A MINI COURSE ON CLIFFORD ANALYSIS

F. SOMMEN

4 HOMO EXPERIMENTALIS

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IX. CLIFFORD ANALYSIS ON NULLCONES

I. THE BASIC SYSTEMS

CAUCHY - RIEMANN SYSTEM

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}_{0,1} = Alg\{i\} = \mathbb{C}$

$(\partial_x + i\,\partial_y) f = 0$

$f = a + ib \Rightarrow \partial_x a - \partial_y b = 0, \ \partial_x b + \partial_y a = 0$

HAMILTON - FUETER SYSTEM

$f : \mathbb{R}^4 \longrightarrow \mathbb{R}_{0,2} = Alg\{e_1, e_2\} \cong \mathbb{H}$

$$(\partial_{x_0} + e_1 \partial_{x_1} + e_2 \partial_{x_2} + e_{12} \partial_{x_{12}}) f = 0$$

Stan. Not. $(\partial_{q_0} + i\,\partial_{q_1} + j\,\partial_{q_2} + k\,\partial_{q_3}) f = 0$

MOISIL - THEODORESCU SYSTEM

$f : \mathbb{R}^3 \longrightarrow \mathbb{R}_3^+ = Alg\{e_{23}, e_{31}, e_{12}\} \cong \mathbb{H}$

$$(\partial_{x_1} e_{23} + \partial_{x_2} e_{31} + \partial_{x_3} e_{12}) f = 0$$

Stan. Not. $(i\,\partial_x + j\,\partial_y + k\,\partial_z) f = 0$

$\times e_{123}$ $\quad (\partial_{x_1} e_1 + \partial_{x_2} e_2 + \partial_{x_3} e_3) f = 0$

$\times (-e_1)$ & $(-e_1 e_2) \to E_1$, $(-e_1 e_3) \longrightarrow E_2$ & $E_j \to e_j$

$$(\partial_{x_0} + e_1 \partial_{x_1} + e_2 \partial_{x_2}) f = 0, \quad f : \mathbb{R}^3 \longrightarrow \mathbb{R}_{0,2}$$

GENERALIZED CAUCHY-RIEMANN SYSTEMS

WEYL - DELANGHE - FUETER

$$f : \mathbb{R} \oplus \mathbb{R}^m = \mathbb{R}^{m+1} \longrightarrow \mathbb{R}_m \quad (\text{or } \mathbb{C}_m)$$

$$\left(\partial_{x_0} + \sum_{j=1}^{m} e_j \partial_{x_j}\right) f = 0$$

$$\llcorner\, D_x = \partial_{x_0} + \partial_{\underline{x}} \; ; \; x = x_0 + \underline{x}$$

$$\overline{D}_x D_x = \sum_{j=0}^{m} \partial_{x_j}^2 = \Delta_x$$

DIRAC OPERATOR

$$\partial_{\underline{x}} = \sum_{j=1}^{m} e_j \partial_{x_j} \; ; \; \partial_{\underline{x}}^2 = -\Delta_{\underline{x}} \; ; \; \underline{x} = \sum_{j=1}^{m} e_j x_j$$

NAMES: FUETER, NEF, DELANGHE, ATIYAH, HESTENES
BRACKX, BDS, IFTIMIE, ...

EQUIVALENCE

DIRAC : $\left(\partial_{\underline{x}} + e_{m+1} \partial_{x_{m+1}}\right) f = 0$

MULT. : $-e_{m+1}$ & $E_j = -e_{m+1} e_j$ & $f : \mathbb{R}^{m+1} \to \mathbb{R}^+_{m+1}$

$$\left(\partial_{x_{m+1}} + \sum_{j=1}^{m} E_j \partial_{x_j}\right) f = 0$$

$$\mathbb{R}^+_{m+1} \cong \mathbb{R}_m \longrightarrow \left(\partial_{x_0} + \sum_{j=1}^{m} e_j \partial_{x_j}\right) f = 0 .$$

DIRAC EVEN DIMENSION : $m = 2n$

$$e_{j+n} = -i\,\varepsilon_j \quad , \quad I_j = \frac{1}{2}(1 - e_j\varepsilon_j)$$

$$f : \mathbb{R}^m \to \mathbb{C}_m I = \mathbb{C}_n I$$

SPECIAL CASE : $f = g^+ I$, $g^+ : \mathbb{R}^m \to \mathbb{C}_n^+$

$$\Rightarrow \varepsilon_j f = g^+ e_j I$$

&

$$\sum_{j=1}^{n} (\partial_{x_j} e_j g^+ - i\, \underbrace{\partial_{x_{j+n}}}_{\partial_{y_j}} g^+ e_j) = 0$$

$$\Rightarrow \quad \partial_{\underline{x}}\, g^+ - i\, g^+ \partial_{\underline{y}} = 0 \longrightarrow \text{S.C.V.}$$

USING WITT BASIS

$$e_j = f_j - f_j^+ \quad ; \quad e_{j+n} = i(f_j + f_j^+)$$

$$\Rightarrow \partial_{\underline{x}} = -\sum_{j=1}^{n} f_j^+ (\partial_{x_j} - i\partial_{y_j}) + \sum_{j=1}^{n} f_j (\partial_{x_j} + i\partial_{y_j})$$

$$\partial_{\underline{Z}} = \sum_{j=1}^{n} f_j^+ \partial_{z_j} \quad \left(\partial_{z_j} = \frac{1}{2}(\partial_{x_j} - i\partial_{y_j})\right)$$

$$\partial_{\underline{x}} = 2(-\partial_{\underline{Z}} + \partial_{\underline{Z}^+}) \quad ; \quad \underline{x} = \underline{Z} - \underline{Z}^+ , \quad \underline{Z} = \sum_{j=1}^{n} z_j f_j$$

↳ HERMITIAN CLIFFORD ANALYSIS

ROCHA-CHAVEZ, SHAPIRO, SOMMEN, I. SABADINI

REFINED SYS. : $\partial_{\underline{Z}} f(\underline{Z}, \underline{Z}^+) = \partial_{\underline{Z}^+} f(\underline{Z}, \underline{Z}^+) = 0$

HODGE SYSTEM

$$f : \mathbb{R}^m \to \mathbb{R}_m^k$$

$$\partial_{\underline{x}} f = 0 \Rightarrow \bar{f}\,\bar{\partial}_{\underline{x}} = 0 \Rightarrow f \partial_{\underline{x}} = 0$$

OR $[\partial_{\underline{x}}, f] = 0$ & $\{\partial_{\underline{x}}, f\} = 0$

OR $\partial_{\underline{x}} \wedge f = 0$ & $\partial_{\underline{x}} \cdot f = 0$

CONVERSELY

$$\partial_{\underline{x}} f = f \partial_{\underline{x}} = 0 \Rightarrow \partial_{\underline{x}} [f]_k = 0$$

NOW: $f : \mathbb{R}^m \to \mathbb{R}_m^{\#} I \rightarrow \in \mathbb{R}_{m,m}$, $f = g I$

$$\Rightarrow \partial_{\dot{\underline{x}}} = \sum_1^m \varepsilon_j \partial_{x_j} \quad \text{(G. BERNARDES)}$$

$$(\partial_{\underline{x}} f = \partial_{\dot{\underline{x}}} f = 0) \Leftrightarrow (\partial_{\underline{x}} g = g \partial_{\underline{x}} = 0)$$

HIGHER-SPIN DIRAC

$$\partial_{\underline{x},j} = \sum_{k=1}^m e_{k,j} \partial_{x_k} \qquad e_{k,j} = e_{k+(j-1)m}$$

System: $\partial_{\underline{x},j} f(\underline{x}) = 0$; $j = 1, \dots, \ell$

INVESTIGATORES: G. BERNARDES, J. BORY-REYES, DYNKIN R. ABREU-BLAYA, R. DELANGHE, F. BRACKX, ...

MORE SYSTEMS: I. SABADINI & F. SOMMEN (ZAA-paper)

II. CAUCHY-KOWALEWSKI EXTENSIONS

SOLVE $(\partial_{x_0} + \partial_{\underline{x}}) f(x_0, \underline{x}) = 0$

PUT $f(x_0, \underline{x}) = e^{-x_0 \partial_{\underline{x}}} f(\underline{x})$

$\partial_{x_0} e^{-x_0 \partial_{\underline{x}}} = -\partial_{\underline{x}} e^{-x_0 \partial_{\underline{x}}}$

MULTIPLY with $f(\underline{x}) \Rightarrow$ JOB DONE

CONVERGENCE

$f(\underline{x})$ real analytic on $K \subset \mathbb{R}^m$

$\hookrightarrow f(x_0, \underline{x}) = \sum_{k=0}^{\infty} \frac{(-x_0)^k}{k!} \partial_{\underline{x}}^k f(\underline{x}) = CK(f(\underline{x}))$

conv. NBD $\widetilde{\Omega}$ of K, $\widetilde{\Omega} \subset \mathbb{R}^{m+1}$ open

$\forall$ $f(0, \underline{x}) = f(\underline{x})$ & UNIQUE

$\mathcal{A}(K) \cong M_l(K)$ & $\mathcal{B}(K) \cong \mathcal{A}'(K) \cong M_r'(K) \cong M_{l,0}(\mathbb{R}^{m+1} \setminus K)$

FOURIER KERNEL

$CK(e^{i\langle \underline{x}, \underline{t}\rangle}) = \left(\operatorname{ch} x_0 |\underline{t}| + \frac{i\underline{t}}{|\underline{t}|} \operatorname{sh} x_0 |\underline{t}|\right) e^{i\langle \underline{x}, \underline{t}\rangle}$

$= E_+(x; \underline{t}) + E_-(x; \underline{t})$, $x = x_0 + \underline{x}$

$E_\pm(x_0, \underline{x}; \underline{t}) = \frac{1}{2}\left(1 \pm \frac{i\underline{t}}{|\underline{t}|}\right) e^{i\langle \underline{x}, \underline{t}\rangle \mp x_0 |\underline{t}|}$

RADON KERNEL

$$\frac{1}{1-\langle \underline{x},\underline{t}\rangle + x_0\underline{t}} = \frac{1-\langle \underline{x},\underline{t}\rangle \cancel{+} x_0\underline{t}}{(1-\langle \underline{x},\underline{t}\rangle)^2 + x_0^2|\underline{t}|^2}$$

BASIC MONOG. POLYN.

$$(\langle \underline{x},\underline{t}\rangle - x_0\underline{t})^k \quad , \; k\in\mathbb{N} \; , \; k\in\mathbb{Z}$$

MONOGENIC PLANE WAVES

$f(x+iy)$ holom $\Rightarrow$ $f(\langle \underline{x},\underline{t}\rangle - x_0\underline{t})$ monogenic.

BIREGULAR CK-EXTENSION

$$(\partial_{x_0}+\partial_{\underline{x}})f(x_0,\underline{x};t_0,\underline{t}) = f(\cdots)(\partial_{t_0}+\partial_{\underline{t}}) = 0$$

SOLUTION: $f(x_0,\underline{x};t_0,\underline{t}) = e^{-x_0\partial_{\underline{x}}} f(\underline{x};\underline{t})\, e^{-t_0\partial_{\underline{t}}}$

BIREGULAR EXPONENTIAL (Sabadini - Sommen - Struppa)

$$E(x_0,\underline{x};t_0,\underline{t}) = e^{-x_0\partial_{\underline{x}}}\, e^{i\langle \underline{x},\underline{t}\rangle}\, e^{-t_0\partial_{\underline{t}}}$$

$$= E_{+,+} + E_{+,-} + E_{-,+} + E_{-,-}$$

USE $e^{i\langle \underline{x},\underline{t}\rangle} = \dfrac{1}{(2\pi)^m}\displaystyle\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} e^{i\langle \underline{x},\underline{u}\rangle}\, e^{-i\langle \underline{u},\underline{v}\rangle}\, e^{i\langle \underline{v},\underline{t}\rangle}\, d\underline{u}\, d\underline{v}$

CK-extension integrand explodes

$$E_{\pm,\pm}(x_0, \underline{x}; t_0, \underline{t})$$

$$= \frac{1}{4(2\pi)^m} \int_{\mathbb{R}^m \times \mathbb{R}^m} \left(1 \pm i\frac{\underline{u}}{|\underline{u}|}\right)\left(1 \pm i\frac{\underline{v}}{|\underline{v}|}\right) e^{i(\langle \underline{x}, \underline{u}\rangle - \langle \underline{u}, \underline{v}\rangle + \langle \underline{v}, \underline{t}\rangle) \mp x_0|\underline{u}| \mp t_0|\underline{v}|}$$

Converges for $x_0 \gtrless 0$ & $t_0 \gtrless 0$

CLIFFORD-HERMITE

$$CK(e^{-|\underline{x}|^2/2}) = \sum_{k=0}^{\infty} \frac{(-x_0)^k}{k!} H_k(\underline{x}) e^{-|\underline{x}|^2/2}.$$

$$H_k(\underline{x}) = e^{|\underline{x}|^2/2}\, \partial_{\underline{x}}^k\, e^{-|\underline{x}|^2/2}$$

$$= \begin{cases} 2^{k/2}\,\Gamma(\frac{k}{2}+1)\, L_{k/2}^{m/2-1}(|\underline{x}|^2/2), & k \text{ even} \\ 2^{\frac{k-1}{2}}\,\Gamma(\frac{k+1}{2})\, \underline{x}\, L_{\frac{k-1}{2}}^{m/2}(|\underline{x}|^2/2), & k \text{ odd} \end{cases}$$

CLIFFORD-BESSEL

$$\mathcal{E}(x_0, \underline{x}) = e^{x_0}\, \mathcal{E}(\underline{x})$$

$$\hookrightarrow \mathcal{E}(\underline{x}) = 2^{m/2-1}\,\Gamma\left(\frac{m}{2}\right)|\underline{x}|^{1-\frac{m}{2}} \left\{ J_{\frac{m}{2}-1}(|\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} J_{\frac{m}{2}}(|\underline{x}|) \right\}$$

$\underline{x}\,\underline{u} = -\langle \underline{x}, \underline{u}\rangle + \underline{x} \wedge \underline{u}$: $x_0 \to \langle \underline{x}, \underline{u}\rangle$, $\underline{x} \to \underline{x} \wedge \underline{u}$, $m \to m-1$

$$\partial_{\underline{x}}\, \mathcal{E}(\langle \underline{x}, \underline{u}\rangle, \underline{x} \wedge \underline{u}) = \mathcal{E}(\langle \underline{x}, \underline{u}\rangle, \underline{x} \wedge \underline{u})\, \partial_{\underline{u}} = 0$$

III. INTEGRAL FORMULAS

DIFFERENTIAL FORM APPROACH

$x_j : j = 1, \ldots, m$ coordinates

↳ dx_j : basic differentials

$$dx_j\, dx_k = -dx_k\, dx_j$$

WORKING ALGEBRA ; $\mathrm{Alg}\{e_j ; x_j ; dx_j : j = 1, \ldots, m\} = W$

$$F = \sum_A F_A\, dx_A \; ; \; dx_A = dx_{\alpha_1} \cdots dx_{\alpha_k}$$

(i) $d(x_j F) = dx_j F + x_j dF$

(ii) $d(dx_j F) = -dx_j\, dF$

(iii) $d(e_j F) = e_j\, dF$

↳ $d \in \mathrm{End}(\mathrm{Alg}\{e_j ; x_j ; dx_j\})$, $d^2 = 0$

IN COORDINATES : $d = \sum_{j=1}^{m} dx_j\, \partial_{x_j}$

STOKES : $\int_\Omega dF = \int_{\partial\Omega} F$.

DERIVATIVES : ∂_{x_j}

$$\partial_{x_j} \in \mathrm{End}(\mathrm{Alg}\{e_j ; x_j ; dx_j\})$$

CONTRACTIONS: $\partial_{x_j} \rfloor$

(i) $\partial_{x_j} \rfloor (x_k F) = x_k \, \partial_{x_j} \rfloor F$

(ii) $\partial_{x_j} \rfloor (dx_k F) = \delta_{jk} F - dx_k \, \partial_{x_j} \rfloor F$

LIE DERIVATIVES

$$\underline{v} = \sum_{j=1}^{m} v_j(x) e_j \longrightarrow v = \sum_{j=1}^{m} \underbrace{v_j}_{\mathbb{C}_m\text{-valued}}(\underline{x}) \, \partial_{x_j}$$

$\underline{v} \in \mathrm{End}(W)$ so $v[F] = \sum_A v[F_A] \, dx_A$

$v \rfloor \in \mathrm{End}(W)$, $v \rfloor = \sum_{j=1}^{m} v_j(x) \partial_{x_j} \rfloor$

$$\mathcal{L}_v F = (d\, v\rfloor + v\rfloor\, d) F$$

IF $v_j = \text{constant} \Rightarrow \mathcal{L}_v F = v[F]$

EXAMPLE $v_j(\underline{x}) = e_j \longrightarrow v = \partial_{\underline{x}} = \sum_{j=1}^{m} e_j \partial_{x_j}$

$$\partial_{\underline{x}} F = \mathcal{L}_{\partial_{\underline{x}}} F = (d\, \partial_{\underline{x}}\rfloor + \partial_{\underline{x}}\rfloor\, d) F$$

$$F = f(x)\, dx_1 \cdots dx_m = f(x)\, dV$$

$$\partial_{\underline{x}} \rfloor dV = \sum_{j=1}^{m} (-1)^{j+1} e_j \, d\hat{x}_j = d\sigma = \underline{n}\, dS$$

$$d(d\sigma\, f) = dV\, \partial_{\underline{x}} f \Rightarrow \int_\Omega (\partial_{\underline{x}} f)\, dV = \int_{\partial\Omega} d\sigma\, f$$

CAUCHY-POMPEJU FORMULA

$$d(f\, d\sigma\, g) = [(f\partial_{\underline{x}})g + f(\partial_{\underline{x}} g)]\, dV$$

$$\hookrightarrow \int_{\partial\Omega} f\, d\sigma\, g = \int_{\Omega} [(f\partial_{\underline{x}})g + f(\partial_{\underline{x}} g)]\, dV$$

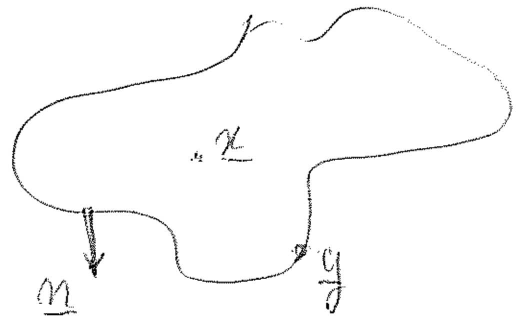
CAUCHY-INTEGRAL FORMULA

$$E(\underline{x}) = -\frac{1}{A_m}\frac{\underline{x}}{|\underline{x}|^m}$$

$$\hookrightarrow \partial_{\underline{x}} E(\underline{x}) = E(\underline{x})\partial_{\underline{x}} = \delta(\underline{x})$$

HENCE

$$\int_{\partial\Omega} E(\underline{y}-\underline{x})\, d\sigma_{\underline{y}}\, g(\underline{y}) = g(\underline{x}) + \int_{\Omega} E(\underline{y}-\underline{x})\, \partial_{\underline{y}}\, g(\underline{y})\, dy$$



$$\partial_{\underline{x}}\, g(\underline{x}) = 0 \text{ in } \overline{\Omega}$$

$$\hookrightarrow g(\underline{x}) = \int_{\partial\Omega} E(\underline{y}-\underline{x})\, d\sigma_{\underline{y}}\, g(\underline{y})$$

F, k-form : $\displaystyle\int_{\partial\Omega} \partial_{\underline{x}} \rfloor F = \int_{\Omega} (\partial_x - \partial_{\underline{x}} \rfloor d) F$

DISTRIBUTIONAL CAUCHY-POMPEJU

$$\chi_\Omega(\underline{x}) = \begin{cases} 1 & : \underline{x} \in \Omega \\ 0 & : \underline{x} \notin \Omega \end{cases}$$

$$\partial_{\underline{x}} \chi_\Omega(\underline{x}) = -\underline{n}(\underline{x})\, \delta_{\partial\Omega}(\underline{x}) \qquad (\text{def.})$$

CLAIM : $\displaystyle\int_{\mathbb{R}^m} \underline{n}(\underline{x})\, \delta_{\partial\Omega}(\underline{x})\, f(\underline{x}) = \int_{\partial\Omega} \underline{n}(\underline{x})\, f(\underline{x})\, dS_{\underline{x}}$

PHASE FUNCTION

$$\varphi_\Omega(\underline{x}) \in C_\infty(\mathbb{R}^m)\,,\quad \varphi_\Omega(\underline{x}) = \begin{cases} \geq 0 \text{ in } \overline{\Omega} \\ < 0 \text{ in } \mathbb{R}^m \setminus \Omega \end{cases}$$

$$\& \quad \sum_{j=1}^{m} (\partial_{x_j} \varphi)^2 = 1$$

THEN $\chi_\Omega(\underline{x}) = \underbrace{Y}_{\text{HEAVISIDE}}(\varphi_\Omega(\underline{x}))$

$$\Rightarrow \partial_{\underline{x}} \chi_\Omega(\underline{x}) = \underbrace{\partial_{\underline{x}} \varphi_\Omega(\underline{x})}_{-\underline{n}(\underline{x})}\; \delta(\varphi_\Omega(\underline{x}))$$

$$0 = \int_{\mathbb{R}^m} \partial_{\underline{x}} [\chi_\Omega(\underline{x})\, g(\underline{x})] = \int_\Omega \partial_{\underline{x}}\, g(\underline{x}) - \int_{\partial\Omega} \underline{n}(\underline{x})\, g(\underline{x})\, dS_{\underline{x}}$$

IV. THE FISCHER DECOMPOSITION

$\mathcal{P} = \mathrm{Alg}\{x_1, \dots, x_m ; e_1, \dots, e_m\}$

FISCHER INNER PRODUCT

$(R(\underline{x}), S(\underline{x})) = [R(\partial_{\underline{x}})^\dagger S(\underline{x})]_0 \Big|_{\underline{x}=0}$ posit. def. Hermitian

$\mathcal{P}_k = \{S(\underline{x}) \in \mathcal{P} : S(\lambda \underline{x}) = \lambda^k S(\underline{x})\}$

$\mathcal{P}_0 \perp \mathcal{P}_1 \perp \mathcal{P}_2 \perp \mathcal{P}_3 \dots$

Vector Variable : $\underline{x} = \sum_{j=1}^{m} x_j e_j \in \mathcal{P}_1$

↳ SUBSPACE

$\underline{x}\,\mathcal{P} = \{\underline{x}\, R(\underline{x}) : R \in \mathcal{P}\}$

(SOLID) SPHERICAL MONOGENICS

$P_k(\underline{x}) \in \mathcal{P}_k$ & $\partial_{\underline{x}} P_k(\underline{x}) = 0$

↳ P_k is spher. mon. deg. k

$M_k \subset \mathcal{P}_k$: SUBSPACE OF SPHER. MON. DEG. k.

THEOREM. (FISCHER DECOMPOSITION)

LET $R_k(\underline{x}) \in \mathcal{P}_k \Rightarrow \exists!\ P_k(\underline{x}) \in M_k, R_{k-1}(\underline{x}) \in \mathcal{P}_{k-1}$

SUCH THAT $R_k(\underline{x}) = P_k(\underline{x}) + \underline{x}\, R_{k-1}(\underline{x})$

IN FACT : $\mathcal{P}_k = M_k \oplus_{\perp} \underline{x}\, \mathcal{P}_{k-1}$

PROOF. $(\underline{x}\, R(\underline{x}), S(\underline{x})) = -(R(\underline{x}), \partial_{\underline{x}} S(\underline{x}))$. ///

THEOREM. (MONOGENIC DECOMPOSITION)

LET $R_k(\underline{x}) \in \mathcal{P}_k \Rightarrow \exists!\ P_{k-s}(\underline{x}) \in M_{k-s}$

SUCH THAT $R_k(\underline{x}) = \sum_{s=0}^{k} \underline{x}^s P_{k-s}(\underline{x})$ ORTHOGONAL

PROOF. RECURSIVE FISCHER $\Rightarrow$ DIRECT SUM

$\partial_{\underline{x}}^{s+1}\left(\underline{x}^s P_l(\underline{x})\right) = 0 \Rightarrow$ ORTHOGONALITY . ///

EULER & GAMMA OPERATOR

$\underline{x}\, \partial_{\underline{x}} = -E_{\underline{x}} - \Gamma_{\underline{x}}$, $E_{\underline{x}} = \sum x_j \partial_{x_j}$: EULER

$\Gamma_{\underline{x}} = -\underline{x} \wedge \partial_{\underline{x}} = -\sum_{j<k} e_{jk} L_{jk}$

$L_{jk} = x_j \partial_{x_k} - x_k \partial_{x_j}$: ANGULAR MOMENTUM OPS.

BASIC IDENTITIES

$$R_k(\underline{x}) \in \mathcal{P}_k \Rightarrow E_{\underline{x}} R_k(\underline{x}) = k R_k(\underline{x})$$

$$P_k(\underline{x}) \in \mathcal{M}_k \Rightarrow \Gamma_{\underline{x}} P_k(\underline{x}) = -k P_k(\underline{x})$$

$$\underline{x}\,\partial_{\underline{x}} + \partial_{\underline{x}}\,\underline{x} = -2E_{\underline{x}} - m$$

$$\Rightarrow \partial_{\underline{x}}\,\underline{x} = -E_{\underline{x}} - m + \Gamma_{\underline{x}}$$

HENCE FOR $P_k(\underline{x}) \in \mathcal{M}_k$:

$$\underline{x}\,\partial_{\underline{x}}(\underline{x} P_k(\underline{x})) = -(2k+m)\,\underline{x} P_k(\underline{x})$$

$$\text{L} -(E_{\underline{x}} + \Gamma_{\underline{x}})(\underline{x} P_k(\underline{x})) = -(k+1+\Gamma_{\underline{x}})(\underline{x} P_k(\underline{x}))$$

$$\Rightarrow \Gamma_{\underline{x}}(\underline{x} P_k(\underline{x})) = (k+m-1)\,\underline{x} P_k(\underline{x})$$

ALSO: $\Gamma_{\underline{x}} |\underline{x}|^2 R(\underline{x}) = |\underline{x}|^2 \Gamma_{\underline{x}} R(\underline{x})$

CONCLUSION

$$E_{\underline{x}}(\underline{x}^s P_k(\underline{x})) = (s+k)\,\underline{x}^s P_k(\underline{x})$$

$$\Gamma_{\underline{x}}(\underline{x}^{2s} P_k(\underline{x})) = -k\,\underline{x}^{2s} P_k(\underline{x})$$

$$\Gamma_{\underline{x}}(\underline{x}^{2s+1} P_k(\underline{x})) = (k+m-1)\,\underline{x}^{2s+1} P_k(\underline{x})$$

RECURSION FORMULAS

$$\partial_{\underline{x}}\left(\underline{x}^{2s} P_k(\underline{x})\right) = -2s\, \underline{x}^{2s-1} P_k(\underline{x})$$

$$\partial_{\underline{x}}\left(\underline{x}^{2s+1} P_k(\underline{x})\right) = -(2s+2k+m)\, \underline{x}^{2s} P_k(\underline{x})$$

WRITE : $\partial_{\underline{x}}\left(\underline{x}^{s} P_k(\underline{x})\right) = B_{s,k}\, \underline{x}^{s-1} P_k(\underline{x})$

SPHERICAL HARMONICS

$S_k(\underline{x}) \in \mathcal{P}_k$ & $\Delta_{\underline{x}} S_k(\underline{x}) = 0$: space H_k

EXAMPLES : Let $P_k \in M_k$ & $P_{k-1} \in M_{k-1}$

$\hookrightarrow P_k(\underline{x})$, $\underline{x} P_{k-1}(\underline{x}) \in H_k$

HARMONIC (FISCHER) DEC. : $(\underline{x}^2 R, S) = -(R, \Delta_{\underline{x}} S)$?

$\hookrightarrow R_k(\underline{x}) = \sum_{2s \leq k} |\underline{x}|^{2s} S_{k-2s}(\underline{x})$

THEOREM. (REFINEMENT)

LET $S_k(\underline{x}) \in H_k \Rightarrow \exists!$ DEC $S_k(\underline{x}) = P_k(\underline{x}) + \underline{x} P_{k-1}(\underline{x})$

$$P_k(\underline{x}) = \frac{k+m-2-\Gamma_{\underline{x}}}{2k+m-2} S_k(\underline{x})$$

$$\underline{x} P_{k-1}(\underline{x}) = \frac{k+\Gamma_{\underline{x}}}{2k+m-2} S_k(\underline{x})$$

///

LAPLACE - BELTRAMI OPERATOR

$$\Delta_{\underline{x}} = -\partial_{\underline{x}}^2 = \partial_{\underline{x}}\, \underline{x}\ \underline{x}/|\underline{x}|^2\, \partial_{\underline{x}}$$

$$= \partial_r^2 + \frac{m-1}{r}\partial_r + \frac{1}{r^2}\tilde{\Delta}_{\underline{x}} \quad , \quad |\underline{x}| = r$$

$$\tilde{\Delta}_{\underline{x}} = \sum_{j<k} L_{jk}^2 = \Gamma_{\underline{x}}\,(m-2-\Gamma_{\underline{x}})$$

ZONAL SPHERICAL MONOGENICS

$$R_k(\underline{x}) = \frac{1}{k!}\langle \underline{x}, \partial_{\underline{u}}\rangle^k R_k(\underline{u})$$

$$= \left(\frac{1}{k!}\langle \underline{x}, \underline{u}\rangle^k , R_k(\underline{u})\right)$$

FISCHER:

$$\frac{1}{k!}\langle \underline{x}, \underline{u}\rangle^k = \sum_{s=0}^{k} \underline{x}^s Z_{k,s}(\underline{x}, \underline{u})\, \underline{u}^s$$

$$Z_{k,s}(\underline{x}, \underline{u}) \in \mathcal{P}_{k-s} \quad \& \quad \partial_{\underline{x}} Z_{k,s}(\underline{x}, \underline{u}) = Z_{k,s}(\underline{x}, \underline{u})\partial_{\underline{u}} = 0$$

IF $R_k(\underline{x}) = \sum_{s=0}^{k} \underline{x}^s P_{k-s}(\underline{x})$

THEN $P_{k-s}(\underline{x}) = Z_{k,s}(\underline{x}, \partial_{\underline{u}})\,\partial_{\underline{u}}^s R_k(\underline{u})$

$$= (-1)^s\left(\underline{u}^s Z_{k,s}(\underline{u}, \underline{x}), R_k(\underline{u})\right)$$

↳ Reproducing kernels.

CALCULATION

TAKING $\partial_{\underline{x}}$ – DERIVATIVE

↳ $$\frac{1}{(k-1)!}\langle \underline{x},\underline{u}\rangle^{k-1}\underline{u} = \sum_{s=1}^{k} B_{s,k-s}\,\underline{x}^{s-1} Z_{k,s}(\underline{x},\underline{u})\,\underline{u}^{s}$$

SO GET

$$Z_{k,s}(\underline{x},\underline{u}) = \frac{1}{B_{s,k-s}} Z_{k-1,s-1}(\underline{x},\underline{u})$$

$$= \frac{1}{B_{s,k-s}\cdots B_{1,k-s}} Z_{k-s,0}(\underline{x},\underline{u})$$

BASIC ZONAL MONOGENIC : $Z_k(\underline{x},\underline{u}) = Z_{k,0}(\underline{x},\underline{u})$

PUT

$$Z_k(\underline{x},\underline{u}) = (|\underline{x}||\underline{u}|)^k\left(A(t) + \frac{\underline{x}\wedge\underline{u}}{|\underline{x}\wedge\underline{u}|}B(t)\right)$$

$|\underline{x}||\underline{u}|$

$t = \cos\theta = \langle \underline{x},\underline{u}\rangle / |\underline{x}||\underline{u}|$

MONOGEN. ⟹ GEGENBAUER SYSTEM

$$kA + (1-t^2)\,\partial_t \overset{B}{A} - (m-1)tB = 0$$

$$kB + (-\partial_t \underset{A}{B}) + (m-2)\underset{B}{A} = 0$$

↳ $$Z_k(\underline{x},\underline{u}) = \frac{\Gamma(\frac{m}{2}-1)}{2^{k+1}\Gamma(k+\frac{m}{2})}(|\underline{x}||\underline{u}|)^k\Big[(k+m-2)\,C_k^{\frac{m}{2}-1}(t)$$

$$+ (m-2)\frac{\underline{x}\wedge\underline{u}}{|\underline{x}\wedge\underline{u}|}\,C_{k-1}^{\frac{m}{2}}(t)\Big]$$

$|\underline{x}||\underline{u}|$

V. SPINGROUPS & SPHERICAL MONOGENICS

POLYNOMIALS ON S^{m-1}

$\underline{x} \longrightarrow r\underline{\omega}$, $\underline{\omega} \in S^{m-1}$, $r = |\underline{x}|$

$R_k(\underline{x}) \in \mathcal{P}_k \Rightarrow R_k(\underline{x}) = r^k R_k(\underline{\omega})$

$\underline{x}^{2s} P_k(\underline{x}) \longrightarrow (-1)^s P_k(\underline{\omega})$: i.s.m.d. k

$\underline{x}^{2s+1} P_k(\underline{x}) \longrightarrow (-1)^s \underline{\omega} P_k(\underline{\omega})$: o.s.m.d. k

HARMONIC EXTENSIONS:

$S_k(\underline{\omega}) \longrightarrow S_k(\underline{x}) = r^k S_k(\underline{\omega})$

$\underline{\omega} P_{k-1}(\underline{\omega}) \longrightarrow \underline{x} P_{k-1}(\underline{x}) \in H_k$

MONOGENIC EXTENSIONS:

$P_k(\underline{\omega}) \longrightarrow r^k P_{k+}(\underline{\omega}) = P_k(\underline{x})$

$\underline{\omega} P_k(\underline{\omega}) \longrightarrow r^{-k-m+1} \underline{\omega} P_k(\underline{\omega}) = \dfrac{\underline{x}}{|\underline{x}|^m} P_k\left(\dfrac{\underline{x}}{|\underline{x}|^2}\right)$

INNER PRODUCT IN $L_2(S^{m-1})$

$$\langle f, g \rangle = \frac{1}{A_m} \int_{S^{m-1}} f^{+}(\underline{\omega})\, g(\underline{\omega})\, d\underline{\omega}$$

ORTHOGONAL DECOMPOSITIONS

$$f(\underline{\omega}) = \sum_{k=0}^{\infty} S_k[f](\underline{\omega})$$

$$f(\underline{\omega}) = \sum_{k=0}^{\infty} \left(P_k[f](\underline{\omega}) + Q_k[f](\underline{\omega})\right)$$

$$Q_k[f](\underline{\omega}) = -\underline{\omega}\, P_k[\underline{\omega}' f(\underline{\omega}')](\underline{\omega})$$

CAUCHY TRANSFORM

$$C[f](\underline{x}) = -\frac{1}{A_m} \int_{S^{m-1}} \frac{\underline{x}-\underline{\omega}}{|\underline{x}-\underline{\omega}|^m}\, \underline{\omega}\, f(\underline{\omega})\, d\underline{\omega}$$

IN L_2-SENSE

$$f(\underline{\omega}) = \lim_{\varepsilon \to 0} \left\{ C[f]((1+\varepsilon)\underline{\omega}) - C[f]((1-\varepsilon)\underline{\omega}) \right\}$$

$$= C^+[f](\underline{\omega}) + C^-[f](\underline{\omega})$$

NOW

$$C^+[f](\underline{\omega}) = \sum_{k=0}^{\infty} P_k[f](\underline{\omega}),$$

$$C^-[f](\underline{x}) = \sum_{k=0}^{\infty} Q_k[f](\underline{\omega}).$$

$\underline{x} = r\underline{\omega}'$

&

$$-\frac{\underline{x}-\underline{\omega}}{|\underline{x}-\underline{\omega}|^m} = \sum_{k=0}^{\infty} \frac{r^k}{m-2} \left[(k+m-2)\, C_k^{\frac{m}{2}-1}(t) + (m-2)\, \underline{\omega}' \wedge \underline{\omega}\, C_{k-1}^{\frac{m}{2}}(t)\right] \underline{\omega}$$

$$= \sum_{k=0}^{\infty} r^k \left[C_k^{\frac{m}{2}}(t) + \underline{\omega}'\underline{\omega}\, C_{k-1}^{\frac{m}{2}}(t)\right] \underline{\omega}$$

MONOGENIC PROJECTIONS

$$P_k[f](\underline{\omega}') = \frac{1}{A_m} \int_{S^{m-1}} \left[C_k^{\frac{m}{2}}(t) + \underline{\omega}'\underline{\omega}\, C_{k-1}^{\frac{m}{2}}(t) \right] f(\underline{\omega})\, d\underline{\omega}$$

$$= \frac{1}{A_m(m-2)} \int_{S^{m-1}} \left[(k+m-2)\, C_k^{\frac{m}{2}-1}(t) + (m-2)\, \underline{\omega}' \wedge \underline{\omega}\, C_{k-1}^{\frac{m}{2}}(t) \right] f(\underline{\omega})\, d\underline{\omega}$$

$$Q_k[f](\underline{\omega}') = -\frac{1}{A_m} \int_{S^{m-1}} \left[C_{k-1}^{\frac{m}{2}}(t) + \underline{\omega}'\underline{\omega}\, C_k^{\frac{m}{2}}(t) \right] f(\underline{\omega})\, d\underline{\omega}$$

$$= \frac{1}{A_m(m-2)} \int_{S^{m-1}} \left[(k+1)\, C_{k+1}^{\frac{m}{2}-1}(t) - (m-2)\, \underline{\omega}' \wedge \underline{\omega}\, C_k^{\frac{m}{2}}(t) \right] f(\underline{\omega})\, d\underline{\omega}$$

SZEGÖ KERNEL

PUT NOW: $C_{m,k}(\underline{x},\underline{u}) = (|\underline{x}||\underline{u}|)^k \left[C_k^{\frac{m}{2}}(t) + \frac{\underline{x}\,\underline{u}}{|\underline{x}||\underline{u}|} C_{k-1}^{\frac{m}{2}}(t) \right]$

↳ $$S(\underline{u},\underline{x}) = \frac{1+\underline{x}\,\underline{u}}{|1+\underline{x}\,\underline{u}|^m} = \sum_{k=0}^{\infty} C_{m,k}(\underline{x},\underline{u})$$

ALSO

$$C_{m,k}(\underline{x},\underline{u}) = \frac{2^k\, \Gamma\left(k+\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2}\right)} Z_k(\underline{x},\underline{u})$$

Fischer

NOW

$$P_k(\underline{x}) = \left(Z_k(\underline{x},\underline{u}), P_k(\underline{u}) \right) = \langle C_k(\underline{\omega},\underline{x}), P_k(\underline{\omega}) \rangle$$

↳ $$(P_{k'}, P_k) = \frac{2^k\, \Gamma\left(k+\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2}\right)} \langle P_{k'}, P_k \rangle$$

FISCHER I.P. HARMONIC POL.

$$(S'(\underline{x}), S(\underline{x})) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} S'^{+}(\underline{x}) S(\underline{x}) e^{-|\underline{x}|^2/2} \, d\underline{x}$$

Spin(m)-REPRESENTATIONS.

$H_0(s) f(\underline{x}) = f(\bar{s}\,\underline{x}\,s)$; $s \in Spin(m)$

$\hookrightarrow \Delta_{\underline{x}} H_0(s) = H_0(s) \Delta_{\underline{x}}$ (Spin 0 repr.)

$L(s) f(\underline{x}) = s f(\bar{s}\,\underline{x}\,s)$: Spin 1/2 - repr.

$H_1(s) f(\underline{x}) = s f(\bar{s}\,\underline{x}\,s)\bar{s}$: Spin 1 - repr.

$\partial_{\underline{x}} L(s) = L(s) \partial_{\underline{x}}$; $\partial_{\underline{x}} H_1(s) = H_1(s) \partial_{\underline{x}}$

INFINITESIMAL : $s = 1 + \varepsilon e_{ij}$

$$dR(e_{ij}) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(R(1+\varepsilon e_{ij}) - 1 \right)$$

$dH_0(e_{ij}) = -2L_{ij}$; $L_{ij} = x_i \partial_{x_j} - x_j \partial_{x_i}$

$dL(e_{ij}) = -2L_{ij} + e_{ij}$; $dH_1(e_{ij}) = -2L_{ij} + [e_{ij}, \cdot]$

CASIMIR OPERATOR

$C(R) = \frac{1}{4} \sum_{i<j} dR(e_{ij})^2$; $C(H_0) = \tilde{\Delta}_{\underline{x}} = \Gamma_{\underline{x}}(m-2-\Gamma_{\underline{x}})$

$C(L) = \tilde{\Delta}_{\underline{x}} + \Gamma_{\underline{x}} - \frac{1}{4}\binom{m}{2}$; $C(H_1) = \tilde{\Delta}_{\underline{x}} + [\Gamma_{\underline{x}}, \cdot] - \frac{1}{4}\tilde{\sigma}$

VI. VEKUA SYSTEMS, FUETER THEOREMS

1. AXIAL SYMMETRY

OPERATOR : $\partial_{x_0} + \partial_{\underline{x}} = D_x$

P. LOUNESTO, P. BERGH

$$f(x_0, \underline{x}) = A(x_0, \rho) + \underline{\omega} B(x_0, \rho) \quad , \rho = |\underline{x}|$$

$$\left(\partial_{x_0} + \underline{\omega}\,\partial_\rho + \frac{1}{\rho}\underline{\omega}\,\Gamma_{\underline{\omega}}\right)(A + \underline{\omega} B) = 0$$

$$\hookrightarrow \left|\begin{array}{l} \partial_{x_0} A - \partial_\rho B = \dfrac{m-1}{\rho} B \\ \partial_{x_0} B + \partial_\rho A = 0 \end{array}\right.$$

IN GENERAL

$$f(x_0, \underline{x}) = \left(A(x_0, \rho) + \underline{\omega} B(x_0, \rho)\right) P_k(\underline{\omega})$$

$$\left(\partial_{x_0} + \underline{\omega}\,\partial_\rho + \frac{1}{\rho}\underline{\omega}\,\Gamma_{\underline{\omega}}\right)\left[(A + \underline{\omega} B) P_k(\underline{\omega})\right] = 0$$

$$\hookrightarrow \left|\begin{array}{l} \partial_{x_0} A - \partial_\rho B = \dfrac{k+m-1}{\rho} B \\ \partial_{x_0} B + \partial_\rho A = \dfrac{k}{\rho} A \end{array}\right.$$

CLASSICAL VEKUA SYSTEMS (G. JANK, S. RUSCHEWEIH)

AXIAL EXPONENTIALS

FACTOR e^{x_0}

$$\mathcal{E}_{k,m}(x_0,\underline{x})P_k(\underline{\omega}) = e^{x_0}\,\mathcal{E}_{k,m}(\underline{x})\,P_k(\underline{\omega}) \quad ; \quad \underline{x} = \rho\,\underline{\omega}$$

$$\mathcal{E}_{k,m}(\underline{x}) = 2^{k+\frac{m}{2}-1}\,\Gamma\left(k+\frac{m}{2}\right)\rho^{1-\frac{m}{2}}\left[J_{k+\frac{m}{2}-1}(\rho) + \underline{\omega}\,J_{k+\frac{m}{2}}(\rho)\right]$$

AXIAL HERMITE POLYNOMIALS

ASSUME : $A(0,\rho) = \rho^k e^{-\rho^2/2}$, $B(0,\rho) = 0$

$$CK\left(e^{-|\underline{x}|^2/2}P_k(\underline{x})\right) = \sum_{\ell=0}^{\infty} \frac{(-x_0)^\ell}{\ell!}\, H_{\ell,m,k}(\underline{x})\, e^{-\frac{|\underline{x}|^2}{2}} P_k(\underline{x})$$

$$H_{\ell,m,k}(\underline{x}) = H_{\ell,m+2k}(\underline{x})$$

FUETER-SCE THEOREM

$m = \text{ODD}$

$$\Box = \partial_{x_0}^2 + \Delta_{\underline{x}} = D_x \overline{D}_x$$

IF $f(x+iy) = g + ih$ is HOLOMORPHIC

PUT $f(x_0+\underline{x}) = g(x_0,\rho) + \underline{\omega}\,h(x_0,\rho)$

THEN : $\Box_x^{\frac{m-1}{2}} f(x_0+\underline{x})$ is MONOGENIC

$m = \text{EVEN}$: TAO QIAN

GENERALIZATION (F. SOMMEN): $m = \text{ODD}$

SAME CONDITION ON f:

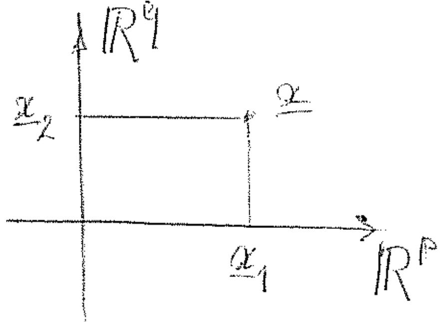
$\hookrightarrow \square^{k+\frac{m-1}{2}}\left(f(x_0+\underline{x})P_k(\underline{x})\right)$ is MONOGENIC

$m = \text{EVEN}$: K.I. KOU - T. QIAN - F. SOMMEN

2. BIAXIAL SYMMETRY

G. JANK, F. SOMMEN, A. COMMON

IDEA: $\underline{x} = \underline{x}_1 + \underline{x}_2 = \rho_1 \underline{\omega}_1 + \rho_2 \underline{\omega}_2$



$p + q = m$

$$\partial_{\underline{x}_1} = \sum_{j=1}^{p} e_j \partial_{x_j} \quad ; \quad \partial_{\underline{x}_2} = \sum_{j=p+1}^{p+q} e_j \partial_{x_j}$$

SOLUTION TYPE

$$F(\underline{x}) = \left\{A(\rho_1,\rho_2) + \underline{\omega}_1 B(\rho_1,\rho_2) + \underline{\omega}_2 C(\rho_1,\rho_2) + \underline{\omega}_1\underline{\omega}_2 D(\rho_1,\rho_2)\right\} P_{k,\ell}(\underline{\omega}_1, \underline{\omega}_2)$$

$$P_{k,\ell}(\lambda \underline{x}_1, \underline{x}_2) = \lambda^k P_{k,\ell}(\underline{x}_1, \underline{x}_2) \; ; \; P_{k,\ell}(\underline{x}_1, \lambda \underline{x}_2) = \lambda^\ell P_{k,\ell}(\underline{x}_1, \underline{x}_2)$$

$$\& \quad \partial_{\underline{x}_1} P_{k,\ell}(\underline{x}_1, \underline{x}_2) = \partial_{\underline{x}_2} P_{k,\ell}(\underline{x}_1, \underline{x}_2) = 0$$

EXAMPLE : $P_{k,l}(\underline{x}_1, \underline{x}_2) = P_k(\underline{x}_1)\, P_l(\underline{x}_2)$

WITH $P_k : \mathbb{R}^p \to \mathbb{R}_p$: s.m.d. k

$P_l : \mathbb{R}^q \to \mathbb{R}_q$: s.m.d. l

$$\partial_{\underline{x}} = \partial_{\underline{x}_1} + \partial_{\underline{x}_2} = \underline{\omega}_1\left(\partial_{\rho_1} + \frac{1}{\rho_1}\Gamma_{\underline{\omega}_1}\right) + \underline{\omega}_2\left(\partial_{\rho_2} + \frac{1}{\rho_2}\Gamma_{\underline{\omega}_2}\right)$$

$\partial_{\underline{x}} F(\underline{x}) = 0$

$$\Rightarrow \left|\begin{array}{l} \left(\partial_{\rho_1} - \frac{k}{\rho_1}\right)A + \left(\partial_{\rho_2} + \frac{l+q-1}{\rho_2}\right)D = 0 \\ \left(\partial_{\rho_1} + \frac{k+p-1}{\rho_1}\right)D - \left(\partial_{\rho_2} - \frac{l}{\rho_2}\right)A = 0 \end{array}\right.$$

$$\left|\begin{array}{l} \left(\partial_{\rho_1} + \frac{k+p-1}{\rho_1}\right)B + \left(\partial_{\rho_2} + \frac{l+q-1}{\rho_2}\right)C = 0 \\ \left(\partial_{\rho_1} - \frac{k}{\rho_1}\right)C - \left(\partial_{\rho_2} - \frac{l}{\rho_2}\right)B = 0 \end{array}\right.$$

ORTHONORMAL BASIS M_k

$$\left(\sum_{j=0}^{k-l-n} \Theta_{j,k,l,n}\, \underline{x}_1^{\,j}\, \underline{x}_2^{\,k-l-n-j}\right) P_l(\underline{x}_1)\, P_n(\underline{x}_2)$$

$\hookrightarrow$ REAL CONSTANTS

G. JANK, F. SOMMEN : JACOBI POLYN. (pure MAGIC)

J. CNOPS : generalised GEGENBAUER

PUTTING

$$S_{k,l,n}(\underline{x}_1,\underline{x}_2)=\sum_{j=0}^{k-l-n}\Theta_{jkln}\,\underline{x}_1^{\,j}\,\underline{x}_2^{\,k-l-n-j}$$

WE HAVE THAT

(i) FOR $k=l+n+2s$

$$S_{k,l,n}(\underline{x}_1,\underline{x}_2)=(\underline{x}_1^2+\underline{x}_2^2)^{s-1}\Big\{(\underline{x}_1^2+\underline{x}_2^2)\,P_s^{n+\frac{q}{2}-1,\,l+\frac{p}{2}-1}\Big(\frac{\underline{x}_1^2-\underline{x}_2^2}{\underline{x}_1^2+\underline{x}_2^2}\Big)$$

$$-\underline{x}_1\underline{x}_2\,P_{s-1}^{n+\frac{q}{2},\,l+\frac{p}{2}}\Big(\frac{\underline{x}_1^2-\underline{x}_2^2}{\underline{x}_1^2+\underline{x}_2^2}\Big)\Big\}$$

(ii) FOR $k=l+n+2s+1$

$$S_{k,l,n}(\underline{x}_1,\underline{x}_2)=(\underline{x}_1^2+\underline{x}_2^2)^{s}\Big\{(n-l-p+1-k)\,\underline{x}_2\,P_s^{n+\frac{q}{2}-1,\,l+\frac{p}{2}}\Big(\frac{\underline{x}_1^2-\underline{x}_2^2}{\underline{x}_1^2+\underline{x}_2^2}\Big)$$

$$-(l-n-q+1-k)\,\underline{x}_1\,P_s^{n+\frac{q}{2},\,l+\frac{p}{2}-1}\Big(\frac{\underline{x}_1^2-\underline{x}_2^2}{\underline{x}_1^2+\underline{x}_2^2}\Big)\Big\}$$

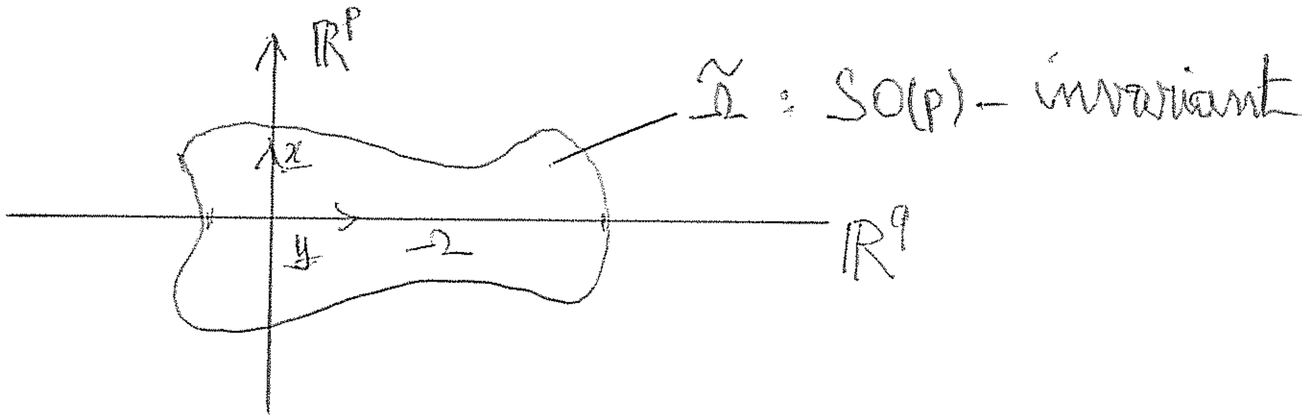
RECURSIVELY : $\mathbb{R}^p=\mathbb{R}^{p_1}\oplus\mathbb{R}^{p_2}$; $\mathbb{R}^q=\mathbb{R}^{q_1}\oplus\mathbb{R}^{q_2}$

$$P_l(\underline{x}_1)=\Big(\sum_{\dots}\Theta_{\dots}\,\underline{x}_{11}^{\dots}\,\underline{x}_{12}^{\dots}\Big)P_{\dots}(\underline{x}_{11})\,P_{\dots}(\underline{x}_{12})$$

$P_n(\underline{x}_2)$ = similar

$\hookrightarrow$ ORTHOGONAL BASIS i.t.o. JACOBI POLYN.

3. GENERALIZED CK-EXTENSION THM.



$P_k(\underline{x})$: fixed s.m.h. k in $\mathbb{R}^p$ ($\mathbb{R}_p^+$ - valued)

$A_0(\underline{y})$: fixed real anal. fct. in Ω ($\mathbb{R}_q$ - valued)

THEN: $\exists$ UNIQUE $A_s(\underline{y})$: real anal. such that

$$f(\underline{x}, \underline{y}) = \sum_{s=0}^{\infty} \underline{x}^s P_k(\underline{x}) A_s(\underline{y})$$

is monogenic, i.e. $(\partial_{\underline{x}} + \partial_{\underline{y}}) f(\underline{x}, \underline{y}) = 0$.

MOREOVER : $P_k(\underline{\xi}) A_0(\underline{y}) = \lim_{\rho \to 0} \frac{1}{\rho^k} f(\underline{x}, \underline{y})$; $\rho = |\underline{x}|$

EXPLICITLY

$$f(\underline{x}, \underline{y}) = \Gamma\left(k + \frac{p}{2}\right) \left(\frac{\rho\sqrt{\Delta_{\underline{y}}}}{2}\right)^{-(k+\frac{p}{2})} \times$$

$$\left\{ \frac{\rho\sqrt{\Delta_{\underline{y}}}}{2} J_{k+\frac{p}{2}-1}\left(\rho\sqrt{\Delta_{\underline{y}}}\right) + \frac{\underline{x}\,\partial_{\underline{y}}}{2} J_{k+\frac{p}{2}}\left(\rho\sqrt{\Delta_{\underline{y}}}\right) \right\} P_k(\underline{x}) A_0(\underline{y}$$

CONVERGENCE IN SOME SO(p)- INVARIANT OPEN SET $\tilde{\Omega}$ WITH $\Omega = \tilde{\Omega} \cap \mathbb{R}^q$.

4. MORE CK-EXTENSIONS

GENERAL $\quad f(x_0, \underline{x}) = \exp(-x_0 \partial_{\underline{x}}) f(\underline{x})$

$$f(\underline{x}) = (A(\rho) + \underline{\omega} B(\rho)) P_k(\underline{\omega}) \quad ; \quad \underline{x} = \rho \underline{\omega}$$

EXTENSION : $(A(x_0,\rho) + \underline{\omega} B(x_0,\rho)) P_k(\underline{\omega})$

$$A(x_0,\rho) = \sum_{l=0}^{\infty} x_0^l A_l(y) \ ; \ B(x_0,\rho) = \sum_{l=0}^{\infty} x_0^l B_l(y)$$

FROM VEKUA:

$$\begin{cases} A_{l+1} = \dfrac{1}{l+1}\left(B_l' + \dfrac{k+m-1}{\rho} B_l\right) \\ B_{l+1} = \dfrac{1}{l+1}\left(-A_l' + \dfrac{k}{\rho} A_l\right) \end{cases}$$

CK-EXTENSION OF $\underline{x}^s P_k(\underline{x})$: $X_k^s(x_0, \underline{x}) P_k(\underline{x})$

$$X_k^s(x_0,\underline{x}) = \lambda_k^s\, r^s \left(C_s^{\frac{m-1}{2}+k}\left(\frac{x_0}{r}\right) + \frac{2k+m-1}{s+2k+m-1}\, C_{s-1}^{\frac{m+1}{2}+k}\left(\frac{x_0}{r}\right) \frac{\underline{x}}{r} \right)$$

WITH $\quad r^2 = x_0^2 + |\underline{x}|^2$

$$\lambda_k^{2l} = \frac{l!\, \Gamma\left(k + \frac{m-1}{2}\right)}{\Gamma\left(l+k+\frac{m-1}{2}\right)} \quad \& \quad \lambda_k^{2l+1} = \frac{2l+2k+m}{2k+m-1}\, \frac{l!\, \Gamma\left(k+\frac{m+1}{2}\right)}{\Gamma\left(l+k+\frac{m+1}{2}\right)}$$

INNER POWER FUNCTIONS

$$A(0,\rho) = \rho^{\alpha}\ , \ B(0,\rho) = 0$$

EXTENSION :

$$P_{\alpha,k,m}(x_0,\underline{x})\,P_k(\underline{\omega}) = \left(A(x_0,\rho) + \underline{\omega}\,B(x_0,\rho)\right)P_k(\underline{\omega})$$

$$A(x_0,\rho) = \rho^{\alpha}\,F\left(1-\frac{k+m+\alpha}{2}\ ;\ \frac{k-\alpha}{2}\ ;\ \frac{1}{2}\ ;\ -\frac{x_0^2}{\rho^2}\right)$$

$$B(x_0,\rho) = (-k-\alpha)\,x_0\,\rho^{\alpha-1}\,F\left(1-\frac{k+m+\alpha}{2}\ ,\ \frac{k-\alpha}{2}\ ;\ \frac{3}{2}\ ;\ -\frac{x_0^2}{\rho^2}\right)$$

OUTER POWER FUNCTIONS

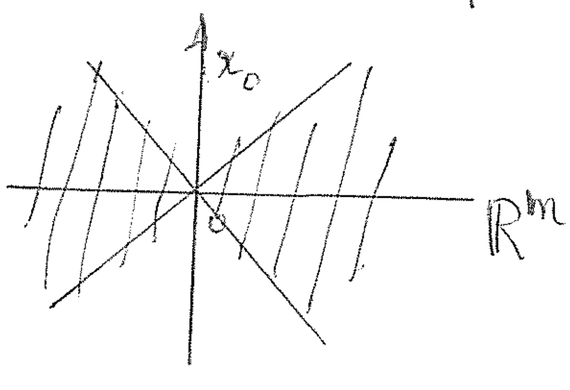
$$A(0,\rho) = 0\ , \ B(0,\rho) = \rho^{\alpha}$$

$$q_{\alpha,k,m}(x_0,\underline{x})\,P_k(\underline{\omega}) = \left(A(x_0,\rho) + \underline{\omega}\,B(x_0,\rho)\right)P_k(\underline{\omega})$$

$$B(x_0,\rho) = \rho^{\alpha}\,F\left(\frac{k-\alpha+1}{2}\ ;\ \frac{1-\alpha-k-m}{2}\ ;\ \frac{1}{2}\ ;\ -\frac{x_0^2}{\rho^2}\right)$$

$$A(x_0,\rho) = (\alpha+k+m-1)\,x_0\,\rho^{\alpha-1}\,F\left(\frac{k-\alpha+1}{2}\ ;\ \frac{1-\alpha-k-m}{2}\ ;\ \frac{3}{2}\ ;\ -\frac{x_0^2}{\rho^2}\right)$$

CONVERGENCE : $|x_0| < \rho$



Axial Exponential Functions

$$\mathcal{E}_k(x_0;\underline{x})\,P_k(\underline{x}) = \big(A(x_0,\rho)+\underline{\omega}\,B(x_0,\rho)\big)\,P_k(\underline{\omega})$$

$$A = e^{x_0}\,a(\rho) \quad ; \quad B = e^{x_0}\,b(\rho) \quad ; \quad \nu = k+\frac{m}{2}$$

$$\mathcal{E}_k(x_0;\underline{x}) = e^{x_0}\,\Gamma(\nu)\sum_{l=0}^{\infty}\frac{(\underline{x}/2)^{2l}}{l!\,\Gamma(\nu+l)}\left(1+\frac{\underline{x}}{2(\nu+l)}\right)$$

$$= e^{x_0}\,2^{\nu-1}\,\Gamma(\nu)\,\rho^{1-\nu}\big(J_{\nu-1}(\rho)+\underline{\omega}\,J_\nu(\rho)\big)$$

Cauchy Type Integrals

Consider $(x_0,\underline{x}) \longrightarrow P_k(\underline{x})$ & $r^2 = x_0^2+\rho^2$

Inversion: $f(x_0,\underline{x}) \longrightarrow \dfrac{x_0-\underline{x}}{|x_0-\underline{x}|^{m+1}}\, f\!\left(\dfrac{x_0}{r^2},\dfrac{\underline{x}}{r^2}\right)$

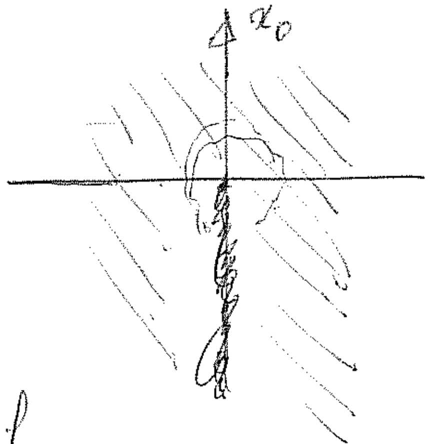
Apply: $\dfrac{x_0-\underline{x}}{|x_0+\underline{x}|^{2k+m+1}}\,P_k(\underline{x})$: Generalized C. Kernel

$f(t)$ function on $\mathbb{R}^+ =]0,+\infty[$

$$\Lambda_k^{\pm} f(x_0,\underline{x}) = \frac{\pm 1}{A_{m+1}}\int_0^{+\infty}\frac{(x_0\pm t)\mp\underline{x}}{|x_0\pm t+\underline{x}|^{2k+m+1}}\,f(t)\,dt$$

$\Lambda_k^{\pm} f(x_0,\underline{x})\,P_k(\underline{x})$ monogenic for $x_0+\underline{x} \in \mathbb{R}^{m+1}\setminus\mathbb{R}^{\mp}$

$\Lambda_k^+ f\, P_k$



$$\Lambda_k^{\pm} f' = \mp\, \partial_{x_0} \Lambda_k^{\pm} f$$

SPECIAL CASE : $f(t) = t^{\alpha}$, $-1 < \mathrm{Re}\,\alpha < 2k+m-1$

$$\Lambda^{\pm}_{\alpha,k}(x_0,\underline{x}) = \frac{\pm 1}{A_{m+1}} \int_0^{+\infty} \frac{(x_0 \pm t - \underline{x})\, t^{\alpha}}{|x_0 \pm t - \underline{x}|^{2k+m+1}}\, dt$$

$$\Lambda^{\pm}_{\alpha,k}(x_0,\underline{x})P_k(\underline{x}) = \pm \frac{1}{\alpha+1}\, \partial_{\underline{x}}\left(\Lambda^{\pm}_{\alpha+1,k}(x)P_k(\underline{x})\right)$$

$\hookrightarrow$ EXTENSION TO $\mathrm{Re}\,\alpha < 2k+m-1$ & $\alpha \notin \{-1,-2,\dots\}$

BOUNDARY VALUES $x_0 \xrightarrow{>} 0$ & $-1 < \mathrm{Re}\,\alpha < 2k+m-1$

$$\Lambda^{\pm}_{\alpha,k}(\underline{x}) = |\underline{x}|^{\alpha-m+1-2k}\left(\sigma_{m,k}(\alpha+1) \mp \underline{\omega}\, \sigma_{m,k}(\alpha)\right)$$

$$\sigma_{m,k}(\alpha) = \frac{1}{2A_{m+1}}\, B\left(\frac{2k+m-\alpha}{2}, \frac{\alpha+1}{2}\right) = \sigma_{m,k}(2k+m-1-\alpha)$$

$\Lambda^{\pm}_{\alpha,k}(\underline{x})P_k(\underline{x}) \longrightarrow$ Generalised Riesz potentials

(see also : F. BRACKX, B. DE KNOCK
H. DE SCHEPPER, D. EELBODE

CONCLUSION : POWERFUNCTIONS IN $\mathbb{R}^{m+1} \setminus \mathbb{R}^{\mp}$

VII. PLANE WAVE DECOMPOSITIONS

FRITZ JOHN : PLANE WAVES AND SPHERICAL MEANS

$$g(\langle \underline{x}, \underline{t}\rangle), \ \underline{t} \in \mathbb{R}^m$$

PLANE WAVE INTEGRAL : $\displaystyle\int_{S^{m-1}} g(\langle \underline{x}, \underline{\omega}\rangle)\, W(\underline{\omega})\, d\underline{\omega}$

TAKE $W(\underline{\omega}) = S_k(\underline{\omega})$ & $\Delta_{\underline{x}} S_k(\underline{x}) = 0$, $\underline{x} = \rho\underline{\omega}$

FUNK-HECKE THEOREM

$$\int_{S^{m-1}} g(\langle \underline{x}, \underline{\omega}\rangle)\, S_k(\underline{\omega})\, d\underline{\omega} = A_{m-1} \underbrace{I_k g(\rho)}_{\int_{-1}^{1} g(\rho s)(1-s^2)^{\frac{m-3}{2}} P_{k,m}(s)\, ds} S_k(\underline{\xi}), \quad \underline{x} = \rho\underline{\xi}$$

MONOGENIC PLANE WAVES

$$G(x_0, \underline{x}; \underline{t}) = g_1(\langle \underline{x}, \underline{t}\rangle, x_0|\underline{t}|) - \frac{\underline{t}}{|\underline{t}|}\, g_2(\langle \underline{x}, \underline{t}\rangle, x_0|\underline{t}|)$$

$$(\partial_{x_0} + \partial_{\underline{x}})\, G(x_0, \underline{x}; \underline{t}) = 0$$

$$\Longleftrightarrow \quad \partial_x g_1 - \partial_y g_2 = 0, \quad \partial_x g_2 + \partial_y g_1 = 0$$

$$\int_{S^{m-1}} G(x_0, \underline{x}; \underline{\omega})\, P_k(\underline{\omega})\, d\underline{\omega} = \int_{S^{m-1}} \left(g_1(\langle \underline{x}, \underline{\omega}\rangle, x_0) - \underline{\omega}\, g_2(\langle \underline{x}, \underline{\omega}\rangle, x_0)\right) P_k(\underline{\omega})\, d\underline{\omega}$$

$$= A_{m-1}\left(\underbrace{I_k g_1(\rho, x_0)}_{\int_{-1}^{1} g_1(\rho s, x_0)\, P_{k,m}(s)(1-s^2)^{\frac{m-3}{2}} ds} - \underline{\xi}\, I_{k+1} g_2(\rho, x_0)\right) P_k(\underline{\xi}) \quad \text{VEKUA}$$

EXPONENTIAL PLANE WAVE INTEGRAL

$$\mathrm{Exp}(x_0, \underline{x}; \underline{t}) = e^{i\langle \underline{x}, \underline{t}\rangle}\left(\mathrm{ch}\, x_0|\underline{t}| - i\frac{\underline{t}}{|\underline{t}|}\,\mathrm{sh}\, x_0|\underline{t}|\right)$$

$$= \mathcal{E}_+(x_0, \underline{x}; \underline{t}) + \mathcal{E}_-(x_0, \underline{x}; \underline{t})$$

$$\mathcal{E}_\pm(x_0, \underline{x}; \underline{t}) = \frac{1}{2}\left(1 \pm i\frac{\underline{t}}{|\underline{t}|}\right) e^{i\langle \underline{x}, \underline{t}\rangle \mp x_0|\underline{t}|}$$

$$\int_{S^{m-1}} \mathcal{E}_+(x_0, \underline{x}, \underline{\omega}) P_k(\underline{\omega})\, d\underline{\omega}$$

$$= \frac{e^{-x_0}}{2} A_{m-1}\left(I_k(e^{i\cdot})(\rho) + i\underline{\xi}\, I_{k+1}(e^{i\cdot})(\rho)\right) P_k(\underline{\xi})$$

$$A_{m-1} I_k(e^{i\cdot})(\rho) = (2\pi)^{\frac{m}{2}} i^k \rho^{1-\frac{m}{2}} J_{k+\frac{m}{2}-1}(\rho)$$

$$= C_k\, \mathcal{E}_k(-x_0, -\underline{x})\, P_k(\underline{x})$$

$$\llcorner \frac{\pi^{k+\frac{m}{2}}}{\Gamma(k+\frac{m}{2})} \frac{1}{(-2\pi i)^k}$$

PLANE GENERALIZED POWERS

$\alpha \in \mathbb{C}$: multivalued function

$$z^\alpha = p^\alpha(z) = |z|^\alpha\left(\cos(\alpha \arg z) + i\sin(\alpha \arg z)\right)$$

not very useful.

FOR $\theta \in \mathbb{R}$: $\arg_\theta z$: BRANCH of $\arg z$ WITH

$$\arg_\theta z \in [\theta - \pi, \theta + \pi[$$

PUT ALSO $\quad \arg_\theta(\underline{x}, \underline{t}) = \arg_\theta(\langle \underline{x}, \underline{t}\rangle + i x_0 |\underline{t}|)$

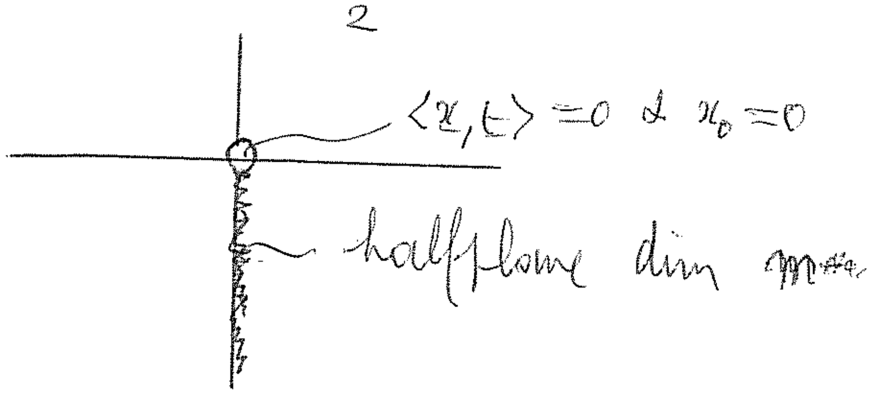
$\hookrightarrow$ FOR $\arg_\theta(\underline{x}, \underline{t}) \neq \theta - \pi$: MONOGENIC PLANE WAVE

$$P_\theta^\alpha(x_0, \underline{x}; \underline{t})$$

$$= \left(\langle \underline{x}, \underline{t}\rangle^2 + x_0^2 |\underline{t}|^2\right)^{\alpha/2} \left\{ \cos(\alpha \arg_\theta(\underline{x}, \underline{t})) - \frac{\underline{t}}{|\underline{t}|} \sin(\alpha \arg_\theta(\underline{x}, \underline{t})) \right\}$$

$$P_\theta^k(x_0, \underline{x}; \underline{t}) = P_k(\underline{x}, \underline{t}) = (\langle \underline{x}, \underline{t}\rangle - x_0 \underline{t})^k, \ k \in \mathbb{Z}.$$

INTERESTING : $\theta = \pm \frac{\pi}{2}$



$$\mathcal{C}_\alpha^\pm(P_k)(x_0, \underline{x}) = \int_{S^{m-1}} P^\alpha_{\pm\frac{\pi}{2}}(x_0, \underline{x}; \underline{\omega}) P_k(\underline{\omega}) \, d\underline{\omega}$$

$$\mathcal{S}_\alpha^\pm(P_k)(x_0, \underline{x}) = \int_{S^{m-1}} P^\alpha_{\pm\frac{\pi}{2}}(x_0, \underline{x}; \underline{\omega}) \, \underline{\omega} \, P_k(\underline{\omega}) \, d\underline{\omega}$$

FUNK-HECKE FOR $x_0 = 0$

$$P^{\alpha}_{\pm\frac{\pi}{2}}(\langle \underline{x}, \underline{t}\rangle) = |\langle \underline{x}, \underline{t}\rangle|^{\alpha}\Big(\cos\alpha\pi\, Y(-\langle \underline{x}, \underline{t}\rangle)$$

$$\mp \frac{\underline{t}}{|\underline{t}|}\sin\alpha\pi\, Y(-\langle \underline{x}, \underline{t}\rangle)\Big)$$

$\Rightarrow$

$$C^{\pm}_{\alpha}(P_k)(\underline{x}) = A_{m-1}|\underline{x}|^{\alpha}\big(C_k(\alpha) \mp S_k(\alpha)\underline{\xi}\big)P_k(\underline{\xi})$$

$$S^{\pm}_{\alpha}(P_k)(\underline{x}) = A_{m-1}|\underline{x}|^{\alpha}\big(C_{k+1}(\alpha)\underline{\xi} \pm S_{k-1}(\alpha)\big)P_k(\underline{\xi})$$

$$C_k(\alpha) = \frac{1+\cos(\alpha-k)\pi}{2^{k+1}}\ \frac{\Gamma(\alpha+1)\,\Gamma\left(\frac{\alpha-k+1}{2}\right)\Gamma\left(\frac{m-1}{2}\right)}{\Gamma(\alpha-k+1)\,\Gamma\left(\frac{\alpha+k+m}{2}\right)}$$

$$S_k(\alpha) = \frac{\sin(k-\alpha)\pi}{2^{k+1}}\ \frac{\Gamma(\alpha+1)\,\Gamma\left(\frac{\alpha-k}{2}+1\right)\Gamma\left(\frac{m-1}{2}\right)}{\Gamma(\alpha-k+1)\,\Gamma\left(\frac{\alpha+k+m+1}{2}\right)}$$

THEOREM.

$$C^{\pm}_{\alpha}(P_k)(x_0, \underline{x}) = \frac{A_{m-1}\,C_k(\alpha)}{\sigma_{m,k}(\alpha+k+m)}\ \Lambda^{\pm}_{\alpha+k+m-1,k,m}(x)\,P_k(\underline{x})$$

WITH
$$\Lambda^{\pm}_{\beta,k,m}(x_0, \underline{x}) = \frac{\pm 1}{A_{m+1}}\int_0^{+\infty}\frac{(x_0 \pm t - \underline{x})\, t^{\beta}\, dt}{|x_0 \pm t - \underline{x}|^{2k+m+1}}$$

$$S^{\pm}_{\alpha}(P_k)(x) = \frac{1}{\alpha+1}\,\partial_{\underline{x}}\, C^{\pm}_{\alpha+1}(P_k)(x)$$

RADON-DECOMPOSITION OF CAUCHY KERNEL

(i) FOR m EVEN & $x_0 \gtrless 0$

$$\frac{1}{A_{m+1}} \frac{x_0 - \underline{x}}{|x_0 - \underline{x}|^{m+1}} = \pm \frac{(-1)^{m/2}(m-1)!}{2(2\pi)^m} \int_{S^{m-1}} \left(\langle \underline{x}, \underline{\omega}\rangle - x_0 \underline{\omega}\right)^{-m} d\underline{\omega}$$

(ii) FOR m ODD

$$\frac{1}{A_{m+1}} \frac{x_0 - \underline{x}}{|x_0 - \underline{x}|^{m+1}} = \frac{(-1)^{\frac{m+1}{2}}(m-1)!}{2(2\pi)^m} \int_{S^{m-1}} \left(\langle \underline{x}, \underline{\omega}\rangle - x_0 \underline{\omega}\right)^{-m} \underline{\omega}\, d\underline{\omega}$$

PLANE WAVES IN GENERAL POSITION

IN $\mathbb{R}^m$: VARIABLE $\underline{x}$, Op. $\partial_{\underline{x}}$

CONSIDER VECTORS , $\underline{t}$, $\underline{s}$ with $|\underline{t}| = |\underline{s}|$ & $\underline{t}\,\underline{s} = -\underline{s}\,\underline{t}$

Put

$$G(\underline{x}, \underline{t}, \underline{s}) = f_1(\underline{z}) - \frac{\underline{t}\,\underline{s}}{|\underline{t}||\underline{s}|} f_2(\underline{z})$$

$$F(\underline{x}, \underline{t}, \underline{s}) = \underline{s} f_1(\underline{z}) + \underline{t} f_2(\underline{z})$$

with $(\underline{z}) = (\langle \underline{x}, \underline{t}\rangle, \langle \underline{x}, \underline{s}\rangle)$ OR SIMPLY

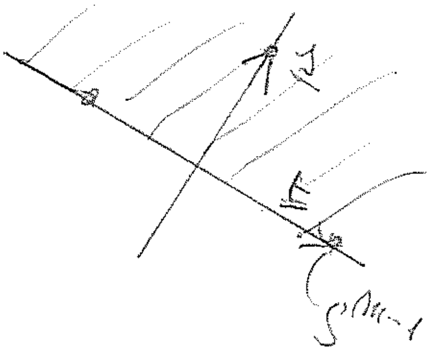
$$F(\underline{x}, \underline{\tau}) = \underline{\tau}\, f(\langle \underline{x}, \underline{\tau}\rangle) \; ; \; \underline{\tau} = \underline{t} + i\underline{s} \in NC.$$

THEOREM. (INVARIANT RADON DECOMPOSITION)

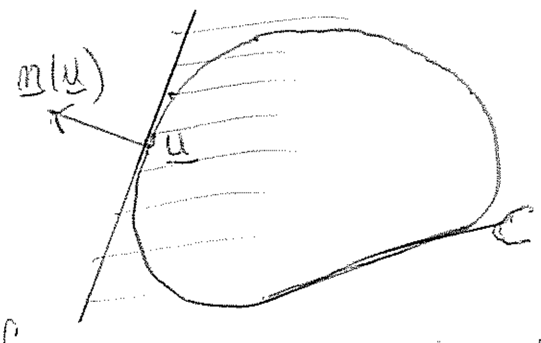
(i) FOR m EVEN & $\langle \underline{x}, \underline{s}\rangle \neq 0$, $\underline{\tau} = \underline{t} + i\underline{s}$

$$-\frac{1}{A_m}\frac{\underline{x}}{|\underline{x}|^m} = \frac{i^m (m-2)!}{2(2\pi)^{m-1}} \int_{S^{m-2}} \langle \underline{x}, \underline{\tau}\rangle^{1-m}\, \underline{\tau}\, d\underline{t}$$

(ii) FOR m ODD & $\langle \underline{x}, \underline{s}\rangle > 0$: SAME FORMULA



MONOGENIC FUNCTIONS IN CONVEX SETS



$$f(\underline{x}) = \frac{1}{A_m}\int_{\partial C} \frac{\underline{x}-\underline{u}}{|\underline{x}-\underline{u}|^m}\, \underline{n}(\underline{u})\, f(\underline{u})\, du$$

FOR CONVEX SETS $\langle \underline{x}-\underline{u}, \underline{n}(\underline{u})\rangle < 0$ TAKE $\underline{s} = -\underline{n}$

$$f(\underline{x}) = \frac{(m-2)!}{2(2\pi)^{m-1}} \int_{\partial C}\int_{\underline{t}\perp\underline{n}} \frac{(1+i\,\underline{n}(\underline{u})\,\underline{t})}{\langle \underline{u}-\underline{x}, \underline{n}(\underline{u})+i\underline{t}\rangle^{m-1}}\, f(u)\, d\underline{t}\, d\underline{u}$$

Cauchy kern

THEOREM. (RADON DECOMPOSITION IN $\tilde{B}(1)$)

$$f(\underline{x}) = \frac{(m-2)!}{2(2\pi)^{m-1}} \int_{S^{m-1}} \int_{S^{m-1}} \frac{\delta(\langle \underline{u}, \underline{t} \rangle)(1 + i\underline{u}\,\underline{t})\, f(\underline{u})}{(1 - \langle \underline{x}, \underline{u} + i\underline{t} \rangle)^{m-1}}\, d\underline{u}\, d\underline{t}$$

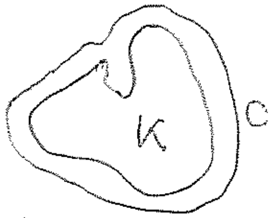
RADON TRANSFORM OF MONOGENIC FUNCTIONALS

K : COMPACT , $M_r(K) = \lim\limits_{\substack{\longrightarrow \\ \Omega \supset K}} \text{ind}\, M_r(\Omega)$

DELANGHE - BRACKX : $M_r'(K) = M_{l,0}(\mathbb{R}^m \setminus K)$, K no-holes

$T \in M_r'(K)$: CAUCHY-FANTAPPIE TRANSFORM

$$\hat{T}(\underline{x}) = T_u[E(\underline{x} - \underline{u})] = \frac{1}{A_m} \int \frac{\underline{x} - \underline{u}}{|\underline{x} - \underline{u}|^m} \mu_T(\underline{u})\, d\underline{u}$$



For $g \in M_r(\overline{C})$:

$$\cancel{\int_{\partial C} \hat{T}(\underline{x})\, d\sigma} \quad \int_{\partial C} g(\underline{x})\, d\sigma_{\underline{x}}\, \hat{T}(\underline{x}) = \int \int_{\partial C} g(\underline{x})\, d\sigma_{\underline{x}} \frac{\underline{u} - \underline{x}}{\dots} \mu_T(\underline{u})\, d\underline{u}$$

$$= \int g(\underline{u})\, \mu_T(\underline{u})\, d\underline{u} = T[g]$$

CLIFFORD-RADON TRF:

$$R(T)(z,\underline{\tau}) = \frac{1}{2\pi} T_u\left[\frac{\underline{\tau}}{\langle \underline{u},\underline{\tau}\rangle - z}\right]$$

EXERCISE: Reconstruct T from $R(T)$

FOURIER-BOREL 1.

$$\mathcal{FB}_1(T)(\underline{\tau}) = T_u\left[\underline{\tau}\, e^{\langle \underline{u},\underline{\tau}\rangle}\right]$$

FOURIER-BOREL 2.

$$\mathcal{E}(\underline{x};\underline{u}) = 2^{\frac{m}{2}-1}\Gamma\left(\tfrac{m}{2}\right) e^{\langle \underline{x},\underline{u}\rangle} |\underline{x}\wedge\underline{u}|^{1-\frac{m}{2}}\left\{J_{\frac{m}{2}-1}(\dots) + \frac{\underline{x}\wedge\underline{u}}{|\underline{x}\wedge\underline{u}|} J_{\frac{m}{2}}(\dots)\right\}$$

$$\mathcal{FB}_2(T)(\underline{x}) = T_{\underline{u}}\left[\mathcal{E}(\underline{x};\underline{u})\right]$$

FOURIER-BOREL 3. (AVEIRO TRANSFORM)

$$\exp\langle \underline{x},\underline{u}\rangle = \sum_{k=0}^{\infty} \frac{\langle \underline{x},\underline{u}\rangle^k}{k!} = \sum_{k=0}^{\infty}\sum_{j=0}^{k} \underline{x}^j \underbrace{Z_{k,j}(\underline{x},\underline{u})}_{\in \mathcal{M}_{k-j}} \underline{u}^j$$

$$= \sum_{j=0}^{\infty} \underline{x}^j E_j(\underline{x},\underline{u})\underline{u}^j$$

$$E(\underline{x};\underline{u}) = E_0(\underline{x},\underline{u}) = \sum_{k=0}^{\infty} Z_k(\underline{x},\underline{u}) = A + \underline{x}\wedge\underline{u}\, B$$

zonal

$$\mathcal{FB}_3(T)(\underline{x}) = T_u\left[E(\underline{x};\underline{u})\right]$$

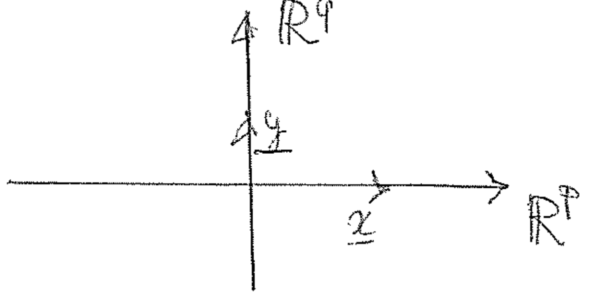
cfr. H. SHAPIRO

BICYCLE INTEGRALS

LET $F(\underline{x}, \underline{y}\,; \underline{t}, \underline{s})$ BE OF THE FORM

$$(\underline{t} + i\underline{s})\, f(\langle \underline{x}, \underline{t}\rangle + i\langle \underline{y}, \underline{s}\rangle)$$

$$(1 + i\,\underline{t}\,\underline{s})\, f(\langle \underline{x}, \underline{t}\rangle + i\langle \underline{y}, \underline{s}\rangle)$$



$\underline{t} \in S^{p-1}$

$\underline{s} \in S^{q-1}$

CONSIDER $$\int_{S^{p-1}} \int_{S^{q-1}} F(\underline{x}, \underline{y}\,; \underline{t}, \underline{s})\, P_{k,l}(\underline{t}, \underline{s})\, d\underline{t}\, d\underline{s}$$

↳ spher. MON.

APPLY FUNK-HECKE TWICE

BIAXIAL EXPONENTIALS

$$\int_{S^{p-1}} \int_{S^{q-1}} (1 + i\,\underline{t}\,\underline{s})\, \exp(\langle \underline{x}, \underline{t}\rangle + i\langle \underline{y}, \underline{s}\rangle)\, P_{k,l}(\underline{t}, \underline{s})\, d\underline{t}\, d\underline{s}$$

$$= (2\pi)^{\frac{m}{2}}\, i^{k+l}\, (-ir)^{1-\frac{p}{2}}\, \rho^{1-\frac{q}{2}}$$

$$\times \left[J_{k+\frac{p}{2}-1}(-ir)\, J_{l+\frac{q}{2}-1}(\rho) + i\,\underline{\omega}\,\underline{\xi}\, J_{k+\frac{p}{2}}(-ir)\, J_{l+\frac{q}{2}}(\rho) \right] P_{k,l}(\underline{\xi}, \underline{\omega})$$

Whereby $\underline{x} = r\underline{\xi}$ & $\underline{y} = \rho\,\underline{\omega}$

GENERALIZED POWERS

$$I_1 = \int_{S^{p-1}} \int_{S^{q-1}} (1 + i\underline{t}\,\underline{s}) \left(\langle \underline{x}, \underline{t}\rangle + i \langle \underline{y}, \underline{s}\rangle\right)^{\alpha} P_{k,l}(\underline{t}, \underline{s})\, d\underline{t}\, d\underline{s}$$

$$I_2 = \int_{S^{p-1}} \int_{S^{q-1}} (\underline{t} + i\underline{s}) \left(\langle \underline{x}, \underline{t}\rangle + i \langle \underline{y}, \underline{s}\rangle\right)^{\alpha} P_{k,l}(\underline{t}, \underline{s})\, d\underline{t}\, d\underline{s}$$

THEOREM. Let p, q be ODD. THEN

$$\frac{1}{A_m} \frac{\underline{x} + \underline{y}}{|\underline{x} + \underline{y}|^m} = \lambda_{p,q} \int_{S^{p-1}} \int_{S^{q-1}} \frac{\underline{t} + i\underline{s}}{\left(\langle \underline{x}, \underline{t}\rangle + i \langle \underline{y}, \underline{s}\rangle\right)^{m-1}}\, d\underline{t}\, d\underline{s}$$

WITH $\lambda_{p,q} = -i^{p-1} \dfrac{(m-2)!}{4(2\pi)^{m-1}}$

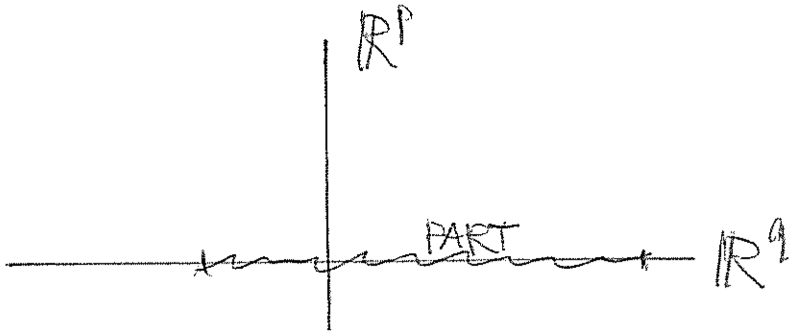
ALSO

$$\ldots = -\frac{\partial_{\underline{x}}^{p-1}\, \partial_{\underline{y}}^{q-1}}{4(2\pi)^{m-1}} \int_{S^{p-1} \times S^{q-1}} \frac{\underline{t} + i\underline{s}}{\langle \underline{x}, \underline{t}\rangle + i \langle \underline{y}, \underline{s}\rangle}\, d\underline{t}\, d\underline{s}$$

USING CYCLES

$$-\frac{1}{A_m} \frac{\underline{u}}{|\underline{u}|^m} = \frac{i^{m-1}(m-2)!}{2(2\pi)^{m-1}} \int_C \underbrace{\langle \underline{u}, \underline{\tau}\rangle^{1-m} \langle \underline{\tau}, \partial_{\underline{\tau}}\rangle 1\, \partial_{\underline{\tau}} 1\, d\tau_1 \ldots d\tau_m}_{\text{closed form in } \underline{\tau}}$$

GENERALIZED CK FROM FUNK-HECKE



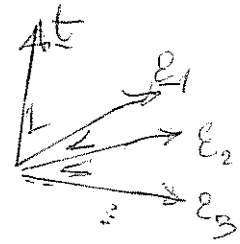
$\underline{t} \in S^{p-1}$ fixed

$f(\underline{y})$: function analytic on PART OF $\mathbb{R}^q$

$\Rightarrow$ CONSIDER : $\underbrace{\underline{t}\langle \underline{t}, \partial_{\underline{x}}\rangle}_{\simeq \partial_{x_0}} + \partial_{\underline{y}} = D_{\underline{t}}$ $\quad x_0 \simeq \langle \underline{x}, \underline{t}\rangle$

$\exp\left(\langle \underline{t}, \underline{x}\rangle \underline{t}\, \partial_{\underline{y}}\right) f(\underline{y})$ satisf. $D_{\underline{t}} F(\langle \underline{x}, \underline{t}\rangle, \underline{y}) = 0$

ALSO $\partial_{\underline{x}} = \sum_{j=1}^{p-1} \underline{\varepsilon}_j \langle \underline{\varepsilon}_j, \partial_{\underline{x}}\rangle + \underline{t}\langle \underline{t}, \partial_{\underline{x}}\rangle$



& $(\partial_{\underline{x}} + \partial_{\underline{y}}) F(\ldots) = D_{\underline{t}} F(\ldots)$

TAKE F : $\mathbb{R}_q$-valued, $P_k(\underline{x})$, $\mathbb{R}_p^+$-valued

$\Rightarrow$ Generalised C.K. :

$$F(\underline{x}; \underline{y}) P_k(\underline{x}) = \left(\frac{1}{A_p} \int_{S^{p-1}} \exp\left(\langle \underline{t}, \underline{x}\rangle \underline{t}\, \partial_{\underline{y}}\right) P_k(\underline{t})\, d\underline{t}\right) f(\underline{y})$$