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ON THE EXISTENCE OF THREE NONTRIVIAL SMOOTH SOLUTIONS FOR NONLINEAR ELLIPTIC EQUATIONS

NIKOLAOS S. PAPAGEORGIOU, EUGÉNIO M. ROCHA, AND VASILE STAICU

Abstract. We consider a nonlinear elliptic problem with a nonsmooth potential function. The nonlinear differential operator includes as special case the $p-$Laplacian. Using a variational approach based on nonsmooth critical point theory, we show the existence of at least three nontrivial smooth solutions. Two of them have constant sign (one is positive and the other is negative).

1. Introduction

Let $Z \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary ∂Z. In this paper, we study the following nonlinear elliptic equation:

$$\begin{cases} -\operatorname{div} a(z, Dx(z)) \in \partial j(z, x(z)) & \text{a.e. on } Z, \\ x|_{\partial Z} = 0. \end{cases} \tag{1.1}$$

The map $a : \bar{Z} \times \mathbb{R}^N \to \mathbb{R}^N$ is strictly monotone in the second variable, satisfying additional regularity conditions (see hypotheses H(a)). In particular, the p-Laplacian differential operator satisfies these conditions. Also, $j : Z \times \mathbb{R} \to \mathbb{R}$ is a measurable function, which is locally Lipschitz in the second variable in general nonsmooth and $\partial j(z,x)$ is the generalized subdifferential of the map $x \mapsto j(z,x)$. If $j(z,\cdot) \in C^1(\mathbb{R})$, then $\partial j(z,x) = \{j'_x(z,x)\}$.

Our goal is to prove a multiplicity result, establishing the existence of at least three nontrivial smooth solutions for problem (1.1). Note that there are elliptic systems, of the form $-\operatorname{div} D\phi(\nabla x) = 0$, with nowhere C^1 solutions, see Müller-Šverák [22]. The first section of this paper presents the background material needed in the subsequent sections. In Section 3, we state the hypotheses for a and j and we obtain two solutions of opposite and constant sign, relying on the $(S)_+$ property of some maximal monotone operator, the coercivity of the corresponding Euler functional and the nonlinear maximum principle of Damascelli [8]. In Section 4, by strengthening one of the hypotheses on j, and using a nonsmooth version of the second deformation theorem, we find a third nontrivial solution. Examples of functions verifying our hypotheses are also given.

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Recently multiplicity results producing three nontrivial solutions for problems driven by the p-Laplacian and with a smooth potential (i.e. $j(z,\cdot) \in C^1(\mathbb{R})$), were proved by Zhang-Li [25], Zhang-Chen-Li [26], Liu-Liu [21], Liu [20], Papageorgiou-Papageorgiou [23] and Carl-Motreanu [3]. On the other hand, problems driven by more general p-Laplacian-like operators were investigated by De Napoli-Mariani [9], Duc-Vu [10], Kristály-Lisei-Varga [16] (all considering a smooth potential) and by Dabuleanu-Radulescu [7], Hu-Papageorgiou [15] (problems with a nonsmooth potential). From the aforementioned works, only De Napoli-Mariani [9] and Kristály-Lisei-Varga [16] prove multiplicity results. The multiplicity result of De Napoli-Mariani [9] (see Theorem 4.1, p.1216) requires symmetry conditions on the map $x \mapsto a(z,x)$ and on the right hand side single-valued nonlinearity $x \mapsto \partial j(z,x) = f(z,x)$. Kristály-Lisei-Varga [16] do not make any symmetry hypothesis on the data of the problem. Instead, they assume that the right hand side single-valued nonlinearity is independent of z and $x \mapsto \partial j(z,x) = f(x)$ is strictly p-sublinear at infinity and strictly p-superlinear at zero. They prove the existence of three solutions, using an abstract multiplicity result of Bonanno [2]. However, among the three solutions, one may be trivial. For a Neumann problem with p-Laplacian type differential operator Gasinski-Papageorgiou [14] obtained two solutions when the potential is bounded.

Here, in contrast to the works of De Napoli-Mariani [9] and Kristály-Lisei-Varga [16], the potential function is nonsmooth, hence the right hand side is multivalued. We do not impose any symmetry conditions and we require that $x \mapsto \partial j(z,x)$ is p-linear both near infinity and near zero. Our multiplicity result establishes the nontriviality of all solutions and we also show that two of them have constant sign (one is positive and the other is negative). Our approach is variational based on nonsmooth critical point theory. A major difficulty that we faced was the lack of a strong maximum principle (nonlinear Hopf's theorem), analogous to the one for the p-Laplacian proved by Vazquez [24].

2. Preliminaries

The nonsmooth critical point theory that we use in the analysis of the problem (1.1), is based on the subdifferential theory of locally Lipschitz functions due to Clarke [4]. For the convenience of the reader, we recall some definitions and facts from this theory.

Let X be a Banach space and X^* its topological dual. By $<\cdot,\cdot>$ we denote the duality brackets for the pair (X, X^*). Given a locally Lipschitz function $\varphi : X \to \mathbb{R}$, the *generalized directional derivative* $\varphi^0(x;h)$ of φ at $x \in X$ in the direction $h \in X$, is defined by

$$\varphi^0(x;h) = \limsup_{\substack{x' \to x \\ \lambda \downarrow 0}} \frac{\varphi(x' + \lambda h) - \varphi(x')}{\lambda}.$$

It is easy to check that the function $h \mapsto \varphi^0(x;h)$ is sublinear continuous and so it is the support function of a nonempty, convex and w^*-compact set, $\partial\varphi(x) \subseteq X^*$, defined by

$$\partial\varphi(x) = \left\{ x^* \in X^* : < x^*, h > \leq \varphi^0(x;h) \text{ for all } h \in X \right\}.$$

The multifunction $x \mapsto \partial\varphi(x)$ is called the *generalized subdifferential* of φ. In particular, if $\varphi : X \to \mathbb{R}$ is continuous and convex, it is locally Lipschitz, and the generalized subdifferential coincides with the subdifferential in the sense of convex analysis $\partial_c\varphi(x)$, defined by

$$\partial_c\varphi(x) = \{x^* \in X^* :< x^*, h > \leq \varphi(x+h) - \varphi(x) \text{ for all } h \in X\}\,.$$

Also, if $\varphi \in C^1(X)$, again φ is locally Lipschitz and we have $\partial\varphi(x) = \{\varphi'(x)\}$.

We say that $x \in X$ is a *critical point* of the locally Lipschitz function $\varphi : X \to \mathbb{R}$, if $0 \in \partial\varphi(x)$. It is easy to see that, if $x \in X$ is a local extremum of φ (i.e. a local minimum or a local maximum), then it is a critical point of φ.

A locally Lipschitz function $\varphi : X \to \mathbb{R}$ satisfies the *Palais-Smale condition at level* $c \in \mathbb{R}$ (the PS_c-condition for short), if every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that

$$\varphi(x_n) \to c \quad \text{and} \quad m(x_n) = \inf\{||x^*|| : x^* \in \partial\varphi(x_n)\} \to 0 \text{ as } n \to +\infty$$

has a strongly convergent subsequence. We say that φ satisfies the PS-condition, if it satisfies the PS_c-condition for every $c \in \mathbb{R}$.

In the minimax characterization of the critical values of φ, the following topological notion plays a crucial role.

Definition 2.1. Let Y be a Hausdorff topological space and E_0, E and D are nonempty, closed subsets of Y with $E_0 \subseteq E$. We say that the pair $\{E_0, E\}$ is a linking with D in Y if and only if

(a) $E_0 \cap D = \emptyset$
(b) for any $\gamma \in C(E, Y)$ such that $\gamma|_{E_0} = id|_{E_0}$, we have $\gamma(E) \cap D \neq \emptyset$.

This topological notion together with the PS-compactness condition, lead to the following general minimax characterization of the critical values of a locally Lipschitz function $\varphi : X \to \mathbb{R}$.

Theorem 2.2. *If X is a Banach space, E_0, E and D are nonempty closed subsets of X such that*

(i) *the pair $\{E_0, E\}$ is linking with D in X;*
(ii) *$\varphi : X \to \mathbb{R}$ is locally Lipschitz and $\sup_{E_0} \varphi < \inf_D \varphi$;*
(iii) *$\Gamma = \{\gamma \in C(E, X) : \gamma|_{E_0} = id|_{E_0}\}$;*
(iv) *$c = \inf_{\gamma\in\Gamma} \sup_{u\in E} \varphi(\gamma(u))$;*
(v) *φ satisfies the PS_c-condition;*

then $c \geq inf_D\varphi$ and c is a critical value of φ.

Remark. From this general minimax principle, by appropriate choices of the linking sets, one can have nonsmooth versions of the mountain pass theorem, of the saddle point theorem and of the generalized mountain pass theorem (see Gasinski-Papageorgiou [12, pp.140–145]).

Definition 2.3. Let Y be a subset of the Banach space X. A *continuous deformation* of Y is a continuous map $h : [0,1] \times Y \to Y$ such that $h(0, y) = y$ for all $y \in Y$. If $V \subseteq Y$, then we say that V is a *weak deformation retract of* Y, if there exists a continuous deformation h of Y such that

$$h(1, Y) \subseteq V \text{ and } h(t, V) \subseteq V \text{ for all } t \in [0, 1].$$

Given a locally Lipschitz function $\varphi : X \to \mathbb{R}$ and $c \in \mathbb{R}$, we define the following sets:

- the *sublevel set of* φ *at* c as
$$\varphi^c = \{x \in X : \varphi(x) \leq c\};$$
- the *strict sublevel set of* φ *at* c as
$$\dot{\varphi}^c = \{x \in X : \varphi(x) < c\}\,;$$
- the *critical set of* φ as
$$K[\varphi] = \{x \in X : 0 \in \partial\varphi(x)\};$$
- *critical set of* φ at c as
$$K_c[\varphi] = \{x \in K[\varphi] : \varphi(x) = c\}.$$

The next theorem is a partial extension to a nonsmooth setting of the so-called *second deformation theorem* (see e.g. Gasinski-Papageorgiou [13, p.628]). The result is due to Corvellec [5]. In fact, the result of Corvellec [5] is formulated in the more general context of metric spaces and continuous functions using the notion of *weak slope*. For our purposes, the following particular version of the result suffices.

Theorem 2.4. *Suppose X is a Banach space, $\varphi : X \to \mathbb{R}$ is locally Lipschitz and satisfies the PS-condition, $a \in \mathbb{R}$, $b \in \mathbb{R} \cup \{+\infty\}$, φ has no critical points in $\varphi^{-1}(a,b)$ and $K_a[\varphi]$ is discrete and contains only local minimizers of φ. Then there exists a continuous deformation $h : [0,1] \times \dot{\varphi}^b \to \dot{\varphi}^b$ such that*

- $h(t,\cdot)|_{K_a[\varphi]} = id|_{K_a[\varphi]}$ *for all* $t \in [0,1]$;
- $h(1,\dot{\varphi}^b) \subseteq \dot{\varphi}^a \cup K_a[\varphi]$;
- $\varphi(h(t,x)) \leq \varphi(x)$ *for all* $t \in [0,1]$ *and all* $x \in \dot{\varphi}^b$.

In particular, the set $\dot{\varphi}^a \cup K_a[\varphi]$ is a weak deformation retract of $\dot{\varphi}^b$.

Remark. In the smooth version of the second deformation theorem, the conclusion is that φ^a is a strong deformation retract of $\varphi^b \backslash K_b[\varphi]$. The set φ^a is a *strong deformation retract* of $\varphi^b \backslash K_b[\varphi]$, if there exists a continuous deformation $h : [0,1] \times (\varphi^b \backslash K_b[\varphi]) \to \varphi^b$ such that $h(t,x) = x$ for all $t \in [0,1]$ and all $x \in \varphi^a$ and
$$h(1, \varphi^b \backslash K_b[\varphi]) \subseteq \varphi^a$$
(see Gasinski-Papageorgiou [13, p.628]).

In the analysis of problem (1.1), we will also need some basic facts about the spectrum of the negative p-Laplacian with Dirichlet boundary condition.

So, let $1 < p < \infty$ and
$$\Delta_p u = \mathrm{div}(||Du||^{p-2} Du)$$
be the p-Laplacian differential operator. We consider the following nonlinear eigenvalue problem:

$$\begin{cases} -\Delta_p\, x(z) = \lambda\, |x(z)|^{p-2} x(z) & \text{a.e. on } Z, \\ x|_{\partial Z} = 0, \end{cases} \tag{2.1}$$

for $\lambda \in \mathbb{R}$. The least real number $\lambda \in \mathbb{R}$, for which the above problem has a nontrivial solution, is the first eigenvalue of $(-\Delta_p, \mathrm{W}_0^{1,p}(Z))$ and it is denoted by λ_1.

We know that $\lambda_1 > 0$ is isolated and simple (namely the corresponding eigenspace is one-dimensional). Moreover, it admits the following variational characterization

$$\lambda_1 = \inf\left\{\frac{||Dx||_p^p}{||x||_p^p} : x \in \mathrm{W}_0^{1,p}(Z) \text{ and } x \neq 0\right\}. \tag{2.2}$$

The infimum in (2.2) is attained in the corresponding one-dimensional eigenspace. If $u_1 \in \mathrm{W}_0^{1,p}(Z)$ is the L^p-normalized eigenfunction, then from (2.2), we see that $|u_1|$ also realizes the infimum. Hence, we may assume that $u_1 \geq 0$. Nonlinear regularity theory (see e.g. Gasinski-Papageorgiou [13, pp.737–738]) implies that

$$u_1 \in C_0^1(\bar{Z}) = \left\{u \in C^1(\bar{Z}) : u|_{\partial Z} = 0\right\}.$$

This is an ordered Banach space with positive (order) cone

$$C_+ = \left\{x \in C_0^1(\bar{Z}) : x(z) \geq 0 \text{ for all } z \in \bar{Z}\right\},$$

and C_+ has a nonempty interior

$$\operatorname{int} C_+ = \left\{x \in C_0^1(\bar{Z}) : x(z) > 0 \text{ for all } z \in Z, \left.\frac{\partial x}{\partial n}\right|_{\partial Z} < 0\right\}.$$

The nonlinear strong maximum principle (nonlinear Hopf's theorem) due to Vazquez [24] implies that $u_1 \in \operatorname{int} C_+$. In fact, it is precisely the lack of such a result for the more general nonlinear differential operator considered here, that causes serious difficulties in the analysis of problem (1.1).

Using the Lusternik-Schnirelmann theory, in addition to $\lambda_1 > 0$, we obtain a whole strictly increasing sequence $\{\lambda_k\}_{k\geq 1} \subseteq \mathbb{R}_+$ of eigenvalues of problem (2.1), such that $\lambda_k \to +\infty$ when $k \to +\infty$. These are the so-called LS-eigenvalues of $(-\Delta_p, \mathrm{W}_0^{1,p}(Z))$. If $p = 2$ (linear eigenvalue problem), then these are all the eigenvalues. For $p \neq 2$ (nonlinear eigenvalue problem), we do not know if this is true. Nevertheless, since $\lambda_1 > 0$ is isolated, we can define

$$\lambda_2^* = \inf\{\lambda : \lambda \text{ is an eigenvalue of (2.1) and } \lambda > \lambda_1\} > \lambda_1.$$

Because the set of eigenvalues of (2.1) is closed, λ_2^* is an eigenvalue of $(-\Delta_p, \mathrm{W}_0^{1,p}(Z))$ (the second eigenvalue). In fact, $\lambda_2^* = \lambda_2$, i.e. the second eigenvalue and the second LS-eigenvalue coincide. Then λ_2 has a variational characterization provided by the Lusternik-Schnirelmann theory. However, for our purposes that characterization is not satisfactory. Instead, we will use an alternative one due to Cuesta-de Figueiredo-Gossez [6]. So, let

$$\partial B_1^p = \{x \in L^p(Z) : ||x||_p = 1\} \text{ and } S = \mathrm{W}_0^{1,p}(Z) \cap \partial B_1^p,$$

furnished with the relative $\mathrm{W}_0^{1,p}(Z)$-topology and

$$\Gamma_0 = \{\gamma_0 \in C([-1,1], S) : \gamma_0(-1) = -u_1 \text{ and } \gamma_0(1) = u_1\}.$$

Then

$$\lambda_2 = \inf_{\gamma_0 \in \Gamma_0} \max_{x \in \gamma_0([-1,1])} ||Dx||_p^p. \tag{2.3}$$

Definition 2.5. Let X be a Banach space and $A : X \to X^*$. We say that A is of type $(S)_+$, if for any sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that

$$x_n \xrightarrow{w} x \text{ in } X \text{ and } \limsup_{n\to+\infty} < A(x_n), x_n - x > \leq 0,$$

one has $x_n \to x$ in X.

In the sequel, by $\xrightarrow{w}$ we denote the weak convergence and by $\to$ the strong convergence. Also, by $< \cdot, \cdot >$ we denote the duality brackets for the pair $(\mathrm{W}_0^{1,p}(Z), \mathrm{W}_0^{1,p}(Z)^*)$, recalling that $\mathrm{W}_0^{1,p}(Z)^* = \mathrm{W}^{-1,p'}(Z)$, $\frac{1}{p} + \frac{1}{p'} = 1$.

3. Solutions of Constant Sign

In this section, we produce two smooth solutions of constant sign for the problem (1.1). More precisely using hypotheses H(a) and H(j)$_1$ stated below, we prove the that problem (1.1) has at least two solutions $\hat{x}_0, \hat{x}_1 \in C^1(\bar{Z})$ such that $\hat{x}_0(z) < 0 < \hat{x}_1(z)$ for all $z \in Z$.

In what follows, we describe the sets of hypotheses H(a) and H(j)$_1$ and give concrete examples of functions that satisfy them. Let $\mathcal{M} = \bar{Z} \times \mathbb{R}^N$ and $\mathcal{M}_0 = \bar{Z} \times (\mathbb{R}^N \backslash \{0\})$. The hypotheses on the map a are the following.

H(a): $a(z, y) = D_y G(z, y)$ where $G \in C^1(\mathcal{M}) \cap C^2(\mathcal{M}_0)$, G(z,0)=0, $a(z, 0) = 0$ for all $z \in \bar{Z}$, and

(i) for every $z \in \bar{Z}$, $y \mapsto a(z, y)$ is strictly monotone;

(ii) for every $(z, y) \in \mathcal{M}_0$, we have

$$||D_y a(z, y)|| \leq c_1 ||y||^{p-2}$$

for some $1 < p < \infty$ and some $c_1 \geq 1$;

(iii) for every $(z, y) \in \mathcal{M}_0$ and $\xi \in \mathbb{R}^N$, we have

$$(D_y a(z, y)y, y)_{\mathbb{R}^N} \geq c_0 ||y||^p \text{ and } (D_y a(z, y)\xi, \xi)_{\mathbb{R}^N} \geq \hat{c}_0 ||y||^{p-2} ||\xi||^2$$

for some $c_0, \hat{c}_0 > 0$.

In the above hypotheses by $|| \cdot ||$ we denote the Euclidean norm in $\mathbb{R}^N$.

Hypothesis H(a)(i) implies that for all $z \in \bar{Z}$, the function $y \mapsto G(z, y)$ is strictly convex. Moreover, observe that for all $(z, y) \in \mathcal{M}$, we have

$$a(z, y) = \int_0^1 \frac{d}{dt} a(z, ty)\, dt = \int_0^1 D_y a(z, ty) y\, dt,$$

so, using hypothesis H(a)(ii),

$$||a(z, y)|| \leq \int_0^1 ||D_y a(z, ty) y||\, dt \leq c_1 ||y||^{p-1} \int_0^1 t^{p-2}\, dt = \frac{c_1}{p-1} ||y||^{p-1}. \tag{3.1}$$

Then, for all $(z,y)\in\mathcal{M}$,

$$G(z,y)=\int_0^1\frac{d}{dt}G(z,ty)\,dt=\int_0^1(a(z,ty),y)_{\mathbb{R}^N}\,dt$$
$$\leq\frac{c_1}{p-1}||y||^p\int_0^1 t^{p-1}\,dt=\frac{c_1}{p(p-1)}||y||^p. \tag{3.2}$$

On the other hand using hypothesis H(a)(iii), for all $(z,y)\in\mathcal{M}$, we have

$$(a(z,y),y)_{\mathbb{R}^N}=\int_0^1(D_ya(z,ty)y,y)_{\mathbb{R}^N}\,dt\geq\int_0^1\frac{c_0}{t^2}t^p||y||^p\,dt=\frac{c_0}{p-1}||y||^p. \tag{3.3}$$

So, for all $(z,y)\in\mathcal{M}$, we have

$$G(z,y)=\int_0^1(a(z,ty),y)_{\mathbb{R}^N}\,dt\geq\frac{c_0}{p-1}||y||^p\int_0^1 t^{p-1}\,dt=\frac{c_0}{p(p-1)}||y||^p. \tag{3.4}$$

Concluding,

$$\frac{c_0}{p(p-1)}||y||^p\leq G(z,y)\leq\frac{c_1}{p(p-1)}||y||^p.$$

Example. The following functions satisfy hypotheses H(a):

(i) $a_1(z,y)=a_1(y)=||y||^{p-2}y$;
(ii) $a_2(z,y)=\Theta(z)||y||^{p-2}y$;
(iii) $a_3(z,y)=\Theta(z)(1+||y||^2)^{(p-2)/2}y$;
(iv) $a_4(z,y)=K(z)y$;

where $\Theta\in C(\bar{Z})$, $\Theta(z)>0$ for all $z\in\bar{Z}$, and $K\in C(\bar{Z},\mathbb{R})$, $K(z)>0$ for all $z\in\bar{Z}$. Note that a_1 corresponds to the p-Laplacian differential operator ($G(y)=\frac{1}{p}||y||^p$) and a_3 corresponds to the generalized mean curvature differential operator

$$\operatorname{div}\left(1+||Dx||^2\right)^{\frac{p-2}{2}}Dx.$$

To obtain the first two nontrivial smooth solutions, which have opposite and constant sign, we will need the following hypotheses on the nonsmooth potential j.

$\mathbf{H(j)_1}$: $j:Z\times\mathbb{R}\to\mathbb{R}$ is a function such that $j(z,0)=0$, $0\in\partial j(z,0)$ a.e. on Z, and

(i) for all $x\in\mathbb{R}$, $z\mapsto j(z,x)$ is measurable;
(ii) for almost all $z\in Z$, $x\mapsto j(z,x)$ is locally Lipschitz;
(iii) for almost $z\in Z$, all $x\in\mathbb{R}$ and all $u\in\partial j(z,x)$, we have
$$|u|\leq\alpha(z)+c|x|^{p-1}$$
with $\alpha\in L^\infty(Z)_+$ and $c>0$;
(iv) there exists $\Theta\in L^\infty(Z)_+$ such that $\Theta(z)\leq\frac{c_0}{p-1}\lambda_1$ a.e. on Z, where the inequality is strict on a set of positive measure and
$$\limsup_{|x|\to\infty}\frac{p\,j(z,x)}{|x|^p}\leq\Theta(z)$$
uniformly for a.a. $z\in Z$;

(v) there exist $\delta > 0$ and $\hat{c} > 0$ such that

$$\frac{c_1}{p-1}\lambda_1|x|^p \leq p\, j(z,x) \leq \hat{c}|x|^p$$

for all a.a. $z \in Z$ and all $|x| \leq \delta$;

(vi) for almost all $z \in Z$, all $x \in \mathbb{R}$ and all $u \in \partial j(z,x)$, we have the sign condition

$$u\, x \geq 0.$$

Remark 3.1. Since hypothesis $H\,(j)_1\,(vi)$ will only be used in Section 4, we denote by $H\,(j)_1'$ hypotheses $H\,(j)_1$ without (vi).

Example. The following function satisfies hypotheses $\mathrm{H(j)}_1$, where for simplicity we have dropped the z-dependence:

$$j(x) = \begin{cases} \frac{c_1}{p(p-1)}\lambda_1|x|^p & \text{if } |x| \leq 1 \\ \frac{\theta}{p(p-1)}|x|^p + \xi \ln(|x|) + \frac{1}{p(p-1)}(\lambda_1 c_1 - \theta) & \text{if } |x| > 1 \end{cases},$$

with $\xi > 0$ and $\theta < \frac{c_0}{p-1}\lambda_1$. Observe that, if $\xi = \frac{1}{p-1}(\lambda_1 c_1 - \theta)$, then we have $j \in C^1(\mathbb{R})$.

Now, in order to prove the Proposition 3.5, we will need to establish several intermediate results. Consider the nonlinear operator $V : \mathrm{W}_0^{1,p}(Z) \to \mathrm{W}^{-1,p'}(Z)$ defined by

$$< V(x), y >= \int_Z (a(z, Dx), Dy)_{\mathbb{R}^N}\, dz$$

for all $x, y \in \mathrm{W}_0^{1,p}(Z)$.

Proposition 3.2. *If hypotheses H(a) hold, then V is a maximal monotone operator of type $(S)_+$.*

Proof. Due to hypothesis H(a)(i), the operator V is monotone. Also, it is easy to see that V is demicontinuous, i.e. $x_n \to x$ in $\mathrm{W}_0^{1,p}(Z)$ implies $V(x_n) \xrightarrow{w} V(x)$ in $\mathrm{W}^{-1,p'}(Z)$. Therefore, V is maximal monotone (see Gasinski-Papageorgiou [13, p.310]).

Next, we show that V is of type $(S)_+$. To this end, let $x_n \xrightarrow{w} x$ in $\mathrm{W}_0^{1,p}(Z)$ and suppose that

$$\limsup_{n\to+\infty} < V(x_n), x_n - x > \leq 0. \tag{3.5}$$

From (3.1), it follows that, for $v_n(\cdot) = a(\cdot, Dx_n(\cdot))$ the sequence

$$\{v_n\}_{n\geq 1} \subseteq L^{p'}(Z) \text{ is bounded.}$$

So, we may assume that $v_n \xrightarrow{w} v$ in $L^{p'}(Z)$, which implies $\operatorname{div} v_n \xrightarrow{w} \operatorname{div} v$ in $\mathrm{W}^{-1,p'}(Z)$. Let $y \in \mathrm{W}_0^{1,p}(Z)$. Due to the monotonicity of $a(z,\cdot)$, we have

$$\begin{aligned}
0 &\leq \int_Z (v_n - a(z,Dy), Dx_n - Dy)_{\mathbb{R}^N}\, dz \\
&= \int_Z (v_n, Dx_n - Dx)_{\mathbb{R}_N}\, dz + \int_Z [(v_n, Dx - Dy)_{\mathbb{R}^N} - (a(z,Dy), Dx_n - Dy)_{\mathbb{R}^N}]\, dz \\
&= < V(x_n), x_n - x > + \int_Z [(v_n, Dx - Dy)_{\mathbb{R}^N} - (a(z,Dy), Dx_n - Dy)_{\mathbb{R}^N}]\, dz,
\end{aligned}$$

hence, using (3.5),

$$0 \leq \int_Z (v - a(z,Dy), Dx - Dy)_{\mathbb{R}^N}\, dz$$

and

$$0 \leq < -\operatorname{div} v - V(y), x - y > . \tag{3.6}$$

Since $y \in \mathrm{W}_0^{1,p}(Z)$ was arbitrary and V is maximal monotone, from the last inequality it follows that

$$-\operatorname{div} v = V(x). \tag{3.7}$$

From the choice of the sequence $\{x_n\}_{n\geq 1} \subseteq \mathrm{W}_0^{1,p}(Z)$ and considering (3.5), we have

$$\limsup_{n\to+\infty} < V(x_n) - V(x), x_n - x > \leq 0. \tag{3.8}$$

On the other hand, due to the monotonicity of V

$$\liminf_{n\to+\infty} < V(x_n) - V(x), x_n - x > \geq 0. \tag{3.9}$$

Therefore, it follows that $\lim_{n\to+\infty} < V(x_n), x_n - x >= 0$, so defining

$$\beta_n(z) = (a(z, Dx_n(z)) - a(z, Dx(z)), Dx_n(z) - Dx(z))_{\mathbb{R}^N}$$

we have

$$\lim_{n\to+\infty} \int_Z \beta_n(z)\, dz = 0, \tag{3.10}$$

and due to the monotonicity of $a(z,\cdot)$, we have $\beta_n \geq 0$ for all $n \geq 1$. Therefore from (3.10), it follows that, at least for a subsequence, we have $\beta_n(z) \to 0$ a.e. on Z and

$$|\beta_n(z)| \leq k(z) \tag{3.11}$$

for a.a. $z \in Z$, all $n \geq 1$, and with $k \in L^1(Z)_+$. Using (3.1), (3.3) and (3.11), we obtain

$$\begin{aligned}
k(z) \geq \beta_n(z) &= (a(z, Dx_n(z)) - a(z, Dx(z)), Dx_n(z) - Dx(z))_{\mathbb{R}^N} \\
&\geq \frac{c_0}{p-1}(||Dx_n(z)||^p + ||Dx(z)||^p) - \frac{c_1}{p-1}||Dx(z)||^{p-1}||Dx_n(z)|| \\
&\quad - \frac{c_1}{p-1}||Dx_n(z)||^{p-1}||Dx(z)||
\end{aligned} \tag{3.12}$$

for a.a. $z \in Z$ and all $n \geq 1$. From the above inequality, it follows that for all $z \in Z \backslash \mathcal{N}$ with $|\mathcal{N}|_N = 0$ (where $|\cdot|_N$ denotes the Lebesgue measure on $\mathbb{R}^N$), the sequence $\{Dx_n(z)\}_{n\geq 1} \subseteq \mathbb{R}^N$ is bounded. Passing to a suitable subsequence (which in general depends on $z \in Z \backslash \mathcal{N}$), we have $Dx_n(z) \to \xi(z)$ for all $z \in Z\backslash\mathcal{N}$, which implies

$$a(z, Dx_n(z)) \to a(z, \xi(z)) \text{ as } n \to +\infty, \text{ with } z \in Z\backslash\mathcal{N}.$$

But, we know that $\beta_n(z) \to 0$ a.e. on Z, so

$$(a(z, \xi(z)) - a(z, Dx(z)), \xi(z) - Dx(z))_{\mathbb{R}^N} = 0 \text{ a.a. } z \in Z. \tag{3.13}$$

Due to the strict monotonicity of $y \mapsto a(z, y)$ for all $z \in \bar{Z}$ (see hypothesis H(a)(i)) and from (3.13), it follows that

$$\xi(z) = Dx(z) \text{ for all } z \in Z\backslash\mathcal{N}.$$

Therefore, for the original sequence we have

$$Dx_n(z) \to Dx(z) \text{ a.e. on } Z. \tag{3.14}$$

On the other hand, from (3.12) it is clear that the sequence

$$\{||Dx_n(\cdot)||^p\}_{n\geq 1} \subseteq L^1(Z) \tag{3.15}$$

is uniformly integrable. From (3.14), (3.15) and Vitali's theorem (see Gasinski-Papageorgiou [12, pp.715]), we infer that

$$||Dx_n||_p^p \to ||Dx||_p^p.$$

Recall that $Dx_n \xrightarrow{w} Dx$ in $L^p(Z, \mathbb{R}^N)$, a uniformly convex function space. So, from the Kadec-Klee property, we conclude that

$$Dx_n \to Dx \text{ in } L^p(Z, \mathbb{R}^N),$$

hence

$$x_n \to x \text{ in } \mathrm{W}_0^{1,p}(Z),$$

and so finally

$$V \text{ is a } (S)_+ \text{ type operator.}$$

□

To characterize the two solutions, we will use truncation maps and the index set $\mathcal{S}$ ={$-1, 0, +1$}. However, with the purpose of simplifying the notation, we use "$+$" for $+1$, "$-$" for -1, and "$\pm$" for the set $\bar{\mathcal{S}}$ ={$-1, +1$}$\subset \mathcal{S}$. So, we use $x_\pm$ to mean "x_s with $s \in \bar{\mathcal{S}}$".

Define

$$\tau_\pm(x) = \begin{cases} 0 & \text{if } \pm x \leq 0 \\ x & \text{if } \pm x > 0 \end{cases},$$

$$j_\pm(z, x) = j(z, \tau_\pm(x))$$

and

$$j_s(z, x) = \begin{cases} j_-(z, x) & \text{if } s = -1 \\ j(z, x) & \text{if } s = 0 \\ j_+(z, x) & \text{if } s = +1 \end{cases}$$

for all $(z, x) \in Z \times \mathbb{R}$ and $s \in \mathcal{S}$. Evidently, both $j_{\pm}$ are measurable in $z \in Z$ and locally Lipschitz in $x \in \mathbb{R}$. Moreover, from the nonsmooth chain rule (see Clarke [4, p.42]), if $\Upsilon_{\pm} : Z \times \mathbb{R} \to 2^{\mathbb{R}}$ are the multifunctions defined by

$$\Upsilon_{\pm}(z, x) = \begin{cases} \{0\} & \text{if } \pm x < 0 \\ \{r\, \partial j(z, 0) : r \in [0, 1]\} & \text{if } x = 0 \\ \partial j(z, x) & \text{if } \pm x > 0 \end{cases},$$

we have

$$\partial j_{\pm}(z, x) \subseteq \Upsilon_{\pm}(z, x). \tag{3.16}$$

We introduce the functional $\varphi : \mathrm{W}_0^{1,p}(Z) \to \mathbb{R}$ defined by

$$\varphi(x) = \int_Z G(z, Dx(z))\, dz - \int_Z j(z, x(z))\, dz,$$

and the functionals $\varphi_s : \mathrm{W}_0^{1,p}(Z) \to \mathbb{R}$, for $s \in \mathcal{S}$, defined by

$$\varphi_s(x) = \int_Z G(z, Dx(z))\, dz - \int_Z j_s(z, x(z))\, dz.$$

Note that φ_0 is precisely φ.

The next Lemma is a direct consequence of the positivity of the principal eigenfunction u_1 of $(-\Delta_p, \mathrm{W}_0^{1,p}(Z))$ and of the variational characterization of $\lambda_1 > 0$ (see (2.2)), so its proof is omitted (see Gasinski-Papageorgiou [12, p.570]).

Lemma 3.3. *If $\Theta \in L^\infty(Z)_+$ and $\Theta(z) \le \frac{c_0}{p-1}\lambda_1$ a.e. on Z with strict inequality on a set of positive measure, then there exists $\mu > 0$ such that*

$$\frac{c_0}{p-1}||Dx||_p^p - \int_Z \Theta(z)|x(z)|^p\, dz \ge \mu||Dx||_p^p$$

for all $x \in W_0^{1,p}(Z)$.

Using the above Lemma, we can prove the following Proposition.

Proposition 3.4. *If hypotheses $H(a)$ and $H\,(j)_1'$ hold, then the functionals φ_s, for $s \in \mathcal{S}$, are locally Lipschitz and coercive.*

Proof. Let $I_G : L^p(Z, \mathbb{R}^N) \to \mathbb{R}$ and $I_j^s : L^p(Z) \to \mathbb{R}$, for $s \in \mathcal{S}$, be the integral functionals defined by

$$I_G(y) = \int_Z G(z, y(z))\, dz,$$
$$I_j^s(x) = \int_Z j_s(z, x(z))\, dz,$$

for $y \in L^p(Z, \mathbb{R}^N)$ and $x \in L^p(Z)$. Evidently, I_G is continuous convex, hence it is locally Lipschitz. Also, I_j^s $(s \in \mathcal{S})$ are Lipschitz continuous on bounded sets, thus locally Lipschitz (see Clarke [4, p.83]). Let $D \in \mathcal{L}(\mathrm{W}_0^{1,p}(Z), L^p(Z, \mathbb{R}^N))$ be the gradient operator and let $\iota : \mathrm{W}_0^{1,p}(Z) \to L^p(Z)$ be the embedding operator, which is a compact operator. Then

$$\varphi_s(x) = (I_G \circ D)(x) - (I_j^s \circ \iota)(x) \text{ with } s \in \mathcal{S}$$

for all $x \in \mathrm{W}_0^{1,p}(Z)$. Hence the functionals φ_s are locally Lipschitz and moreover, we have

$$\begin{aligned} \partial\varphi_s(x) &= \nabla(I_G \circ D)(x) - \partial(I_j^s \circ \iota)(x) \\ &\subseteq -\operatorname{div}(\nabla I_G(Dx)) - \iota^* \partial I_j^s(x) \\ &\subseteq V(x) - \iota^* \partial I_j^s(x) \end{aligned} \tag{3.17}$$

for all $x \in \mathrm{W}_0^{1,p}(Z)$ (see Clarke [4, p.39 and p.45] and recall that $D^* = -\operatorname{div}$). By virtue of hypothesis $\mathrm{H(j)}_1$(iv), given $\epsilon > 0$, we can find $M_1 \equiv M_1(\epsilon) > 0$ such that

$$j(z,x) \leq \frac{1}{p}\left(\Theta(z) + \epsilon\right)|x|^p \tag{3.18}$$

for a.a. $z \in Z$ and all $|x| \geq M_1$. On the other hand, from hypothesis $\mathrm{H(j)}_1$(iii) and the mean value theorem for locally Lipschitz functions (see Clarke [4, p.41]), we have

$$|j(z,x)| \leq c_2 \tag{3.19}$$

for a.a. $z \in Z$, all $|x| \leq M_1$ and for some $c_2 > 0$. From (3.18) and (3.19), it follows that

$$j(z,x) \leq \frac{1}{p}(\Theta(z) + \epsilon)|x|^p + c_2 \tag{3.20}$$

for a.a. $z \in Z$ and all $x \in \mathbb{R}$. Hence, noting that $j_\pm(z,x) = 0$ a.a. $z \in Z$ and all $\pm x \leq 0$, we also have

$$j_s(z,x) \leq \frac{1}{p}(\Theta(z) + \epsilon)|x|^p + c_2 \text{ with } s \in \mathcal{S}, \tag{3.21}$$

for a.a. $z \in Z$ and all $x \in \mathbb{R}$. Then, for every $x \in \mathrm{W}_0^{1,p}(Z)$ and some $c_3 > 0$, using (3.21), we get

$$\begin{aligned} \varphi_s(x) &= \int_Z G(z, Dx(z))\, dz - \int_Z j_s(z, x(z))\, dz \\ &\geq \int_Z G(z, Dx(z))\, dz - \frac{1}{p}\int_Z \Theta(z)|x|^p\, dz - \frac{\epsilon}{p}||x||_p^p - c_3. \end{aligned}$$

Therefore, using (3.4) and (3.20),

$$\varphi_s(x) \geq \frac{c_0}{p(p-1)}||Dx||_p^p - \frac{1}{p}\int_Z \Theta(z)|x|^p\, dz - \frac{\epsilon}{p}||x||_p^p - c_3,$$

and, applying Lemma 3.3,

$$\varphi_s(x) \geq \frac{1}{p}\left(\mu - \frac{\epsilon}{\lambda_1}\right)||Dx||_p^p - c_3. \tag{3.22}$$

In particular, choosing $\epsilon < \lambda_1\mu$, it follows that the functionals φ_s $(s \in \mathcal{S})$ are coercive. $\square$

We have now all the necessary tools to prove the main result of this section.

Proposition 3.5. *If hypotheses $H(a)$ and $H\,(j)_1'$ hold, then problem (1.1) has at least two solutions $\hat{x}_0, \hat{x}_1 \in C^1(\bar{Z})$ such that $\hat{x}_0(z) < 0 < \hat{x}_1(z)$ for all $z \in Z$.*

Proof. We will use the index set $s \in \bar{\mathcal{S}} = \{-1, +1\} \subset \mathcal{S}$ to simultaneously find the solutions $\hat{x}_0 = x_s|_{s=-1}$ and $\hat{x}_1 = x_s|_{s=+1}$.

Choose arbitrarily $s \in \bar{\mathcal{S}}$. The convexity of $G(z,\cdot)$ implies that the integral functional I_G is weakly lower semicontinuous, hence the functionals $x \mapsto \varphi_s(x)$ are weakly lower semicontinuous and coercive (see Proposition 3.4). So, by the theorem of Weierstrass, we can find $x_s \in \mathrm{W}_0^{1,p}(Z)$ such that

$$\varphi_s(x_s) = \inf\left\{\varphi_s(x) : x \in \mathrm{W}_0^{1,p}(Z)\right\}.$$

Then,

$$0 \in \partial\varphi_s(x_s)$$

which implies $V(x_s) = u_s$ for

$$u_s \in N_s(x_s) = \left\{u \in L^{p'}(Z) : u(z) \in \partial j_s(z, x_s(z)) \text{ a.e. on } Z\right\} \tag{3.23}$$

(see (3.17) and Clarke [4, p.83]). Let $y_s = \max\{s\, x_s, 0\}$. If on (3.23) we act with $-y_s \in \mathrm{W}_0^{1,p}(Z)$, then using (3.16), we obtain

$$\int_Z (a(z, Dx_s), -Dy_s)_{\mathbb{R}^N}\, dz = 0.$$

Now, using hypothesis H(a)(iii),

$$\frac{c_0}{p-1}||Dy_s||_p^p \leq 0$$

so, $y_s(z) = 0$ for a.a. $z \in Z$. Therefore, $x_s = \tau_s(x_s)$ and

$$\hat{x}_0 = x_s|_{s=-1} \leq 0 \text{ and } \hat{x}_1 = x_s|_{s=+1} \geq 0. \tag{3.24}$$

Also from (3.23), we obtain

$$-\operatorname{div} a(z, Dx_s(z)) = u_s(z) \text{ a.e. on } Z \text{ and } x_s|_{\partial Z} = 0. \tag{3.25}$$

Note that

$$u_s \in \partial j_s(z, x_s(z)) = \partial j(z, x_s(z)) \text{ a.e. on } \{s\, x_s > 0\}. \tag{3.26}$$

From Stampacchia's theorem (see Gasinski-Papageorgiou [13, p.195–196]), we have $Dx_s(z) = 0$ a.e. on $\{s\, x_s > 0\}$. Since $a(z, 0) = 0$ for all $z \in \bar{Z}$, it follows that

$$u_s(z) = 0 \in \partial j(z, 0) \text{ a.e on } \{s\, x_s > 0\}. \tag{3.27}$$

Because $s\, x_s \geq 0$, we have

$$Z = \{s\, x_s > 0\} \cup \{x_s = 0\}.$$

Hence, from (3.26) and (3.27), it follows that

$$u_s(z) \in \partial j(z, x_s(z)) \text{ a.e. on } Z. \tag{3.28}$$

Now, (3.25) and (3.28) imply that $x_s \in \mathrm{W}_0^{1,p}(Z)$ is a solution of problem (1.1). From Theorem 7.1 of Ladyzhenskaya-Uraltseva [18, p.286], we have $x_s \in L^\infty(Z)$. Therefore, from Theorem 1 of Lieberman [19], we infer that $x_s \in C_0^1(\bar{Z})$, $s\, x_s \geq 0$ with $s \in \bar{\mathcal{S}}$. Invoking the nonlinear maximum principle of Damascelli [8, p.507], we have two possibilities: $x_s = 0$ for all $z \in Z$; or $s\, x_s(z) > 0$ for all $z \in Z$.

Now, recall that $u_1 \in \operatorname{int} C_+$ is the L^p-normalized principal eigenfunction of $(-\Delta_p, \mathrm{W}_0^{1,p}(Z))$. Let $t > 0$ be small and such that

$$|tu_1(z)| \le \delta \text{ for all } z \in \bar{Z}. \tag{3.29}$$

Then using (3.2), for $s \in \bar{\mathcal{S}}$,

$$\varphi_s(tu_1) = \int_Z G(z, tDu_1(z))\, dz - \int_Z j_s(z, tu_1(z))\, dz \tag{3.30}$$

$$\le \frac{c_1 t^p}{p(p-1)} ||Du_1||_p^p - \int_Z j_s(z, tu_1(z))\, dz. \tag{3.31}$$

Hence, by hypothesis $\mathrm{H(j)}_1$(v) and (3.29),

$$\varphi_s(tu_1) \le \frac{c_1 t^p}{p(p-1)} ||Du_1||_p^p - \frac{c_1 t^p}{p(p-1)} \lambda_1 ||u_1||_p^p$$

thus, considering (2.2),

$$\varphi_s(tu_1) \le 0 = \varphi_s(0). \tag{3.32}$$

In the case $x_s = 0$, we have

$$\varphi_s(x_s) = \varphi_s(0) = 0 = \varphi_s(tu_1)$$

for all $t > 0$ small, which implies that each point in the set

$$\{tu_1 : t \in [0, \epsilon] \text{ and } \epsilon > 0\}$$

is a minimizer of φ_s ($s \in \bar{\mathcal{S}}$). Therefore, we have a continuum of solutions, with constant sign equal to $\operatorname{sign}(s)$, for problem (1.1), and so we are done.

Otherwise (i.e. $s\, x_s > 0$), for each $s \in \bar{\mathcal{S}}$, $x_s \in C_0^1(\bar{Z})$ is a solution of problem (1.1) with constant sign equal to $\operatorname{sign}(s)$ for all $z \in Z$. Therefore, $\hat{x}_0 = x_s|_{s=-1} \equiv x_- < 0$ and $\hat{x}_1 = x_s|_{s=+1} \equiv x_+ > 0$ are solutions of problem (1.1). $\square$

4. Three Nontrivial Solutions

In this, section we obtain an additional solution for problem (1.1). We point out that the nonlinear Hopf's theorem does not apply to the general differential operators considered in this work. In particular, there is no analog of the theorem of Vazquez [24] for our setting. Therefore, we are not able to locate local minimizers of φ in $\operatorname{int} C_+$ and then appeal to the nonsmooth analog of the result of Azorero-Manfredi-Alonso [11] (see also Gasinski-Papageorgiou [12, p.685] and Kyritsi-Papageorgiou [17]). However, we manage to overcome this difficulty with the help of the next Proposition.

Proposition 4.1. *If hypotheses H(a) and H(j)$_1$ hold, then every local minimizer x_s of φ_s ($s \in \bar{\mathcal{S}}$), with $s\, x_s(z) > 0$ for all $z \in Z$, is also a local minimizer of φ.*

Proof. Choose arbitrarily $s \in \bar{\mathcal{S}} = \{-1, +1\}$. Suppose that $x_n \to x_s$ in $\mathrm{W}_0^{1,p}(Z)$. We will show that

$$\varphi(x_n) \ge \varphi(x_s) \tag{4.1}$$

for some $n_0 \geq 1$ and all $n \geq n_0$. Write $x_n = \zeta^+(x_n) - \zeta^-(x_n)$ where $\zeta^s(x_n) = \max\{s\, x_n, 0\}$. Then using hypotheses $\mathrm{H(j)_1}$(iv)-(v)

$$\varphi(x_n) = \int_Z G(z, D\zeta^+(x_n))\, dz + \int_Z G(z, -D\zeta^-(x_n))\, dz \tag{4.2}$$

$$- \int_Z j_+(z, x_n)\, dz - \int_Z j_-(z, x_n)\, dz \tag{4.3}$$

$$\geq \varphi_s(\zeta^s(x_n)) + \frac{c_0}{p(p-1)}||D\zeta^{-s}(x_n)||_p^p - c_4||\zeta^{-s}(x_n)||_p^p \tag{4.4}$$

for some $c_4 > 0$. Since, by hypotheses $s\, x_s(z) > 0$ for all $z \in Z$, we have

$$\zeta^s(x_n) \to x_s \text{ and } \zeta^{-s}(x_n) \to 0 \text{ (both in } \mathrm{W}_0^{1,p}(Z)).$$

If $\zeta^{-s}(x_n) = 0$ eventually (say for $n \geq \hat{n}_0$ and with $\hat{n}_0 \geq n_0$), then from (4.2)

$$\varphi(x_n) \geq \varphi_s(\zeta^s(x_n)) \geq \varphi_s(x_s) = \varphi(x_s)$$

for all $n \geq \hat{n}_0$. Otherwise, for every $n \geq 1$, we set

$$Z_n = \{z \in Z : s\, x_n(z) < 0\}.$$

Claim 1. Let $|\cdot|_N$ stand for the Lebesgue measure on $\mathbb{R}^N$. Then $|Z_n|_N \to 0$ as $n \to +\infty$.

From the regularity of the Lebesgue measure, given $\epsilon > 0$, we can find a compact set $K_\epsilon \subseteq Z$ such that

$$|K_\epsilon^c|_N = |Z \backslash K_\epsilon|_N \leq \epsilon. \tag{4.5}$$

Then, since $s\, x_n < 0$ on Z_n by definition,

$$||x_n - x_s||_p^p = \int_Z |x_n - x_s|^p\, dz \geq \int_{K_\epsilon \cap Z_n} |x_n - x_s|^p\, dz \geq \int_{K_\epsilon \cap Z_n} |x_s|^p\, dz,$$

we have

$$||x_n - x_s||_p^p \geq m_\epsilon\, |K_\epsilon \cap Z_n|_N, \tag{4.6}$$

where $m_\epsilon = \inf\{x_s(z) \in \mathbb{R} : z \in K_\epsilon\}$ and $s\, m_\epsilon > 0$. From (4.6) and since $x_n \to x_s$ in $\mathrm{W}_0^{1,p}(Z)$, we have

$$|K_\epsilon \cap Z_n|_N \to 0 \text{ as } n \to +\infty. \tag{4.7}$$

Note that $Z_n \subseteq (K_\epsilon \cap Z_n) \cup K_\epsilon^c$, together with (4.5), implies

$$|Z_n|_N \leq |K_\epsilon \cap Z_n|_N + |K_\epsilon^c|_N \leq |K_\epsilon \cap Z_n|_N + \epsilon.$$

Hence, from (4.7),

$$\limsup_{n \to +\infty} |Z_n|_N \leq \epsilon.$$

Because $\epsilon > 0$ was arbitrary, we conclude that

$$|Z_n|_N \to 0 \text{ as } n \to +\infty,$$

concluding the proof of Claim 1.

Claim 2. There exists $n_0 \geq 1$ such that

$$\frac{c_0}{p(p-1)}||D\zeta^{-s}(x_n)||_p^p > c_4||\zeta^{-s}(x_n)||_p^p$$

for all $n \geq n_0$.

We argue indirectly. Suppose the above claim is not true. Then, by passing to a subsequence if necessary, we may assume that

$$\frac{c_0}{p(p-1)}||D\zeta^{-s}(x_n)||_p^p \leq c_4||\zeta^{-s}(x_n)||_p^p \text{ for all } n \geq 1. \tag{4.8}$$

Let

$$y_n = \frac{\zeta^{-s}(x_n)}{||\zeta^{-s}(x_n)||_p}, n \geq 1.$$

By virtue of (4.8), $\{y_n\}_{n\geq 1} \subseteq \mathrm{W}_0^{1,p}(Z)$ is bounded. So, we may assume that

$$y_n \xrightarrow{w} y \text{ in } \mathrm{W}_0^{1,p}(Z) \text{ and } y_n \to y \text{ in } L^p(Z) \text{ as } n \to +\infty.$$

Note that $||y||_p = 1$, $y \neq 0$, $y \geq 0$. Therefore, we can find $\eta > 0$ small such that, if $Z_\eta = \{z \in Z : y(z) \geq \eta\}$, then

$$|Z_\eta|_N > 0. \tag{4.9}$$

We have

$$\begin{aligned} ||y_n - y||_p^p &= \int_Z |y_n - y|^p \, dz \geq \int_{Z_\eta \setminus Z_n} |y_n - y|^p \, dz \\ &= \int_{Z_\eta \setminus Z_n} |y|^p \, dz \geq \eta^p |Z_\eta \setminus Z_n|_N \geq \eta^p \left(|Z_\eta|_N - |Z_n|_N\right). \end{aligned}$$

Passing to the limit as $n \to +\infty$ and using Claim 1, we obtain $|Z_\eta|_N = 0$, which contradicts (4.9). This ends the proof of Claim 2.

Now, returning to (4.2) and using Claim 2, we have

$$\varphi(x_n) \geq \varphi_s(\zeta^s(x_n)) \text{ for all } n \geq n_0,$$

hence, by increasing $n_0 \geq 1$ if necessary,

$$\varphi(x_n) \geq \varphi_s(x_s) = \varphi(x_s).$$

So this proves (4.1) and completes the proof of the Proposition. □

The next Proposition is a consequence of Proposition 3.4.

Proposition 4.2. *If hypotheses $H(a)$ and $H(j)_1$ hold, then all φ_s ($s \in \mathcal{S}$) satisfy the PS-condition.*

Proof. Choose arbitrarily $s \in \mathcal{S} \equiv \{-1, 0, +1\}$. Let $\{x_n\}_{n\geq 1} \subseteq \mathrm{W}_0^{1,p}(Z)$ be a sequence such that, for some $M > 0$,

$$\forall_{n\geq 1} \, |\varphi_s(x_n)| \leq M \text{ and } m(x_n) \to 0 \text{ as } n \to +\infty. \tag{4.10}$$

Since $\partial\varphi_s(x_n) \subseteq \mathrm{W}^{-1,p'}(Z)$ is weakly compact and the norm functional in a Banach space is lower semicontinuous, from the theorem of Weierstrass, we can find $x_n^* \in \partial\varphi_s(x_n)$ such that

$$\|x_n^*\|_{W^{-1,p'}(Z)} = m\,(x_n) \text{ for all } n \geq 1$$

where $p' = \frac{p}{p-1}$ is the conjugate exponent of p. We know that

$$x_n^* = V(x_n) - u_n \tag{4.11}$$

with

$$u_n \in \left\{ u \in L^{p'}(Z) : u(z) \in \Upsilon_s(z, x_n(z)) \text{ a.e. on } Z \right\}.$$

Because of (4.10) and since φ_s is coercive (see proposition 3.4), we have that $\{x_n\}_{n\geq 1} \subseteq \mathrm{W}_0^{1,p}(Z)$ is bounded. Therefore, we may assume that

$$x_n \xrightarrow{w} x \text{ in } \mathrm{W}_0^{1,p}(Z) \text{ and } x_n \to x \text{ in } L^p(Z) \text{ as } n \to +\infty. \tag{4.12}$$

From $H\,(j)_1\,(iii)$ we have

$$|u_n\,(z)|^{p'} \leq 2^{p'}\,\|a\|_\infty^{p'} + 2^{p'}c^{p'}\,|x_0\,(z)|^p \text{ a.e. on } Z.$$

From this inequality and (4.12) it follows that $\{u_n\}_{n\geq 1}$ is bounded in $L^{p'}\,(Z)$.

If on (4.11) we act with $x_n - x \in \mathrm{W}_0^{1,p}(Z)$ and consider (4.10), we get

$$\left| < V(x_n), x_n - x > - \int_Z u_n(x_n - x)\,dz \right| \leq \epsilon_n ||x_n - x|| \text{ with } \epsilon_n \downarrow 0. \tag{4.13}$$

Thus, if we pass to the limit as $n \to \infty$ in (4.13), we obtain

$$< V(x_n), x_n - x > \to 0. \tag{4.14}$$

But from Proposition 3.2, we know that V is an $(S)_+$-type operator. So, considering (4.14), we infer that

$$x_n \to x \in \mathrm{W}_0^{1,p}(Z).$$

Since $s \in \mathcal{S}$ wa arbitrarily chosen, this means that each φ_s ($s \in \mathcal{S}$) satisfies the PS-condition. ☐

In order to produce a third nontrivial smooth solution for the problem (1.1), we need to strengthen hypothesis H(j)$_1$(v) concerning the growth of the nonsmooth potential j near the origin. So, the new hypotheses are the following:

H(j)$_2$: $j : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that $j(z, 0) = 0$, $0 \in \partial j(z, 0)$ a.e. on Z, and

(i) for all $x \in \mathbb{R}$, $z \mapsto j(z, x)$ is measurable;

(ii) for almost all $z \in Z$, $x \mapsto j(z, x)$ is locally Lipschitz;

(iii) for almost $z \in Z$, all $x \in \mathbb{R}$ and all $u \in \partial j(z, x)$, we have

$$|u| \leq \alpha(z) + c|x|^{p-1}$$

with $\alpha \in L^\infty(Z)_+$ and $c > 0$;

(iv) there exists $\Theta \in L^\infty(Z)_+$ such that $\Theta(z) \leq \frac{c_0}{p-1}\lambda_1$ a.e. on Z, where the inequality is strict on a set of positive measure and

$$\limsup_{|x|\to\infty} \frac{p\,j(z, x)}{|x|^p} \leq \Theta(z)$$

uniformly for a.a. $z \in Z$;

(v) there exist $\delta > 0$ and $\eta > \lambda_2$ such that

$$\frac{c_1\,\eta}{p-1}||x||^p \leq j(z,x)$$

for all a.a. $z \in Z$ and all $|x| \leq \delta$;

(vi) for almost all $z \in Z$, all $x \in \mathbb{R}$ and all $u \in \partial j(z,x)$, we have the sign condition

$$u\,x \geq 0.$$

Now, we are ready to prove the main result of this work, namely the existence of three nontrivial smooth solutions of problem (1.1).

Theorem 4.3. *If hypotheses H(a) and H(j)$_2$ hold, then problem (1.1) has at least three nontrivial solutions $\hat{x}_0, \hat{x}_1, \hat{x}_2 \in C_0^1(\bar{Z})$ and*

$$\hat{x}_0(z) < 0 < \hat{x}_1(z) \textit{ for all } z \in Z.$$

Proof. From Proposition 3.5, we already have two solutions $\hat{x}_0, \hat{x}_1 \in C_0^1(\bar{Z})$ such that

$$\hat{x}_0(z) < 0 < \hat{x}_1(z) \text{ for all } z \in Z.$$

By virtue of Proposition 3.4, $x_{\pm}$ is a minimizer of $\varphi_{\pm}$. Proposition 4.1 implies that both $x_{\pm}$ are local minimizers of φ. Without any loss of generality, we may assume that both solutions are isolated local minimizers of φ, otherwise, we can have a whole sequence of distinct nontrivial solutions of problem (1.1). Then, as in Aizicovici-Papageorgiou-Staicu [1, proof of Proposition 29], we can find $r > 0$ such that

$$\varphi(x_{\pm}) < \inf\left\{\varphi(x) : ||x - x_{\pm}|| = r\right\}.$$

Without any loss of generality, we may assume that $\varphi(x_-) \leq \varphi(x_+) < 0$ (see the proof of Proposition 3.5). Then, define

$$\begin{aligned}
E_0 &= \{x_{\pm}\},\\
E &= \left\{x \in \mathrm{W}_0^{1,p}(Z) : x_-(z) \leq x(z) \leq x_+(z) \text{ a.e. on } Z\right\},\\
\partial B_r(x_+) &= \left\{x \in \mathrm{W}_0^{1,p}(Z) : ||x - x_+|| = r\right\}.
\end{aligned}$$

Is easy to verify that the pair $\{E_0, E\}$ is linking with $\partial B_r(x_+)$ in $\mathrm{W}_0^{1,p}(Z)$ (see Gasinski-Papageorgiou [13, p.642]). Also φ satisfies the PS-condition (Proposition 4.2). So, we can apply Theorem 2.2 and obtain $\hat{x}_2 \in \mathrm{W}_0^{1,p}(Z)$, a critical point of φ, such that

$$\varphi(x_-) \leq \varphi(x_+) < \inf\left\{\varphi(x) : ||x - x_+|| = r\right\} = c_r \leq \varphi(\hat{x}_2). \tag{4.15}$$

From (4.15), we see that $\hat{x}_2 \neq x_-$ and $\hat{x}_2 \neq x_+$. Moreover, since $\hat{x}_2$ is a critical point of φ, we have $0 \in \partial\varphi(\hat{x}_2)$ so

$$V(\hat{x}_2) = u_0 \text{ with } u_0 \in \left\{u \in L^{p'}(Z) : u(z) \in \partial j(z, \hat{x}_2(z)) \text{ a.e. on } Z\right\}. \tag{4.16}$$

As before, from (4.16), it follows that $\hat{x}_2$ solves (1.1) and, by the nonlinear regularity theory, we have $\hat{x}_2 \in C_0^1(\bar{Z})$.

It remains to show that $\hat{x}_2$ is nontrivial. To this end, observe that

$$\varphi(\hat{x}_2) = \inf_{\gamma\in\Gamma} \max_{t\in[-1,1]} \varphi(\gamma(t)) \tag{4.17}$$

with the set of paths Γ defined by

$$\Gamma = \left\{\gamma \in C([-1,1], \mathrm{W}_0^{1,p}(Z)) : \gamma(-1) = x_- \text{ and } \gamma(1) = x_+\right\}.$$

If we can produce a path $\gamma_0 \in \Gamma$ such that $\varphi|_{\gamma_0} < 0$, then from (4.17) we have

$$\varphi(\hat{x}_2) < 0 = \varphi(0), \text{ thus } \hat{x}_2 \neq 0.$$

In what follows, we concentrate our effort in producing the path $\gamma_0 \in \Gamma$. Recall that we have defined the set $S = \mathrm{W}_0^{1,p}(Z) \cap \partial B_1^p$, furnished with the relative $\mathrm{W}_0^{1,p}(Z)$-topology. Also, let $S_c = S \cap C_0^1(\bar{Z})$ endowed with the relative $C_0^1(\bar{Z})$-topology. Evidently, S_c is dense in S. Therefore, if we set

$$\Gamma_0 = \{\gamma_0 \in C([-1,1], S) : \gamma(-1) = -u_1 \text{ and } \gamma_0(1) = u_1\},$$
$$\Gamma_0^c = \{\gamma_0 \in C([-1,1], S_c) : \gamma(-1) = -u_1 \text{ and } \gamma_0(1) = u_1\},$$

then we have that Γ_0^c is dense in Γ_0. Thus, given $\delta_0 > 0$, due to (2.3) we can find $\hat{\gamma}_0 \in \Gamma_0^c$ such that

$$\max\left\{||Dx||_p^p : x \in \hat{\gamma}_0([-1,+1])\right\} \leq \lambda_2 + \delta_0. \tag{4.18}$$

If $\eta > \lambda_2$ is as in hypothesis $\mathrm{H(j)_2(v)}$, we choose $\delta_0 > 0$ such that

$$\lambda_2 + 2\delta_0 < \eta. \tag{4.19}$$

Also, we can find $\epsilon > 0$ small such that

$$|\epsilon x(z)| \leq \delta \tag{4.20}$$

for all $z \in \bar{Z}$ and all $x \in \hat{\gamma}_0([-1,+1])$, where $\delta > 0$ is as in hypothesis $\mathrm{H(j)_2(v)}$. Therefore, if $x \in \hat{\gamma}_0([-1,+1])$, then

$$\begin{aligned}\varphi(\epsilon x) &= \int_Z G(z, \epsilon Dx(Z))\,dz - \int_Z j(z, \epsilon x(z))\,dz \\ &\leq \int_Z G(z, \epsilon Dx(z))\,dz - \frac{\eta c_1 \epsilon^p}{p(p-1)}||x||_p^p.\end{aligned}$$

by using (4.20) and hypothesis $\mathrm{H(j)_2(v)}$. Now, considering (3.2), (4.18), (4.19) and recalling that $||x||_p = 1$,

$$\begin{aligned}\varphi(\epsilon x) &\leq \frac{c_1\epsilon^p}{p(p-1)}\left(||Dx||_p^p - \eta||x||_p^p\right) \\ &\leq \frac{c_1\epsilon^p}{p(p-1)}\left(\lambda_2 + \delta_0 - \eta\right) \\ &< 0. \end{aligned} \tag{4.21}$$

Hence $\gamma_0 = \epsilon\hat{\gamma}_0$, so γ_0 is a continuous path joining $-\epsilon u_1$ to ϵu_1, and because of (4.21), we have

$$\varphi|_{\gamma_0} < 0. \tag{4.22}$$

Next, we use Theorem 2.4 (the nonsmooth second deformation theorem) to produce a continuous path joining $\pm\epsilon u_1$ to $x_\pm$, along which $\varphi_\pm$ is negative. For this purpose, let

$$a = \varphi_\pm(x_\pm) = \varphi(x_\pm) < 0 = \varphi_\pm(0) = \varphi(0) = b. \tag{4.23}$$

We may assume that $\{0, x_\pm\}$ are the only critical points of $\varphi_\pm$. Indeed, if $\hat{u}_0 \in \mathrm{W}_0^{1,p}(Z)$ is another critical point of $\varphi_\pm$ distinct from 0 and $x_\pm$, then as before using the nonlinear regularity theory, we can show that $\hat{u}_0 \in C_0^1(\bar{Z})$, solves problem (1.1), and $\pm\hat{u}_0(z) > 0$ for all $z \in Z$ (see Damascelli [8]). Therefore, we have produced a third nontrivial smooth solution and so we are done. Otherwise, by virtue of Theorem 2.4, we can find a continuous deformation $h : [0,1] \times \dot{\varphi}_\pm^b \to \dot{\varphi}_\pm^b$ such that

(a) $h(t,\cdot)|_{K_a} = id|_{K_a}$ for all $t \in [0,1]$;
(b) $h(1, \dot{\varphi}_\pm^b) \subseteq \dot{\varphi}_\pm^a \cup K_a$;
(c) $\varphi_\pm(h(t,x)) \leq \varphi_\pm(x)$ for all $t \in [0,1]$ and all $x \in \dot{\varphi}_\pm^b$,

where

$$K_a = \{x \in \mathrm{W}_0^{1,p}(Z) : 0 \in \partial\varphi_\pm(x) \text{ and } \varphi_\pm(x) = a\}$$

(recall that $\varphi_\pm$ satisfies the PS-condition, see Proposition 4.2). We now consider the continuous path $\gamma_\pm : [0,1] \to \dot{\varphi}_\pm^b$ defined by

$$\gamma_\pm(t) = h(t, \pm\epsilon u_1) \text{ for all } t \in [0,1].$$

Observe that, by definition of deformation, $\gamma_\pm(0) = h(0, \pm\epsilon u_1) = \pm\epsilon u_1$. On the other hand, since $\dot{\varphi}_\pm^a = \emptyset$ and $K_a = \{x_\pm\}$ (see (b)), $\gamma_\pm(1) = h(1, \pm\epsilon u_1) = x_\pm$. Further,

$$\varphi_\pm(\gamma_\pm(t)) = \varphi_\pm(h(t, \pm\epsilon u_1)) \leq \varphi_\pm(\pm\epsilon u_1) < 0 \tag{4.24}$$

for all $t \in [0,1]$, see (c) and (4.22). Therefore, the continuous path $\gamma_\pm$ joins $\pm\epsilon u_1$ to $x_\pm$, and because of (4.24), we can say that

$$\varphi_\pm|_{\gamma_\pm} < 0. \tag{4.25}$$

Due to hypothesis $\mathrm{H(j)}_2$(vi) (the sign condition), we have

$$j(z,x) \geq 0$$

for all a.a. $z \in Z$ and all $x \in \mathbb{R}$. Hence

$$j(z,x) \geq j_\pm(z,x)$$

for all a.a. $z \in Z$ and all $x \in \mathbb{R}$, which implies

$$\varphi \leq \varphi_\pm.$$

Consequently, considering (4.25),

$$\varphi|_{\gamma_\pm} \leq \varphi_\pm|_{\gamma_\pm} < 0. \tag{4.26}$$

This proves that $\hat{x}_2 \neq 0$ (see (4.17)), $\hat{x}_2 \in C^1(\bar{Z})$ (by the nonlinear regularity theory), and it is a solution of problem (1.1). $\square$

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References

[1] S. Aizicovici, N.S. Papageorgiou and V. Staicu, *Degree theory for operators of monotone type and nonlinear elliptic equations with inequality constraints*, Mem. Amer. Math. Soc. **196** (2008), pp. – pp.

[2] G. Bonanno, *Some remarks on a three critical points theorem*, Nonlinear Anal. **54** (2003), 651–655.

[3] S. Carl and D. Motreanu, *Constant sign and sign-changing solutions for nonlinear eigenvalue problems*, Nonlinear Anal. (to appear).

[4] F. Clarke, *Optimization and Nonsmooth Analysis*, Wiley New York, 1983.

[5] J.-M. Corvellec, *On the second deformation lemma*, Topol. Methods Nonlinear Anal. **17** (2001), 55–56.

[6] M. Cuesta, D. de Figueiredo and J.-P. Gossez, *The beginning of the Fučik spectrum for the p-Laplacian*, J. Differential Equations **159** (1999), 212–238.

[7] S. Dabuleanu and V. Radulescu, *Multivalued boundary value problems involving Leray-Lions operators and discontinuous nonlinearities*, Rend. Circ. Mat. Palermo **52** (2003), 57–69.

[8] L. Damascelli, *Comparision theorems for some quasilinear degenerate elliptic operators and applications to symmetry and monotonicity results*, Ann. Inst. H. Poincaré Anal. Non Lineaire **15** (1998), 493–516.

[9] P. De Napoli, and M. C. Mariani, *Mountain pass solutions to equations of p-Laplacian type*, Nonlinear Anal. **54** (2003), 1205–1219.

[10] D. M. Duc and N. T. Vu, *Nonuniformly elliptic equations of p-Laplacian type*, Nonlinear Anal. **61** (2005), 1483–1495.

[11] J. Garcia Azorero, J. J. Manfredi and I. Peral Alonso, *Sobolev versus Hölder local minimizers and global multiplicity for some quasilinear elliptic equations.* Comm. Contemp. Math. **2** (2000), 385–404.

[12] L. Gasinski and N. S. Papageorgiou, *Nonsmooth Critical Point Theory and Nonlinear Boundary Value Problems*, Chapman & Hall / CRC, Boca Raton, 2005.

[13] L. Gasinski and N.S. Papageorgiou, *Nonlinear Analysis*, Chapman & Hall / CRC, Boca Raton, 2006.

[14] L. Gasinski and N. S. Papageorgiou, *Existence and Multiplicity of Solutions for Neumann p-Laplacian type equations*, Advanced Nonlinear Studies **8** (2008), 843–870.

[15] S. Hu and N. S. Papageorgiou, *Positive solutions for nonlinear hemivariational inequalities*, J. Math. Anal. Appl. **310** (2005), 161–176.

[16] A. Kristály, H. Lisei and C. Varga, *Multiple solutions for p-Laplacian type operator*, Nonlinear Anal. (to appear doi:10.1016/j.na.2006.12.031).

[17] S. Kyritsi and N. S. Papageorgiou, *Multiple solutions for constant sign for nonlinear nonsmooth eigenvalue problems near resonance*, Calc. Var. and Partial Differential Equations **20** (2004), 1–24.

[18] C. Ladyzhenskaya and N. Uraltseva, *Linear and Quasilinear Elliptic Equations*, Academic Press, New York, 1968.

[19] G. Lieberman, *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Anal. **12** (1988), 1203–1219.

[20] S. Liu, *Multiple solutions for coercive p-Laplacian equations*, J. Math Anal. Appl. **316** (2006), 229–236.

[21] J. Liu and S. Liu, *The existence of multiple solutions to quasilinear elliptic equations*, Bull. London Math. Soc. **37** (2005), 592–600.

[22] S. Müller and V. Šverák, *Convex integration for Lipschitz mapping and counterexamples to regularity*, Ann. of Math. **157** (2003), 715–742.

[23] E. Papageorgiou and N. S. Papageorgiou, *A multiplicity theorem for problems with the p-Laplacian*, J. Funct. Anal. **244** (2007), 63–77.

[24] J. L. Vazquez, *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim. **12** (1984), 191–202.

[25] Z. Zhang and S. Li, *On sign-changing and multiple solutions for the p-Laplacian*, J. Funct. Anal. **197** (2003), 447–468.

[26] Z. Zhang, J. Chen, and S. Li, *Construction of pseudogradient vector field and multiple solutions involving p-Laplacian*, J. Differential Equations **201** (2004), 287–303.

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Nikolaos S. Papageorgiou
Department of Mathematics, National Technical University, Zografou Campus, Athens 15780, Greece
E-mail address: `npapg@math.ntua.gr`

Eugénio M. Rocha
Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal
E-mail address: `eugenio@ua.pt`

Vasile Staicu
Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal
E-mail address: `vasile@ua.pt`