A MINI COURSE ON CLIFFORD ALGEBRA

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I. THE STANDARD CLIFFORD ALGEBRA

CONSTRUCTION

$e_1, e_2, \ldots, e_m$: SYMBOLIC GENERATORS

CONSIDER $\mathbb{R}_m = \mathrm{Alg}\{e_1, \ldots, e_m\}$

- ASSOCIATIVE
- SUBJECT TO "DEFINING RELATIONS"

$$e_j e_k + e_k e_j = -2\delta_{jk}$$

i.e. $e_j^2 = -1$ & $e_j e_k = -e_k e_j$ for $j \neq k$

LET $\beta_1, \ldots, \beta_s \in \{1, \ldots, m\}$

THEN EXISTS $\alpha_1, \ldots, \alpha_k \in \{1, \ldots, m\}$

$k \leq s$ & $\alpha_1 < \alpha_2 < \ldots < \alpha_k$, $k \leq m$

s.t. $e_{\beta_1} \ldots e_{\beta_s} = \pm e_{\alpha_1} \ldots e_{\alpha_k}$

$A = \{\alpha_1, \ldots, \alpha_k\}$ IS UNIQUE

BASIS : $A \subset \{1, \dots, m\}$; $A = \{\alpha_1, \dots, \alpha_k\}$

$$e_A = e_{\alpha_1 \dots \alpha_k} = e_{\alpha_1} e_{\alpha_2} \cdots e_{\alpha_k}$$

$\dim \mathbb{R}_m = 2^m$? SURELY $\leq 2^m$

CHEVALLEY : UNIVERSAL CLIFFORD ALGEBRA
(CONSISTENCY ABOVE CONSTRUCTION)

T_m = TENSOR ALGEBRA V. $e_1, \dots, e_m$
= FREE ASSOC. ALG. GEN. $e_1, \dots, e_m$

ELEM. $e_{j_1} \otimes e_{j_2} \otimes \dots \otimes e_{j_s}$, 1

I_m = 2-SIDED IDEAL IN T_m
GENERATED BY $\alpha_{jk} = e_j \otimes e_k + e_k \otimes e_j + 2\delta_{jk}$

DEFINE $\mathbb{R}_m = T_m / I_m$

$\mathbb{R}_m$ IS ASSOC. ALGEBRA

UNIVERSALITY : $\mathcal{A}$ ALGEBRA , $a_1 \dots, a_m \in \mathcal{A}$
$$a_j a_k + a_k a_j = -2\delta_{jk}$$
$\Rightarrow \exists$ EPIMORPHISM $\phi : \mathbb{R}_m \to \mathrm{Alg}\{a_1, \dots a_m\}$: $\phi(e_j) = a_j$

EXAMPLES

$m=1$: BASIS : $1, e_1$ & $e_1^2=-1$

$$\Phi : \mathbb{R}_1 \to \mathbb{C} \quad , \quad \Phi(e_1)=i$$

$$\Rightarrow \mathbb{R}_1 \cong \mathbb{C}$$

$m=2$: BASIS : $1;\ e_1, e_2;\ e_{12}=e_1e_2$

$\Phi : \mathbb{R}_2 \to \mathbb{H}$ HAMILTON QUATERNIONS

$$\phi(e_1)=i \quad , \quad \phi(e_2)=j$$

$$i^2=j^2=-1 \quad , \quad ij=k=-ji$$

$$\phi(e_{12})=\phi(e_1)\,\phi(e_2)=ij=k$$

$$\mathbb{H}=\operatorname{span}\{1,i,j,k\}$$

REMARK : $i \to j \to k$ CYCLIC IN $\mathbb{H}$

$\hookrightarrow$ VECTORS

$\mathbb{R}_2$:

1	SCALAR
e_1, e_2	VECTORS
$e_1e_2=e_{12}$	BIVECTOR

$m = 3$: $\mathbb{R}_3$

BASIS : SCALAR : 1

VECTORS : e_1, e_2, e_3

BIVECTORS : $e_{12}, e_{23}, e_{31} = -e_{13}$

TRIVECTOR : $e_{123} = e_M$ (PSEUDOSCALAR)

CENTER : $\text{span}\{1, e_{123}\}$

EVEN SUBALGEBRA : $\mathbb{R}_3^+$

$$\mathbb{R}_3^+ = \text{span}\{1, e_{23}, e_{31}, e_{12}\} \cong \mathbb{H}$$

ISOMORPHISM
$$e_{23} \longrightarrow i$$
$$e_{31} \longrightarrow j$$
$$e_{12} \longrightarrow k$$

PROJECTORS $E_\pm = \frac{1}{2}(1 \pm e_{123})$

$E_+ + E_- = 1$, $E_+ E_- = 0$, $E_\pm$ CENTRAL

$$\hookrightarrow \mathbb{R}_3 = \underset{\displaystyle E_+ \mathbb{R}_3^+}{\underset{\|?}{E_+ \mathbb{R}_3}} \oplus \underset{\displaystyle E_- \mathbb{R}_3^+}{\underset{\|?}{E_- \mathbb{R}_3}} \simeq \mathbb{H} \oplus \mathbb{H} = {}^2\mathbb{H}$$

I. PORTEOUS, P. LOUNESTO

MULTIVECTOR STRUCTURE

$$a \in \mathbb{R}_m : a = \sum_{A \subset M} a_A e_A = \sum_{k=0}^{m} \sum_{|A|=k} \underset{\substack{\cap \\ \mathbb{R}}}{a_A} e_A$$

$$\mathbb{R}_m^k = \operatorname{span}\{ e_A = e_{\alpha_1} \dots e_{\alpha_k} ; \alpha_1 < \dots < \alpha_k \}$$

k- VECTORS

$$\mathbb{R}_m = \underset{\substack{\| \\ \mathbb{R}}}{\mathbb{R}_m^0} \oplus \underset{\substack{\| \wr \\ \mathbb{R}^m}}{\mathbb{R}_m^1} \oplus \mathbb{R}_m^2 \oplus \dots \oplus \underset{\substack{\| \\ e_M \mathbb{R}}}{\mathbb{R}_m^m}$$

SCALAR: 1 ; PSEUDOSCALAR: e_M

VECTORS : $\mathbb{R}_m^1 = \operatorname{span}\{e_1, \dots, e_m\} \longrightarrow m$

BIVECTORS : $\mathbb{R}_m^2 = \operatorname{span}\{ e_j e_k ; j < k \} \longrightarrow \binom{m}{2}$

$$2^m = 1 + \binom{m}{1} + \binom{m}{2} + \dots + \binom{m}{m}$$

$$a \in \mathbb{R}_m : a = [a]_0 + [a]_1 + \dots + [a]_m$$

$$[\cdot]_k : \mathbb{R}_m \longrightarrow \mathbb{R}_m^k$$

CENTER

$\mathbb{R}$ IN CASE $m = 2n$ EVEN

$\mathbb{R} + \underset{\substack{\| \\ e_M}}{e_{1 \dots m}} \mathbb{R}$ IN CASE $m = 2n+1$ ODD

EVEN SUBALGEBRA : $\mathbb{R}_m^+$

$$\mathbb{R}_m^+ = \mathbb{R}_m^0 \oplus \mathbb{R}_m^2 \oplus \mathbb{R}_m^4 \oplus \cdots$$

$$\mathbb{R}_m^- = \mathbb{R}_m^1 \oplus \mathbb{R}_m^3 \oplus \cdots$$

$a \in \mathbb{R}_m : \ a = [a]_+ + [a]_- \ ; [a]_\pm \in \mathbb{R}_m^\pm$

THEOREM. $\mathbb{R}_m^+ \cong \mathbb{R}_{m-1}$

PROOF. $E_1 = e_1 e_m , \ldots , E_{m-1} = e_{m-1} e_m$

THEN $\mathbb{R}_m^+ = \mathrm{Alg}\{E_1, \ldots, E_{m-1}\}$

$$E_j E_k + E_k E_j = -2\delta_{jk}$$

$\dim \mathbb{R}_m^+ = 2^{m-1} = \dim \mathbb{R}_{m-1}$. /// (UNIVERSALITY)

ODD CASE : $m = 2n+1$

$e_M = e_1 \ldots e_m = e_{1\ldots m}$ CENTRAL

$m = 4l+1 : \ e_M^2 = -1$

$$\hookrightarrow \mathbb{R}_{4l+1} \cong \mathbb{C} \otimes \mathbb{R}_{4l+1}^+ \cong \mathbb{C} \otimes \mathbb{R}_{4l} = \mathbb{C}_{4l}$$

$m = 4l+3 : e_M^2 = +1 : E_\pm = \frac{1}{2}(1 \pm e_M)$

$$\hookrightarrow \mathbb{R}_{4l+1} \cong E_+ \mathbb{R}_{4l+3}^+ \oplus E_- \mathbb{R}_{4l+3}^+ \cong {}^2\mathbb{R}_{4l+2}$$

COMPLEXIFICATION

$$\mathbb{C}_m = \mathbb{C} \otimes \mathbb{R}_m \qquad \text{BASIS } e_A \, ; \, i e_A = i \otimes e_A$$

$$= \{ a + ib : a, b \in \mathbb{R}_m \}$$

CASE $m = 2n+1$

FOR $m = 4l+1$ PUT $\omega = i e_M$

FOR $m = 4l+3$ PUT $\omega = e_M$

$\hookrightarrow$ WITH $E_\pm = \frac{1}{2}(1 \pm \omega)$

$$\mathbb{C}_{2n+1} = E_+ \mathbb{C}^+_{2n+1} \oplus E_- \mathbb{C}^+_{2n+1} \cong {}^2\mathbb{C}_{2n}$$

INVOLUTIONS

(i) MAIN INVOLUTION

$$(ab)^\sim = \widetilde{ab} = \tilde{a}\,\tilde{b} \quad \& \quad \tilde{e}_j = -e_j$$

$$a = [a]_+ + [a]_- \, , \quad [a]_\pm \in \mathbb{R}^\pm_m$$

$$\hookrightarrow \quad \tilde{a} = [a]_+ - [a]_-$$

(ii) CONJUGATION

$$\overline{ab} = \bar{b}\,\bar{a} \quad \& \quad \bar{e}_j = -e_j$$

$\bar{1} = 1$

$\bar{e}_1 = -e_1$

$\bar{e}_{12} = \bar{e}_2\,\bar{e}_1 = e_2 e_1 = -e_1 e_2$

$\bar{e}_{123} = \bar{e}_3\,\bar{e}_2\,\bar{e}_1 = -e_3 e_2 e_1 = -e_2 e_1 e_3 = e_{123}$

etc.

$a = [a]_0 + [a]_1 + [a]_2 + [a]_3 + \dots$

$\Rightarrow \bar{a} = [a]_0 - [a]_1 - [a]_2 + [a]_3 + [a]_4 - [a]_5 - \dots$

(iii) REVERSION

$(ab)^* = b^* a^* \quad \& \quad e_j^* = e_j$

Vb $e_{1234}^* = e_4 e_3 e_2 e_1 = e_{1234}$

$a^* = \tilde{\bar{a}}$

$a^* = [a]_0 + [a]_1 - [a]_2 - [a]_3 + \dots$

(iv) HERMITIAN CONJUGATION $(\mathbb{C}_m)$

$c = a + ib \; ; \; a, b \in \mathbb{R}_m$

$\Rightarrow c^\dagger = \bar{a} - i\bar{b}$

REM. $\bar{c} = \bar{a} + i\bar{b}$

INNER PRODUCTS ($\mathbb{R}_m$ & $\mathbb{C}_m$)

ORTHOGONAL

$$\langle a, b\rangle = [\bar{a} b]_0 = [a \bar{b}]_0 = [b \bar{a}]_0$$

$$= \sum_A a_A b_A$$

$$|a| = \sqrt{\langle a, a\rangle} \quad (\text{in } \mathbb{R}_m)$$

$$|ab| \leq 2^m |a||b|$$

HERMITIAN

$$(a, b) = [a^\dagger b]_0 = [b a^\dagger]_0$$

$$= \sum_A a_A^c b_A$$

$$|a| = \sqrt{(a, a)} \quad (\text{in } \mathbb{C}_m)$$

OTHER SIGNATURES : $\mathbb{R}_{p,q}$

$$\mathbb{R}_{p,q} = \mathrm{Alg}\{\varepsilon_1, \dots, \varepsilon_p ; e_1, \dots, e_q\}$$

$$\varepsilon_j^2 = +1 \; ; \; e_j^2 = -1 \quad \text{\& ANTICOMMUTATIVE}$$

$$j \neq k : e_j e_k = -e_k e_j \; ; \; \varepsilon_j \varepsilon_k = -\varepsilon_k \varepsilon_j \;\; \& \;\; e_j \varepsilon_k = -\varepsilon_k e_j, \; \forall j, k$$

$$\mathbb{R}_{p,q} \subset \mathbb{C} \otimes \mathbb{R}_{p,q} = \mathbb{C}_{p+q} \quad (\text{gen. } e_1, \dots, e_{p+q})$$

PUT $\varepsilon_j = i\, e_{j+q}$; $j = 1, \dots, p$.

II. ENDOMORPHISMS, SPINORS

$\mathrm{End}(\mathbb{R}^m)$

BASIC ENDOMORPHISMS

$a \in \mathbb{R}_m$

$e_j \circ : a \longrightarrow e_j a$

$\circ e_j : a \longrightarrow a e_j$

? IS $\mathrm{Alg}\{e_j \circ, \circ e_j\} = \mathrm{End}(\mathbb{R}_m)$? NO

$e_j | : a \longrightarrow \tilde{a} e_j$

& DENOTE $e_j \circ = e_j$

$e_j | e_k | = - e_k | e_j |$ $j \neq k$

$e_j | e_j | [a] = \widetilde{\tilde{a} e_j}\, e_j = - a e_j e_j = a$

$\Rightarrow e_j |^2 = +1$

$e_j e_k | [a] = e_j \tilde{a} e_k = - \widetilde{(e_j a)} e_k = - e_k | e_j [a]$

CONCLUSION $\mathrm{Alg}\{e_1, \dots, e_m ; e_1 |, \dots, e_m |\}$ repr. $\mathbb{R}_{m,m}$

$\mathbb{R}_{m,m} = \mathrm{Alg}\{\varepsilon_1, \dots, \varepsilon_m ; e_1, \dots, e_m\}$

SO $\phi : \mathbb{R}_{m,m} \longrightarrow \mathrm{End}(\mathbb{R}_m)$

$$e_j \longrightarrow e_j \circ \ (=_{\text{not}} e_j)$$
$$\varepsilon_j \longrightarrow e_j \rfloor$$

ALSO $\dim \mathbb{R}_{m,m} = \dim \mathrm{End}(\mathbb{R}_m) = 2^{2m}$

HENCE ONTO $\Longleftrightarrow$ INTO

REASONINGS

(i) ϕ ONTO : DIRECT VERIF.

(ii) FOR $p+q$ EVEN , $\phi : \mathbb{R}_{p,q} \to A$

ALWAYS INJECTIVE

IN PARTICULAR $\mathbb{R}_{m,m} = \mathbb{R}(2^m)$

(cfr. CLASSIFICATION TABLE)

<u>THEOREM.</u> $\mathrm{End}(\mathbb{R}_m) \cong \mathbb{R}_{m,m}$

COMPLEXIFICATION : $\mathbb{C}_{2m} \cong \mathbb{C} \otimes \mathbb{R}_{m,m} \cong \mathbb{C}(2^m)$

RECALL $\mathbb{C}_{2m+1} \cong {}^2\mathbb{C}_{2m} \cong {}^2\mathbb{C}(2^m)$

THEOREM. FOR m even $e_j\circ$ & $\circ e_j$ GENERATE

$\mathrm{End}(\mathbb{R}_m)$

FOR m ODD; $e_j\circ$ & $\circ e_j$ GENERATE

$\mathrm{End}(\mathbb{R}_m^+)$

PROOF. m even.

$$e_1\cdots e_m\, e_j\, e_m\cdots e_1 = -e_1\cdots e_m^2\cdots e_1\, e_j = -e_j$$

$$\& \quad e_M\, a\, b\, e_M^* = e_M\, a\, e_M^*\; e_M\, b\, e_M^*$$

CONCLUSION: $\tilde{a} = e_1\cdots e_m\, a\, e_m\cdots e_1$

So $e_j l \in \mathrm{Alg}\{\circ e_j,\, e_j\circ\}$

m ODD : $a\in\mathbb{R}_m^+ \Rightarrow \circ e_j\, a = e_j l a$

$\hookrightarrow$ SURELY GET $\mathrm{End}(\mathbb{R}_m^+)$ $\square$

NOW $m = 4l+3$: $e_M^2 = +1$

$e_j\circ$ & $\circ e_j$ GENERATE $\mathrm{End}\left(\frac{1+e_M}{2}\mathbb{R}_m^+\right) \oplus \mathrm{End}\left(\frac{1-e_M}{2}\mathbb{R}_m^+\right)$

FOR $m = 4l+1$: $e_M \simeq i$

$e_j\circ$ & $\circ e_j$ GENERATE $\mathrm{End}_{\mathbb{C}}(\mathbb{C}_m^+) \cong \mathbb{C}(2^{m-1})$

SPINOR SPACES

PUT $I_j = \frac{1}{2}(1 + e_j \varepsilon_j)$

$$e_j I_j = \varepsilon_j I_j \quad , I_j^2 = I_j$$

$$I_j I_k = I_k I_j$$

PUT $I = I_1 \dots I_m$ PRIMITIVE IDEMPOTENT

CONVERSION RELATIONS: $e_j I = \varepsilon_j I$

SO $e_{\alpha_1} \dots e_{\alpha_k} I = \varepsilon_{\alpha_1} \dots \varepsilon_{\alpha_k} I$

$\hookrightarrow e_A \varepsilon_B I = e_A e_B I$

$\forall a \in \mathbb{R}_{m,m}, \exists \hat{a} \in \mathbb{R}_m : aI = \hat{a} I$

LEMMA. $\hat{a}$ IS UNIQUE

PROOF. LET $P: \mathbb{R}_{m,m} \to \mathbb{R}_m$

CANONICAL PROJECTOR $P(e_A \varepsilon_B) = \delta_{B\emptyset} e_A$

$\hookrightarrow P(e_A I) = 2^{-m} e_A$

HENCE $e_A I$ LINEARLY INDEP.

OR $\dim \mathbb{R}_m I = 2^m$

& $\hat{a} I = 0 \Rightarrow \hat{a} = 0$. ■

$S = \mathbb{R}_{m,m} I = \mathbb{R}_m I$: SPINOR SPACE

$\qquad\qquad\qquad\qquad \llcorner$ MINIMAL LEFT IDEAL

$\dim S = 2^m$

$\mathrm{End}\, S \cong \mathrm{End}(\mathbb{R}^m) \cong \mathbb{R}_{m,m}$

BY LEFT MULTIPLICATION:

$e_j e_A I = e_j \cdot [e_A] I$ TRIVIAL

$\varepsilon_j e_A I = (-1)^k e_A \varepsilon_j I$

$\qquad = \tilde{e}_A e_j I = e_j \lrcorner [e_A] I$

IN GENERAL $b \in \mathbb{R}_{m,m}$, $\Phi(b) \in \mathrm{End}(\mathbb{R}_m)$

$$b\, a\, I = \Phi(b)[a]\, I$$

WITT BASIS.

$$f_j = \tfrac{1}{2}(e_j - \varepsilon_j) \; ; \; f_j^{\dagger} = \tfrac{1}{2}(e_j + \varepsilon_j)$$

GRAẞMANN

$$f_j f_k = -f_k f_j \;\&\; f_k^{\dagger} f_j^{\dagger} = -f_j^{\dagger} f_k^{\dagger}$$

DUALITY: $f_j^{\dagger} f_k + f_k f_j^{\dagger} = -\delta_{jk}$

$\mathrm{Alg}\{\mathfrak{f}_1,\dots,\mathfrak{f}_m\} = \Lambda_m$ GRAßMANN

$\mathrm{Alg}\{\mathfrak{f}_1^\dagger,\dots,\mathfrak{f}_m^\dagger\} = \Lambda_m^\dagger$ DUAL GRAßMANN

CHEVALLEY SPINORS: $e_j I = \varepsilon_j I = \mathfrak{f}_j^\dagger I$

$$\mathfrak{f}_j I = 0$$

So: $S = \mathbb{R}_m I = \Lambda_m^\dagger I$ Spinor SPACE

BASIS $\mathfrak{f}_A^\dagger I = \mathfrak{f}_{\alpha_1}^\dagger \cdots \mathfrak{f}_{\alpha_k}^\dagger I$

IN CASE $|B| = |A| = k$:

$$\mathfrak{f}_A \mathfrak{f}_B^\dagger I = \pm \delta_{AB} I$$

HENCE FOR $a \in \mathbb{R}_m^k$, $T \in \mathrm{End}(\mathbb{R}_m)$

$$T[a] I = \sum_{|D|=k} c_{BD}\, \mathfrak{f}_B^\dagger \mathfrak{f}_D\, a_A \mathfrak{f}_A^\dagger I$$

PROBLEM

IN GENERAL $a = [a]_0 + [a]_1 + \dots$

? $[\cdot]_k$ in TERMS OF $\mathbb{R}_{m,m}$

DEGREE OP. $\mathbb{B}[a] = \sum_{j=1}^{m} e_j \tilde{a} e_j$

$B \in \mathbb{R}_{m,m}$, $B = \sum_{j=1}^{m} e_j \varepsilon_j$ BIVECTOR

$$B[e_k] = -\sum e_j e_k e_j = (2-m) e_k$$

$$B[e_k e_l] = +\sum_j e_j e_k e_l e_j = (4-m) e_k e_l$$

$$B[e_A] = (2k-m) e_A$$

EIGENSPACES : $\mathbb{R}_m^k$

EIGENVAL. : $2k-m$ ALL DIFFERENT

$\hookrightarrow$ $[\cdot]_k = \mathrm{Pol}_k(B) \in \mathbb{R}_{m,m}$

CONCLUSION $\phi : \mathbb{R}_{m,m} \to \mathrm{End}(\mathbb{R}_m)$ ONTO

$T \in \mathrm{End}(\mathbb{R}_m)$

$$T = \sum_{k=0}^{m} \sum_{|D|=k} C_{BD}^{k} \hat{f}_B^{\dagger} \hat{f}_D \mathrm{Pol}_k(B)$$

COMPLEXIFICATION. $\varepsilon_j = i e_{j+m}$

$$\hat{f}_j = \frac{1}{2}(e_j - i e_{j+m}) , \quad \hat{f}_j^{+} = -\hat{f}_j'$$

$$\hat{f}_j \hat{f}_k^{+} + \hat{f}_k^{+} \hat{f}_j = +\delta_{jk}$$ HERM. DUAL.

START FROM GRAßMANN

$$\hat{f}_1, \dots, \hat{f}_m \in \mathbb{C}_{2m}$$

RELATIONS $\hat{f}_j \hat{f}_k = -\hat{f}_k \hat{f}_j$

& $\hat{f}_j \hat{f}_k^+ + \hat{f}_k^+ \hat{f}_j = \delta_{jk}$

REDEFINE REAL BASIS

$$e_j = \hat{f}_j - \hat{f}_j^+ \; ; \; e_{j+m} = -\frac{1}{i}\left(\hat{f}_j + \hat{f}_j^+\right)$$

$$\underline{x} = \sum_{j=1}^{m} \left(e_j x_j + e_{j+m} y_j\right) = \sum_{j=1}^{m} (\hat{f}_j - \hat{f}_j^+) x_j + i(\hat{f}_j + \hat{f}_j^+) y_j$$

$$= \sum_{j=1}^{m} \hat{f}_j z_j - \sum_{j=1}^{m} \hat{f}_j^+ \bar{z}_j \quad ; \; z_j = x_j + i y_j$$

$$= \underline{z} - \underline{z}^+ \quad ; \; \underline{z} = \sum_{j=1}^{m} \hat{f}_j z_j$$

INNER PRODUCT $\mathbb{C}_m$, S

$$a \in \mathbb{C}_{2m} = \mathrm{End}(\mathbb{C}_m)$$

$$\hookrightarrow (aI)^+ = I^+ a^+ = I a^+$$

$$I a I = I a[1] I = I \left[a[1]\right]_0 I = [\overset{1}{a}]_0 I$$

$$\underset{\Lambda_m^{\prime}}{\uparrow}$$

EXPECTATION VALUE

$$\langle a \rangle = [\hat{a}]_0$$

$$\rho_0 : I\, a\, I = \langle a \rangle I$$

INNER PRODUCT ON S

$$a, b \in \mathbb{C}_m : (aI)^{\dagger} bI = I a^{\dagger} b I$$

$$= [a^{\dagger} b]_0 I = (a, b) I \in \hat{\mathbb{C}}_m$$

IDEA OF METRODYNAMICS

$$e_j = f_j + \sum_k g_{jk} f'_k \quad \& \; g_{jk} = g_{kj}$$

$$\hookrightarrow e_j e_k + e_k e_j = -2 g_{jk}$$

$\hookrightarrow$ VARIABLE

EINSTEIN :

$$f'_k \xrightarrow[\text{NOT.}]{} f^{\dagger k}$$

$$e_j = f_j + g_{jk} f^{\dagger k} = f_j + f^{\dagger}_k$$

ALSO FOR NONSYMM. g.

SYMPLECTIC CASE $g_{jk} = -g_{kj}$; $e_j = f_j + g_{jk} f^{\dagger k}$

$$\varepsilon_j = i\,(f_j - g_{jk} f^{\dagger k})$$

III. GEOMETRIC ALGEBRA, SPINGROUPS

D. HESTENES, G. SOBCZYK: Geometric CALCULUS
F. SOMMEN: RADIAL ALGEBRA

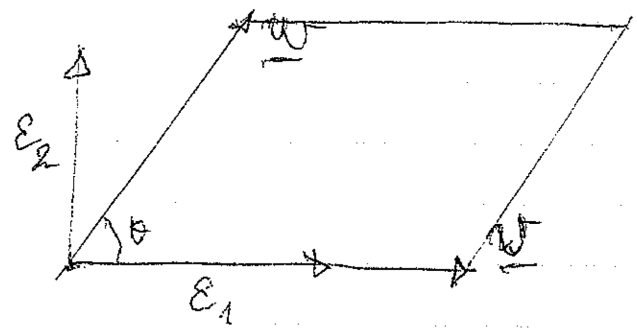
$$\underline{v} = \sum_{j=1}^{m} v_j e_j \quad , \quad \underline{w} = \sum_{j=1}^{m} w_j e_j$$

$$\underline{v}\,\underline{w} = [\underline{v}\,\underline{w}]_0 + [\underline{v}, \underline{w}]_2 = \underline{v} \cdot \underline{w} + \underline{v} \wedge \underline{w}$$

$$= -\langle \underline{v}, \underline{w} \rangle + \underline{v} \wedge \underline{w}$$

$$\underline{v} \cdot \underline{w} = \frac{1}{2}\{\underline{v}, \underline{w}\} = -\cos\theta\, |\underline{v}|\,|\underline{w}|$$

$$\underline{v} \wedge \underline{w} = \frac{1}{2}[\underline{v}, \underline{w}] = \sin\theta\, |\underline{v}|\,|\underline{w}|\, \underline{\varepsilon}_1 \wedge \underline{\varepsilon}_2$$



$$|\underline{v}+\underline{w}|^2 + |\underline{v}-\underline{w}|^2 = -(\underline{v}+\underline{w})^2 - (\underline{v}-\underline{w})^2$$

$$= -\underline{v}^2 - \underline{w}^2 - \{\underline{v}, \underline{w}\} - \underline{v}^2 - \underline{w}^2 + \{\underline{v}, \underline{w}\}$$

$$= 2(|\underline{v}|^2 + |\underline{w}|^2)$$

REDEFINE GRAßMANN PRODUCT

$$\underline{v}_1 \wedge \dots \wedge \underline{v}_k = \frac{1}{k!} \sum_{\pi} \operatorname{sgn}\pi \; v_{\pi(1)} \cdots v_{\pi(k)}$$

ANTISYMMETRIC IN INDICES.

$\underline{v}_1, \dots, \underline{v}_k$ LINEARLY DEP. $\Rightarrow \underline{v}_1 \wedge \dots \wedge \underline{v}_k = 0$

$\sum \lambda_j \underline{v}_j = 0$ & e.g. $\lambda_k \neq 0$

$\Rightarrow \quad \underline{v}_k = \sum_{1}^{k-1} \alpha_j \underline{v}_j$

$\wedge$ IS MULTILINEAR & $\underline{v}_1 \wedge \dots \wedge \underline{v}_{k-1} \wedge v_j = 0$, $j = 1, \dots, k-1$

ALSO USING $\underline{v}_j = \sum \lambda^j_k e_k$

$\underline{v}_1 \wedge \dots \wedge \underline{v}_k =$ $\qquad e_{l_k}$

$$= \sum_{l_1 \dots l_k} \lambda^1_{l_1} \cdots \lambda^k_{l_k} e_{l_1} \wedge \dots \wedge e_{l_k} \in \mathbb{R}^k_m$$

GRAMM-SCHMIDT: $\underline{v}_1 = \lambda_1 \underline{\varepsilon}_1$, $\underline{v}_2 = \lambda_2 \underline{\varepsilon}_2 + \underline{w}_1$

$\dots$, $\underline{v}_k = \lambda_k \underline{\varepsilon}_k + \underline{w}_{k-1} \in \operatorname{span}\{\underline{v}_1, \dots, \underline{v}_{k-1}\}$

THEN $\underline{v}_1 \wedge \dots \wedge \underline{v}_k = \lambda_1 \lambda_2 \cdots \lambda_k \, \underline{\varepsilon}_1 \wedge \dots \wedge \underline{\varepsilon}_k$

k-BLADE

FOR $\underline{v} \in \mathbb{R}_m^1$, $a \in \mathbb{R}_m^k$

$$\underline{v}\, a = \underline{v} \cdot a + \underline{v} \wedge a$$

$$\underline{v} \cdot a = [\underline{v}\, a]_{k-1} = \frac{1}{2}\left(\underline{v}\, a - (-1)^k a\, \underline{v}\right)$$

$$\underline{v} \wedge a = [\underline{v}\, a]_{k+1} = \frac{1}{2}\left(\underline{v}\, a + (-1)^k a\, \underline{v}\right)$$

$a = e_1 \cdots e_k$, $\underline{v} = \underline{v}_{\parallel} + \underline{v}_{\perp}$

$\underline{v}_{\parallel}$: span$\{e_1, \ldots, e_k\}$; $\underline{v}_{\perp}$: $\perp e_1, \ldots, e_k$

$$\Rightarrow [\underline{v}\, a]_{k-1} = [\underline{v}_{\parallel}\, a]_{k-1} = \underline{v}_{\parallel}\, a = (-1)^{k-1} a\, \underline{v}_{\parallel}$$

$$[\underline{v}\, a]_{k+1} = \underline{v}_{\perp}\, a = (-1)^k a\, \underline{v}_{\perp}$$

ALSO $\underline{v}_1 \wedge \ldots \wedge \underline{v}_k = \underline{v}_1 \wedge (\underline{v}_2 \wedge \ldots \wedge \underline{v}_k)$ (multilin)

$$= [\underline{v}_1 \ldots \underline{v}_k]_k$$

$$= \frac{1}{2}\left(\underline{v}_1 \wedge \ldots \wedge \underline{v}_{k-1}\, \underline{v}_k + (-1)^{k-1}\, \underline{v}_k\, \underline{v}_1 \wedge \ldots \wedge \underline{v}_{k-1}\right)$$

IN GENERAL FOR $k + l \leq m$ & $a \in \mathbb{R}_m^k$, $b \in \mathbb{R}_m^l$

$$a \wedge b = [a\, b]_{k+l}$$

$(\mathbb{R}_m, \wedge)$ ASSOCIATIVE ALGEBRA of GRASSMANN

GRASSMANN MANIFOLD

$$\tilde{G}(m,k) = \{ \omega = \underline{v}_1 \wedge \ldots \wedge \underline{v}_k \, ; \, \omega^2 = \pm 1 \}$$

$$= \{ \text{ORIENTED } k\text{-subspaces } \mathbb{R}^m \}$$

$$G(m,k) = \{ (\lambda \omega \mid \lambda \neq 0) \, ; \, \omega \in \tilde{G}(m,k) \}$$

$$= \{ k\text{-subspaces } \mathbb{R}^m \}$$

$$= \{ (\lambda \omega \mid \lambda \neq 0) \, ; \, \omega = \underbrace{\underline{v}_1 \wedge \ldots \wedge \underline{v}_k}_{\text{lin. indep.}} \}$$

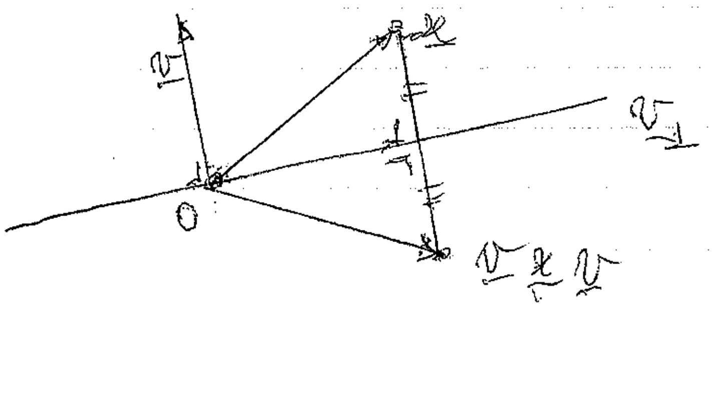
THE SPINGROUP Spin(m)

Consider $\underline{v} \in \mathbb{R}^1_m$, $|\underline{v}| = 1$

$$\underline{x} = \underline{x}_{\parallel} + \underline{x}_{\perp} \quad ; \quad \underline{x}_{\parallel} = \langle \underline{x}, \underline{v} \rangle \underline{v}$$

$$\underline{x}_{\perp} = \underline{v}(\underline{x} \wedge \underline{v})$$

$$\underline{v}\,\underline{x}\,\underline{v} = -\underline{x}_{\parallel} + \underline{x}_{\perp} = R_{\underline{v}}(\underline{x})$$



THM. HAMILTON

$$T \in SO(m) \Rightarrow \exists\, \underline{v}_1, \dots, \underline{v}_{2\ell} : T = R_{\underline{v}_1} \cdots R_{\underline{v}_{2\ell}}$$

IN OTHER WORDS

$$T[\underline{x}] = \underline{v}_{2\ell} \cdots \underline{v}_1\, \underline{x}\, \underline{v}_1 \cdots \underline{v}_{2\ell}$$

$$= s\,\underline{x}\,\bar{s} \;;\; s = \underline{v}_{2\ell} \cdots \underline{v}_1$$

$$\mathrm{Spin}(m) = \{ s = \underline{\omega}_1 \cdots \underline{\omega}_{2\ell} : \underline{\omega}_j^2 = -1 \}$$

$$h : \mathrm{Spin}(m) \to SO(m) : s \to h(s) : \underline{x} \to s\,\underline{x}\,\bar{s}$$

ALSO $\quad h(s)[\underline{x}] = s\,\underline{x}\,\bar{s}$

$$h(s)[a] = s\,a\,\bar{s}$$

$$h(s) : \mathbb{R}_m^k \to \mathbb{R}_m^k$$

$$h(s)[a\,b] = \cancel{s}\,a\,b\,\bar{s} = s\,a\,\bar{s}\,s\,a\,\bar{s}$$

$$= h(s)[a]\,h(s)[b]$$

i.e. $h(s) \in \mathrm{Hom}(\mathbb{R}_m)$

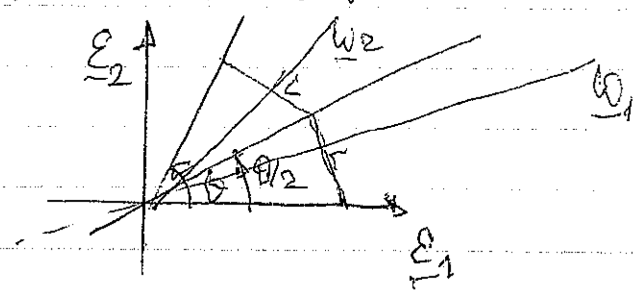
$$h(s)[\underline{v}_1 \wedge \dots \wedge \underline{v}_k] = h(s)[\underline{v}_1] \wedge \dots \wedge h(s)[\underline{v}_k]$$

i.e. $h(s) \in \mathrm{Hom}(\Lambda_m)$

$$h(s_1) = h(s_2) \iff s_1 = \pm s_2$$

i.e. $SO(m) = Spin(m)/\mathbb{Z}_2$

PLANE ROTATION θ



$$R(\theta, \underline{\varepsilon}_1 \underline{\varepsilon}_2)[\underline{x}] = s\, \underline{x}\, \bar{s}$$

$$s = \exp\left(\theta \frac{\underline{\varepsilon}_1 \underline{\varepsilon}_2}{2}\right)$$

$$s\, \underline{\varepsilon}_1\, \bar{s} = s\, \underline{\varepsilon}_1 \exp\left(\theta \frac{\underline{\varepsilon}_1 \underline{\varepsilon}_2}{2}\right) = s \exp\left(\frac{\theta\, \underline{\varepsilon}_1 \underline{\varepsilon}_2}{2}\right) \underline{\varepsilon}_1$$

$$= (\cos\theta + \underline{\varepsilon}_1 \underline{\varepsilon}_2 \sin\theta)\, \underline{\varepsilon}_1$$

$$= (\cos\theta\, \underline{\varepsilon}_1 + \sin\theta\, \underline{\varepsilon}_2)$$

$$s\, \underline{\varepsilon}_2\, \bar{s} = (\cos\theta + \underline{\varepsilon}_1 \underline{\varepsilon}_2 \sin\theta)\, \underline{\varepsilon}_2$$

$$= -\sin\theta\, \underline{\varepsilon}_1 + \cos\theta\, \underline{\varepsilon}_2$$

$T \in SO(m) : \exists\ \underline{\varepsilon}_1 \underline{\varepsilon}_2,\ \underline{\varepsilon}_3 \underline{\varepsilon}_4 \cdots \underline{\varepsilon}_{2l-1} \underline{\varepsilon}_{2l}$

$$\underline{\underline{\varepsilon}}_{ij}\, \underline{\underline{\varepsilon}}_{kl} = \underline{\underline{\varepsilon}}_{kl}\, \underline{\underline{\varepsilon}}_{ij}$$

$$\& \quad T = R(\theta_1, \underline{\varepsilon}_1 \underline{\varepsilon}_2) \cdots R(\theta_l, \underline{\varepsilon}_{2l-1} \underline{\varepsilon}_{2l})$$

$$s = \exp\left(\frac{\theta_1}{2} \underline{\varepsilon}_1 \underline{\varepsilon}_2\right) \cdots \exp\left(\frac{\theta_l}{2} \underline{\varepsilon}_{2l-1} \underline{\varepsilon}_{2l}\right)$$

$$= \exp\left(\frac{1}{2}\left(\theta_1 \underline{\varepsilon}_1 \underline{\varepsilon}_2 + \cdots + \theta_l \underline{\varepsilon}_{2l-1} \underline{\varepsilon}_{2l}\right)\right)$$

$\hookrightarrow$ HAMILTON & $SO(m) = Spin(m)/\mathbb{Z}_2$

& $Spin(m) \subset \exp \mathbb{R}^2_m$

LIE ALGEBRA $= \mathbb{R}^2_m$

$$s = 1 + \varepsilon a = \underline{\omega}_1 \underline{\omega}'_1 \cdots \underline{\omega}_l \underline{\omega}'_l$$

$$\underline{\omega}'_j = -\underline{\omega}_j + \varepsilon \underline{a}_j$$

$$\underline{\omega}'^2_j = -1 - \varepsilon \{\underline{\omega}_j, \underline{a}_j\} + \cancel{\varepsilon^2 a^2} = -1$$

$$\hookrightarrow \{\underline{\omega}_j, \underline{a}_j\} = 0 \text{ OR } \underline{\omega}_j \perp \underline{a}_j$$

& $s = (1 + \varepsilon\, \underline{\omega}_1 \underline{a}_1) \cdots (1 + \varepsilon\, \underline{\omega}_l \underline{a}_l)$

$$= 1 + \varepsilon\, \underbrace{(\underline{\omega}_1 \underline{a}_1 + \cdots + \underline{\omega}_l \underline{a}_l)}_{\in \mathbb{R}^2_m}$$

IN GENERAL

$$s = \exp \underline{b} \ ; \ b \in \mathbb{R}^2_m$$

LIE BRACKET : $[\underline{b}_1, \underline{b}_2]$

CONJUGATION

$$\bar{s} = \exp \bar{\underline{b}} = \exp(-\underline{b}) = s^{-1}$$

GROUP: $\{s \in \mathbb{R}_m^+ : s\bar{s} = \bar{s}s = 1\} = S(m)$

LIE ALG: $s = 1 + \varepsilon a \quad , \bar{s} = 1 + \varepsilon \bar{a}$

$$s\bar{s} = 1 + \varepsilon(a + \bar{a}) = 1$$

$$\Rightarrow a + \bar{a} = 0$$

$\mathbb{R}_m^+$: $[a]_0 + [a]_2 + [a]_4 + [a]_6 + \ldots$

$$\bar{a} = [a]_0 - [a]_2 + [a]_4 - [a]_6 + \ldots$$

$$a + \bar{a} = 0 \Longrightarrow a = [a]_2 + [a]_6 + [a]_{10} + \ldots$$

FOR $m < 6$: $\mathrm{Spin}(m) = S(m)$, FOR $m \geq 6$ no MORE

EIGENVALUE DECOMPOSITION $\mathbb{R}_m^2$

$$\underline{b} = \theta_1 \underline{e}_1 \underline{e}_2 + \theta_2 \underline{e}_3 \underline{e}_4 + \cdots + \theta_l \underline{e}_{2l-1} \underline{e}_{2l}$$

WITH (i) $2l \leq m$

(ii) $\theta_1 \geq \theta_2 \cdots \geq \theta_l \geq 0$

(iii) $\underline{e}_1 \perp \underline{e}_2 \perp \ldots \perp \underline{e}_{2l}$

i.e. $\underline{\underline{b}} = \theta_1 \underline{\underline{b}}_1 + \cdots + \theta_\ell \underline{\underline{b}}_\ell$

$\underline{\underline{b}}_j = \underline{\varepsilon}_{2j-1} \underline{\varepsilon}_{2j}$; UNIT 2-BLADE

$\underline{\underline{b}}_j \underline{\underline{b}}_k = \underline{\underline{b}}_k \underline{\underline{b}}_j$

$\theta_1 \geq \cdots \geq \theta_\ell \geq 0$: POLYRADIUS $(\theta_1, \ldots, \theta_\ell)$

$\underline{\underline{b}}^2 = [\underline{\underline{b}}^2]_0 + [\underline{\underline{b}}^2]_4$

$[\underline{\underline{b}}^2]_4 = 0 \iff \underline{\underline{b}} = \theta \underline{\varepsilon}_1 \underline{\varepsilon}_2$; 2-BLADE

↳ EQUATION FOR $G(m, 2)$

EXAMPLE: THE KLEIN QUADRIC

$m = 4$: $\underline{\underline{b}} = b_1 e_{14} + b_2 e_{24} + b_3 e_{34} + b_4 e_{23} + b_5 e_{31} + b_6 e_{12}$

$[\underline{\underline{b}}]_4 = (b_1 b_4 + b_2 b_5 + b_3 b_6)\, e_{1234}$

PLÜCKER COORDINATES OF PROJECTIVE LINE

$[\underline{\underline{b}}]_4 = 0 \Rightarrow \underline{\underline{b}} = \underline{v} \wedge \underline{w} = \sum_{j<k} (v_j w_k - v_k w_j)\, e_{jk}$

$b_{jk} = v_j w_k - w_j v_k$; 6 PLÜCKER COORD.

SATISFY KLEIN QUADRIC

INCIDENCE GEOMETRY OF SUBSPACES

V_k, V_l SUBSPACES of $\dim k, l$; $l \leq k$

$j \leq l$; I_{klj} : $\dim V_k \cap V_l = j$

BLADES : $V_k = \underline{v}_1 \wedge \ldots \wedge \underline{v}_k$

$$V_l = \underline{w}_1 \wedge \ldots \wedge \underline{w}_l$$

$$V_k V_l = [V_k V_l]_{k-l} + [V_k V_l]_{k-l+2} + \ldots + [V_k V_l]_{k+l}$$

$$(V_k, V_l) \in I_{klj} \Leftrightarrow [V_k V_l]_{k+l-2i} = 0 \;, \; 0 \leq i \leq j-1$$

$$V_k = \underline{\varepsilon}_1 \cdots \underline{\varepsilon}_k$$

V_l : $\underline{w}_1, \ldots, \underline{w}_j \in \operatorname{span}\{\underline{\varepsilon}_1, \ldots, \underline{\varepsilon}_k\}$

$\dim \operatorname{span}\{\underline{w}_{j+1}, \ldots, \underline{w}_l ; \underline{\varepsilon}_1, \ldots, \underline{\varepsilon}_k\} = k+l-j$

$$\underline{\varepsilon}_1 \cdots \underline{\varepsilon}_k \underline{w}_1 \cdots \underline{w}_j = \underline{\nu}_1 \cdots \underline{\nu}_{k-j}$$

$$V_k V_l = \underline{\nu}_1 \cdots \underline{\nu}_{k-j} \underline{w}_{j+1} \cdots \underline{w}_l$$

& $[V_k V_l]_{k+l-2j} = \underline{\nu}_1 \wedge \ldots \wedge \underline{\nu}_{k-j} \wedge \underline{w}_{j+1} \wedge \ldots \wedge \underline{w}_l \neq 0$

PROBLEM : INTEGRAL GEOMETRY I_{klj}

ALSO $V_k e_M V_l = \ldots = V_k V_l^{\perp}$

IV. CLASSIFICATION

$\mathbb{R}_{p,q}$: $\varepsilon_1, \dots, \varepsilon_p \,;\, e_1, \dots, e_q$

LOW DIMENSIONAL CASES

$$\mathbb{R}_{0,0} = \mathbb{R} \,,\; \mathbb{R}_{0,1} = \mathbb{C} \,,\; \mathbb{R}_{0,2} = \mathbb{H} \,,\; \mathbb{R}_{0,3} = {}^2\mathbb{H}$$

$$\mathbb{R}_{1,0} = \mathrm{Alg}\{\varepsilon_1\} = \tfrac{1}{2}(1+\varepsilon_1)\mathbb{R} \oplus \tfrac{1}{2}(1-\varepsilon_1)\mathbb{R} = {}^2\mathbb{R}$$

Matrix:
$$\varepsilon_1 \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \,,\; 1 \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\mathbb{R}_{2,0}$: PAULI MATRICES

$$\varepsilon_1 \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \;;\; \varepsilon_2 \longrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\varepsilon_1^2 = \varepsilon_2^2 = +1 \,,\; \varepsilon_1\varepsilon_2 = -\varepsilon_2\varepsilon_1 \longrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\Rightarrow \mathbb{R}_{2,0} = \mathbb{R}(2)$$

$$\mathbb{R}_{1,1} = \mathrm{Alg}\{\varepsilon_1, e_1\}$$

$$\varepsilon_1 \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \;;\; e_1 \longrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\varepsilon_1 e_1 \longrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\Rightarrow \mathbb{R}_{1,1} \cong \mathbb{R}_{2,0} \cong \mathbb{R}(2)$$

DIMENSION REDUCTION PRINCIPLE

$$\mathbb{R}_{p+1,q+1} \cong \mathbb{R}_{p,q} \otimes \mathbb{R}_{1,1} \cong \mathbb{R}_{p,q} \otimes \mathbb{R}(2)$$

PRF. $\mathbb{R}_{p+1,q+1} = \mathrm{Alg}\{\varepsilon_1, \dots, \varepsilon_{p+1}; e_1, \dots, e_{q+1}\}$

$b = \varepsilon_{p+1} e_{q+1}$

$\Rightarrow b^2 = 1$ & b commutes with $\varepsilon_1, \dots, \varepsilon_p; e_1, \dots, e_q$

Putt $\check{\varepsilon}_j = b\varepsilon_j \quad ; \quad j = 1, \dots, p$

$\check{e}_j = b e_j \quad ; \quad j = 1, \dots, q$

THENN $\check{\varepsilon}_j^{\,2} = 1, \ \check{e}_j^{\,2} = -1$, ANTICOMM

$\Rightarrow \mathbb{R}_{p,q} = \mathrm{Alg}\{\check{\varepsilon}_j, \check{e}_k : j = 1, \dots, p; k = 1, \dots, q\}$

ALSO $\check{\varepsilon}_j, \check{e}_j$ commute with ε_{p+1} and e_{q+1}

HENCE $\mathbb{R}_{p+1,q+1} = \mathrm{Alg}\{\check{\varepsilon}_j, \check{e}_k, \varepsilon_{p+1}, e_{q+1}\}$

$= \mathrm{Alg}\{\check{\varepsilon}_j, \check{e}_k\} \otimes \mathrm{Alg}\{\varepsilon_{p+1}, e_{q+1}\} = \mathbb{R}_{p,q} \otimes \mathbb{R}_{1,1}$.

COROLLARY.

$$\mathbb{R}_{m,m} = \mathbb{R}_{m-1,m-1} \otimes \mathbb{R}(2) = \bigotimes_1^m \mathbb{R}(2) = \mathbb{R}(2^m).$$

SPACE-TIME ALGEBRAS

DIRAC ALGEBRA : $\mathbb{R}_{1,3} = \mathbb{R}_{0,2} \otimes \mathbb{R}_{1,1} = \mathbb{H} \otimes \mathbb{R}(2) = \mathbb{H}(2)$

MAJORANA ALGEBRA :

$$\mathbb{R}_{3,1} = \mathbb{R}_{2,0} \otimes \mathbb{R}_{1,1} = \mathbb{R}(2) \otimes \mathbb{R}(2) = \mathbb{R}(4).$$

SWAP PROPERTY

$$\mathbb{R}_{p+1,q} \cong \mathbb{R}_{q+1,p}$$

GENBASIS $\mathbb{R}_{p+1,q}$:

$\varepsilon_1, \dots, \varepsilon_p, \varepsilon_{p+1}$

$e_1, \dots, e_q$

GENBASIS $\mathbb{R}_{q+1,p}$:

$e_1 \varepsilon_{p+1}, \dots, e_q \varepsilon_{p+1}, \varepsilon_{p+1}$

$\varepsilon_1 \varepsilon_{p+1}, \dots, \varepsilon_p \varepsilon_{p+1}$

IN PARTICULAR : $\mathbb{R}_{p+1,0} \cong \mathbb{R}_{1,p} \cong \mathbb{R}_{p-1} \otimes \mathbb{R}(2)$

ALREADY : $\mathbb{R}_{p,q} \longrightarrow (\mathbb{R}_{m,0} \text{ or } \mathbb{R}_{0,m}) \longrightarrow \mathbb{R}_{0,m} = \mathbb{R}_m$

THE ELLIPTIC SWAP

$$\mathbb{R}_{p,q+2} \cong \mathbb{R}_{q,p} \otimes \mathbb{R}_{0,2} \cong \mathbb{R}_{q,p} \otimes \mathbb{H}$$

GENBASIS : $\varepsilon_1, \dots, \varepsilon_p ; e_1, \dots, e_q, e_{q+1}, e_{q+2}$

$b = e_{q+1} e_{q+2}$

$\Rightarrow b^2 = -1$ & b commutes with $\varepsilon_1, \dots, \varepsilon_p ; e_1, \dots, e_q$

$\mathbb{H} = \mathrm{Alg}\{e_{q+1}, e_{q+2}\}$

$\mathbb{R}_{q,p} = \mathrm{Alg}\{e_1 b, \dots, e_q b ; \varepsilon_1 b, \dots, \varepsilon_p b\}$

Both commute & generate $\mathbb{R}_{p,q+2}$. ///

FUNDAMENTAL EXAMPLES

$\mathbb{R}_4 = \mathbb{R}_{0,4} = \mathbb{H} \otimes \mathbb{R}_{2,0} = \mathbb{H}(2)$

$\mathbb{R}_{3,0} = \mathbb{R}_{1,2} = \mathbb{R}_{0,1} \otimes \mathbb{R}(2) = \mathbb{C}(2)$

$\mathbb{R}_{4,0} = \mathbb{R}_{1,3} = \mathbb{R}_{0,2} \otimes \mathbb{R}(2) = \mathbb{H}(2)$

$\mathbb{R}_{5,0} = \mathbb{R}_{0,3} \otimes \mathbb{R}(2) = {}^2\mathbb{H}(2)$

$\mathbb{R}_{6,0} = \mathbb{R}_{0,4} \otimes \mathbb{R}(2) = \mathbb{H}(4)$

$\mathbb{C} \otimes \mathbb{H} = \mathrm{Alg}\{i, e_1, e_2\} = \mathrm{Alg}\{i, ie_1, e_2\} = \mathbb{C} \otimes \mathbb{R}_{1,1} = \mathbb{C}(2)$

$\mathbb{H} \otimes \mathbb{H} = \mathbb{R}_{0,2} \otimes \mathbb{R}_{0,2} = \mathbb{R}_{2,2} = \mathbb{R}(4)$

$\mathbb{R}_5 = \mathbb{C} \otimes \mathbb{R}_4 = \mathbb{C} \otimes \mathbb{H}(2) = \mathbb{C}(4)$

$\mathbb{R}_6 = \mathbb{R}_{0,2} \otimes \mathbb{R}_{4,0} = \mathbb{H} \otimes \mathbb{H}(2) = \mathbb{R}(8)$

$\mathbb{R}_{7,0} = \mathbb{R}_{0,5} \otimes \mathbb{R}(2) = \mathbb{C}(8)$

$\mathbb{R}_7 = {}^2\mathbb{R}_6 = {}^2\mathbb{R}(8)$

PERIODICITY: $\mathbb{R}_{p+8,q} \cong \mathbb{R}_{p,q+8} \cong \mathbb{R}_{p,q} \otimes \mathbb{R}(16)$