t \in T$;
(vi) there exist $\theta > r - 1$ and $M > 0$, such that for almost all $t \in T$ and all $|x| \geqslant M$, we have

$$\widehat{c}\,|x|^{\theta} \leqslant -j^0(t, x; -x) - 2j(t, x), \qquad \text{with } \widehat{c} > 0.$$

Remark 2 We no longer employ the AR-condition. Instead, we have hypothesis $H(j)_2$ (vi). As the next examples illustrate, there are functions which satisfy $H(j)_2$ (vi), but not the AR-condition.

Example 2 We consider the following functions. For the sake of simplicity, we drop the t-dependence.

$$j_1(x) = \begin{cases} 1/5\,|x|^5 & \text{if } |x| \leqslant 1 \\ x^2 \ln|x| + c\,|x| + 1/5 - c & \text{if } |x| > 1 \end{cases}, \qquad \text{with } c \geqslant 0$$

and

$$j_2(x) = x^2 \ln(|x| + 1).$$

Note that $j_2 \in C^1(\mathbb{R})$, while, if $c = 1/5$, then $j_1 \in C^1(\mathbb{R})$, too. Both these functions satisfy hypotheses $H(j)_2$, but not the AR-condition.

The Euler functional $\varphi_\lambda : W_0^{1,2}(0, b) \to \mathbb{R}$ for problem Eq. 3.37 is defined by

$$\varphi_\lambda(x) = 1/2\,\|x'\|_2^2 - \lambda/2 \int_0^b a x^2 dt - \int_0^b j(t, x(t))\, dt, \qquad \text{for all } x \in W_0^{1,2}(0, b).$$

We know that φ_λ is Lipschitz continuous on bounded sets, hence it is locally Lipschitz (see Clarke [5], p. 83).

Proposition 5 *If hypotheses $H(j)_2$ hold and $\lambda_k \leqslant \lambda < \lambda_{k+1}$, for some $k \geqslant 1$, then φ_λ satisfies the C-condition.*

Proof Let $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,2}(0,b)$ be a sequence such that

$$|\varphi_\lambda(x_n)| \leqslant M_2, \qquad \text{for some } M_2 > 0, \text{all } n \geqslant 1 \tag{3.38}$$

and

$$(1+\|x_n\|)\, m_\lambda(x_n) = \inf\left[\|x^*\| : x^* \in \partial\varphi_\lambda(x_n)\right] \to 0, \qquad \text{as } n \to \infty. \tag{3.39}$$

As before, we can find $x_n^* \in \partial\varphi_\lambda(x_n)$, satisfying $m_\lambda(x_n) = \|x_n^*\|$, for all $n \geqslant 1$. We know that

$$x_n^* = A(x_n) - \lambda a x_n - u_n, u_n \in N(x_n),$$

with $A \in \mathcal{L}\left(W_0^{1,2}(0,b), W^{-1,2}(0,b)\right)$ defined by

$$\langle A(x), y\rangle = \int_0^b x'y'dt, \qquad \text{for all } x, y \in W_0^{1,2}(0,b)$$

and, for all $x \in W_0^{1,2}(0,b)$,

$$N(x) = \left\{u \in L^{\frac{r+1}{r}}(0,b) : u(t) \in \partial j(t, x(t)) \text{ a.e. on } (0,b)\right\}.$$

Claim The sequence $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,2}(0,b)$ is bounded.

We argue indirectly. So, suppose that the claim is not true. By passing to a subsequence, if necessary, we may assume that $\|x_n'\|_2 \to \infty$. From Eqs. 3.38 and 3.39, we have

$$|2\varphi_\lambda(x_n)| = \left|\|x_n'\|_2^2 - \lambda\int_0^b a x_n^2 dt - \int_0^b 2j(t, x_n)\,dt\right| \leqslant 2M_2, \tag{3.40}$$

for all $n \geqslant 1$ and

$$|\langle x_n^*, x_n\rangle| = \left|\|x_n'\|_2^2 - \lambda\int_0^b a x_n^2 dt - \int_0^b u_n x_n dt\right| \leqslant \varepsilon_n, \tag{3.41}$$

for all $n \geqslant 1$, with $\varepsilon_n \downarrow 0$. From Eqs. 3.40 and 3.41, it follows that

$$\int_0^b (u_n x_n - 2j(t, x_n))\,dt \leqslant M_3 \text{ for some } M_3 > 0, \qquad \text{all } n \geqslant 1,$$

hence

$$\int_0^b \left[-j^0(t, x_n; -x_n) - 2j(t, x_n)\right] dt \leqslant M_3,$$

therefore

$$\int_{\{|x_n|\geqslant M\}}\left[-j^0(t,x_n;-x_n)-2j(t,x_n)\right]dt$$
$$+\int_{\{|x_n|<M\}}\left[-j^0(t,x_n;-x_n)-2j(t,x_n)\right]dt\leqslant M_3,$$

so

$$\widehat{c}\int_{\{|x_n|\geqslant M\}}|x_n|^{\theta}\,dt\leqslant M_4,\qquad \text{for some } M_4>0,\qquad \text{and all } n\geqslant 1$$

(see $H(j)_2$ (iii) , (vi)), hence

$$\{x_n\}_{n\geqslant 1}\subseteq L^{\theta}(0,b)\ \text{ is bounded.} \tag{3.42}$$

We consider the orthogonal direct sum decomposition

$$W_0^{1,2}(0,b)=H_-\oplus E(\lambda_k)\oplus H_+,$$

We can write, in an unique way,

$$x_n=\overline{x}_n+x_n^0+\widehat{x}_n,\qquad \text{with } \overline{x}_n\in H_-, x_n^0\in E(\lambda_k), \widehat{x}_n\in H_+, n\geqslant 1.$$

From Eq. 3.39, we have

$$\left|\langle x_n^*,u\rangle\right|\leqslant\frac{\varepsilon_n}{1+\|x_n\|}\|u\|,\qquad \text{for all } u\in W_0^{1,2}(0,b).$$

Let $u=\widehat{x}_n$. Exploiting the orthogonality of the component spaces, we have

$$\left|\langle x_n^*,\widehat{x}_n\rangle\right|=\left|\left\|\widehat{x}_n'\right\|_2^2-\lambda\int_0^b a\widehat{x}_n^2dt-\int_0^b u_n\widehat{x}_ndt\right|\leqslant\varepsilon_n. \tag{3.43}$$

Since, by hypothesis, $\lambda\in\left[\lambda_k,\lambda_{k+1}\right)$, from Lemma 1$(a)$, we have

$$\xi_0\left\|\widehat{x}_n'\right\|_2^2\leqslant\left\|\widehat{x}_n'\right\|_2^2-\lambda\int_0^b a\widehat{x}_n^2dt. \tag{3.44}$$

Also, we have

$$\int_0^b u_n\widehat{x}_ndt\leqslant\int_0^b|u_n|\,|\widehat{x}_n|\,dt\leqslant c_{27}\left\|\widehat{x}_n'\right\|_2\int_0^b|u_n|\,dt,\qquad \text{for some } c_{27}>0$$
$$\leqslant c_{27}\left\|\widehat{x}_n'\right\|_2\int_0^b\left(a_0(t)+c_0|x_n|^r\right)dt\ \ (\text{see hypothesis } H(j)\,(\text{iii}))$$
$$\leqslant c_{28}\left\|\widehat{x}_n'\right\|_2+c_{29}\left\|\widehat{x}_n'\right\|_2\|x_n\|_r^r,\qquad \text{for some } c_{28},c_{29}>0.$$

Assuming without any loss of generality that $\theta\leqslant r$ and using the interpolation inequality (see, for example, Gasinski–Papageorgiou [10], p. 905), we have

$$\|x_n\|_r\leqslant\|x_n\|_{\theta}^{1-t}\|x_n\|_{\infty}^{t},\qquad \text{where } t\in(0,1),\ \frac{1-t}{\theta}=1/r,$$

so

$$\|x_n\|_r^r \leqslant c_{30} \left\|x_n'\right\|^{tr}, \qquad \text{for some } c_{30} > 0, \qquad \text{all } n \geqslant 1 \text{ (see (3.42))}$$

Therefore, we have

$$\int_0^b u_n \widehat{x}_n dt \leqslant c_{28} \left\|\widehat{x}_n'\right\|_2 + c_{31} \left\|\widehat{x}_n'\right\| \left\|x_n'\right\|^{tr}, \qquad \text{for some } c_{31} > 0, \qquad \text{all } n \geqslant 1. \quad (3.45)$$

Returning to Eq. 3.43 and using Eqs. 3.44 and 3.45, we obtain

$$\xi_0 \left\|\widehat{x}_n'\right\|_2^2 \leqslant \varepsilon_n + c_{28} \left\|\widehat{x}_n'\right\|_2 + c_{31} \left\|\widehat{x}_n'\right\| \left\|x_n'\right\|^{tr},$$

hence

$$\xi_0 \frac{\left\|\widehat{x}_n'\right\|_2^2}{\|x_n\|^2} \leqslant \frac{\varepsilon_n}{\|x_n\|^2} + \frac{c_{32}}{\|x_n\|} + c_{33} \frac{1}{\|x_n\|^{1-tr}},$$

for some $c_{32}, c_{33} > 0$, all $n \geqslant 1$. Note that $\theta > r - 1$ is equivalent to $tr < 1$. So, passing to the limit as $n \to \infty$, we obtain

$$\frac{\left\|\widehat{x}_n'\right\|_2}{\|x_n\|} \to 0, \qquad \text{as } n \to \infty,$$

therefore

$$\frac{\|\widehat{x}_n\|}{\|x_n\|} \to 0, \qquad \text{as } n \to \infty. \quad (3.46)$$

In a similar fashion, using as a test function $u = \overline{x}_n$ and Lemma 1(b), we show that

$$\frac{\|\overline{x}_n\|}{\|x_n\|} \to 0, \qquad \text{as } n \to \infty. \quad (3.47)$$

Let $y_n = \frac{x_n}{\|x_n\|}$, $n \geqslant 1$. Then $\|y_n\| = 1$, for all $n \geqslant 1$, and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,2}(0, b) \text{ and } y_n \to y \text{ in } C\left(T, \mathbb{R}^N\right).$$

From Eqs. 3.46 and 3.47, it follows that $y \in E(\lambda_k)$. Also, due to Eq. 3.42, we have $y = 0$. Then the finite dimensionality of the space H_0 implies

$$y_n^0 = \frac{x_n^0}{\|x_n\|} \to 0 \text{ in } W_0^{1,2}(0, b). \quad (3.48)$$

Combining Eqs. 3.46, 3.47 and 3.48, we have

$$1 = \|y_n\| \leqslant \frac{\|\overline{x}_n\| + \left\|x_n^0\right\| + \|\widehat{x}_n\|}{\|x_n\|} \to 0, \qquad \text{as } n \to \infty,$$

a contradiction. This proves the claim.

Because of the Claim and by passing to a suitable subsequence, if necessary, we may assume that

$$x_n \xrightarrow{w} x \text{ in } W_0^{1,2}(0, b) \text{ and } x_n \to x \text{ in } C(T). \quad (3.49)$$

From Eq. 3.39, we have

$$\left| \langle A(x_n), x_n - x \rangle - \lambda \int_0^b a x_n (x_n - x)\, dt - \int_0^b u_n (x_n - x)\, dt \right| \leqslant \varepsilon_n. \tag{3.50}$$

Because of Eq. 3.49, we have

$$\int_0^b a x_n (x_n - x)\, dt \to 0 \text{ and } \int_0^b u_n (x_n - x)\, dt \to 0, \qquad \text{as } n \to \infty.$$

So, from Eq. 3.50, we have

$$\langle A(x_n), x_n - x \rangle \to 0, \qquad \text{as } n \to \infty.$$

From this, as in the proof of Proposition 5, we conclude that $x_n \to x$ in $W_0^{1,2}(0, b)$ and, so, φ_λ satisfies the C-condition. □

We, also, check the case when $\lambda < \lambda_1$.

Proposition 6 *If hypotheses $H(j)_2$ hold and $\lambda < \lambda_1$, then φ_λ satisfies the C-condition.*

Proof We know that, in this case,

$$|x|^2 = \|x'\|_2^2 - \lambda \int_0^b a x^2 dt$$

is an equivalent norm for the Sobolev space $W_0^{1,2}(0, b)$. Let $\{x_n\}_{n \geqslant 1} \subseteq W_0^{1,2}(0, b)$ be a C-sequence. Arguing as in proof of Proposition 5 and using hypothesis $H(j)_2$ (vi), we show that $\{x_n\}_{n \geqslant 1} \subseteq L^\theta(Z)$ is bounded. Also, we have

$$\left|\langle x_n^*, u \rangle\right| \leqslant \frac{\varepsilon_n}{1 + \|x_n\|} \|u\|, \qquad \text{for all } u \in W_0^{1,2}(0, b), \qquad \text{with } \varepsilon_n \downarrow 0.$$

Choose $u = x_n$. Then

$$\left| |x_n|^2 - \int_0^b u_n x_n dt \right| \leqslant \varepsilon_n, \qquad \text{for all } n \geqslant 1. \tag{3.51}$$

From Eq. 3.45 (with $\widehat{x}_n$ replaced by x_n), we have

$$\int_0^b u_n x_n dt \leqslant c_{34} |x_n| + c_{35} |x_n|^{tr+1}, \qquad \text{for some } c_{34}, c_{35} > 0, \qquad \text{all } n \geqslant 1. \tag{3.52}$$

We use Eq. 3.52 in Eq. 3.51 and obtain

$$|x_n|^2 \leqslant \varepsilon_n + c_{34} |x_n| + c_{35} |x_n|^{tr+1}. \tag{3.53}$$

Recall that $tr + 1 < 2$. Hence, from Eq. 3.53, it follows that $\{x_n\}_{n \geqslant 1} \subseteq W_0^{1,2}(0, b)$ is bounded. Continuing as in the proof of Proposition 5, we conclude that φ_λ satisfies the C-condition. □

Using hypothesis $H(j)_2$ (iv) and arguing as in the proof of Proposition 3, we show that:

Proposition 7 *If hypotheses $H(j)_2$ hold and $\lambda_k \leqslant \lambda < \lambda_{k+1}$, for some $k \geqslant 1$, then we can find $\rho > 0$ and $\beta > 0$, such that*

$$\varphi(x) \geqslant \beta > 0, \qquad \textit{for all } x \in H_+, \qquad \textit{with } \|x\| = \rho.$$

We choose $e \in E(\lambda_{k+1})$, with $\|e\| = 1$ and set $Y = V \oplus \mathbb{R}e$, with $V = H_- \oplus E(\lambda_k)$. We consider the following cylinder:

$$E = \{y = v + re : v \in V,\ \ \|v\| \leqslant R,\ 0 \leqslant r \leqslant R\},$$

with $R > \rho$ (see Proposition 3) to be fixed in the process of the proof. We set

$$E_0 = \partial E = E_1 \cup E_2 \cup E_3,$$

with

$E_1 = \{v \in V = H_- \oplus E(\lambda_k) : \|v\| \leqslant R\}$ (the lower base of the cylinder),
$E_2 = \{y = v + Re : \|v\| \leqslant R\}$ (the upper base of the cylinder),
$E_3 = \{y = v + re : 0 < r < R,\ \ \|v\| = R\}$ (the lateral surface of the cylinder).

We want to estimate the values of φ_λ on E_0. To this end, we shall need the following lemma. We state the result for the more general Sobolev space $H_0^1(Z)$. In what follows, by $|\cdot|_N$, we denote the Lebesgue measure on $\mathbb{R}^N$.

Lemma 3 *If $V \subseteq H_0^1(Z)$ is a nontrivial finite dimensional subspace, then there exists $\eta > 0$, such that*

$$|\{z \in Z : |v(z)| \geqslant \eta \|v\|\}|_N \geqslant \eta, \qquad \textit{for all } v \in V,\ x \neq 0.$$

Proof We argue by contradiction. So, suppose that the lemma is not true. Then we can find $\{v_n\}_{n \geqslant 1} \subseteq V$, $v_n \neq 0$, such that

$$|\{z \in Z : |v_n(z)| \geqslant 1/n \|v_n\|\}|_N < 1/n, \qquad \text{for all } n \geqslant 1.$$

We set $w_n = \frac{v_n}{\|v_n\|}$, for all $n \geqslant 1$. Then

$$|\{z \in Z : |w_n(z)| \geqslant 1/n\}|_N < 1/n,$$

hence

$$w_n \to 0 \text{ in the Lebesgue measure.}$$

So, passing to a suitable subsequence, if necessary, we may assume that $w_n(z) \to 0$ a.e. on Z. Note that $\|w_n\| = 1$ and $\{w_n\}_{n \geqslant 1} \subseteq V$, with V finite dimensional. So, it follows that $w_n \to 0$ in $H_0^1(Z)$, a contradiction to the fact that $\|w_n\| = 1$, for all $n \geqslant 1$. □

With the help of this lemma, we can now estimate $\varphi_\lambda|_{E_0}$.

Proposition 8 *If hypotheses* $H(j)_2$ *hold and* $\lambda_k \leqslant \lambda < \lambda_{k+1}$*, for some* $k \geqslant 1$*, then* $\varphi_\lambda|_{E_0} < 0$.

Proof By virtue of hypothesis $H(j)_2$ (v), given $\xi > 0$, we can find $M_5 = M_5(\xi) > 0$, such that

$$j(t, x) \geqslant \xi x^2 \text{ for a.a. } t \in T \text{ and all } |x| \geqslant M_5.$$

For any function $y \in Y = V \oplus \mathbb{R}e$, we have $y = v + re$, with $v \in V$, $r \in \mathbb{R}$ and so

$$\begin{aligned}\varphi_\lambda(y) &= 1/2\,\|v'\|_2^2 + r^2/2\,\|e'\|_2^2 - \lambda/2\int_0^b av^2dt - \lambda r^2/2\int_0^b ae^2dt - \int_0^b j(t, y)\,dt \\ &\leqslant r^2/2\,\|e'\|_2^2 - \lambda r^2/2\int_0^b ae^2dt - \int_0^b j(t, y)\,dt \text{ (see (2.3))}. \end{aligned} \tag{3.54}$$

Lemma 3 implies that there exists $\eta > 0$, such that

$$\left|D_\eta\right|_1 = |\{t \in (0, b) : |y(t)| \geqslant \eta\,\|y\|\}|_1 \geqslant \eta \tag{3.55}$$

for all $y \in Y = V \oplus \mathbb{R}e$, $y \neq 0$. Recall that $\xi > 0$ was arbitrary. So, we choose $\xi \geqslant 1/\eta^3$. Then we have

$$j(t, y(t)) \geqslant \xi\eta^2\,\|y\|^2 \tag{3.56}$$

for a.a. $t \in D_\eta$ and all $y \in Y$, with $\|y\| \geqslant M_5/\eta$. Choose $R \geqslant M_5/\eta$. Then for $y \in Y = V \oplus \mathbb{R}e$ with $\|y\| = R$, we have

$$\begin{aligned}\varphi_\lambda(y) &\leqslant r^2/2\,\|e'\|_2^2 - \lambda r^2/2\int_0^b ae^2dt - \int_0^b j(t, y(t))\,dt \text{ (see (3.54))} \\ &= r^2/2\,\|e'\|_2^2 - \lambda r^2/2\int_0^b ae^2dt - \int_{D_\eta} j(t, y(t))\,dt - \int_{D_\eta^c} j(t, y(t))\,dt \\ &\leqslant r^2/2\,\|e'\|_2^2 - \lambda r^2/2\int_0^b ae^2dt - \xi\eta^2\,\|y\|^2\left|D_\eta\right|_1 \text{ (see (3.56) and recall } j \geqslant 0)) \\ &\leqslant r^2/2\,\|e'\|_2^2 - \lambda r^2/2\int_0^b ae^2dt - \xi\eta^3\,\|y\|^2 \text{ (since } \left|D_\eta\right|_1 \geqslant \eta, \text{ see (3.55))} \end{aligned} \tag{3.57}$$

Now, let $y \in E_1$. Then $r = 0$ and so Eq. 3.57 becomes

$$\varphi_\lambda(y) \leqslant 0,$$

hence

$$\varphi_\lambda(y)\,|_{E_1} \leqslant 0. \tag{3.58}$$

Next let $y \in E_2$. Then $y = v + \mathbb{R}e$, with $\|v\| \leqslant R$ and so, from Eq. 3.57, we have

$$\begin{aligned}
\varphi_\lambda(y) &\leqslant R^2/2 \left\|e'\right\|_2^2 - \lambda R^2/2 \int_0^b ae^2 dt - \xi\eta^3 \|v\|^2 - \xi\eta^3 R^2 \|e\|^2 \\
&\leqslant R^2/2 \left\|e'\right\|_2^2 - \lambda R^2/2 \int_0^b ae^2 dt - \|v\|^2 - R^2 \text{ (since } \xi \geqslant 1/\eta^3 \text{ and } \|e\| = 1) \\
&= R^2/2 \left(1 - \frac{\lambda}{\lambda_{k+1}}\right) \left\|e'\right\|_2^2 - R^2 \text{ (see (2.3))} \\
&< R^2/2 - R^2 \text{ (since } \lambda_k \leqslant \lambda < \lambda_{k+1} \text{ and } \|e\| = 1) \\
&< 0.
\end{aligned}$$

So

$$\varphi_\lambda(y)\,|_{E_2} < 0. \tag{3.59}$$

Finally, let $y \in E_3$. We have that $y = v + te$, with $\|v\| = R, 0 < t < R$. From Eq. 3.57, we have

$$\begin{aligned}
\varphi_\lambda(y) &\leqslant t^2/2 \left(1 - \frac{\lambda}{\lambda_{k+1}}\right) \left\|e'\right\|_2^2 - \|v\|^2 - t^2 \text{ (see (2.3) and recall } \xi \geqslant 1/\eta^3,\ \|e\| = 1) \\
&\leqslant t^2/2 - t^2 \text{ (since } \lambda_k \leqslant \lambda < \lambda_{k+1},\ \|e\| = 1) \\
&< 0,
\end{aligned}$$

therefore

$$\varphi_\lambda(y)\,|_{E_3} < 0. \tag{3.60}$$

Combining Eqs. 3.58, 3.59 and 3.60, we conclude that $\varphi_\lambda(y)\,|_{E_0} \leqslant 0$. □

Now, we are ready for the existence result, concerning problem Eq. 3.37.

Theorem 5 *If hypotheses $H(j)_2$ hold, then, for every $\lambda \in \mathbb{R}$, problem Eq. 3.37 has a nontrivial solution $x \in C_0^1[0, b]$.*

Proof First, we assume that $\lambda_k \leqslant \lambda < \lambda_{k+1}$, for some $k \geqslant 1$. We can check that the sets $\{E, E_0\}$ and $D = H_+ \cap \partial B_\rho$ link in $W_0^{1,2}(0, b)$ (see Gasinski–Papageorgiou [10], p. 643). So, because of Propositions 5, 7 and 8, we can apply Theorem 1 and obtain $x \in W_0^{1,2}(0, b)$, such that

$$\varphi_\lambda(x) \geqslant \beta > 0 = \varphi_\lambda(0) \tag{3.61}$$

and

$$0 \in \partial\varphi_\lambda(x) \text{ (i.e., } x \text{ is a critical point of } \varphi_\lambda). \tag{3.62}$$

From Eq. 3.61, we see that $x \neq 0$, while from Eq. 3.62 we infer that $x \in C_0^1\left[0, b\right]$ is a solution of Eq. 3.37. Next, let $\lambda < \lambda_1$. We know that, in this case,

$$|x|^2 = \left\|x'\right\|_2^2 - \lambda \int_0^b ax^2 dt$$

is an equivalent norm for the Sobolev space $W_0^{1,2}(0, b)$. Let $u \in E(\lambda_1)$, $u \neq 0$ and $\mu > 0$. Then

$$\varphi_\lambda(\mu u) = \mu^2 |u|^2 - \int_0^b j(t, \mu u)\, dt. \tag{3.63}$$

By virtue of $H(j)_2$ (iii) and (v), given $\xi > 0$, we can find $\gamma_\xi \in L^1(T)_+$, such that

$$\xi |x|^2 - \gamma_\xi(t) \leqslant j(t, x) \text{ for a.a. } t \in T \text{ and all } x \in \mathbb{R}.$$

Using this in Eq. 3.63, we obtain

$$\begin{aligned}\varphi_\lambda(\mu u) &\leqslant \mu^2 |u|^2 - \xi\mu^2 \|u\|_2^2 + c_{36}, \qquad \text{for some } c_{36} > 0\\ &\leqslant \mu^2(1 - \xi c_{37})|u|^2 + c_{36}, \qquad \text{for some } c_{37} > 0\end{aligned} \tag{3.64}$$

(since all norms are equivalent on $E(\lambda_1)$). Since $\xi > 0$ was arbitrary, we choose $\xi > 1/c_{37}$ and so, from Eq. 3.64, we infer that

$$\varphi_\lambda(\mu u) \to -\infty \text{ as } \mu \to +\infty. \tag{3.65}$$

Note that hypotheses $H(j)_2$ (iii) and (iv), imply that given $\varepsilon > 0$, we can find $c_{38} = c_{38}(\varepsilon) > 0$, such that

$$j(t, x) \leqslant \varepsilon x^2 + c_{38}|x|^\tau, \qquad \text{with } \tau > 2.$$

Then, choosing $\varepsilon > 0$ small, we see that, for all $x \in W_0^{1,2}(0, b)$, we have

$$\varphi_\lambda(x) \geqslant c_{39}\|x\|^2 - c_{40}\|x\|^\tau, \tag{3.66}$$

for some $c_{39}, c_{40} > 0$, all $x \in W_0^{1,2}(0, b)$. Because $\tau > 2$, from Eq. 3.66 it follows that, if $\rho \in (0, 1)$ is small, then

$$\varphi_\lambda|_{\partial B_\rho} \geqslant \beta_0 > 0 = \varphi_\lambda(0). \tag{3.67}$$

Then Eqs. 3.65, 3.67 and Proposition 6 permit the application of Theorem 2, which gives $x \in W_0^{1,2}(0, b)$ satisfying Eqs. 3.61, 3.62. Then $x \in C_0^1\left[0, b\right]$, $x \neq 0$ and solves problem Eq. 3.37. □

4 Multiple Solutions for Resonant Problems

In this section, we prove a multiplicity result for problem Eq. 1.1, when $\lambda = \lambda_1$. So, the problem under consideration is:

$$\begin{cases} -\Delta x(z) - \lambda_1 a(z)\, x(z) \in \partial j(z, x(z)) \text{ a.e. on } Z \\ x|_{\partial Z} = 0. \end{cases} \tag{4.1}$$

The hypotheses on the nonsmooth potential are the following:

H(j)$_3$: j: $Z \times \mathbb{R} \to \mathbb{R}$ is a function, such that $j(z,0) = 0$ a.e. on Z and

(i) for all $x \in \mathbb{R}$, $z \to j(z,x)$ is measurable;
(ii) for almost all $z \in Z$, $x \to j(z,x)$ is locally Lipschitz;
(iii) for almost all $z \in Z$, all $x \in \mathbb{R}$ and all $u \in \partial j(z,x)$

$$|u| \leqslant a_0(z) + c_0 |x|, \text{ with } a_0 \in L^\infty(Z)_+, c_0 > 0;$$

(iv) $\lim\limits_{|x|\to\infty} \frac{j(z,x)}{x^2} = 0$ and $\lim\limits_{|x|\to\infty} j(z,x) = -\infty$, uniformly for a.a. $z \in Z$;
(v) there exist $\delta > 0$ and $\theta \in L^\infty(Z)_+$, $\theta(z) \leqslant \lambda_2 - \lambda_1$ a.e. on Z, $\theta \neq \lambda_2 - \lambda_1$, such that

$$0 \leqslant j(z,x) \leqslant \frac{\theta(z)}{2} a(z) x^2 \text{ for a.a. } z \in Z \text{ and all } |x| \leqslant \delta.$$

Remark 3 These hypotheses imply that we have resonance both at zero and at infinity.

Example 3 The following function satisfy hypotheses $H(j)_3$. For the sake of simplicity, we drop the z-dependence.

$$j(x) = \begin{cases} \dfrac{\theta}{2} x^2 & \text{if } |x| \leqslant 1 \\ \xi \ln|x| - c|x| + \dfrac{\theta + 2c}{2} & \text{if } |x| > 1 \end{cases}, \qquad \text{with } 0 < \theta < \lambda_2 - \lambda_1, \xi, c > 0.$$

Note that, if $\xi \geqslant \lambda_1$ and $c = \xi - \theta > 0$, then $j \in C^1(\mathbb{R})$.

We consider the Euler functional φ: $H_0^1(Z) \to \mathbb{R}$ for problem Eq. 4.1, defined by

$$\varphi(x) = 1/2 \|Dx\|_2^2 - \lambda_1/2 \int_Z a x^2 dz - \int_Z j(z, x(z))\, dz \qquad \text{for all } x \in H_0^1(Z).$$

We know that φ is Lipschitz continuous on bounded sets, hence it is locally Lipschitz.

Proposition 9 *If hypotheses $H(j)_3$ hold, then φ is coercive (i.e., $\varphi(x) \to +\infty$, as $\|x\| \to \infty$).*

Proof We argue indirectly. So, suppose that φ is not coercive. Then we can find $\{x_n\}_{n\geqslant 1} \subseteq H_0^1(Z)$, such that

$$\|x_n\| \to \infty \text{ and } \varphi(x_n) \leqslant M_5, \qquad \text{for some } M_5 > 0, \qquad \text{all } n \geqslant 1. \tag{4.2}$$

Let $y_n = \frac{x_n}{\|x_n\|}$, $n \geqslant 1$. Then $\|y_n\| = 1$, for all $n \geqslant 1$, and, so, we may assume that

$$y_n \xrightarrow{w} y \text{ in } H_0^1(Z), \; y_n \to y \text{ in } L^2(Z), \; y_n(z) \to y(z) \text{ a.e. on } Z$$

and

$$|y_n(z)| \leqslant k(z) \text{ for a.a. } z \in Z, \qquad \text{all } n \geqslant 1, \qquad \text{with } k \in L^2(Z)_+. \tag{4.3}$$

By virtue of hypotheses $H(j)_3$ (iii) and (iv), we can find $c_{36} > 0$, such that

$$j(z, x) \leqslant c_{36} \text{ for a.a. } z \in Z, \qquad \text{all } x \in \mathbb{R}. \tag{4.4}$$

Then, from Eqs. 4.2 and 4.4, we have

$$\frac{M_5}{\|x_n\|^2} \geqslant 1/2 \, \|Dy_n\|_2^2 - \lambda_1/2 \int_Z a y_n^2 dz - \frac{c_{37}}{\|x_n\|^2}, \qquad \text{with } c_{37} = c_{36} \, |Z|_N \, , \tag{4.5}$$

so

$$\lambda_1 \int_Z a y^2 dz \geqslant \|Dy\|_2^2 \text{ (see (4.3))}.$$

From Eq. 2.2, it follows that

$$\|Dy\|_2^2 = \lambda_1 \int_Z a y^2 dz,$$

therefore

$$y \in E(\lambda_1) \, . \tag{4.6}$$

If $y = 0$, then from Eq. 4.5, it is clear that $\|Dy_n\|_2 \to 0$ and so $y_n \to 0$ in $H_0^1(Z)$, a contradiction to the fact $\|y_n\| = 1$, for all $n \geqslant 1$. Hence $y \neq 0$. Because $y \in E(\lambda_1)$ (see Eq. 4.6), we have $|y(z)| > 0$, for all $z \in Z$ and so $|x_n(z)| \to +\infty$, for all $z \in Z$, as $n \to \infty$, hence $j(z, x_n(z)) \to -\infty$, for a.a. $z \in Z$, as $n \to \infty$. Then, by Fatou's Lemma (see Eq. 4.4), we have

$$\lim_{n\to\infty} \int_Z j(z, x_n(z)) \, dz = -\infty,$$

which contradicts Eq. 4.2. This proves that φ is coercive. □

Corollary 1 *If hypotheses $H(j)_3$ hold, then φ is bounded below and satisfies the PS-condition.*

Proof Since φ is coercive (see Proposition 9), it is bounded below. Also, let $\{x_n\}_{n\geqslant 1} \subseteq H_0^1(Z)$ be such that

$$|\varphi(x_n)| \leqslant M_6 \text{ for some } M_6 > 0, \qquad \text{all } n \geqslant 1 \tag{4.7}$$

and

$$m(x_n) \to 0 \qquad \text{as } n \to \infty. \tag{4.8}$$

Because of Eq. 4.7 and the coercivity of φ (see Proposition 9), we have that $\{x_n\}_{n\geqslant 1} \subseteq H_0^1(Z)$ is bounded. So, we may assume that

$$x_n \xrightarrow{w} x \text{ in } H_0^1(Z) \text{ and } x_n \to x \text{ in } L^2(Z) \, .$$

From Eq. 4.8, we have

$$\left| \langle A(x_n), x_n - x \rangle - \lambda_1 \int_Z a x_n (x_n - x) \, dz - \int_Z u_n (x_n - x) \, dz \right| \leqslant \varepsilon_n \|x_n - x\| \, , \tag{4.9}$$

with $\varepsilon_n \downarrow 0$. Evidently, we have

$$\int_Z ax_n (x_n - x)\, dz \to 0 \text{ and } \int_Z u_n (x_n - x)\, dz \to 0, \qquad \text{as } n \to \infty.$$

So, from Eq. 4.9, it follows that

$$\lim_{n\to\infty} \langle A(x_n), x_n - x\rangle = 0,$$

from which, as before, it follows that $x_n \to x$ in $H_0^1(Z)$. Therefore φ satisfies the PS-condition. □

We consider the orthogonal direct sum decomposition

$$H_0^1(Z) = E(\lambda_1) \oplus V, \qquad \text{with } V = E(\lambda_1)^{\perp}.$$

Proposition 10 *If hypotheses $H(j)_3$ hold, then we can find $r > 0$, such that*

$$\begin{cases} \varphi(x) \leqslant 0 \text{ if } x \in E(\lambda_1), & \|x\| \leqslant r \\ \varphi(x) \geqslant 0 \text{ if } x \in V, & \|x\| \leqslant r \end{cases}. \tag{4.10}$$

Proof Let $x \in E(\lambda_1)$. Since $E(\lambda_1) \subseteq C^1\left(\overline{Z}\right)$ and all norms on $E(\lambda_1)$ are equivalent (since $\dim E(\lambda_1) = 1$), we can find $r_1 > 0$, such that $x \in E(\lambda_1)$, with $\|x\| \leqslant r_1$, we have $|x(z)| \leqslant \delta$ for all $z \in \overline{Z}$. Hence, by virtue of hypothesis $H(j)_3$ (v), we have

$$0 \leqslant j(z, x(z)) \text{ a.e. on } Z. \tag{4.11}$$

Therefore, if $x \in E(\lambda_1)$, with $\|x\| \leqslant r_1$, then

$$\varphi(x) \leqslant 1/2\, \|Dx\|_2^2 - \lambda_1/2 \int_Z ax^2 dz \text{ (see (4.10))},$$

so

$$\varphi(x) \leqslant 0, \text{ for all } x \in E(\lambda_1), \qquad \text{with } \|x\| \leqslant r_1. \tag{4.12}$$

On the other hand, by virtue of hypotheses $H(j)_3$ (v) and (iii), we have

$$j(z, x) \leqslant \frac{\theta(z)}{2} a(z)\, x^2 + c_{38}\, |x|^{\tau}, \tag{4.13}$$

for a.a. $z \in Z$, all $x \in \mathbb{R}$, with $c_{38} > 0$, $2 < \tau$. Therefore, if $x \in V$, then

$$\begin{aligned} \varphi(x) &= 1/2\, \|Dx\|_2^2 - \lambda_1/2 \int_Z ax^2 dz - \int_Z j(z, x)\, dz \\ &\geqslant 1/2\, \|Dx\|_2^2 - 1/2 \int_Z (\lambda_1 + \theta(z))\, x(z)^2\, dz - c_{39}\, \|Dx\|_2^{\tau} \text{ for some } c_{39} > 0, \\ &\geqslant c_{40}\, \|Dx\|_2^2 - c_{39}\, \|Dx\|_2^{\tau}, \qquad \text{for some } c_{40} > 0 \text{ (see Lemma 1 } (a)\text{)}. \end{aligned} \tag{4.14}$$

Since $\tau > 2$, from Eq. 4.14, we see that we can find $r_2 > 0$ such that

$$\varphi(x) \geqslant 0, \qquad \text{for all } x \in V, \qquad \text{with } \|x\| \leqslant r_2.$$

Finally, if we set $r = \min\{r_1, r_2\}$, then we see that Eq. 4.10 holds. □

Theorem 6 *If hypotheses $H(j)_3$ hold, then problem Eq. 4.1 has, at least, two nontrivial solutions $x, y \in C_0^1(\overline{Z})$.*

Proof Note that $\inf \varphi \leqslant 0$. If $\inf \varphi = 0$, then, by virtue of Eq. 4.10, all $x \in E(\lambda_1)$ with $0 < \|x\| \leqslant r$ are nontrivial critical points of φ, hence solutions of Eq. 4.1, which also belong in $C_0^1(\overline{Z})$ (regularity theory). If $\inf \varphi < 0$, then, by virtue of Corollary 1 and Proposition 10, we can apply Theorem 3 and produce $x, y \in H_0^1(Z)$, two nontrivial critical points of φ.Then x, y are nontrivial solutions of Eq. 4.1 and, also, $x, y \in C_0^1(\overline{Z})$ (regularity theory). $\square$

References

1. Barletta, G., Marano, S.: Some remarks on critical point theory for locally Lipschitz functions. Glasgow Math. J. **45**, 131–141 (2003)
2. Brézis, H., Nirenberg, L.: Remarks on finding critical points. Comm. Pure Appl. Math. **44**, 939–963 (1991)
3. Carl, S., Le, V.K., Motreanu, D.: Nonsmooth Variational Problems and their Inequalities: Comparision Principles and Applications. Springer, New York (2007)
4. Cirstea, F.S., Radulescu, V.: Multiplicity of solutions for a class of nonsymmetric eigenvalue hemivariational inequalities. J. Global Optim. **17**, 43–54 (2000)
5. Clarke, F.H.: Optimization and Nonsmooth Analysis. Wiley, New York (1983)
6. Denkowski, Z., Migorski, S., Papageorgiou, N.S.: An Introduction to Nonlinear Analysis: Applications. Kluwer Academic, Boston (2003)
7. Gasinski, L., Papageorgiou, N.S.: Existence of solutions and of multiple solutions for eigenvalue problems of hemivariational inequalities. Adv. Math. Sci. Appl. **11**, 437–464 (2001)
8. Gasinski, L., Papageorgiou, N.S.: Solutions and multiple solutions for quasilinear hemivariational inequalities at resonance. Proc. Roy. Soc. Edinburgh Sect. A. **131**, 1091–1111 (2001)
9. Gasinski, L., Papageorgiou, N.S.: Nonsmooth Critical Point Theory and Nonlinear Boundary Value Problems. Chapman Hall and CRC Press, Boca Raton (2005)
10. Gasinski, L., Papageorgiou, N.S.: Nonlinear Analysis. Chapman Hall and CRC Press, Boca Raton (2006)
11. Goeleven, D., Motreanu, D., Panagiotopoulos, P.: Multiple solutions for a class of eigenvalue problems in hemivariational inequalities. Nonlinear Anal. **29**, 9–26 (1997)
12. Kandilakis, D., Kourogenis, N., Papageorgiou, N.S.: Two nontrivial critical points for nonsmooth functionals via local linking and applications. J. Global Optim. **34**, 219–244 (2006)
13. Kourogenis, N., Papageorgiou, N.S.: Nonsmooth critical point theory and nonlinear elliptic equations at resonance. J. Aust. Math. Soc. Ser. A **69**, 245–271 (2000)
14. Marano, S., Bisci, G.M., Motreanu, D.: Multiple solutions for a class of elliptic hemivariational inequalities. J. Math. Anal. Appl. **337**, 85–97 (2008)
15. Motreanu, D., Motreanu, V., Papageorgiou, N.S.: Multiple nontrivial solutions for nonlinear eigenvalue problems. Proc. AMS **135**, 3649–3658 (2007)
16. Motreanu, D., Panagiotopoulos, P.: An eigenvalue problem for a hemivariational inequality involving a nonlinear compact operator. Set-Valued Anal. **3**, 157–166 (1995)
17. Motreanu, D., Panagiotpoulos, P.: On the eigenvalue problem for hemivariational inequalities: existence and multiplicity of solutions. J. Math. Anal. Appl. **197**, 75–89 (1996)
18. Motreanu, D., Radulescu, V.: Variational and Nonvariational Methods in Nonlinear Analysis and Boundary Value Problems. Kluwer, Dordrecht (2003)
19. Motreanu, D., Radulescu, V.: Eigenvalue problems for degenerate nonlinear elliptic equations in anisotropic media. Bound. Value Probl. **2**, 107–127 (2005)
20. Naniewicz, Z., Panagiotopoulos, P.: Mathematical Theory of Hemivariational Inequalities and Applications. Marcel Dekker, New York (1995)
21. Papageorgiou, E., Papageorgiou, N.S.: A multiplicity theorem for problems with the $p-$Laplacian. J. Funct. Anal. **244**, 63–77 (2007)
22. Rabinowitz, P.: Minimax Methods in Critical Point Theory with Applications to Differential Equations Vol. 65 of CBMS Regional Conference Series in Math. AMS, Providence (1986)