Infinite dimensional analysis, non commutative stochastic distributions and applications

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Introduction

Setting

Will discuss connections between five topics:

1. Complex analysis.
2. Theory of linear systems (signal processing).
3. Stochastic processes.
4. Infinite dimensional analysis.
5. Free analysis.

Toolbox:

- Positive definite functions and kernels, and the associated reproducing kernel Hilbert spaces, play a key role in the exposition.
- Locally convex topological vector spaces, nuclear spaces.
- Fock spaces and second quantization.

Introduction

Specific aims:

1. A new class of topological algebras.
2. Models for stochastic and generalized stochastic processes, and stochastic integrals
3. Non-commutative spaces of distributions and non-commutative generalized stochastic processes, and associated stochastic integrals.
4. Linear stochastic systems.

Credits

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- Most of the new material is joint work with Guy Salomon (mathematics department BGU).
- Linear stochastic systems using white noise space was developed with David Levanony (EE, BGU) and graduate students (Haim Attia, Alon Bublil, Alon Kipnis, Ariel Pinhas).
- Part of the material is also joint work with Palle Jorgensen (Iowa City).
- Quaternionic Fock space was developed with Fabrizio Colombo, Irene Sabadini (Politecnico di Milano) and Guy Salomon.

Contents of lectures

Outline

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Outline

3. Lecture 3
   - Bochner and Bochner-Sazonov (Bochner-Minlos) theorem.
   - Hida's white noise space and the Wick product
   - Construction of certain stochastic processes with stationary increments
4. Lecture 4
   - Kondratiev space of stochastic distributions.
   - Some facts on duals of locally convex topological vector spaces.
   - Linear stochastic systems

Contents of lectures

Outline

5. Lecture 5
   - Fock spaces (full and symmetric)
   - Kondratiev space of non commutative stochastic distributions.
   - The free case
6. Lecture 6
   - Strong algebras

First lecture

Outline

1. Positive definite functions
2. Some reminders on Hermite polynomials and functions.

Preliminaries: Reproducing kernel Hilbert spaces and positive definite functions

Definition:

A Hilbert space $\mathcal{H}$ of functions defined on a set Ω is called a reproducing kernel Hilbert space if the point evaluations

$$f \mapsto f(w), \quad w \in \Omega$$

are bounded.

Reproducing kernel

By Riesz's representation theorem there exists a uniquely determined function $K(z,w)$ defined on $\Omega\times\Omega$, and with the following properties:
(1) For every $w\in\Omega$, the function

$$K_w \,:\, z\mapsto K(z,w)$$

belongs to $\mathcal{H}$, and
(2) For every $f\in\mathcal{H}$ and $w\in\Omega$,

$$\langle f, K_w\rangle_{\mathcal{H}} = f(w).$$

Reproducing kernel

Reproducing kernel

The function $K(z,w)$ is called the reproducing kernel of the space $\mathcal{H}$.

Inner product on the reproducing kernel

$$\langle K(\cdot,w)\,,\,K(\cdot,z)\rangle_{\mathcal{H}} = K(z,w)$$

Reproducing kernel

The reproducing kernel is positive definite, in the sense that for every $N \in \mathbb{N}$, every choice of $w_1, \ldots, w_N \in \Omega$, and every choice of $c_1, \ldots, c_N \in \mathbb{C}$

$$\sum_{j,k=1}^{N} \overline{c_j} K(w_j, w_k) c_k \geq 0.$$

Remark:

The terminology *positive definite* is a bit misleading since the inequality is not strict.

Axiom of choice:

To build Hilbert spaces of functions which are not reproducing kernel Hilbert spaces one needs to use the axiom of choice

A first example

Note

The space $\mathbf{L}_2(\mathbb{R}, dx)$ *is not* a space of functions, let alone a reproducing kernel Hilbert space. But, let Ω be the set of measurable sets.

$$F(\omega) = \int_\omega f(x)dx$$

is a function, and uniquely determines f. The set of such F with norm $\|F\| = \|f\|$ is the reproducing kernel Hilbert space with reproducing kernel

$$K(\omega, \nu) = \int_{\omega \cap \nu} dx \quad \text{(the Lebesgue measure of } \omega \cap \nu\text{)}.$$

Elaborating on the previous example

Inner products

Let $\mathcal{H}$ be a Hilbert space. The inner product

$$K(f,g) = \langle f, g \rangle$$

is positive definite on $\mathcal{H}$.

Elaborating on the previous example

Two important cases later

Case 1: Commutative case. There is an isomorphism $f \mapsto Q_f$ between $\mathbf{L}_2(\mathbb{R}, dx)$ and a probability space (Hida's white noise space):

$$\langle f, g \rangle = \langle Q_f, Q_g \rangle$$

Case 2: Non commutative case. There is a von Neumann algebra $\mathcal{M}_{\mathcal{H}}$ with trace τ, and a map $h \mapsto T_f$ from $\mathcal{H}$ into $\mathcal{M}_{\mathcal{H}}$ such that

$$\langle f, g \rangle = \tau(Q_f^* Q_g)$$

The finite dimensional case

Formula for the reproducing kernel in the finite dimensional case

Let $\mathscr{H}$ denote a finite dimensional space of complex-valued (or more generally $\mathbb{C}^N$-valued) functions, defined on some set Ω, let $f_1, \ldots, f_n$ denote a basis of the space and let P denote the $n \times n$ matrix with (ℓ, k) entry given by

$$P_{\ell,k} = \langle f_k, f_\ell \rangle_{\mathscr{H}}.$$

Then, P is strictly positive. Furthermore, let F denote the $\mathbb{C}^{N \times n}$-valued matrix-valued function

$$F(z) = \begin{pmatrix} f_1(z) & f_2(z) & \cdots & f_n(z) \end{pmatrix}.$$

The reproducing kernel is then given by the formula:

$$K(z, w) = F(z) P^{-1} F(w)^*$$

Special case

$n = 1$

When $n = 1$,

$$K(z, w) = \frac{f(z)f(w)^*}{p}$$

where f is a basis of $\mathscr{H}$ and $p = \langle f, f \rangle_{\mathscr{H}}$.

For instance, let $w \in \mathbb{C}_+$ and let

$$\mathscr{B}_w(z) = \frac{z - w}{z - \overline{w}}.$$

Then, one has the well-known formula

$$\frac{1 - \mathscr{B}_w(z)\overline{\mathscr{B}_w(v)}}{z - \overline{v}} = \frac{-2i\mathrm{Im}\ w}{(z - \overline{w})(\overline{v} - w)}, \quad z, v \in \mathbb{C}_+.$$

Finite Blaschke products

When the Blaschke factor $\mathscr{B}_w$ is replaced by a finite Blaschke product:

Let $m \in \mathbb{N}$ and $w_1, \ldots, w_m \in \mathbb{C}$ be such that

$$w_j \neq \overline{w_k}, \quad \forall j, k \in \{1, \ldots, N\},$$

and set

$$B(z) = \prod_{j=1}^{m} \frac{z - w_j}{z - \overline{w_j}}.$$

Let P denote the $m \times m$ matrix with jk entry equal to

$$P_{jk} = \frac{1}{-i(w_k - \overline{w_j})}, \quad j, k = 1, \ldots, m.$$

Then it holds that

$$\frac{1 - B(z)\overline{B(w)}}{-i(z-w)} =$$

$$= \begin{pmatrix} \frac{1}{-i(z-\overline{w_1})} & \frac{1}{-i(z-\overline{w_2})} & \cdots & \frac{1}{-i(z-\overline{w_m})} \end{pmatrix} P^{-1} \begin{pmatrix} \frac{1}{i(\overline{w}-w_1)} \\ \frac{1}{i(\overline{w}-w_2)} \\ \vdots \\ \frac{1}{i(\overline{w}-w_m)} \end{pmatrix}.$$

Structured matrices

Let $J \in \mathbb{C}^{n\times n}$ be invertible and such that $J = J^* = J^{-1}$. Let $(C, A) \in \mathbb{C}^{n\times m} \times \mathbb{C}^{m\times m}$ such that

$$C(I_m - zA)^{-1} f \equiv 0 \quad \Longrightarrow f = 0.$$

Let $P \in \mathbb{C}^{m\times m}$ be an Hermitian invertible matrix, and define

$$\Theta(z) = I_n + izC(I_m - zA)^{-1} P^{-1} C^* J.$$

It holds that

$$C(I_m - zA)^{-1} P^{-1} (I_m - wA)^{-*} C^* = \frac{J - \Theta(z) J \Theta(w)^*}{-i(z - \overline{w})}, \quad z, w \in \mathbb{C},$$

if and only if P is a solution of the equation

$$PA - A^* P = iC^* JC$$

J-unitary rational functions

The function Θ satisfies

$$\Theta(z)J\Theta(z)^* = J, \quad z \in \mathbb{R} \cap \rho(A)$$

where $\rho(A)$ denotes the resolvent set of A

A remark on rational functions

In the setting of the present talks one replaces $\mathbb{C}$ by (possibly non commutative rings) of stochastic distributions, to study rational functions (and linear stochastic systems). In quaternionic setting, similar problems when $\mathbb{C}$ is replaced by the quaternions (see papers A-Colombo-Sabadini)

Formula for the reproducing kernel

Take $(f_i)_{i\in I}$ an orthonormal basis of $\mathcal{H}$. Then,

$$K(z,w)=\sum_{i\in I} f_i(z)\overline{f_i(w)}$$

Positive definite functions and reproducing kernel spaces

Positive definite kernel:

Let Ω be a set. There is a one-to-one correspondence between positive definite kernels on Ω and reproducing kernel Hilbert spaces of functions defined on Ω.

Covariance:

Let Ω be a set. There is a one-to-one correspondence between positive definite functions on Ω and covariances of Gaussian processes indexed by Ω (see M. Loève, J. Neveu).

Various aspects of positivity

1. Functional analysis (RKHS)
2. Covariance function (probability)

Main properties of positive definite functions

The sum of two positive definite functions is positive definite

Let $K_1(z,w)$ and $K_2(z,w)$ be two functions positive on a given set Ω. The reproducing kernel Hilbert space $\mathcal{H}(K_1+K_2)$ is the sum of the reproducing kernel Hilbert spaces $\mathcal{H}(K_1)$ and $\mathcal{H}(K_2)$:

$$\mathcal{H}(K_1+K_2)=\mathcal{H}(K_1)+\mathcal{H}(K_2).$$

Furthermore

$$\|f\|^2_{\mathcal{H}(K_1+K_2)}=\min_{\substack{f=f_1+f_2\\ f_i\in\mathcal{H}(K_i),\, i=1,2}}\|f_1\|^2_{\mathcal{H}(K_1)}+\|f_2\|^2_{\mathcal{H}(K_2)},$$

and the sum is direct and orthogonal if and only if $\mathcal{H}(K_1)\cap\mathcal{H}(K_2)=\{0\}$.

Main properties of positive definite functions

The product of two positive definite functions is positive definite

Usual proof uses Hadamard's lemma on entry-wise product of positive matrices. Another way is:

$$\begin{aligned} K_1(z,w)K_2(z,w) &= \left(\sum_{i\in I} f_i(z)\overline{f_i(w)}\right)\left(\sum_{j\in J} g_j(z)\overline{g_j(w)}\right) \\ &= \sum_{(i,j)\in I\times J} (f_i(z)g_j(z))\overline{(f_i(w)g_j(w))} \end{aligned}$$

Factorization

Theorem

A $\mathbb{C}$-valued function $K(z,w)$ defined on a set Ω is positive definite on Ω if and only if it can be written as

$$K(z,w) = \langle f(w)\,,\, f(z)\rangle$$

where $z \mapsto f(z)$ is a $\mathcal{H}$-valued function ($\mathcal{H}$ some Hilbert space).

Generalizations:

Far-reaching, to operator-valued case and case of functions with values operators from certain topological vector space into their anti-dual.

Positive definite kernels. First examples

Covariance example

Let α be a continuous positive function defined on $[0, \infty)$. The kernel

$$K(t,s) = \int_0^{t \wedge s} \alpha(u)du$$

is positive definite on $[0, \infty)$. ($\alpha \equiv 1$ corresponds to the Brownian motion)

Complex variable

The kernel $e^{z\overline{w}}$ is positive definite in $\mathbb{C}$. Corresponding space is the Fock space (the symmetric Fock space associated to $\mathbb{C}$)

Positive definite kernels. First examples

Infinite dimensional analysis

The kernel $e^{-\frac{\|f-g\|^2}{2}}$ is positive definite on $\mathbf{L}_2(\mathbb{R})$ (real-valued)

Generalized stochastic processes

The kernel

$$F(s-t)=e^{\left(\int_{\mathbb{R}}(e^{i(s(x)-t(x))}-1)dx\right)}$$

is positive definite on $\mathscr{S}_{\mathbb{R}}$.

Reproducing kernel spaces

(at least) three descriptions:

- One geometric
- One analytic
- One via the kernel

and others (for instance, for Fock space, the Bargmann transform, or the fact that $M_z^* = \frac{\partial}{\partial z}$)

Note:

Sometimes it is not so obvious that a given kernel is positive definite.

Example 1:

$$K(t,s)=e^{-|t-s|},\quad t,s\in\mathbb{R}$$

is positive definite in $\mathbb{R}$.

Example 2: Fractional Brownian motion covariance

$$K(t,s)=\frac{1}{2}\left(|t|^{2H}+|s|^{2H}-|t-s|^{2H}\right),\quad t,s\in\mathbb{R},$$

(where $H\in(0,1)$ is fixed) is positive definite in $\mathbb{R}$.

Binomial coefficients

The function

$$K(m,n)=\binom{m+n}{n}$$

is positive definite on $\mathbb{N}_0$.

To see this use the Chu-VanderMonde formula

$$\binom{m+n}{n}=\sum_{\ell=0}^{m\wedge n}\binom{n}{\ell}\binom{m}{\ell}$$

Related spaces and analysis in a preprint A-Jorgensen in Arxiv.

Positive definite functions

Definition:

Let G be an Abelian group. The function

$$f(v) \;:\; G \;\longrightarrow\; \mathbb{C}$$

is positive definite if the kernel $K(u, v) = f(u - v)$ is positive definite on G.

Example

The function $e^{-|t|}$ is positive definite on $\mathbb{R}$ since

$$K(t, s) = e^{-|t-s|}, \quad t, s \in \mathbb{R}$$

is positive definite in $\mathbb{R}$.

Positive definite functions

Two important cases:

Case $V = \mathbb{R}$

$G = \mathbb{R}$. Bochner's theorem gives the structure of such functions as Fourier transforms of positive measures, like

$$e^{-|t|} = \int_{\mathbb{R}} e^{-itu} \frac{du}{\pi(u^2+1)}$$

and

$$e^{-\frac{t^2}{2}} = \int_{\mathbb{R}} e^{-itu} \frac{e^{-\frac{u^2}{2}} du}{\sqrt{2\pi}}.$$

Positive definite functions

Case of Fréchet nuclear spaces

G is a Fréchet nuclear space on the real numbers (for instance the real-valued Schwartz functions $\mathscr{S}'$). Then a theorem of Sazanov (which is also called Bochner-Minlos) gives the structure of such functions as Fourier transforms of positive measures on the strong dual of G.

Example

$G = \mathscr{S}$ and

$$f(s) = e^{-\int_{\mathbb{R}} \widehat{s}(u)|^2 d\sigma(u)}$$

where $\int_{\mathbb{R}} \frac{d\sigma(u)}{(u^2+1)^n} < \infty$ for some $n \in \mathbb{N}$. (see A-Jorgensen for applications)

Summary

We will consider Bochner-Minlos and applications in the following lectures. Now turn to a different topic.

Preliminaries: Hermite polynomials (formulas from Hille; other variations exist)

Hermite polynomials

$$H_n(z) = (-1)^n e^{z^2} \left(e^{-z^2}\right)^{(n)}, \quad n = 0, 1 \ldots$$

Some properties

$$e^{2uz-u^2} = \sum_{n=0}^{\infty} \frac{H_n(z)}{n!} u^n, \quad z, u \in \mathbb{C}.$$

Hermite polynomials: properties

Some properties

$$\begin{aligned} H_n'(z) - 2nH_{n-1}(z) &= 0, \\ H_n''(z) - 2zH_n'(z) + 2nH_n(z) &= 0, \\ H_n(z) - 2zH_{n-1}(z) + 2(n-1)H_{n-2}(z) &= 0. \end{aligned}$$

$$\int_{\mathbb{R}} e^{-u^2} H_n(u)H_m(u)du = \sqrt{\pi}2^n n!\delta_{m,n}$$

Preliminaries: Hermite functions

Hermite functions (formulas from Hille; other variations exist)

$$\xi_n(u) = \frac{(-1)^n}{\sqrt[4]{\pi}2^{\frac{n}{2}}} e^{\frac{u^2}{2}} \left(e^{-u^2}\right)^{(n)}.$$

or, normalized:

$$\eta_n(u) = \frac{1}{\sqrt[4]{\pi}2^{\frac{n}{2}}\sqrt{n!}} e^{\frac{u^2}{2}} \left(e^{-u^2}\right)^{(n)}, \quad n = 0, 1, \ldots$$

Orthogonality

It holds that

$$\langle \xi_n, \xi_m \rangle_{\mathbf{L}_2(\mathbb{R}, dx)} = \delta_{m,n} n!$$

Orthogonal basis

In fact the ξ_n form an orthogonal basis of $\mathbf{L}_2(\mathbb{R}, dx)$.

Mehler's formula (1866)

Mehler's formula

$$\sum_{n=0}^{\infty} t^n \frac{\xi_n(z)\xi_n(w)}{n!} = \frac{1}{\sqrt{\pi(1-t^2)}} e^{\frac{4zwt-(z^2+w^2)(1+t^2)}{2(1-t^2)}}, \quad |t| < 1.$$

Reproducing kernel:

This is really a reproducing kernel formula:

$$\sum_{n=0}^{\infty} t^n \frac{\xi_n(z)\overline{\xi_n(w)}}{n!} = \frac{1}{\sqrt{\pi(1-t^2)}} e^{\frac{4z\overline{w}t-(z^2+\overline{w}^2)(1+t^2)}{2(1-t^2)}}, \quad |t| < 1.$$

A reproducing kernel Hilbert space of entire functions

A reproducing kernel Hilbert space of entire functions

Let $t = 1/2^p$ and $p \in \mathbb{N}$, and $K_p = \frac{2^{1-p}}{\sqrt{\pi(1-2^{-2p})}}$. The space $\mathscr{G}_p$ of entire functions such that

$$\|f\|^2_{\mathscr{G}_p} = K_p \iint_{\mathbb{C}} |f(z)|^2 \, e^{\frac{1-2^{-p}}{1+2^{-p}}x^2 - \frac{1+2^{-p}}{1-2^{-p}}y^2} dxdy < \infty$$

is the reproducing kernel Hilbert space with reproducing kernel

$$\sum_{n=0}^{\infty} t^n \frac{\xi_n(z)\overline{\xi_n(w)}}{n!} = \frac{1}{\sqrt{\pi(1-t^2)}} e^{\frac{4z\overline{w}t-(z^2+\overline{w}^2)(1+t^2)}{2(1-t^2)}}.$$

A reproducing kernel Hilbert space of entire functions

Remarks:

The dual of the Fréchet space $\cap_{p\in\mathbb{N}}\mathscr{G}_p$ is an example of a strong algebra (as explained toward the end of the second lecture). See A-Salomon IDA-QP, 2012.

The Hilbert spaces with reproducing kernel

$$\sum_{n=0}^{\infty} t^n \frac{\xi_n(z)\overline{\xi_n(w)}}{n!}$$

(with $t \in (0,1)$) were considered in a number of places:

Eijndhoven and Meyers, JMAA, 1987,

Ali, Górska, Horzela, and Szafraniec, Journal of Mathematical Physics, 2014.

Hermite functions as eigenfunctions

First some definitions: Positive operator in Hilbert space and Hilbert-Schmidt operators.

Positive operator

Let $\mathcal{H}$ be a Hilbert spaces. The linear map

$$T \;:\; \mathcal{H} \;\rightarrow\; \mathcal{H}$$

is positive if

$$\langle Tf, f\rangle \geq 0, \quad \forall f \in \mathrm{Dom}\, T$$

Hermite functions as eigenfunctions

Hilbert-Schmidt maps in a Hilbert space

Let $\mathcal{H}$ be a Hilbert space. The positive map

$$T \; : \; \mathcal{H} \; \rightarrow \; \mathcal{H}$$

is nuclear (resp. Hilbert-Schmidt) if it is compact and if its non-zero eigenvalues $\lambda_1, \lambda_2, \ldots$ (counted with multiplicity) satisfy

$$\sum_{n=0}^{\infty} \lambda_n < \infty \quad (\text{resp. } \sum_{n=0}^{\infty} \lambda_n^2 < \infty)$$

Hermite functions as eigenfunctions

A positive operator:

Let

$$Hf(x) = -f'' + (x^2+1)f.$$

Then, H (with appropriate domain) is self-adjoint, positive, and the operator H^{-1} has a Hilbert-Schmidt extension.

Eigenfunctions

$$-\xi_n''(z) + (z^2+1)\xi_n(z) = 2(n+1)\xi_n(z)$$

$$H\xi_n = 2(n+1)\xi_n, \quad , n = 0, 1, \ldots$$

Another definition:

Weight $\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$

In the construction of the white noise space the polynomials

$$h_n(x) = 2^{-\frac{n}{2}}H_n(\frac{x}{\sqrt{2}})$$

intervene. They satisfy

$$\int_{\mathbb{R}} h_n(x)h_m(x)\frac{e^{-\frac{x^2}{2}}dx}{\sqrt{2\pi}} = \delta_{n,m}n!.$$

Second lecture: Nuclear Fréchet spaces and a first example of strong algebra

Outline

1. Topological vector spaces
2. Fréchet spaces and nuclear Fréchet spaces.
3. Schwartz space and some spaces of entire functions.
4. A first example of strong algebra.

Topological vector space

Definition

Let $\mathcal{V}$ be a vector space (over the complex numbers, or the real numbers), endowed with a topology. It is called a topological vector space if the maps

$$(\lambda, v) \mapsto \lambda v, \quad \mathbb{C} \times \mathcal{V} \longrightarrow \mathcal{V},$$

(or from $\mathbb{R} \times \mathcal{V}$ into $\mathcal{V}$), and

$$(v, w) \mapsto v + w, \quad \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V},$$

are continuous with respect to this topology.

Absorbing sets and balanced sets

The continuity of the vector space operations forces special properties of the neighborhoods.

Absorbing and balanced sets

A subset $A \subset \mathcal{V}$ is called absorbing if for every $v \in \mathcal{V}$ there exists $t_v > 0$ such that

$$|z| \leq t_v \implies zv \in A.$$

It is called balanced if for every $v \in A$ and $z \in \overline{\mathbb{D}}$ the vector $zv \in A$.

Let p be a semi-norm on a vector space $\mathcal{V}$. Then the set

$$\{v \in \mathcal{V}\,;\, p(v) \leq 1\}$$

is absorbing, convex and balanced.

Barrels

Let $\mathcal{V}$ be a topological vector space. Then:
(1) Every neighborhood V of the origin is absorbing, and contains a balanced neighborhood, and $\lambda V \in \mathcal{N}(0)$ for every $\lambda > 0$.
(2) Every neighborhood contains a closed neighborhood.
(3) There is a basis of neighborhoods of the origin made of closed balanced sets.

Barrel

Let $\mathcal{V}$ be a topological vector space. An absorbing balanced closed convex set is called a barrel.

Locally convex topological vector spaces

The definition of a topological space is too general, in the sense that the topological dual of the space may be reduced to the zero functional.

Locally convex topological vector space

A vector space whose topology is Hausdorff and defined by a family of semi-norms is called locally convex. Equivalently, a vector space is locally convex if it admits a basis of neighborhoods of the origin which are convex. In fact, it has a basis of neighborhoods of sets which are much more structured than convex sets, namely barrels.

Not every topological vector space is locally convex.

Fréchet space

Fréchet space

A locally convex topological vector space which is metrizable and complete is called a Fréchet space.

Definitions:

Locally convex topological vector space: Topology given by a family of norms.

Metrizable: The family is (at most) countable and a metric is given by

$$d(v,w)=\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{p_n(v-w)}{1+p_n(v-w)}$$

Compatible norms

Two norms are called compatible (or pairwise coordinated) if a Cauchy sequence both the norms converges to zero in one of these norms, then it also converges to zero in the other.

Compatible norms

Example (from Gelfand-Shilov's book)

Take $V = C^\infty[0,1]$.

$$\|f\|_1 = \max_{x\in[0,1]} |f(x)| \quad \text{and} \quad \|f\|_2 = \max_{x\in[0,1]} (|f(x)| + |f'(x)|)$$

are compatible norms, but not the norms

$$\|f\|_1 = \max_{x\in[0,1]} |f(x)| \quad \text{and} \quad \|f\|_3 = \max_{x\in[0,1]} |f(x)| + |f'(0)|$$

Explanation:

Take for instance $f_n(x) = \frac{1}{n}e^{-nx}$, $x \in [0,1]$. Then $f_n'(0) = -e^{-1}$ and (f_n) is a Cauchy sequence with respect to the three norms. It converges to 0 in the first two norms, but not with respect to the third one since $f_n'(0) = -e^{-1}$ and so $\|f_n\|_3 \geq e^{-1}$.

Compatible norms

Let p_1, p_2 be two norms given on the vector space $\mathcal{V}$, and assume that $p_1 \leq p_2$ and that p_1 and p_2 are compatible. Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be the respective completion of $\mathcal{V}$ with respect to p_1 and p_2. Let $(v_n)_{n\in\mathbb{N}}$ be a Cauchy sequence with respect to p_2, with limit $\ell_2 \in \mathcal{V}_2$, and $\ell_1 \in \mathcal{V}_1$. Then, the map $\ell_2 \mapsto \ell_1$ is linear, continuous and one-to-one.

Countably normed space

Countably normed space (Gelfand-Shilov terminology)

A countably normed space Φ is a locally convex space whose topology is defined using a countable set of compatible norms $(\|\cdot\|_n)$ i.e. norms such that if a sequence (x_n) that is a Cauchy sequence in the norms $\|\cdot\|_p$ and $\|\cdot\|_q$ converges to zero in one of these norms, then it also converges to zero in the other. $(\|\cdot\|_n)$ can be always assumed to be non-decreasing.

Completeness

Denoting by Φ_p the completion of Φ with respect to the norm $\|\cdot\|_p$, we obtain a sequence of Banach spaces

$$\Phi_1 \supseteq \Phi_2 \supseteq \cdots \supseteq \Phi_p \supseteq \cdots .$$

Φ is complete if and only if $\Phi = \bigcap \Phi_p$, and Φ is a Banach space if and only if there exists some p_0 such that $\Phi_p = \Phi_{p_0}$ for all $p > p_0$.

Dual

A remark

The dual of a countably normed space is not metrizable but under certain hypothesis, has a very nice structure, in particular behavior on sequences.

Nuclear maps

Nuclear maps in a Hilbert space

Let $\mathcal{H}$ be a Hilbert spaces. The positive map

$$f \;:\; \mathcal{H} \;\to\; \mathcal{H}$$

is nuclear if it is compact and if its non-zero eigenvalues $\lambda_1, \lambda_2, \ldots$ satisfy

$$\sum_{n=0}^{\infty} \lambda_j < \infty.$$

Nuclear maps between Hilbert spaces

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces. The positive map

$$f \;:\; \mathcal{H}_1 \;\to\; \mathcal{H}_2$$

is nuclear if $(ff^*)^{1/2}$ is nuclear from $\mathcal{H}_2$ into itself.

Nuclear spaces

Definition

The complete countably normed space $\Phi = \bigcap \Phi_p$ is called nuclear if for every $p \in \mathbb{N}$ there is a $q > p$ such that the inclusion from Φ_q into Φ_p is nuclear.

A first example: Schwartz functions and tempered distributions.

Schwartz functions:

$f \in C^\infty(\mathbb{R})$ and $\lim_{x\to\pm\infty} x^m f^{(n)}(x) = 0$ for all $n, m \in \mathbb{N}_0$.

Equivalent definition

Define

$$\mathcal{H}_p = \ell^2(\mathbb{N}, (n+1)^{2p}), \quad p \in \mathbb{Z}.$$

Then

$$\mathscr{S} = \cap_{p=0}^\infty \mathcal{H}_p \subset \mathbf{L}_2(\mathbb{R}, dx) \subset \mathscr{S}' = \cup_{p=0}^\infty \mathcal{H}_{-p}$$

($s(x) = \sum_{n=1}^\infty a_n \eta_n(x)$, with $\eta_1, \eta_2, \ldots$ the (normalized) Hermite functions)

Topology of $\mathscr{S}$

Recall:

$$\mathcal{H}_p = \ell^2(\mathbb{N}, (n+1)^{2p}) = \mathrm{Dom}\, H^p,$$

where

$$Hf(x) = -f''(x) + (x^2+1)f(x).$$

Topology of the Schwartz space

$\mathscr{S} = \cap_{p=0}^{\infty} \mathrm{Dom}\, H^p$ is a Fréchet nuclear space.
Topology defined by the norms of the spaces $\mathcal{H}_p$

A second example: exponential weights (toward the definition of strong algebras).

$$\mathcal{G}_p = \ell^2(\mathbb{N}, 2^{np}), \quad p \in \mathbb{Z}.$$

and

$$\mathscr{G} = \cap_{p=0}^{\infty} \mathcal{G}_p \subset \mathbf{L}_2(\mathbb{R}, dx) \subset \mathscr{G}' = \cup_{p=0}^{\infty} \mathcal{G}_{-p}$$

Geometric characterization of $\mathscr{G}$

Using Mehler's formula with $t = 1/2$ we have

Geometric characterization of $\mathscr{G}_p$

$\mathscr{G}_p$ is the space of all entire functions $f(z)$ such that

$$\iint_{\mathbb{C}} |f(z)|^2 \, e^{\frac{1-2^{-p}}{1+2^{-p}}x^2 - \frac{1+2^{-p}}{1-2^{-p}}y^2} dxdy < \infty$$

Theorem

$\mathscr{G}$ is the space of all entire functions $f(z)$ such that

$$\iint_{\mathbb{C}} |f(z)|^2 \, e^{\frac{1-2^{-p}}{1+2^{-p}}x^2 - \frac{1+2^{-p}}{1-2^{-p}}y^2} dxdy < \infty \quad \text{for all } p \in \mathbb{N}.$$

See A-Salomon, IDA-QP, 2012.

A first example of strong algebra

$\mathscr{S}'$ versus $\mathscr{G}'$

What is the difference between these two spaces? Both are closed under convolution of coefficients: $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$ belong to $\mathscr{S}'$ (resp. $\mathscr{G}'$) then $(c_n)_{n\in\mathbb{N}_0}$ belongs to $\mathscr{S}'$ (resp. $\mathscr{G}'$) where

$$c_n = \sum_{u=0}^{n} a_u b_{n-u}$$

The relations between the norms of a, b and c are completely different in $\mathscr{S}'$ and $\mathscr{G}'$.

Inequalities

In $\mathscr{S}'$:

Let $a \in \mathcal{H}_p$ and $b \in \mathcal{H}_q$, with $p \geq q+2$, and let $a \star b$ be their convolution. Then $a \star b \in \mathcal{H}_{2p}$ and

$$\|a \star b\|_{2p} \leq A(p-q)\|a\|_p\|b\|_q$$

where $A(p-q) = \sum_{u=0}^{\infty}(u+1)^{q-p}$.

In $\mathscr{G}'$:

Let $a \in \mathcal{G}_p$ and $b \in \mathcal{G}_q$, with $p \geq q+2$, and let $a \star b$ their convolution. Then $a \star b \in \mathcal{G}_p$, and

$$\|a \star b\|_p \leq A(p-q)\|a\|_p\|b\|_q$$

where $A(p-q) = \sum_{u=0}^{\infty} 2^{u(q-p)}$.

A more general case:

Theorem

Let $\alpha = (\alpha_n)_{n\in\mathbb{N}_0}$ be a sequence of strictly positive numbers such that

$$\alpha_{n+m} \le \alpha_n\alpha_m, \quad \forall n, m \in \mathbb{N}_0,$$

and such that $\sum_{n=0}^{\infty} \alpha_n^{-d} < \infty$ for some $d \in \mathbb{N}$. Let

$$\mathcal{K}_p(\alpha) = \left\{ a = (a_n)_{n\in\mathbb{N}_0}, \ ; \|a\|_{-p}^2 = \sum_{n=0}^{\infty} \alpha_n^{-2p}|a_n|^2 < \infty \right\}, \quad p \in \mathbb{N}.$$

Let $a \in \mathcal{K}_{-p}(\alpha)$ and $b \in \mathcal{K}_{-q}(\alpha)$ with $p, q \in \mathbb{N}$ such that $p - q \ge d$. Then the convolution $a \star b \in \mathcal{K}_{-p}(\alpha)$ and

$$\|a \star b\|_{-p} \le A(p-q)\|a\|_{-p}\|b\|_{-q},$$

where $A(p-q) = \left(\sum_{n=0}^{\infty} \alpha_n^{2(q-p)}\right)^{\frac{1}{2}}$.

Remark

For necessity of the condition

$$\alpha_{n+m} \leq \alpha_n \alpha_m, \quad \forall n, m \in \mathbb{N}_0,$$

to obtain conditions on the norms, see A-Salomon IDA-QP.

Strong algebras

Strong algebras (A-Salomon)

Let $\Phi = \bigcup_p \Phi_p'$ be a dual of a complete countably normed space. We call Φ a strong algebra if it satisfies the property that there exists a constant d such that for any q and for any $p > q + d$ there exists a positive constant $A_{p,q}$ such that for any $a \in \Phi_q'$ and $b \in \Phi_p'$,

$$\|ab\|_p \leq A_{p,q}\|a\|_q\|b\|_p \quad \text{and} \quad \|ba\|_p \leq A_{p,q}\|a\|_q\|b\|_p.$$

Remark:

More examples in the sequel. A more general definition (inductive limit of Banach spaces) and properties will be given in the last lecture.

Third lecture: Hida's white noise

Outline

1. Bochner and Bochner-Sazonov (Bochner-Minlos) theorem.
2. Hida's white noise space.
3. Stationary increment stochastic processes

Bochner's theorem

The key to construct the white noise space is an extension of Bochner's theorem.

Bochner's theorem

Let F be a complex-valued continuous function which satisfies

$$\sum_{i,j=0}^{N} F(x_i - x_j)c_i\bar{c}_j \geq 0 \quad \forall N \quad \forall c_i \in \mathbb{C} \quad \forall x_1, x_2, ..., x_n \in \mathbb{R}$$

Then there exists a positive measure μ such that

$$F(u) = \int_{\mathbb{R}} e^{-ixu} d\mu(x)$$

$F(u)$ is the Fourier transform of a positive measure.

Bochner's theorem. A counterexample

A counterexample

Bochner's theorem does not hold in infinite dimensional space. For instance the function

$$F(s_1 - s_2) = e^{-\frac{\|s_1 - s_2\|^2}{2}} = e^{-\frac{\|s_1\|^2}{2}} e^{\langle s_1, s_2 \rangle} e^{-\frac{\|s_2\|^2}{2}}$$

is positive on $\mathbf{L}_2(\mathbb{R})$, but F cannot be written as

$$F(s) = e^{-\frac{\|s\|^2}{2}} = \int_{\mathbf{L}_2(\mathbb{R})} e^{i\langle s, u \rangle_{\mathbf{L}_2(\mathbb{R})}} dP(u)$$

Bochner's theorem, continuation

To see that, take $s = u_n$ an orthonormal basis and apply the dominated convergence theorem to

$$F(u_n) = e^{-\frac{\|u_n\|^2}{2}} = e^{-1/2} = \int_{\mathbf{L}_2(\mathbb{R})} e^{i\langle u_n, u\rangle_{\mathbf{L}_2(\mathbb{R})}} dP(u)$$

taking into account that P would be, if it exists, a probability measure.

Sazonov (Bochner-Minlos) theorem

A counterpart of Bochner's theorem holds for Fréchet nuclear spaces (Sazonov theorem), and in particular, for the Schwartz space $\mathscr{S}$.

Theorem

Let $\mathcal{V}$ be a *real* Fréchet nuclear space, and let $F(v)$ be a complex-valued function positive definite in $\mathcal{V}$ (that is, the kernel $K(v_1, v_2) = F(v_1 - v_2)$ is positive definite on $\mathcal{V}$) and continuous (with respect ot the topology of $\mathcal{V}$). Then there exists a measure P on the Borel sets of $\mathcal{V}'$ such that

$$F(v) = \int_{\mathcal{V}'} e^{i\langle v', v\rangle} dP(v').$$

The white noise space

The function

$$F(s_1 - s_2) = e^{-\frac{\|s_1 - s_2\|^2}{2}} = e^{-\frac{\|s_1\|^2}{2}} e^{\langle s_1, s_2 \rangle} e^{-\frac{\|s_2\|^2}{2}}$$

restricted to $\mathscr{S}$ is still positive.
By Sazonov' theorem,

$$e^{-\frac{\|s\|^2}{2}} = \int_{\mathscr{S}'} e^{i\langle s', s\rangle} dP(s')$$

Hida's white noise space

The white noise space is the probability space

$$\mathscr{W} \stackrel{\text{def}}{=} (\mathscr{S}'(\mathbb{R}), \mathcal{B}(\mathscr{S}'(\mathbb{R})), dP)$$

An isometry. The Gaussian Q_s

Take $\epsilon \geq 0$.

$$e^{-\frac{\epsilon^2\|s\|^2}{2}} = \int_{\mathscr{S}'} e^{-i\epsilon\langle s',s\rangle} dP(s')$$

Let Q_s denote the linear functional $Q_s(s') = \langle s', s\rangle_{\mathscr{S}',\mathscr{S}}$
Comparing powers of ϵ (or noticing that Q_s is Gaussian) we get:

The Gaussian Q_s

$$E(Q_s) = 0 \quad \text{and} \quad E(Q_s^2) = \|s\|^2_{\mathbf{L}_2(\mathbb{R})}.$$

An isometry, continuation

Thus the map $s \to Q_s$ is an isometry from $S \subset \mathbf{L}_2(\mathbb{R})$ into $\mathbf{L}_2(\mathscr{W})$.

Theorem

$s \mapsto Q_s$ extends to an isometry from $\mathbf{L}_2(\mathbb{R})$ into $\mathbf{L}_2(\mathscr{W})$, and

$$\langle Q_{1_{[0,t]}}, Q_{1_{[0,s]}} \rangle_{\mathbf{L}_2(\mathbb{R})} = \min(t, s).$$

Covariance of the Brownian motion. So $Q_{1_{[0,t]}}$ can be seen as a Brownian motion.

Motivation for the white noise space approach

Question

There exist easier and more elementary ways to build the Brownian motion. Why consider this one?

Some motivation

- We will consider $\mathbf{L}_2(\mathscr{W})$ as part of a triple

$$(\mathcal{K}_1, \mathbf{L}_2(\mathscr{W}), \mathcal{K}_{-1})$$

analogous to the triple

$$(\mathscr{S}, \mathbf{L}_2(\mathbb{R}, dx), \mathscr{S}')$$

- $\mathcal{K}_{-1}$ has a structure (dual of Fréchet nuclear space and strong algebra) which allows to use functional analysis tools.

Reminder: the polynomials h_n

In the construction of a basis the white noise space the polynomials

$$h_n(x) = 2^{-\frac{n}{2}} H_n(\frac{x}{\sqrt{2}})$$

will intervene. They satisfy

$$\int_{\mathbb{R}} h_n(x) h_m(x) \frac{e^{-\frac{x^2}{2}} dx}{\sqrt{2\pi}} = \delta_{n,m} n!.$$

Reminder: the Hermite functions

Hermite functions (formulas from Hille; other variations exist)

$$\xi_n(u) = \frac{(-1)^n}{\sqrt[4]{\pi}2^{\frac{n}{2}}} e^{\frac{u^2}{2}} \left(e^{-u^2}\right)^{(n)}.$$

or, normalized:

$$\eta_n(u) = \frac{1}{\sqrt[4]{\pi}2^{\frac{n}{2}}\sqrt{n!}} e^{\frac{u^2}{2}} \left(e^{-u^2}\right)^{(n)}, \quad n = 0, 1, \ldots$$

Orthogonality

It holds that

$$\langle \xi_n, \xi_m \rangle_{\mathbf{L}_2(\mathbb{R}, dx)} = \delta_{m,n} n!$$

Orthogonal basis

In fact the ξ_n form an orthogonal basis of $\mathbf{L}_2(\mathbb{R}, dx)$.

A basis

Among all orthogonal Hilbert bases of $\mathbf{L}_2(\mathscr{W})$, one plays a special role, and is built in terms of the Hermite functions η_n and Hermite polynomials h_n. This basis $(\mathbf{H}_\alpha)$ is indexed by the set ℓ of sequences $(\alpha_1, \alpha_2, \ldots)$ with $\alpha_i \in \mathbb{N} \cup \{0\}$ and only a finite number of $\alpha_i \neq 0$.

Theorem

The functions (random variables)

$$\mathbf{H}_\alpha = \prod_{k=1}^{\infty} h_{\alpha_k}(Q_{\eta_k})$$

form an orthgonal basis of the white noise space and

$$\|\mathbf{H}_\alpha\|_{\mathscr{W}}^2 = \alpha!$$

The Wick product

Definition

The Wick product is defined by

$$\mathbf{H}_\alpha \Diamond \mathbf{H}_\beta = \mathbf{H}_{\alpha+\beta} \quad \alpha = (\alpha_1, \alpha_2, \ldots).$$

Remark

The Wick product is a convolution on the index (also called Cauchy product)

Remark

The Wick product is not stable in the white noise space. Need to look at larger (or smaller) spaces where it is stable. The Kondratiev spaces of stochastic test functions and stochastic distributions.

Stationary increments processes

Definition

A zero mean Gaussian process $\{X(t)\}$ on a probability space is said to be *stationary increment* if the mean-square expectation of the increment $X(t) - X(s)$ is a function only of the time difference $t - s$.

Theorem

There is a measure σ such that the *covariance function* of such a process is of the form

$$E[X(t)X(s)^*] = K_\sigma(t,s) = \int_{\mathbb{R}} \chi_t(u)\chi_s(u)^* d\sigma(u), \quad t,s \in \mathbb{R},$$

where E is expectation, and

$$\chi_t(u) = \frac{e^{itu} - 1}{u}.$$

Stationary increments processes

The spectral measure

The positive measure $d\sigma$ is called the *spectral measure*, and is subject to the restriction

$$\int_{\mathbb{R}} \frac{d\sigma(u)}{u^2+1} < \infty.$$

The covariance function $K_\sigma(t,s)$ can be rewritten as

$$K_\sigma(t,s) = r(t) + \overline{r(s)} - r(t-s),$$

where

$$r(t) = -\int_{\mathbb{R}} \left\{ e^{itu} - 1 - \frac{itu}{u^2+1} \right\} \frac{d\sigma(u)}{u^2},$$

Even measures

When σ is even, r is real and takes the simpler form

$$r(t) = \int_{\mathbb{R}} \frac{1-\cos(tu)}{u^2} d\sigma(u).$$

Stationary increments processes

Question:

Using the white noise space we build stochastic processes with a covariance function of the form

$$r(t) + \overline{r(s)} - r(t-s)$$

The important point, to be elaborated upon later, is that these processes admit a derivative in a larger space.

Example: The fractional Brownian motion

$$t^{2H} + s^{2H} - |t-s|^{2H}, \quad t, s \in \mathbb{R}$$

A class of processes

Theorem (A-Attia-Levanony, SPA (2010))

For m a positive function such that $\int_{\mathbb{R}} \frac{m(u)}{1+u^2} du < \infty$ define

$$\widehat{T_m f}(u) \overset{\text{def.}}{=} \sqrt{m(u)} \widehat{f}(u),$$

where $\widehat{f}$ denotes the Fourier transform of f. Let

$$X_m(t) = Q_{T_m(1_{[0,t]})} \quad \text{and} \quad r(t) = -\int_{\mathbb{R}} \left\{ e^{itu} - 1 - \frac{itu}{u^2+1} \right\} \frac{m(u)du}{u^2}.$$

Then,

$$E(X_m(t)X_m(s)^*) = r(t) + \overline{r(s)} - r(t-s) = \int_{\mathbb{R}} \frac{e^{itu}-1}{u} \frac{e^{-isu}-1}{u} m(u)du,$$

A class of processes

The isometry $s \mapsto Q_s$ allows to build models for processes X_m and study related stochastic integration and calculus. The case

$$m(u) = \frac{1}{2\pi}|u|^{1-2H}du, \quad H \in (0,1),$$

corresponds to the fractional Brownian motion B_H with Hurst parameter H, such that

$$E(B_H(t)B_H(s)) = V_H \left\{|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right\},$$

where

$$V_H = \frac{\Gamma(2-2H)\cos(\pi H)}{\pi(1-2H)H},$$

A class of processes

Theorem

In the sense of distribution it holds that (with $d\sigma(u) = m(u)du$)

$$\frac{\partial^2}{\partial t \partial s} K_r(t,s) = r''(t-s) = \widehat{d\sigma}(s-t)$$

The derivative of X_m exists as a process in a space of stochastic distributions (see Lecture 4).

See A-Attia-Levanony, *On the characteristics of a class of Gaussian processes within the white noise space setting*. Stochastic Processes and their Applications, vol. 120 (issue 7), pp. 1074-1104 (2010).

Singular measures and general measures

This case has been considered in A-Jorgensen-Levanony (JFA, 2011)

Theorem

Let σ be a positive measure subject to

$$\int_{\mathbb{R}} \frac{d\sigma(u)}{1+u^2} < \infty,$$

and assume that $\dim \mathbf{L}_2(d\sigma) = \infty$. There exists a possibly unbounded operator V from $\mathbf{L}_2(d\sigma)$ into $\mathbf{L}_2(\mathbb{R}, dx)$, with domain containing the Schwartz space, such that

$$K_\sigma(t,s) = \langle V(1_{[0,t]}), V(1_{[0,s]}) \rangle_{\mathbf{L}_2(\mathbb{R},dx)},$$

and

$$X_\sigma(t) = Q_{T(1_{[0,t]})}.$$

$\mathcal{C}$ positive Borel measures σ on $\mathbb{R}$ subject to

$$\int_{\mathbb{R}} \frac{d\sigma(u)}{(1+u^2)^p} < \infty$$

for some $p \in \mathbb{N}_0$. Such a measure σ is the spectral function of a homogeneous generalized stochastic field in the sense of Gelfand.

.

We associate to $\sigma\in\mathcal{C}$ four natural objects:
(a) The quadratic form $q_\sigma(\psi)=\int_{\mathbb{R}}|\widehat{\psi}(u)|^2d\sigma(u)$ on the Schwartz space.
(b) A linear operator V_σ such that

$$q_\sigma(\psi)=\|V_\sigma\psi\|^2_{\mathbf{L}_2(\mathbb{R},dx)},\quad \psi\in\mathcal{S}_{\mathbb{R}}.$$

(c) A generalized stochastic process $\{X_\sigma(\psi),\psi\in\mathcal{S}_{\mathbb{R}}\}$ such that

$$E[X_\sigma(\psi_1)\overline{X_\sigma(\psi_2)}]=\int_{\mathbb{R}}\widehat{\psi_1}(u)\overline{\widehat{\psi_2}(u)}d\sigma(u),$$

(d) The fourth object associated to σ is a probability measure $d\mu_\sigma$ on $\mathcal{S}'_{\mathbb{R}}(\mathbb{R})$ such that

$$e^{-\frac{\|\widehat{\psi}\|^2_{\mathbf{L}_2(d\sigma)}}{2}}=\int_\Omega e^{i\langle\omega,\psi\rangle}d\mu_\sigma(\omega),\quad \psi\in\mathcal{S}_{\mathbb{R}}(\mathbb{R})$$

Theorem

Let $\sigma \in \mathcal{C}$. Then, there exists a probability measure μ_σ on $\mathcal{S}'_{\mathbb{R}}(\mathbb{R})$, an element $V_\sigma \in \mathbf{L}(\mathcal{S}_{\mathbb{R}}, \mathbf{L}_2(\mathbb{R}^n, dx))$ and a generalized Gaussian stochastic process $\{X_\sigma(\psi)\}_{\psi \in \mathcal{S}_{\mathbb{R}^n}}$ such that:

$$X_\sigma(\psi) = Q_{V_\sigma(\psi)}, \quad \forall \psi \in \mathcal{S}_{\mathbb{R}}$$

and

$$\int_\Omega e^{i\langle \omega, X_\sigma(\psi)\rangle} d\mu_W(\omega) = \int_\Omega e^{i\langle \omega, \psi\rangle} d\mu_\sigma(\omega) = e^{-\frac{\|\widehat{\psi}\|^2_{\mathbf{L}_2(\sigma)}}{2}}.$$

Fourth lecture:Kondratiev spaces of stochastic test functions and stochastic distributions. Linear stochastic systems

Outline

1. Kondratiev space of stochastic distributions.
2. Some facts on duals of locally convex topological vector spaces.
3. Linear stochastic systems.

The Kondratiev space of stochastic distributions

The white noise space is not stable under the Wick product.

Definition

- For $k \in \mathbb{N}$, we denote by $\mathcal{M}_k$ the Hilbert space of formal series of the form

$$f(\omega) = \sum_{\alpha \in \ell} c_\alpha \mathbf{H}_\alpha(\omega), \quad \text{with} \quad ||f||_k = (\sum_{\alpha \in \ell} c_\alpha^2 (2\mathbb{N})^{-k\alpha})^{1/2} < \infty.$$

  where $(2\mathbb{N})^\alpha = 2^{\alpha_1}(2 \times 2)^{\alpha_2}(2 \times 3)^{\alpha_3} \cdots$.

- The Kondratiev space of stochastic distributions is the inductive limit

$$\mathcal{S}_{-1} = \bigcup_{k \in \mathbb{N}} \mathcal{M}_k.$$

The Kondratiev spaces of stochastic test functions

Definition

Define the Kondratiev space of stochastic test functions

$$S_1 = \bigcap_{k\in\mathbb{N}} \mathcal{G}_k$$

where $\mathcal{G}_k$ the Hilbert space of series of the form

$$f(\omega) = \sum_{\alpha\in\ell} c_\alpha \mathbf{H}_\alpha(\omega), \quad \text{with} \quad ||f||_k = (\sum_{\alpha\in\ell} (\alpha!)^2 c_\alpha^2 (2\mathbb{N})^{k\alpha})^{1/2} < \infty.$$

Note

Analysis in $(\mathcal{S}_1, \mathbf{L}_2(\mathscr{W}), \mathcal{S}_{-1})$.

Wick product and Våge inequalities

- The Kondratiev space is stable under the Wick product.
- Våge's inequality (1996). Let $k > l + 1$.

$$\|h\lozenge u\|_k \leq A(k-l)\|h\|_l\|u\|_k,$$

where

$$A(k-l) = \sum_{\alpha\in\ell}(2\mathbb{N})^{(l-k)\alpha}.$$

The above inequality expresses the fact that the multiplication operator

$$T_h \,:\, u \mapsto h\lozenge u$$

is a bounded map from the Hilbert space $\mathcal{H}_k$ into itself.

Dual of a countably normed space

Bounded sets

A subset A of a topological vector space is called bounded if for every neighborhood V of the origin there exists a positive number $M > 0$ such that

$$A \subset MV.$$

When the original space is countably normed, the dual is not metrizable (strong or weak topology).

Strong dual

Defined in terms of the bounded sets in $\mathcal{V}$: Let B bounded and $\epsilon > 0$. Then,

$$N_B(\epsilon) = \left\{ v' \in \mathcal{V}'; \sup_{v \in B} |v'(v)| < \epsilon \right\}.$$

$f \in \mathcal{V}'$ if and only if there is a $p \in \mathbb{N}$ such that

$$|f(v)| \leq K_p \|v\|_p.$$

Montel spaces

Definition:

A Montel space is a locally convex topological vector space which is Hausdorff and barreled, and in which every bounded set is relatively compact.

Properties:

The strong dual of a Montel space is a Montel space

Remark:

In the setting of Fréchet spaces, Gelfand and Shilov called these spaces *perfect*.

Theorem

Assume that $\mathcal{V}$ is a complete countably normed Hilbert space, $\mathcal{V} = \cap_{n=1}^{\infty} \mathcal{V}_n$, and that for every n there exists $m \geq n$ such that the injection is compact. Then, $\mathcal{V}$ is a Montel space (in fact, $\mathcal{V}$ is a particular case of a Schwartz space).

Definition:

A locally convex Hausdorff space $\mathcal{V}$ is a Schwartz space if for every balanced closed convex neighborhood U of 0, there exists a neighborhood V of the origin with image in $\mathcal{V}_U$ is precompact ($\mathcal{V}_U$ defined from the semi-norm associated to U).

$\mathcal{S}_{-1}$ is not metrizable, but convergence of sequences has a nice property: A sequence converges in $\mathcal{S}_{-1}$ if and only if it converges in one of the spaces $\mathcal{H}_k$ (where $\mathcal{S}_{-1} = \bigcup_{k\in\mathbb{N}} \mathcal{H}_k$)

Same remarks valid also in the non-commutative setting.

Under some growth properties, the derivative of X_m exists as a process in the space $\mathcal{S}_{-1}$ and can define stochastic integrals via

$$\int_a^b x(t)\lozenge X_m'(t)dt$$

Theorem

Assume that the function m satisfies a bound of the type:

$$m(u) \leq \begin{cases} K\,|u|^{-b} & \text{if} \quad |u| \leq 1, \\ K' & \text{if} \quad |u| > 1, \end{cases}$$

where $b < 2$ and $0 < K, K' < \infty$. Then,

$$|(T_m\eta_n)(u)| \leq \tilde{C}_1 n^{\frac{5}{12}} + \tilde{C}_2,$$

where $\tilde{C}_1$ and $\tilde{C}_2$ are constants independent of n.

The m white noise

Definition

The m-white noise $W_m(t)$ is defined by

$$W_m(t) = \sum_{k=1}^{\infty} (T_m \eta_k)(t) \mathbf{H}_{\epsilon^{(k)}}.$$

Theorem

For every real t we have that $W_m(t) \in S_{-1}$, and it holds that

$$X_m(t) = \int_0^t W_m(s) ds, \quad t \in \mathbb{R}.$$

L

et $Y(t)$, $t \in [a,b]$ be an S_{-1}-valued function, continuous in the strong topology of S_{-1}. Then, there exists a $p \in \mathbb{N}$ such that the function $t \mapsto Y(t) \Diamond W_m(t)$ is $\mathcal{H}'_p$-valued, and

$$\int_a^b Y(t,\omega) \Diamond W_m(t) dt = \lim_{|\Delta| \to 0} \sum_{k=0}^{n-1} Y(t_k,\omega) \Diamond \left(X_m(t_{k+1}) - X_m(t_k) \right),$$

where the limit is in the $\mathcal{H}'_p$ norm, with $\Delta : a = t_0 < t_1 < \cdots < t_n = b$ a partition of the interval $[a,b]$ and $|\Delta| = \max_{0 \le k \le n-1} (t_{k+1} - t_k)$.

See A-Attia-Levanony SPA (2010), absolutely continuous measures
A-Jorgensen-Levanony JFA (2011), singular measures

Theorem

Let (E, d) be a compact metric space, and let f be a continuous function from E into the dual of a countably normed perfect space $\Phi = \cap_{n=1}^{\infty} \mathcal{H}_n$, endowed with the strong topology. Then there exists a $p \in \mathbb{N}$ such that $f(E) \subset \mathcal{H}_p'$, and f is uniformly continuous from E into $\mathcal{H}_p'$, the latter being endowed with its norm induced topology.

What is a linear system

Formally,

$$\text{“}Tf(x) = \int_{\mathbb{R}} K(x,y)f(x)dx\text{”}$$

where input and output are in $\mathbf{L}_2(\mathbb{R}, dx)$.

Identity cannot be written in such a way!

What is a linear system: Schwartz' kernel theorem

Zemanian, Yger: A linear continuous map T from (say) the Schwartz space $\mathscr{S}$ (*test functions*) into its dual, and so by Schwartz' kernel theorem, it is a distribution in two variables, K:

$$\langle Tf, g\rangle_{\mathscr{S},\mathscr{S}'} = \langle K, f(x)g(y)\rangle_{\mathscr{S}(\mathbb{R}^2),(\mathscr{S}(\mathbb{R}^2))'}$$

i.e. (with a big abuse of notation)

$$\langle Tf, g\rangle_{\mathscr{S},\mathscr{S}'} = \iint_{\mathbb{R}^2} K(x,y)f(x)g(y)dxdy,$$

or (still with a big abuse of notation)

$$Tf(y) = \int_{\mathbb{R}} K(x,y)f(x)dx.$$

So, input-output linear continuous relation (with respect to appropriate topologies), which allows to model various situations in engineering.

$$\text{``}Tf(x) = \int_{\mathbb{R}} K(x,y)f(x)dx\text{''}$$

When

$$\boxed{K(x,y) = k(x-y)}$$

(time-invariance), obtain a *convolution* and the Laplace tranform of k is called the *tranfer function*.

Linear time invariant systems (LTI): The classical case

- The output of an LTI system is given by

$$y_n = \sum_{m=0}^{n} h_m u_{n-m}, \quad n \in \mathbb{N},$$

- Taking Z-transforms, we get

$$\mathscr{Y}(z) = \mathscr{H}(z)\mathscr{U}(z),$$

where

$$\mathscr{H}(z) = \sum_{m=0}^{\infty} h_m z^m.$$

is the *transfer function* is the Z-transform of the impulse response (h_m):

- Properties of $\mathscr{H}$ $\Leftrightarrow$ Properties of the system.

Linear time invariant systems (LTI): The classical case

Dissipativity: Want

$$\sum_n |y_n|^2 \leq \sum_n |u_n|^2.$$

A time-invariant, causal linear system is dissipative if and only if its transfer function $\mathscr{H}$ is analytic and contractive in the open unit disk, or equivalently iff the kernel

$$K_{\mathscr{H}}(z,w) = \frac{1 - \mathscr{H}(z)\mathscr{H}(w)^*}{1 - zw^*}$$

is positive definite in the open unit disk.

Positive definite kernel

All the matrices

$$\begin{pmatrix} K_{\mathscr{H}}(w_1, w_1) & \cdots & K_{\mathscr{H}}(w_1, w_N) \\ K_{\mathscr{H}}(w_2, w_1) & \cdots & K_{\mathscr{H}}(w_2, w_N) \\ \vdots & \vdots & \vdots \\ K_{\mathscr{H}}(w_N, w_1) & \cdots & K_{\mathscr{H}}(w_N, w_N) \end{pmatrix} \geq 0.$$

Schur functions

$$K_{\mathscr{H}}(z,w)=\frac{1-\mathscr{H}(z)\mathscr{H}(w)^*}{1-zw^*}$$

is positive definite in the open unit disk if and only if the operator of multiplication by $\mathscr{H}$ is a contraction from the Hardy space $\mathbf{H}_2$ into itself.

$$\mathbf{H}_2=\left\{\sum_{n=0}^{\infty}a_nz^n\,;\,\sum_{n=0}^{\infty}|a_n|^2<\infty\right\}$$

Schur functions

A function $\mathscr{H}$, analytic and contractive in the open unit disk (and its generalizations to various settings such as SCV, hyperholomorphic, ...) can be seen as:
(1) A contractive multiplication operator from the Hardy space into itself.
(2) The characteristic operator function of a contractive operator (and gives then a "model" for this operator).
(3) The transfer function of a dissipative time-invariant linear system.
(4) The scattering function of a certain medium or network.

State space equations: The classical case

- State space equations (Kalman)

$$\begin{aligned} x_{n+1} &= Ax_n + Bu_n, \quad n = 0, 1, \ldots \\ y_n &= Cx_n + Du_n, \end{aligned}$$

- Taking the Z transform, assuming $x_0 = 0$, we have

$$\begin{aligned} \mathscr{X}(z) &= zA\mathscr{X}(z) + zB\mathscr{U}(z) \\ \mathscr{Y}(z) &= C\mathscr{X}(z) + D\mathscr{U}(z), \end{aligned}$$

- The transfer function $\mathscr{H}(z)$ is a rational function

$$\mathscr{H}(z) \stackrel{\text{def}}{=} \frac{\mathscr{Y}(z)}{\mathscr{X}(z)} = D + zC(I_N - zA)^{-1}B.$$

Linear system theory on linear commutative rings

Can assume in

$$y_n = \sum_{m=0}^{n} h_m u_{n-m}, \quad n \in \mathbb{N},$$

and in

$$\begin{aligned} x_{n+1} &= Ax_n + Bu_n, \quad n = 0, 1, \ldots \\ y_n &= Cx_n + Du_n, \end{aligned}$$

the various quantities to belong to a commutative ring (then no realization $D + zC(I - zA)^{-1}B$ in general)

Rational functions of one complex variable

Let f be a matrix–valued function analytic in a neighborhood of the origin. The fact that it is rational can be translated into a number of equivalent definitions, each one catching a specific aspect of the notion of rationality.

Rational functions of one complex variable

Rational functions: Characterizations

Def. 1: *In terms of quotient of polynomials:*

$$f(z) = \frac{Q(z)}{q(z)}, \quad q(0) \neq 0.$$

Def. 2: *In terms of realizations:*

$$f(z) = D + zC(I - zA)^{-1}B.$$

Def. 3: *In terms of Taylor coefficients of the series $f(z) = f_0 + zf_1 + \cdots$:*

$$f_n = CA^{n-1}B, \quad n = 2, 3, \ldots$$

Rational functions of one complex variable

Rational functions: Characterizations

Def. 4: *In terms of a Hankel operator:*

$$\text{rank}\begin{pmatrix} f_1 & f_2 & f_3 & \cdots \\ f_2 & f_3 & \cdots & \\ f_3 & f_4 & \cdots & \\ \cdot & \cdot & \cdots & \end{pmatrix} < \infty.$$

Def. 5: *In terms of backward–shift invariant spaces.*

$$R_0 f(z) = \frac{f(z) - f(0)}{z}$$

f analytic in a neighbourhood of 0 is rational iff

$$\text{linear span }\left\{R_0 f, R_0^2 f, \ldots\right\} < \infty.$$

Rational functions of one complex variable

Last definition seems a triviality, will be very important in the hyperholomorphic case.

Rational functions: Yet another characterization

Def 6: *In terms of restriction.* $R(x+iy)$ *will be analytic in the variable* z *iff* $R(x)$ *is rational from* $\mathbb{R}$ *into* $\mathbb{R}^2$.

First summary

Have introduced linear time-invariant systems, Schur functions, state space equations and realizations of rational functions $(D + zC(I - zA)^{-1}B)$.

The backward-shift operator

$$R_0 f(z) = \frac{f(z)-f(0)}{z} = \int_0^1 \frac{\partial}{\partial z} f(tz)dt$$

plays an important role.

First summary (cont.)

Want to study similar notions when, in

$$y_n = \sum_{m=0}^{n} h_m u_{n-m}, \quad n = 0, 1, 2, \ldots$$

one allows randomness in the various quantities.

We will be in the setting of linear system theory in a certain (very specific) commutative ring

Linear stochastic systems

WNS state space equations

$$\begin{aligned} x_{n+1} &= A\Diamond x_n + B\Diamond u_n \quad n = 0, 1, \ldots \\ y_n &= C\Diamond x_n + D\Diamond u_n, \end{aligned}$$

where

$$A \in (\mathcal{S}_{-1})^{N\times N}, B \in (\mathcal{S}_{-1})^{N\times q}, C \in (\mathcal{S}_{-1})^{p\times N}, D \in (\mathcal{S}_{-1})^{p\times q},$$
$$x_n \in (\mathcal{S}_{-1})^{N}, y_n \in (\mathcal{S}_{-1})^{p} \quad \textit{and} \quad u_n \in (\mathcal{S}_{-1})^{q}.$$

Transfer function

Transfer function

$$H(z,\omega) = D + zC(I-zA)^{-1}B = \underbrace{D_0 + zC_0(I-zA_0)^{-1}B_0}_{\text{Non random term}} +$$

$$\underbrace{+ \sum_{\alpha \neq (0,0,\ldots)} \mathbf{H}_\alpha(\omega)(D_\alpha + zC_\alpha(I-zA_\alpha)^{-1}B_\alpha}_{\text{random terms}}$$

This ends the part on the white noise space setting.

- Allow randomness in input and in impulse response and keep Gaussian input-output relations.
- We work in the setting of the white noise space

$$y_n = \sum_{m=0}^{n} h_m \lozenge u_{n-m}.$$

- When h_m or u_{n-m} is deterministic, the Wick product $h_m \lozenge u_{n-m}$ becomes the usual product.
- White noise space analysis has been developed by Hida since 1975; include much wider setting than one discussed here (Poisson processes, generalized processes as in Gelfand-Vilenkin ...).

Fifth Lecture: The free setting and non-commutative stochastic distributions

Outline

- Fock spaces (full and symmetric)
- Kondratiev space of non commutative stochastic distributions.
- The free case

Fock spaces

Fock spaces

$\mathcal{H}$ general separable Hilbert space, and $\mathcal{H}^{\otimes n} = \underbrace{\mathcal{H} \otimes \mathcal{H} \otimes \cdots \otimes \mathcal{H}}_{n \text{ tensor factors}}$.

$$\Gamma(\mathcal{H}) = \mathbb{C}\Omega \otimes \mathcal{H} \oplus \mathcal{H}^{\otimes 2} \oplus \cdots$$

and $\Gamma_{\rm sym}(\mathcal{H})$ when considering symmetrized tensor products.

Commutative versus non-commutative

Define $\mathcal{H}_p = \ell^2(\mathbb{N}, (n+1)^p)$ Then,

$$\mathscr{S} = \cap_{p=0}^{\infty} \mathcal{H}_p \subset \mathbf{L}_2(\mathbb{R}, dx) \subset \mathscr{S}' = \cup_{p=0}^{\infty} \mathcal{H}_{-p}$$

($s(x) = \sum_{n=1}^{\infty} a_n \eta_n(x)$, with $\eta_1, \eta_2, \ldots$ the Hermite functions)

$$\underbrace{\mathcal{S}_1 = \cap_{p=0}^{\infty} \Gamma_{\rm sym}(\mathcal{H}_p)}_{\substack{\text{Kondratiev}\\ \text{stochastic}\\ \text{test functions}}} \subset \underbrace{\Gamma_{\rm sym}(\mathbf{L}_2(\mathbb{R}, dx))}_{\substack{\text{commutative}\\ \text{white noise}\\ \text{space}}} \subset \underbrace{\mathcal{S}_{-1} = \cup_{p=0}^{\infty} \Gamma_{\rm sym}(\mathcal{H}_{-p})}_{\substack{\text{Kondratiev space}\\ \text{of stochastic}\\ \text{distributions}}}$$

$$\underbrace{\widetilde{\mathcal{S}_1} = \cap_{p=0}^{\infty} \Gamma(\mathcal{H}_p)}_{\substack{\text{Stochastic}\\ \text{non commutative}\\ \text{test functions}}} \subset \underbrace{\Gamma(\mathbf{L}_2(\mathbb{R}, dx))}_{\substack{\text{Non commutative}\\ \text{white noise}\\ \text{space}}} \subset \underbrace{\widetilde{\mathcal{S}_{-1}} = \cup_{p=0}^{\infty} \Gamma(\mathcal{H}_{-p})}_{\substack{\text{Stochastic}\\ \text{non commutative}\\ \text{distributions}}}$$

Commutative versus non-commutative

The inequalities on the norms for $\mathcal{S}_{-1}$

$\mathcal{S}_{-1} = \cup_{p=0}^{\infty}\Gamma_{\rm sym}(\mathcal{H}_{-p})$ is an algebra with the Wick product (denoted here by $\lozenge$) and Våge's inequality (1996) holds. Let $p > q+1$.

$$\|f\lozenge g\|_p \leq A(p-q)\|f\|_p\|g\|_q,$$

where

$$A(p-q) = \sum_{\alpha\in\ell}(2\mathbb{N})^{(p-q)\alpha}.$$

The inequalities on the norms for $\widetilde{\mathcal{S}_{-1}}$

$\widetilde{S_{-1}} = \cup_{p=0}^{\infty}\Gamma(\mathcal{H}_{-p})$ is an algebra with $\otimes$ and:

$$\|f\otimes g\|_q \leq B(p-q)\|f\|_p\|g\|_q, \quad q \geq p+2$$

A remark

A general study of such algebras and their applications has been done (A-Salomon, IDA-QP (2012), JFA, SPA (2013) and Arxiv preprint). They combine ideas from Banach spaces and nuclear spaces (recall that a Banach space is not nuclear in the infinite dimensional case)

In commutative case, the stochastic processes were $\Gamma_{\rm sym}(\mathbf{L}_2)$-valued and their derivatives were $\mathcal{S}_{-1}$-valued. Now the stochastic processes are bounded operators from $\Gamma(\mathbf{L}_2)$ into itself and their derivatives are continuous operators from $\widetilde{\mathcal{S}_1}$ (space of non commutative test functions) into $\widetilde{\mathcal{S}_{-1}}$ (space of non commutative distributions)

The semi-circle law

For these slides we take as source the book *Free random variables* by Voiculescu, Dykema and Nica (AMS, 1992).

The semi-circle law

Let $a \in \mathbb{R}$ and $r > 0$. The semi-circle law is the distribution on $\mathbb{C}[X]$ defined by

$$\gamma_{a,r}(P) = \frac{2}{\pi r^2} \int_{a-r}^{a+r} P(t)\sqrt{r^2 - (t-a)^2} dt.$$

Remark

As in the Gaussian case, uniquely determined by moments of order one and two.

$$\gamma_{a,r}(X) = a \quad \text{and} \quad \gamma_{a,r}(X^2) = a^2 + \frac{r^2}{4}.$$

A pause: Tchebycheff polynomials of the second kind

Definition

The Tchebycheff polynomials of the second kind are an orthonormal basis of the space $\mathbf{L}_2([-1,1],\sqrt{1-x^2}dx)$. They are defined by

$$U_n(x)=\frac{\sin(n+1)\theta}{\sin\theta},\quad \text{with}\quad x=\cos\theta.$$

Orthogonality

We have

$$\frac{2}{\pi}\int_{-1}^{1}U_n(x)U_m(x)\sqrt{1-x^2}dx=\delta_{mn},$$

where δ_{mn} is Kronecker's symbol.

A pause: Tchebycheff polynomials of the second kind

Theorem

Assume $m \geq n$. Then the following linearization formula holds:

$$U_m U_n = \sum_{k=0}^{n} U_{m-n+2k}$$

Non commutative probability space

von Neumann algebra

Subalgebra of $B(\mathcal{H})$ which is weakly closed

Non commutative probability space

Unital algebra with a linear function sending the identity to 1. Random variables are elements in the algebra.

Distribution of a random variable f

$$\mu_f(P) = \phi(P(f)), \quad P \in \mathbb{C}[X].$$

Self-adjoint case in C^*-algebra case (and then ϕ is a state)

$$\mu_f(P) = \int_{\mathbb{R}} P(t) d\mu_f(t), \quad P \in \mathbb{C}[X].$$

Freeness

Let $(\mathcal{A}, \phi)$ be a non commutative probability space. A family of unital subalgebras $(\mathcal{A}_i)_{i\in I}$ is called free if

$$\phi(a_1 a_2 \ldots a_n) = 0$$

when $a_j \in A_{i_j}$ and $i_1 \neq i_2$,$i_2 \neq i_3$, ... implies that $\phi(a_{i_j}) = 0$ for all j.

Some generalities

$\mathcal{H}$ general separable Hilbert space, and $\mathcal{H}^{\otimes n} = \underbrace{\mathcal{H} \otimes \mathcal{H} \otimes \cdots \otimes \mathcal{H}}_{n \text{ tensor factors}}$.

$$\Gamma(\mathcal{H}) = \mathbb{C}\Omega \otimes \mathcal{H} \oplus \mathcal{H}^{\otimes 2} \oplus \cdots$$

$$\ell_h(f) = h \otimes f \quad \text{and} \quad X_h = \ell_h + \ell_h^*$$

$\mathcal{M}_{\mathcal{H}}$: von Neumann algebra generated by the operators T_h. It is a type II_1 algebra, with trace

$$\tau(F) = \langle \Omega, F\Omega \rangle_{\Gamma(\mathcal{H})}$$

$$\tau(X_h^* X_k) = \langle h, k \rangle$$

Preliminary results

Let $h \in \Gamma(\mathcal{H})$ of norm 1. Then $2T_h$ has as its distribution a semi-circle law $C_{0,1}$.

Let $\mathcal{H}_1, \mathcal{H}_2, \ldots$ be pairwise orthogonal Hilbert subspaces of $\mathcal{H}$. Then, the family of algebras $\mathcal{M}_{\mathcal{H}_1}, \mathcal{M}_{\mathcal{H}_2}, \ldots$ is free.

First comparison with white noise space setting

Commutative setting

Recall in WNS setting we had:

$$Q_s(u) = \langle u, s \rangle$$

and

$$\langle Q_h, Q_k \rangle_{WNS} = \langle h, k \rangle$$

Non commutative setting

Now one has (with $X_h = \ell_h + \ell_h^*$)

$$\tau(X_h^* X_k) = \langle h, k \rangle$$

A basis

$\mathcal{M}_{\mathcal{H}}$: von Neumann algebra generated by the operators T_h.

The map $F \mapsto F\Omega$ is one-to-one onto from the von Neumann algebra $\mathcal{M}_{\mathcal{H}}$ onto $\Gamma(\mathcal{H})$.

$\mathbf{L}_2(\tau)$: closure of $\mathcal{M}_{\mathcal{H}}$ with respect to τ.

A basis of $\mathbf{L}_2(\tau)$ is obtained using the Tchebycheff polynomials of the second kind. The semi circle law and freeness are then used.

Recall: If $U_0, U_1, \ldots$ are the Tchebycheff polynomials of the second kind;

$$\frac{2}{\pi}\int_{-1}^{1} U_n(x)U_m(x)\sqrt{1-x^2}dx = \delta_{mn}$$

An orthonormal basis

In the statement the indices are as follows: The space $\widetilde{\ell}$ denotes the free monoid generated by $\mathbb{N}_0$. We write an element of $1 \neq \alpha \in \widetilde{\ell}$ as

$$\alpha = z_{i_1}^{\alpha_1} z_{i_2}^{\alpha_2} \cdots z_{i_k}^{\alpha_k},$$

where $\alpha_1, i_1, \ldots \in \mathbb{N}$ and $i_1, \ldots i_k \in \mathbb{N}_0$ are such that

$$i_1 \neq i_2 \neq \cdots \neq i_{k-1} \neq i_k.$$

Theorem

Let $h_0, h_1, h_2, \ldots$ be any orthonormal basis of $\mathbf{L}_2(d\sigma)$. The functions

$$U_\alpha = U_{\alpha_1}(T_{h_{i_1}}) \cdots U_{\alpha_k}(T_{h_{i_k}}), \tag{8}$$

where $\alpha = z_{i_1}^{\alpha_1} z_{i_2}^{\alpha_2} \cdots z_{i_k}^{\alpha_k} \in \widetilde{\ell}$ form an orthonormal basis for $\mathbf{L}_2(\tau)$.

Non commutative Brownian motion

Recall $T_h = \ell_h + \ell_h^*$ and $\tau(T_k^* T_h) = \langle h, k \rangle$.

Let $t \mapsto f_t$ be an $\mathbf{L}_2(\mathbb{R}, dx)$-valued function. Let $Y_t = X_{f_t}$. Then,

$$\tau(Y_s^* Y_t) = \int_{\mathbb{R}} f_t(x) f_s(x) dx$$

Case $f_t(x) = 1_{[0,t]}(x)$ corresponds to

$$\tau(Y_s^* Y_t) = t \wedge s$$

non commutative Brownian motion (Bozejko-Lytvynov)

Non commutative stochastic distributions

Define $\mathcal{H}_p = \ell^2(\mathbb{N}, (n+1)^p)$ Then,

$$\mathscr{S} = \cap_{p=0}^{\infty} \mathcal{H}_p \subset \mathbf{L}_2(\mathbb{R}, dx) \subset \mathscr{S}' = \cup_{p=0}^{\infty} \mathcal{H}_{-p}$$

$$\underbrace{\widetilde{\mathcal{S}_1} = \cap_{p=0}^{\infty} \Gamma(\mathcal{H}_p)}_{\substack{\text{Stochastic} \\ \text{non commutative} \\ \text{test functions}}} \subset \underbrace{\Gamma(\mathbf{L}_2(\mathbb{R}, dx))}_{\substack{\text{Non commutative} \\ \text{white noise} \\ \text{space}}} \subset \underbrace{\widetilde{\mathcal{S}_{-1}} = \cup_{p=0}^{\infty} \Gamma(\mathcal{H}_{-p})}_{\substack{\text{Stochastic} \\ \text{non commutative} \\ \text{distributions}}}$$

Inequalities on norms

The inequalities on the norms for $\mathcal{S}_{-1}$

$\widetilde{S_{-1}} = \cup_{p=0}^{\infty}\Gamma(\mathcal{H}_{-p})$ is an algebra with $\otimes$ and:

$$\|f \otimes g\|_q \leq B(p-q)\|f\|_p\|g\|_q, \quad q \geq p+2$$

Non commutative stationary increments processes

More generally, can build processes such that

$$\tau(Y_s^* Y_t) = r(t) + \overline{r(s)} - r(t-s)$$

for a wide class of r; includes non commutative version of the fractional Brownian motion.

Take $m(x) \geq 0$ with appropriate bounds at 0 and infinity, and define

$$\widehat{T_m f} = \sqrt{m}\widehat{f}.$$

The choice $f_t = T_m 1_{[0,t]}$ leads to

$$\tau(Y_s^* Y_t) = \int_{\mathbb{R}} \frac{e^{itx} - 1}{x} \frac{e^{-sx} - 1}{x} m(x) dx = r(t) + \overline{r(s)} - r(t-s)$$

with $r(t) = -\int_{\mathbb{R}} (e^{itx} - 1 - \frac{itx}{x^2+1}) m(x) dx$.

Derivatives

Let $m(x) \geq 0$ with appropriate bounds at 0 and infinity. Let $Y_t = X_{T_m 1_{[0,t]}}$. There exists a $L(\widetilde{\mathcal{S}_1}, \widetilde{\mathcal{S}_{-1}})$-valued function $W_m(t)$such that for all $f \in \widetilde{\mathcal{S}_1}$

$$\frac{d}{dt} Y_t f = W_m(t) f$$

in the topology of $\widetilde{\mathcal{S}_{-1}}$

A table comparing the commutative and free cases

The setting	Commutative	Free setting
The underlying space	Symmetric Fock space	Full Fock space
Concrete realization	via Bochner-Minlos $\mathbf{L}_2(\mathcal{S}', dP)$	$\mathbf{L}_2(\tau)$
Polynomials	Hermite polynomials	Tchebycheff polynomials of the second kind
The building blocks	Functions $\mathbf{H}_\alpha$	$U_\alpha = \mathbf{U}_{\alpha_1}(T_{h_{i_1}})\cdots$
Distribution law	Gaussian	Semi-circle

Other Gelfand triples

One can replace the space $\mathcal{H}_p = \ell^2(\mathbb{N}, (n+1)^p)$ by other weights $\mathcal{K}_p = \ell^2(a_n^p)$ satisfying $a_{n+m} \leq a_n a_m$ and

$$\sum_{n=1}^{\infty} a_n^{-d} < 1$$

for some $d \in \mathbb{N}$.

$\cup_p \mathcal{K}_p$ is nuclear and the space $\cup_p \Gamma(\mathcal{K}_p)$ has the property that

$$\|f \otimes g\|_q \leq M_{pq} \|f\|_p \|g\|_q, \quad q \geq p + d$$

with $M_{pq}^2 = \frac{1}{1 - \sum_{n=1}^{\infty} a_n^{p-q}}$.

Strong algebras

Let G be a locally compact topological group with a left Haar measure μ. The convolution of two measurable functions f and g is defined by

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)d\mu(y).$$

$L_1(G,\mu)$ is a Banach algebra with the convolution product, while $L_2(G,\mu)$ is usually not closed under the convolution.

Theorem (Rickert, 1968)

$L_2(G,\mu)$ is closed under convolution if and only if G is compact.

In this case G is unimodular (i.e. its left and right Haar measure coincide) and it holds that

$$\|f * g\| \leq \sqrt{\mu(G)}\|f\|\|g\|, \qquad \text{for any } f,g \in L_2(G,\mu).$$

Hence

$L_2(G,\mu)$ is a convolution algebra which is also a Hilbert space.

Strong algebras

New convolution algebras which behave locally as Hilbert spaces, rather than being Banach spaces, even when the group G is not compact.

Strong algebras

Let (μ_p) be a sequence of measures on G such that

$$\mu \gg \mu_1 \gg \mu_2 \gg \cdots ,$$

(where μ is the left Haar measure of G) and let $S \subseteq G$ be a Borel semi-group with the following property: There exists a non negative integer d such that for any $p > q + d$ and any $f \in L_2(S, \mu_q)$ and $g \in L_2(S, \mu_p)$, the products $f * g$ and $g * f$ belong to $L_2(S, \mu_p)$ and

$$\|f * g\|_p \leq A_{p,q}\|f\|_q\|g\|_p \quad \text{and} \quad \|g * f\|_p \leq A_{p,q}\|f\|_q\|g\|_p,$$

where $\|\cdot\|_p$ is the norm associated to $L_2(S, \mu_p)$, where $A_{p,q}$ is a positive constant

Strong algebras

More generally, we consider dual of reflexive Fréchet spaces of the form $\bigcup_{p\in\mathbb{N}}\Phi'_p$, where the Φ'_p are (dual of) Banach spaces with associated norms $\|\cdot\|_p$), which are moreover algebras and carry inequalities of the form

$$\|ab\|_p \le A_{p,q}\|a\|_q\|b\|_p \quad \text{and} \quad \|ba\|_p \le A_{p,q}\|a\|_q\|b\|_p$$

for $p > q + d$, where d is preassigned and $A_{p,q}$ is a constant. We call these spaces *strong algebras*.

Properties

Strong algebras are topological algebras. Furthermore, the multiplication operators

$$L_a : x \mapsto ax \text{ and } R_a : x \mapsto xa$$

are bounded from the Banach space Φ'_p into itself where $a \in \Phi'_q$ and $p > q + d$.

Functional calculus

If $\sum_{n=0}^{\infty} c_n z^n$ converges in the open disk with radius R, then for any $a \in \Phi'_q$ with $\|a\|_q < \frac{R}{A_{q+d+1,q}}$, we obtain

$$\sum_{n=0}^{\infty} |c_n| \|a^n\|_{q+d+1} \leq \sum_{n=0}^{\infty} |c_n| (A_{p,q} \|f\|_q)^n < \infty,$$

and hence $\sum_{n=0}^{\infty} c_n a^n \in \Phi'_{q+d+1}$.

Example: $\Phi'_p = L_2(S, \mu_p)$

Theorem

Assume that for every $x, y \in S$ and for every $p \in \mathbb{N}$

$$\frac{d\mu_p}{d\mu}(xy) \le \frac{d\mu_p}{d\mu}(x)\frac{d\mu_p}{d\mu}(y),$$

Then, for every choice of $f \in L_2(S, \mu_q)$ and $g \in L_2(S, \mu_p)$,

$$\begin{gathered} \|f * g\|_p \le \left(\int_S \frac{d\mu_p}{d\mu_q} d\mu\right)^{\frac{1}{2}} \|f\|_q \|g\|_p \\ \text{and} \\ \|g * f\|_p \le \left(\int_S \frac{d\mu_p}{d\mu_q} d\widetilde{\mu}\right)^{\frac{1}{2}} \|f\|_q \|g\|_p, \end{gathered} \tag{9}$$

where $\widetilde{\mu}$ is the right Haar measure.

In particular, if there exists a non negative integer d such that

$$\int_S \frac{d\mu_p}{d\mu_q} d\mu < \infty \quad \text{and} \quad \int_S \frac{d\mu_p}{d\mu_q} d\widetilde{\mu} < \infty$$

for every positive integers p, q such that $p > q + d$, then $\bigcup_p L_2(S, \mu_p)$ is a strong convolution algebra (with $A_{p,q}^2 = \max\left(\int_S \frac{d\mu_p}{d\mu_q} d\mu, \int_S \frac{d\mu_p}{d\mu_q} d\widetilde{\mu}\right)$).

Let $\mathcal{K}_0$ be a separable Hilbert space, and let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{K}_0$. Furthermore, let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers greater than or equal to 1. For any $p \in \mathbb{Z}$, we denote

$$\mathcal{K}_p = \left\{ \sum_{n=1}^{\infty} f_n e_n : \sum_{n=1}^{\infty} |f_n|^2 a_n^p < \infty \right\} \cong \mathbf{L}^2(\mathbb{N}, a_n^p).$$

We note that

$$\cdots \subseteq \mathcal{K}_2 \subseteq \mathcal{K}_1 \subseteq \mathcal{K}_0 \subseteq \mathcal{K}_{-1} \subseteq \mathcal{K}_{-2} \subseteq \cdots ,$$

where the embedding $T_{q,p} : \mathcal{K}_q \hookrightarrow \mathcal{K}_p$ satisfies

$$\| T_{q,p} a_n^{-q/2} e_n \|_p = a_n^{-(q-p)/2} \| a_n^{-p/2} e_n \|_q,$$

and hence

$$\| T_{q,p} \|_{HS} = \sqrt{\sum_{n\in\mathbb{N}} a_n^{-(q-p)}}.$$

The dual of a Fréchet space is nuclear if and only if the initial space is nuclear. Thus, $\bigcup_{p\in\mathbb{N}} \mathcal{K}_{-p}$ is nuclear if and only if $\bigcap_{p\in\mathbb{N}} \mathcal{K}_p$ is nuclear. This is turn will hold if and only if for any p there is some $q > p$ such that

Let $T : \mathcal{H}_1 \to \mathcal{H}_2$ be a compact contraction operator between two separable Hilbert spaces with

$$Tf = \sum_{n=1}^{\infty} \lambda_n \langle f, e_n \rangle h_n$$

where $(e_n)_{n\in\mathbb{N}}$ and $(h_n)_{n\in\mathbb{N}}$ are orthonormal basis of $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively and where (λ_n) is a non-negative sequence converging to zero.

Theorem

Let $\Gamma(T)$ be its second quantization. Then,

(a) It holds that

$$\Gamma(T)f = \sum_{\alpha\in\widetilde{\ell}} \lambda_{\mathbb{N}}^{\alpha}\langle f, e_\alpha\rangle h_\alpha,$$

where $(e_\alpha)_{\alpha\in\widetilde{\ell}}$ and $(h_\alpha)_{\alpha\in\widetilde{\ell}}$ are orthonormal basis of $\Gamma(\mathcal{H}_1)$ and $\Gamma(\mathcal{H}_2)$ respectively.

(b) If furthermore T is an Hilbert-Schmidt operator, i.e. $(\lambda_n)\in\ell^2(\mathbb{N})$, then

$$\|\Gamma(T)\|_{HS}^2 = \sum_{n=0}^{\infty} \|T\|_{HS}^{2n}.$$

In particular, $\Gamma(T)$ is a Hilbert-Schmidt operator if and only if T is a Hilbert-Schmidt operator with $\|T\|_{HS} < 1$ and in this case we obtain

$$\|\Gamma(T)\|_{HS} = \frac{1}{\sqrt{1-\|T\|_{HS}^2}}$$

Non commutative stochastic distributions

Define $\mathcal{H}_p = \ell^2(\mathbb{N}, (n+1)^p)$ Then,

$$\mathscr{S} = \cap_{p=0}^{\infty} \mathcal{H}_p \subset \mathbf{L}_2(\mathbb{R}, dx) \subset \mathscr{S}' = \cup_{p=0}^{\infty} \mathcal{H}_{-p}$$

$$\underbrace{\widetilde{\mathcal{S}_1} = \cap_{p=0}^{\infty} \Gamma(\mathcal{H}_p)}_{\text{Stochastic non commutative test functions}} \subset \underbrace{\Gamma(\mathbf{L}_2(\mathbb{R}, dx))}_{\text{Non commutative white noise space}} \subset \underbrace{\widetilde{\mathcal{S}_{-1}} = \cup_{p=0}^{\infty} \Gamma(\mathcal{H}_{-p})}_{\text{Stochastic non commutative distributions}}$$

Inequalities on norms

The inequalities on the norms for $\mathcal{S}_{-1}$

$\widetilde{S_{-1}} = \cup_{p=0}^{\infty}\Gamma(\mathcal{H}_{-p})$ is an algebra with $\otimes$ and:

$$\|f \otimes g\|_q \leq B(p-q)\|f\|_p\|g\|_q, \quad q \geq p+2$$

Strong algebras

These are algebras which are inductive limits of Banach spaces and carry inequalities which are counterparts of the inequality for the norm in a Banach algebra.

Definition

Let $\mathcal{A}$ be an algebra which is also the inductive limit of a family of Banach spaces $\{X_\alpha : \alpha \in A\}$ directed under inclusion. It is a *strong algebra* if for any $\alpha \in A$ there exists $h(\alpha) \in A$ such that for any $\beta \geq h(\alpha)$ there is a positive constant $A_{\beta,\alpha}$ for which

$$\|ab\|_\beta \leq A_{\beta,\alpha}\|a\|_\alpha\|b\|_\beta, \quad \text{and} \quad \|ba\|_\beta \leq A_{\beta,\alpha}\|a\|_\alpha\|b\|_\beta.$$

for every $a \in X_\alpha$ and $b \in X_\beta$.

The case of a Banach algebra corresponds to the case where the set of indices A is a singleton.

First remarks and properties

The strong algebras are topological algebras in the sense of Naimark, i.e. they are locally convex, and the multiplication is separately continuous (this follows from the universal property of inductive limits)

If any bounded set in a strong algebra is bounded in some of the X_α, then the multiplication is jointly continuous.

The inequalities on the norms express the fact that for any $\alpha \in A$, each of the spaces $\{X_\beta : \beta \geq h(\alpha)\}$ "absorbs" X_α from both sides. Due to this property, one may evaluate (with elements of $\mathcal{A}$) power series, and therefore, consider invertible elements.

A special family of such algebras, which are inductive limits of $\mathbf{L}^2$ spaces of measurable functions over a locally compact group. Examples include the algebra of germs of holomorphic functions at the origin, the Kondratiev space of Gaussian stochastic distributions, the algebra of functions $f : [1, \infty) \to \mathbb{C}$ for which $f(x)/x^p$ belongs to $\mathbf{L}^2([1, \infty))$ (this gives relations with the theory of Dirichlet series), and a new space of non-commutative stochastic distributions.

Wiener algebra

Let $\mathcal{A} = \varinjlim X_\alpha$ be a strong algebra. Let Y_α to be the space of periodic functions

$$a(t) = \sum_{n \in \mathbb{Z}} a_n e^{int}, \quad a_n \in X_\alpha,$$

on $-\pi \le t < \pi$ to $\mathcal{A}$, with

$$\|a\|_\alpha = \sum_{n \in \mathbb{Z}} \|a_n\|_\alpha < \infty.$$

The inductive limit $\varinjlim Y_\alpha$ of the Banach spaces Y_α is a a strong algebra *the Wiener algebra associated to* $\mathcal{A}$.

This family extends the case of Wiener algebras of functions with values in a Banach algebra.

Inductive limit of normed spaces

Definition

Let $\{X_\alpha : \alpha \in A\}$ be a family of subspaces of a vector space X such that $X_\alpha \neq X_\beta$ for $\alpha \neq \beta$, directed under inclusion, satisfying $X = \bigcup_\alpha X_\alpha$, where A is directed under $\alpha \leq \beta$ if $X_\alpha \subseteq X_\beta$. Moreover, on each X_α $(\alpha \in A)$, a norm $\|\cdot\|_\alpha$ is given, such that whenever $\alpha \leq \beta$, the topology induced by $\|\cdot\|_\beta$ on X_α is coarser than the topology induced by $\|\cdot\|_\alpha$. Then X, topologized with the inductive limit topology is called *the inductive limit of the normed spaces* $\{X_\alpha : \alpha \in A\}$.

The inductive limit has the following universal property. Given any locally convex space Y, a linear map f from X to Y is continuous if and only if each of the restrictions $f|_{X_\alpha}$ is continuous with respect to the topology of X_α. This property allows to take full advantage of the inequalities in the definition of a strong algebra given now.

Let $\{X_\alpha : \alpha \in A\}$ be a family of Banach spaces directed under inclusions, and let $\mathcal{A} = \bigcup X_\alpha$ be its inductive limit. We call $\mathcal{A}$ a strong algebra if it is an algebra satisfying the property that for any $\alpha \in A$ there exists $h(\alpha) \in A$ such that for any $\beta \geq h(\alpha)$ there is a positive constant $A_{\beta,\alpha}$ for which

$$\|ab\|_\beta \leq A_{\beta,\alpha}\|a\|_\alpha\|b\|_\beta, \quad \text{and} \quad \|ba\|_\beta \leq A_{\beta,\alpha}\|a\|_\alpha\|b\|_\beta.$$

for every $a \in X_\alpha$ and $b \in X_\beta$.

Bornological tvs

A locally convex space X is called bornological if every balanced, convex subset $U \subseteq X$ that absorbs every bounded set in X is a neighborhood of 0. Equivalently, a bornological space is a locally convex space on which each semi-norm that is bounded on bounded sets, is continuous.

Barreled tvs

A topological vector space is said to be barreled if each convex, balanced, closed and absorbent set is a neighborhood of zero. Equivalently, a barreled space is a locally convex space on which each semi-norm that is semi-continuous from below, is continuous.

These properties are stable under inductive limits

Hence

An inductive limit of Banach spaces (and in particular a strong algebra) is bornological and barrelled.

The term "topological algebra" is sometimes refer to topological vector space together with a (jointly) continuous multiplication $(a, b) \mapsto ab$. However, M.A. Naimark gives the following definition for a topological algebra.

Definition

$\mathcal{A}$ is called topological algebra if:

(a) $\mathcal{A}$ is an algebra;

(b) $\mathcal{A}$ is a locally convex topological linear space;

(c) the product ab is a continuous function of each of the factors a, b provided the other factor is fixed.

A strong algebra is a topological algebra in the sense of Naimark.

Let $\mathcal{A} = \bigcup_\alpha X_\alpha$ be a strong algebra and let $a \in \mathcal{A}$. Then the linear mappings $L_a : x \mapsto ax$, $R_a : x \mapsto xa$ are continuous. Thus, it is topological algebra in the sense of Naimark.

Theorem

If in a strong algebra $\mathcal{A} = \bigcup X_\alpha$, any set is bounded if and only if it is bounded in some of the X_α, then the multiplication is jointly continuous.

Boundedness assumption

There are several natural cases in which the assumption of the previous theorem holds. Namely, in each of the following instances of inductive limit of Banach spaces, any bounded set of $\bigcup X_\alpha$ is bounded in some of the X_α. Thus, when a strong algebra $\mathcal{A}$ is of one of these forms, then in particular the multiplication is jointly continuous.

Sufficient condition for boundedness hypothesis to hold

Special cases

(i) The set of indices A is the singleton $\{0\}$, and hence $\bigcup X_\alpha = X_0$ is a Banach space.

(ii) The set of indices A is $\mathbb{N}$, and $\bigcup X_n$ is the *strict* inductive limit of the X_n (and is then called an LB-space), that is, for any $m \geq n$ the topology of X_n induced by X_m, is the initial topology of X_n.

(iii) The set of indices A is $\mathbb{N}$, and the embeddings $X_n \hookrightarrow X_{n+1}$ are compact.

(iv) The set of indices A is $\mathbb{N}$, and the inductive limit is a dual of reflexive Fréchet space. More precisely, let $\Phi_1 \supseteq \Phi_2 \supseteq \ldots$ be a decreasing sequence of Banach spaces, and assume that the corresponding countably normed space $\bigcap \Phi_n$ is reflexive. Then, $\bigcup \Phi_n'$, the strong dual of $\bigcap \Phi_n$ is the same as the inductive limit of the spaces $\Phi_1' \subseteq \Phi_2' \subseteq \ldots$ (as a topological vector space).

In fact, see N. Bourbaki, the following theorem is true:

Theorem

Let E_1 and E_2 be two reflexive Fréchet spaces, and let G a locally convex Hausdorff space. For $i = 1, 2$, let F_i be the strong dual of E_i. Then every separately continuous bilinear mapping $u : F_1 \times F_2 \to G$ is continuous.

This gives another proof for the continuity of the multiplication in case (iv).

There are some cases where the topology on an inductive limit (that is, the inductive topology) is the finest topology such that the mappings $X_\alpha \hookrightarrow \bigcup X_\alpha$ are continuous (instead of the finest locally convex topology such that they are continuous). One example is when X is the inductive limit of a sequence of Banach spaces $\{X_n : n \in \mathbb{N}\}$, and the embeddings $X_n \hookrightarrow X_{n+1}$ are compact. In this case, we have the following sufficient condition on mappings $X \to X$ tbe continuous.

Theorem

Let X be the inductive limit of the family X_α, where its topology is the finest topology such that the mappings $X_\alpha \hookrightarrow \bigcup X_\alpha$ are continuous. Then any map (not necessarily linear) f from an open set $W \subseteq X$ to X which satisfies the property that for any α there is β such that $f(W \cap X_\alpha) \subseteq X_\beta$ and $f|_{W \cap X_\alpha}$ is continuous with respect to the topologies of $W \cap X_\alpha$ at the domain and X_β at the range, is a continuous function $W \to X$.

Proof

In this case, U is open in X if and only if for any $U \cap X_\alpha$ is open in X_α for every α. Let U be an open set of X, and let $\alpha \in A$. By the assumption, there is β such that $f(W \cap X_\alpha) \subseteq X_\beta$ and $f^{-1}(U) \cap X_\alpha = f|_{W \cap X_\alpha}^{-1}(U \cap X_\beta)$ is open in $W \cap X_\alpha$. In particular, $f^{-1}(U) \cap X_\alpha$ is open in X_α, so $f^{-1}(U)$ is open in X and hence in W.

Whenever a strong algebra $\mathcal{A}$ satisfies the assumption of the theorem then the set of invertible elements is open and that $a \mapsto a^{-1}$ is continuous.

There is a "well behaved" family of linear maps from an inductive limit of Banach spaces into itself, which we call admissible operators. These maps are in particular continuous, and sometimes all continuous linear maps are of this form.

Let X be the inductive limit of the Banach spaces X_α. Then a linear map $T : X \to X$ which satisfies the property that for any α there is β such that $T(X_\alpha) \subseteq X_\beta$ and $T|_{X_\alpha}$ is continuous with respect to the topologies of X_α for the domain and X_β for the range, will be called an admissible operator of X. For an admissible operator $T : X \to X$ we denote by $\|T\|_\beta^\alpha$ the norm of $T|_{X_\alpha}$ when the range is restricted to X_β, whenever it makes sense, and otherwise we set $\|T\|_\beta^\alpha = \infty$.

By the universal property of inductive limits, we conclude:

Any admissible operator is continuous.

Note that if any bounded set in X is bounded in some X_α, then any continuous linear map is admissible.

Note that if $\|S\|_\beta^\alpha < \infty$ and $\|T\|_\gamma^\beta < \infty$, then

$$\|TS\|_\gamma^\alpha \leq \|T\|_\gamma^\beta \|S\|_\beta^\alpha.$$

Let $T : X \to X$ be a admissible operator such that there exists α for which $\|T\|_\alpha^\alpha < 1$ then $I - T$ is invertible, and

$$\|(I-T)^{-1}\|_\alpha^\alpha \leq \frac{1}{1 - \|T\|_\alpha^\alpha}.$$

Power series and invertible elements

Here, $\mathcal{A}$ is a unital strong algebra.

Assuming $\sum_{n=0}^{\infty} c_n z^n$ converges in the open disk with radius R, then for any $a \in \mathcal{A}$ such that there exist α, β with $\beta \geq h(\alpha)$ and $A_{\beta,\alpha}\|a\|_\alpha < R$ it holds that

$$\sum_{n=0}^{\infty} c_n a^n \in X_\beta \subseteq \mathcal{A}.$$

Proof

This follows from

$$\sum_{n=0}^{\infty} |c_n| \|a^n\|_\beta \leq \sum_{n=0}^{\infty} |c_n| (A_{\beta,\alpha}\|a\|_\alpha)^n \|1\|_\beta < \infty.$$

Invertible elements

Theorem

Let $a \in \mathcal{A}$ be such that there exists α, β with $\beta \geq h(\alpha)$ and $A_{\beta,\alpha}\|a\|_\alpha < 1$ then $1 - a$ is invertible (from both sides) and it holds that

$$\|(1-a)^{-1}\|_\beta \leq \frac{\|1\|_\beta}{1 - A_{\beta,\alpha}\|a\|_\alpha}, \quad \|1-(1-a)^{-1}\|_\beta \leq \frac{A_{\beta,\alpha}\|a\|_\alpha\|1\|_\beta}{1 - A_{\beta,\alpha}\|a\|_\alpha},$$

where

$$(1-a)^{-1} = \sum_{n=0}^{\infty} a^n.$$

Theorem

If $a \in \mathcal{A}$ has a left inverse $a' \in X_\alpha \subseteq \mathcal{A}$ (i.e. $a'a = 1$), then for any $\beta \geq h(\alpha)$ and $b \in X_\beta$ such that there exists $\gamma \geq h(\beta)$ with $A_{\gamma,\beta}A_{\beta,\alpha}\|a'\|_\alpha\|b\|_\beta < 1$, it holds that $a - b$ has a left inverse $(a-b)' \in X_\gamma$, where

$$(a-b)' = a'\sum_{n=0}^{\infty}(ba')^n.$$

and

$$\|(a-b)' - a'\|_\gamma \leq A_{\gamma,\alpha}\|a'\|_\alpha \frac{A_{\gamma,\beta}A_{\beta,\alpha}\|a'\|_\alpha\|b\|_\beta\|1\|_\gamma}{1 - A_{\gamma,\beta}A_{\beta,\alpha}\|a'\|_\alpha\|b\|_\beta}$$

Proof

We note that

$$A_{\gamma,\beta}\|ba'\|_\beta \leq A_{\gamma,\beta}A_{\beta,\alpha}\|a'\|_\alpha\|b\|_\beta < 1.$$

Thus, $1 - ba'$ is invertible, and

$$a'(1-ba')^{-1}(a-b) = a'(1-ba')^{-1}(1-ba')a = 1.$$

Now, note that

$$\begin{aligned}\|(a-b)' - a'\|_\gamma &= \|a'(1-ba')^{-1} - a'\|_\gamma \\ &\leq A_{\gamma,\alpha}\|a'\|_\alpha\|(1-ba')^{-1} - 1\|_\gamma \\ &\leq A_{\gamma,\alpha}\|a'\|_\alpha\frac{A_{\gamma,\beta}\|ba'\|_\beta\|1\|_\gamma}{1 - A_{\gamma,\beta}\|ba'\|_\beta} \\ &\leq A_{\gamma,\beta}\|a'\|_\alpha\frac{A_{\gamma,\beta}A_{\beta,\alpha}\|a'\|_\alpha\|b\|_\beta\|1\|_\gamma}{1 - A_{\gamma,\beta}A_{\beta,\alpha}\|a'\|_\alpha\|b\|_\beta}.\end{aligned}$$

A Wiener algebra associated to a strong algebra and a strong algebra version of the Wiener theorem

Definition

Let $\mathcal{A} = \bigcup X_\alpha$ be a strong algebra. Let Y_α be the space of periodic functions

$$a(t) = \sum_{n\in\mathbb{Z}} a_n e^{int}, \quad a_n \in X_\alpha,$$

on $-\pi \leq t < \pi$ to $\mathcal{A}$, with

$$\|a\|_\alpha = \sum_{n\in\mathbb{Z}} \|a_n\|_\alpha < \infty.$$

The inductive limit $\mathcal{U} = \bigcup Y_\alpha$ of the Banach spaces Y_α is called the Wiener algebra associated to $\mathcal{A}$.

Theorem

$\mathcal{U} = \bigcup_\alpha Y_\alpha$ is a strong algebra.

Proof

For any $\alpha \in A$ and $\beta \geq h(\alpha)$ and for any $a \in Y_\alpha$ and $b \in Y_\beta$, it holds that

$$\begin{aligned} \|ab\|_\beta &= \sum_{n\in\mathbb{Z}} \left\| \sum_{m\in\mathbb{Z}} a_m b_{n-m} \right\|_\beta \\ &\leq \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} A_{\beta,\alpha} \|a_m\|_\alpha \|b_{n-m}\|_\beta \\ &\leq A_{\beta,\alpha} \|a\|_\alpha \|b\|_\beta. \end{aligned}$$

Similarly, $\|ba\|_\beta \leq A_{\beta,\alpha} \|a\|_\alpha \|b\|_\beta$.

Wiener-Levy like theorem

Theorem

For any $a \in \mathcal{U}$, a is left invertible if and only if $a(t)$ is left invertible for every t.