Rational Orthonormal Functions on the Unit Circle and on the Imaginary Axis, with Applications in System Identification

Tomás Oliveira e Silva[1]

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Abstract

In this report we present a collection of results concerning two families of rational orthonormal functions, one on the unit circle, and another on the imaginary axis. We also describe in detail an interesting link between the two families. Special cases of these rational orthonormal functions include the Laguerre and Kautz orthonormal functions, as well as the orthonormal functions recently introduced by Heuberger, Van den Hof and Bosgra. Among the results presented herein are completeness and uniform boundedness conditions, and their respective proofs, for the above mentioned families of rational orthonormal functions, as well as some interpolatory properties of truncated orthonormal expansions based on these functions. We will only discuss the case of (rational) functions orthonormal with respect to a unit weighting function. The general case of a non constant (non-negative) weighting function is much more involved and will be described in a future report.

We derive the rational orthonormal functions by two methods. The first one is essentially the classical approach, and is based on the Gram-Schmidt orthonormalization of a set of exponentials. This method was the one used by Lee and Wiener, Kautz, and others, and is based on certain elementary properties of contour integrals of rational functions. The second one, which may be called the modern approach, is based on balanced realizations of rational all-pass transfer functions. To the best of our knowledge this method was developed by Roberts and Mullis and is at the root of the Heuberger *et al.* family of rational orthonormal functions.

Some interesting elementary properties of linear models constructed from rational orthonormal functions are also presented. Special cases of these models include the very well known FIR model and the recently studied Laguerre and "two-parameter" Kautz models, as well as the Heuberger *et al.* model. Curiously, these models have many properties in common with the FIR model, such as the theoretical upper bound of the condition number of the auto-correlation matrix of their internal signals. Together with the inherent structure of these models this leads to a simple LMS-like algorithm to adapt its weights that can use the *same* adaptation step size as the standard LMS algorithm for transversal filters.

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Chapter 1

Introduction

As the title suggests, this report is concerned with the problem of constructing rational orthonormal functions on the unit circle and on the imaginary axis. We intend to show that this type of orthonormal functions is the basis of some system identification models with very desirable properties. Not surprisingly, rational orthonormal functions on the unit circle will be related with discrete-time models and those on the imaginary axis with continuous-time models. As we will see, the discrete- and continuous-time models have many properties in common. This is a consequence of the existence of a simple relation between the rational orthonormal functions on the unit circle and those on the imaginary axis.

The models constructed from these rational orthonormal functions include, in the discrete-time case, the FIR (finite impulse response) filters, as well as some of the IIR (infinite impulse response) filters, these last in a very interesting and, apparently, not yet explored form. These IIR models have a similar counterpart in continuous-time. For the discrete-time case, it is even possible to devise an LMS-like algorithm to adapt the weights of the filter. A particular feature of this adaptation algorithm is that it can use the adaptation step size (usually denoted by μ) of the standard LMS algorithm for transversal (i.e., FIR) filters.

The majority of the works cited in this report are in the fields of Digital Signal Processing, Automatic Control, and Circuits and Systems. However, we have made an attempt to trace the origin of the main results presented in this report in the mathematical and engineering literatures. This "search for the origins" is far from being complete, with the consequence that we may have failed to provide the original source of some of the results. Shallow as this search was, we were nonetheless able to uncover some old and important references concerning rational orthonormal functions that appear to be less well known in the Digital Signal Processing and Automatic Control fields.

The level of mathematical rigor in this presentation will be moderate, and the exposition will sometimes be repetitive. This was done (on purpose) to emphasize the similarities between the discrete-time and the continuous-time cases, and also to provide an exhaustive separate account of each of these two related problems.

This report is organized as follows. In chapter 2 we present some elementary facts about the Hilbert and Hardy spaces that we will need. More specifically, we describe some Hilbert and Hardy spaces related with discrete-time signals and others related with continuous-time signals, as well as a useful relation between them. In chapter 3 we present the classical approach to the generation of the rational orthonormal functions, which is based on a Gram-Schmidt orthonormalization of a set of exponential functions. Among the topics discussed in

that chapter are the completeness and/or uniform boundedness of the constructed rational orthonormal functions, and the interpolatory properties of truncated series expansions based on them. In chapter 4 we present the modern approach to the generation of the rational orthonormal functions, which is based on the cascade of balanced realizations of all-pass transfer functions. In chapter 5, and more specifically in section 5.2, we describe a general system identification model based on the rational orthonormal functions developed in chapters 3 and 4, and present some of its most interesting properties. Since the study of this model will be made using a stochastic framework, in section 5.1 we describe the basic facts about wide-sense stochastic processes needed for that study. In section 5.2 we also present a short description of an LMS-like algorithm for the proposed class of models, which shares many properties with the standard LMS algorithm for transversal filters. In that section we also propose a simple and hopefully effective off-line method to select a good set of poles for the model. Lastly, in the appendix we present some of the notation used in this report. We advise the reader to read it first before advancing to chapter 2 (it starts on page 77). Some concepts, definitions, and side issues will be presented in the form of footnotes.

Note that some of the references presented at the end of this report were not cited in the text. They were included because they describe work which is related to (parts of) this report. For the reader's convenience we have placed after each entry in the bibliography the page number(s) where it was cited.

Acknowledgments

I want to take this opportunity to thank Prof. Bo Wahlberg and Per Bodin, of the Royal Institute of Technology (KTH), Stockholm, Sweden, for bringing to my attention their work and the important book of Roberts and Mullis [1987], in which certain interconnections of balanced all-pass transfer functions are shown to remain balanced and all-pass. Without their mention of that book the entire chapter 4 would not have existed! (However, section 4.6 would probably have been placed somewhere else in a somewhat modified form.) In fact, it was that book that prompted me to write this report.

I want to extend my thanks to Dr. Ir. Bert den Brinker and to Harm Belt, of the Eindhoven University of Technology (TUE), Eindhoven, The Netherlands. Their kind hospitality allowed me to conduct a first bibliographic search that provided a large part of the base material for this report.

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Chapter 2

Some Hilbert and Hardy spaces

The main purpose of this chapter is to present to an engineering audience some basic properties of Hilbert spaces, and at the same time introduce some necessary notation and concepts required to understand the following chapters. The interested reader may find a quick and elementary introduction to Hilbert space theory in [Young, 1988]. A more advanced treatment of Hilbert spaces and related subjects can be found in [Rudin, 1987][1]. Advanced treatments of the theory of Hardy spaces can be found in [Duren, 1970; Koosis, 1980]. Other books related to these subjects are [Riesz and Sz.-Nagy, 1990; Hoffman, 1988; Kreyszig, 1978; Garnett, 1981].

As stated in [Young, 1988], "a Hilbert space is an inner product space which is a complete metric space with respect to the metric induced by its inner product." That is, a Hilbert space is a linear vector space, possibly of infinite dimension, with an inner product operation defined between any two of its elements (an inner product space), which is denoted by $\langle \cdot, \cdot \rangle$. This inner product is used to define the norm of an element of that space (norm induced by the inner product), which is simply the square root of the inner product of that element with itself, and is denoted by $\|\cdot\|_2$. This norm is used in turn to define a distance (metric) between any two elements of that space (a metric space), that is the norm of the difference between these two elements. The remaining and very important characteristic of a Hilbert space is that the metric space is closed (complete). This means that any convergent Cauchy sequence[2] of elements of that space converges to an element of that space.

A word of caution is in order here. Obviously, each Hilbert space has its own inner product. For notational simplicity we will use the same symbol, viz. $\langle \cdot, \cdot \rangle$, to denote any inner product, without specifying the underlying Hilbert space. This can be done without any confusion because the reader will be able to infer the Hilbert space being used from the context in which the inner product appears. The same simplification of notation applies to norms induced by inner products.

Two elements of a Hilbert space are said to be orthogonal if their inner product is zero. An element of a Hilbert space is said to be normal if its norm is equal to one. A set of elements $\{X_i\}$ of a Hilbert space H is said to be orthogonal if the elements of that set are orthogonal to each other. If in addition these elements are normal then the set is called an orthonormal set. $\{X_i\}$ is a closed or complete set in H (also called a basis of H) if its closed linear span, i.e., linear combinations of a finite number of elements of $\{X_i\}$ and their closure,

[1] This book is considered to be a master piece by a large number of mathematicians. It is not recommended for a first introduction to Hilbert space theory.

[2] A sequence $(x_k)_{k=1}^{\infty}$ is a Cauchy sequence if for every $\epsilon > 0$ there exists an integer k_ϵ such that $i, j \geq k_\epsilon$ implies that $\|x_i - x_j\|_2 < \epsilon$.

is H itself. In other words, $\{X_i\}$ is complete (or closed)[3] in H if and only if any element of H can be approximated arbitrarily well, in the norm induced by the inner product of H, by a linear combination of a finite number of elements of $\{X_i\}$. If $\{X_i\}$ is also orthonormal then this set is called a complete orthonormal set of H, or an orthonormal basis of H (these two designations are equivalent). The only element that is simultaneously orthogonal to all elements of a complete set of a Hilbert space is the null (zero) element of that space.

In the remaining part of this chapter we introduce the Hilbert and Hardy spaces that will be used in this report, and some of theirs most important properties. In the process we also define the Fourier transform of sequences and functions, the z transform, and the bilateral Laplace transform, together with their inverse transforms and Parseval formulas. Some useful relations between several Hilbert and Hardy spaces will also be described in detail.

2.1 Hilbert and Hardy spaces related to sequences

We denote by $\ell^2(S)$ the Hilbert space of square summable sequences with support in $S \subseteq \mathbb{Z}$, equipped with the inner product

$$\langle f, g \rangle = \sum_{k \in S} f(k) g^*(k). \tag{2.1}$$

Recall that a sequence $f(k)$ is said to be square summable if

$$\sum_{k \in S} |f(k)|^2 < \infty.$$

(In engineering terms such sequences are said to have finite energy.) Hence, by the Cauchy-Schwarz inequality,[4] the inner product between any two elements of $\ell^2(S)$ is a well defined finite complex number. In this report we will use extensively the Hilbert space $\ell^2(\mathbb{N}_0)$ of causal[5] sequences with finite energy, and, occasionally, the Hilbert space $\ell^2(\mathbb{Z})$.

We denote by $L^2(\mathbb{T})$ the Hilbert space of square integrable functions on $\mathbb{T} = \{z : |z| = 1\}$, equipped with the inner product[6]

$$\langle F, G \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{\mathrm{i}\omega}) G^*(e^{\mathrm{i}\omega}) \, d\omega. \tag{2.2}$$

[3] The concepts of completeness and closure are equivalent in separable Hilbert spaces.

[4] The Cauchy-Schwarz inequality states that [Hardy, Littlewood and Pólya, 1934]

$$\left| \sum_{k \in S} x_k y_k \right| \leq \left[\sum_{k \in S} |x_k|^2 \right]^{1/2} \left[\sum_{k \in S} |y_k|^2 \right]^{1/2}.$$

S may have an infinite (but countable) number of elements.

[5] We will use the term "causal" to characterize sequences and functions that vanish (almost everywhere in the case of functions) for all negative values of the time variable. See the next footnote for the reason of the "almost everywhere" when working with functions.

[6] Here and in the rest of this report all integrals are used in the sense of Lebesgue. Contrary to the Riemann integral, in the Lebesgue integral the value of the integrand in a single point is not important. What is important is the value of the integrand in sets of strictly positive measure (length). In the sense of Lebesgue two functions are equivalent (for integration purposes) if they only differ on a set of null measure, i.e., if they are equal almost everywhere (a.e.). This has the implication that Hilbert spaces with inner products defined by an integral are actually sets of classes of equivalence of functions instead of being sets of functions. By an abuse of language, the elements of those sets are usually called functions, omitting the "classes of equivalence of" prefix. We will follow this convention in this report.

Recall that a function F defined on $\mathbb{T}$ is square integrable (or has finite energy) if

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |F(e^{\mathrm{i}\omega})|^2\, d\omega < \infty.$$

As is well known, all separable Hilbert spaces[7] (with the same cardinality) are isomorphic to each other.[8] In fact, it is very easy to construct an isomorphism between two such Hilbert spaces. It is enough to define a one to one mapping that carries each element of a complete orthonormal set of the first Hilbert space to an element of a complete orthonormal set of the second Hilbert space. Sometimes some of this kind of mappings have particularly simple forms. This is the case for the two Hilbert spaces $\ell^2(\mathbb{Z})$ and $L^2(\mathbb{T})$ introduced previously. It turns out that the Fourier transform (to be defined shortly) of sequences of $\ell^2(\mathbb{Z})$ defines an isomorphism between $\ell^2(\mathbb{Z})$ and $L^2(\mathbb{T})$.

The Fourier transform $F(e^{\mathrm{i}\omega})$ of a sequence $f(k)$ belonging to $\ell^2(\mathbb{Z})$ is defined by[9]

$$F(e^{\mathrm{i}\omega}) = \sum_{k=-\infty}^{\infty} f(k)\, e^{-\mathrm{i}\omega k} \tag{2.3}$$

where the equality is to be interpreted as convergence in the mean. It is a well known fact that the function $F(e^{\mathrm{i}\omega})$ is a Lebesgue measurable square integrable function on $\mathbb{T}$ [Rudin, 1987]. It is also well known that the inverse Fourier transform is given by

$$f(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{\mathrm{i}\omega})\, e^{\mathrm{i}\omega k}\, d\omega, \tag{2.4}$$

and that the Parseval theorem for this version of the Fourier transform is expressed by

$$\sum_{k=-\infty}^{\infty} f(k)g^*(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{\mathrm{i}\omega})G^*(e^{\mathrm{i}\omega})\, d\omega. \tag{2.5}$$

It is possible to verify that to any sequence $f(k)$ belonging to $\ell^2(\mathbb{Z})$ corresponds one and only one function $F(e^{\mathrm{i}\omega})$ belonging to $L^2(\mathbb{T})$ and vice versa,[10] and this defines a one to one mapping between $\ell^2(\mathbb{Z})$ and $L^2(\mathbb{T})$. That this mapping is also an isomorphism follows immediately from Parseval's theorem, which implies that $\langle f, g\rangle = \langle F, G\rangle$.

[7] A Hilbert space is said to be separable if it has a basis with at most a countable number of elements. All Hilbert spaces used in this report are separable.

[8] For Hilbert spaces, an isomorphism is a one to one mapping that preserves inner products. So, isomorphisms also preserve norms, i.e., they are also isometries. (In fact, they are unitary operators.) In Hilbert spaces the converse is also true if the isometry is a one to one mapping. Due to the polarization identity, which is [Young, 1988]

$$4\,\langle x, y\rangle = \|x+y\|_2 + \mathrm{i}\,\|x+\mathrm{i}y\|_2 - \|x-y\|_2 - \mathrm{i}\,\|x-\mathrm{i}y\|_2 = \sum_{n=0}^{3} \mathrm{i}^n\,\|x+\mathrm{i}^n y\|_2\,,$$

a one to one isometry between two Hilbert spaces actually preserves inner products.

[9] This definition also holds if $\ell^2(\mathbb{Z})$ is replaced by $\ell^2(S)$, with an obvious change in the range of the summation. Alternatively, the sequences of $\ell^2(S)$ can be regarded as sequences of $\ell^2(\mathbb{Z})$ that vanish for $k \notin S$.

[10] We recall that the elements of $L^2(\mathbb{T})$ are actually classes of equivalence of functions that are equal a.e. on $\mathbb{T}$. The utilization of these classes of equivalence is fundamental to make the correspondence between elements of $\ell^2(\mathbb{Z})$ and $L^2(\mathbb{T})$ well defined and one to one.

It is quite interesting to verify that the Fourier transform of a sequence belonging to $\ell^2(\mathbb{Z})$ can be seen to be the z transform of that sequence evaluated on $\mathbb{T}$. This follows immediately from the definition of the z transform, which is[11]

$$F(z) = \sum_{k=-\infty}^{\infty} f(k) z^{-k} \tag{2.6}$$

and from the definition of the Fourier transform [compare (2.3) with (2.6)]. The inverse z transform is of course given by

$$f(k) = \frac{1}{2\pi \mathsf{i}} \oint_{\mathbb{T}} z^k F(z) \, \frac{dz}{z}, \tag{2.7}$$

which is a generalization of (2.4).[12] The Parseval theorem for z transforms of square summable sequences is expressed by

$$\sum_{k=-\infty}^{\infty} f(k) g^*(k) = \frac{1}{2\pi \mathsf{i}} \oint_{\mathbb{T}} F(z) G^*(1/z^*) \, \frac{dz}{z} = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{\mathsf{i}\omega}) G^*(e^{\mathsf{i}\omega}) \, d\omega, \tag{2.8}$$

and so it also agrees with (2.5). Of course the line integral (actually, a contour integral) is to be interpreted in the Lebesgue sense. If at least one of $F(z)$ and $G(z)$ is analytic on $\mathbb{T}$ this contour integral can be evaluated using the residue theorem (see [Riesz and Sz.-Nagy, 1990] for details).

A particularly interesting and useful (from an engineering perspective) proper subset of $\ell^2(\mathbb{Z})$ is the Hilbert space $\ell^2(\mathbb{N}_0)$ of causal square summable sequences. The usefulness of this Hilbert space is related in part to the fact than the unit pulse response of all stable and causal discrete-time systems belong to it (but the converse is not true). This is so because stability, that can be expressed by the condition[13]

$$\sum_{k=0}^{\infty} |h(k)| < \infty$$

where $h(k)$ is the unit pulse response of a causal system, implies the square summability of $h(k)$.[14] Unfortunately the converse is not always true, since a square summable sequence does not need to be absolutely summable.[15]

The Fourier transform given by (2.3) defines an isomorphism between $\ell^2(\mathbb{N}_0)$ and a proper subset of $L^2(\mathbb{T})$, which is also a Hilbert space. This proper subset is sometimes denoted by $\mathcal{H}^2(\mathbb{E})$, which is the Hardy space of the functions analytic on $\mathbb{E}$ that are square integrable

[11] This definition should be complemented with a description of the region of convergence of $F(z)$. Here we are interested in convergence in the mean and so for square summable sequences $F(z)$ will converge at least on $\mathbb{T}$.

[12] Setting $z = e^{\mathsf{i}\omega}$ in (2.7) gives (2.4).

[13] This condition is for the widely used bounded-input bounded-output stability. Sequences that satisfy it are said to be absolutely summable and belong to the Banach space $\ell^1(\mathbb{N}_0)$.

[14] The proof of this fact is very simple [Oppenheim and Schafer, 1975], and so it is left as an exercise for the interested reader.

[15] A simple example is $h(k) = 1/(1+k)$, for $k \in \mathbb{N}_0$.

on $\mathbb{T}$.[16,17] Note that this particular Hardy space is equipped with an inner product that is defined in exactly the same way as the one of $L^2(\mathbb{T})$. To see why we decided to use the Hardy space defined over $\mathbb{E}$ and not over $\mathbb{T}$ observe that the z transform of bounded causal sequences, and this includes all sequences that belong to $\ell^2(\mathbb{N}_0)$, is analytic for $z > 1$, including the point at infinity,[18] i.e., it is analytic on $\mathbb{E}$. The additional condition that it is also square integrable on $\mathbb{T}$ comes from the fact that the Fourier transform of $\ell^2(\mathbb{Z})$ sequences is the z transform of those sequences evaluated on $\mathbb{T}$. To conclude the argument, observe also that if a function is (not) analytic on $\mathbb{E}$ then its inverse z transform is (not) a causal sequence, and so there is a one to one relation between elements of $\ell^2(\mathbb{N}_0)$ and elements of $\mathcal{H}^2(\mathbb{E})$.

2.2 Hilbert and Hardy spaces related to functions

The structure of this section is very similar to the structure of the previous one. This was done on purpose and not by accident. The reason for this similarity will be explained in section 2.4.

We denote by $L^2(S)$ the Hilbert space of square integrable functions with support in $S \subseteq \mathbb{R}$, equipped with the inner product

$$\langle f, g \rangle = \int_S f(t) g^*(t)\, dt. \tag{2.9}$$

Recall that a function $f(t)$ is said to be square integrable if

$$\int_S |f(t)|^2\, dt < \infty.$$

(In engineering terms such functions are said to have finite energy.) Hence, by the Schwarz inequality,[19] the inner product between any two elements of $L^2(S)$ is a well defined finite complex number. In this report we will use extensively the Hilbert space $L^2(\mathbb{R}^+)$ of causal functions with finite energy, and, occasionally, the Hilbert space $L^2(\mathbb{R})$.

We denote by $L^2(\mathsf{i}\mathbb{R})$ the Hilbert space of square integrable functions on $\mathsf{i}\mathbb{R}$, equipped with the inner product

$$\langle F, G \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\mathsf{i}\omega) G^*(\mathsf{i}\omega)\, d\omega. \tag{2.10}$$

The Fourier transform of a function of $L^2(\mathbb{R})$ (to be defined shortly) defines one isomorphism between $L^2(\mathbb{R})$ and $L^2(\mathsf{i}\mathbb{R})$.

[16] Actually, this proper subset of $L^2(\mathbb{T})$ should be denoted by $\mathcal{H}^2(\mathbb{T})$, which is the Hilbert space of square integrable functions on $\mathbb{T}$ with vanishing negative Fourier moments. However, in this report we will use $\mathcal{H}^2(\mathbb{E})$ instead of $\mathcal{H}^2(\mathbb{T})$ since they are equivalent (see next footnote) and the former will be more useful to us.

[17] Let $f(z)$ be a function of $\mathcal{H}^2(\mathbb{E})$. Then $\lim_{r \to 1^+} f(re^{\mathsf{i}\omega})$ converges for almost all values of $\omega \in [-\pi, \pi]$ to $f(e^{\mathsf{i}\omega})$. This is a property of all Hardy spaces defined on $\mathbb{E}$ [Hoffman, 1988; Duren, 1970]. (Actually, only the tangential limits may not exist a.e. on $\mathbb{T}$.) The functions of $\mathcal{H}^2(\mathbb{E})$ may then be viewed as the unique analytic extension of the corresponding functions of $\mathcal{H}^2(\mathbb{T})$.

[18] A function $f(z)$ is said to be analytic at the point at infinity if $f(1/z)$ is analytic at the origin.

[19] The Schwarz inequality, which is a particular case of Hölder's inequality [Hardy et al., 1934], states that

$$\left| \int_S x(t) y(t)\, dt \right| \leq \left[\int_S |x(t)|^2\, dt \right]^{1/2} \left[\int_S |y(t)|^2\, dt \right]^{1/2}.$$

S may have finite or infinite measure, as long as it is a subset of $\mathbb{R}$.

The Fourier transform $F(\mathsf{i}\omega)$ of a function $f(t)$ belonging to $L^2(\mathbb{R})$ is defined by[20]

$$F(\mathsf{i}\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-\mathsf{i}\omega t}\, dt \tag{2.11}$$

where the equality is to be interpreted as convergence in the mean. Note that this definition of the Fourier transform is slightly non standard because the argument of the transform is imaginary ($\mathsf{i}\omega$) instead of being real (ω), as it is commonly defined. This difference is just a question of notation, that has the advantage of allowing the direct comparison of this transform with the bilateral Laplace transform that will be introduced shortly. It is a well known fact that the function $F(\mathsf{i}\omega)$ is a Lebesgue measurable square integrable function on $\mathsf{i}\mathbb{R}$ [Rudin, 1987]. It is also well known that the inverse Fourier transform is given by

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\mathsf{i}\omega)\, e^{\mathsf{i}\omega t}\, d\omega, \tag{2.12}$$

and that the Parseval theorem for this version of the Fourier transform is expressed by

$$\int_{-\infty}^{\infty} f(t) g^*(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\mathsf{i}\omega) G^*(\mathsf{i}\omega)\, d\omega. \tag{2.13}$$

It is possible to verify that to any function $f(t)$ belonging to $L^2(\mathbb{R})$ corresponds one and only one function $F(\mathsf{i}\omega)$ belonging to $L^2(\mathsf{i}\mathbb{R})$ and vice versa, and this defines a one to one mapping between $L^2(\mathbb{R})$ and $L^2(\mathsf{i}\mathbb{R})$.[21] That this mapping is also an isomorphism follows immediately from Parseval's theorem, which implies that $\langle f, g\rangle = \langle F, G\rangle$.

It is quite interesting to verify that the Fourier transform of a function belonging to $L^2(\mathbb{R})$ can be seen to be the bilateral Laplace transform of that function evaluated on the imaginary axis. This follows immediately from the definition of the bilateral Laplace transform, which is[22]

$$F(s) = \int_{-\infty}^{\infty} f(t) e^{-st}\, dt, \tag{2.14}$$

and from the definition of the Fourier transform [compare (2.11) with (2.14)]. The inverse bilateral Laplace transform for $L^2(\mathbb{R})$ functions may be given by

$$f(t) = \frac{1}{2\pi\mathsf{i}} \int_{\mathsf{i}\mathbb{R}} F(s) e^{st}\, ds, \tag{2.15}$$

which is clearly a generalization of (2.12). The Parseval theorem for bilateral Laplace transforms of $L^2(\mathbb{R})$ functions is expressed by

$$\int_{-\infty}^{\infty} f(t) g^*(t)\, dt = \frac{1}{2\pi\mathsf{i}} \int_{\mathsf{i}\mathbb{R}} F(s) G^*(-s^*)\, ds = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\mathsf{i}\omega) G^*(\mathsf{i}\omega)\, d\omega, \tag{2.16}$$

[20] This definition also holds if $L^2(\mathbb{R})$ is replaced by $L^2(S)$, with an obvious change in the range of the integration. Alternatively, the functions of $L^2(S)$ can be regarded as functions of $L^2(\mathbb{R})$ that vanish a.e. for $t \notin S$.

[21] As a side remark note that our non standard definition of the Fourier transform makes possible the utilization of a different notation for $L^2(\mathbb{R})$ and for $L^2(\mathsf{i}\mathbb{R})$. If we had adhered to the standard definition, we would have had to deal with two Hilbert spaces of square integrable functions on $\mathbb{R}$ with two slightly different inner products, which would have required a much more complex notation.

[22] This definition should be complemented with a description of the region of convergence of $F(s)$. Here we are interested in convergence in the mean and so for square integrable functions $F(s)$ will converge at least a.e. on $\mathsf{i}\mathbb{R}$.

and so it agrees with (2.13). If at least one of $F(s)$ and $G(s)$ is analytic on $\mathrm{i}\mathbb{R}$ this integral can be evaluated using the residue theorem by closing the contour with a semi-circle of infinite radius on either side of the complex plane.[23]

An interesting proper subset of $L^2(\mathbb{R})$ is the Hilbert space $L^2(\mathbb{R}^+)$ of causal square integrable functions. Unfortunately, not all functions in this set are absolutely integrable and it is this second condition that is satisfied by the impulse response of stable and causal continuous-time systems.[24]

The Fourier transform given by (2.11) defines an isomorphism between $L^2(\mathbb{R}^+)$ and a proper subset of $L^2(\mathrm{i}\mathbb{R})$, which is also a Hilbert space. This proper subset is sometimes denoted by $\mathcal{H}^2(\mathbb{C}^+)$, which is the Hardy space of the functions analytic on $\mathbb{C}^+$ that vanish at infinity and that are square integrable on $\mathrm{i}\mathbb{R}$.[25,26] Note that this particular Hardy space is equipped with an inner product that is defined in exactly the same way as the one of $L^2(\mathrm{i}\mathbb{R})$. To see why we decided to use the Hardy space defined over $\mathbb{C}^+$ and not over $\mathrm{i}\mathbb{R}$ observe that the bilateral Laplace transform of causal square integrable functions is analytic for $\mathrm{Re}[s] > 0$ (i.e., on $\mathbb{C}^+$) and that it vanishes at "infinity" (here "infinity" means $r\,e^{\mathrm{i}\theta}$ with $r \to \infty$ and $-\pi < 2\theta < \pi$). The additional condition that it is also square integrable on $\mathrm{i}\mathbb{R}$ comes from the fact that the Fourier transform of $L^2(\mathbb{R})$ functions is the bilateral Laplace transform of those functions evaluated on $\mathrm{i}\mathbb{R}$. To conclude the argument, observe also that if a function is not analytic on $\mathbb{C}^+$ or if it does not vanish at "infinity" then its inverse bilateral Laplace transform is not a square integrable causal function, and so there is a one to one relation between elements of $L^2(\mathbb{R}^+)$ and elements of $\mathcal{H}^2(\mathbb{C}^+)$. Note that for causal functions the bilateral Laplace transform becomes the much more commonly used unilateral Laplace transform.

2.3 Orthonormal expansions

One of the main reasons of the great popularity of Hilbert spaces is that the problem of best approximation, in the norm induced by the inner product, of an element of one such space by a linear combination of other elements of the same space has a unique solution, and to the fact that this solution is not difficult to find using a simple and intuitive geometric line of reasoning.

Let U be a linear closed subspace of a Hilbert space H and V its orthogonal complement, i.e., V is composed by all elements of H that are simultaneously orthogonal to all elements of U. One of the fundamental theorems about Hilbert spaces is the so-called projection theorem, which states that the direct sum of U and V is H itself.[27] One of the consequences of this

[23] In this report we will always close the path of integration using a semi-circle on the left half of the complex plane. Note that closing it on the other side makes the path of integration clockwise.

[24] A function $f(t)$ with support in S is absolutely integrable if $\int_S |f(t)|\,dt < \infty$, and if so it belongs to the Banach space $L^1(\mathbb{R}^+)$.

[25] Actually, this proper subset of $L^2(\mathrm{i}\mathbb{R})$ should be denoted by $\mathcal{H}^2(\mathrm{i}\mathbb{R})$, which is the Hilbert space of square integrable functions on $\mathrm{i}\mathbb{R}$ with inverse Fourier transforms that vanish a.e. on the negative real axis. However, in this report we will use $\mathcal{H}^2(\mathbb{C}^+)$ instead of $\mathcal{H}^2(\mathrm{i}\mathbb{R})$ since they are equivalent (see next footnote) and the former will be more useful to us.

[26] Let $f(s)$ be a function of $\mathcal{H}^2(\mathbb{C}^+)$. Then $\lim_{x\to 0^+} f(x + \mathrm{i}y)$ converges for almost all values of $y \in \mathbb{R}$ to $f(\mathrm{i}y)$. This is a property of all Hardy spaces defined on $\mathbb{C}^+$ [Hoffman, 1988; Duren, 1970]. (Actually, only the tangential limits may not exist a.e. on $\mathrm{i}\mathbb{R}$.) The functions of $\mathcal{H}^2(\mathbb{C}^+)$ may then be viewed as the unique analytic extension of the corresponding functions of $\mathcal{H}^2(\mathrm{i}\mathbb{R})$.

[27] The direct sum between two closed linear subspaces X and Y of a Hilbert space H is the closed linear

theorem is the principle of orthogonality, that provides the solution to the best approximation problem stated in the previous paragraph. This principle states that the best approximation to a given element x of a given Hilbert space H by an element of the linear closed subspace U of that Hilbert space is given by the projection of x into that subspace. In other words, the error of the approximation is orthogonal to U, i.e., belongs to the orthogonal complement V of U. This follows easily from the decomposition $x = u + v$ where $u \in U$ and $v \in V$, which is guaranteed to exist and to be unique due to the projection theorem. Since any approximation y to x belongs to U and because U and V are orthogonal, we have $\|x - y\|_2^2 = \|u - y\|_2^2 + \|v\|_2^2$, which is minimized for $y = u$. In this case, the error of the approximation, which is v, is obviously orthogonal to y.

Assume that $\{\phi_i\}_{i=0}^{\infty}$ is an orthonormal basis of an infinite dimensional Hilbert space H. Then, each element x of H can be expressed in the form

$$x = \sum_{i=0}^{\infty} c_i \phi_i.$$

(As usual the equality has to be interpreted as convergence in the mean.) Due to the orthonormality and completeness of the functions ϕ_i it is simple to verify that

$$c_i = \langle x, \phi_i \rangle ,$$

and that

$$\langle x, x \rangle = \sum_{i=0}^{\infty} |c_i|^2.$$

A very interesting and important fact is that if $\sum_{i=0}^{\infty} |c_i|^2 < \infty$ then $\sum_{i=0}^{\infty} c_i \phi_i$ converges to an element of H. This constitutes the so-called Riesz-Fisher theorem.

Assume now than we want to approximate an element x of a Hilbert space by the best linear combination of the first $n+1$ elements of a complete orthonormal set $\{\phi_i\}_{i=0}^{\infty}$. In terms of the orthogonality principle stated above U_n, which is the subspace where the approximation to x lies, will be the linear span of the set $\{\phi_i\}_{i=0}^{n}$. The orthogonal complement V_n of U_n will then be the closed linear span of the set $\{\phi_i\}_{i=n+1}^{\infty}$. Since

$$x = \underbrace{\sum_{i=0}^{n} \langle x, \phi_i \rangle \, \phi_i}_{u_n \in U_n} + \underbrace{\sum_{i=n+1}^{\infty} \langle x, \phi_i \rangle \, \phi_i}_{v_n \in V_n}$$

it is easy to conclude that the best approximation to x by an element of U_n is u_n. (There are, of course, many other ways to obtain this elementary result.) A particularly important property of the solution of this approximation problem is that the coefficients of the best linear combination of elements of the orthonormal set $\{\phi_i\}_{i=0}^{n}$ are independent of n, i.e., these coefficients can be determined individually and do not depend on each other.

We will now present some examples of complete orthonormal sets for the Hilbert spaces introduced previously. With these orthonormal sets we can use an orthonormal expansion to represent any element of the corresponding Hilbert space.

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subspace Z of H whose elements are linear combinations of a finite number of elements of X and of Y and their closure. X and Y need not be orthogonal to each other.

- [$\ell^2(\mathbb{Z})$ and $\ell^2(\mathbb{N}_0)$] The sequences of the standard or canonical complete orthonormal sets of $\ell^2(\mathbb{Z})$ and $\ell^2(\mathbb{N}_0)$ are given by $\phi_i(k) = \delta_{ik}$, with $i \in \mathbb{Z}$ and $i \in \mathbb{N}_0$, respectively. These sequences are usually called unit pulse sequences, because their general form is

$$(\ldots, 0, \ldots, 0, \underbrace{1}_{k=i}, 0, \ldots, 0, \ldots),$$

  i.e., they vanish except for $k = i$.

- [$L^2(\mathbb{T})$ and $\mathcal{H}^2(\mathbb{E})$] The functions of the standard or canonical complete orthonormal sets of $L^2(\mathbb{T})$ and $\mathcal{H}^2(\mathbb{E})$ are given by $\phi_i(z) = z^{-i}$, with $i \in \mathbb{Z}$ and $i \in \mathbb{N}_0$, respectively. Notice that these orthonormal functions are merely the z transforms of the sequences of the standard orthonormal sequences of $\ell^2(\mathbb{Z})$ and $\ell^2(\mathbb{N}_0)$.

- [$L^2(\mathbb{R})$ and $L^2(\mathbb{R}^+)$] These Hilbert spaces do not have a standard or canonical complete orthonormal set. One of the simplest complete orthonormal sets of $L^2(\mathbb{R}^+)$ is the set of the Laguerre functions, which are obtained by orthonormalizing the functions $t^i e^{-pt}$, $t \in \mathbb{R}^+$, $i \in \mathbb{N}_0$. These functions are closely related to the Laguerre polynomials [Szegö, 1975], and are given by [Lee, 1960][28]

$$l_i(t) = (-1)^i \sqrt{2p}\, e^{-pt} \sum_{j=0}^{i} C_j^i \frac{(-2pt)^j}{j!}, \qquad t \in \mathbb{R}^+, \quad i \in \mathbb{N}_0.$$

  The parameter p is a strictly positive real number that defines the time scale of these functions. Most old definitions of these functions use $p = 1/2$. For obvious reasons, the set

$$\{\ldots, l_1(-t), l_0(-t), l_0(t), l_1(t), \ldots\}$$

  is a complete orthonormal set of $L^2(\mathbb{R})$. Notice that "half" of the Laguerre functions are time reversed.

- [$L^2(\mathrm{i}\mathbb{R})$ and $\mathcal{H}^2(\mathbb{C}^+)$] These Hilbert spaces do not have a standard or canonical complete orthonormal set. One of the simplest complete orthonormal sets of $\mathcal{H}^2(\mathbb{C}^+)$ is the set

$$\left\{ \frac{\sqrt{2p}}{p+s} \left(\frac{p-s}{p+s} \right)^i \right\}_{i=0}^{\infty},$$

  which is just the set of the Laplace transforms of the Laguerre functions. The corresponding complete orthonormal set of $L^2(\mathrm{i}\mathbb{R})$ is the set

$$\left\{ \frac{\sqrt{2p}}{p+\mathrm{i}\omega} \left(\frac{p-\mathrm{i}\omega}{p+\mathrm{i}\omega} \right)^i \right\}_{i=-\infty}^{\infty},$$

  which is just the set of the Fourier transforms of the Laguerre functions and their time reversed versions [Lee, 1933; Steiglitz, 1965a].

[28] Most modern definitions of these functions omit the factor $(-1)^i$ in the definition given here.

Figure 2.1: Diagram of the isomorphisms between the Hilbert spaces described in sections 2.1 and 2.2. Each number inside curly brackets represents the equation number that defines the corresponding isomorphism.

2.4 A family of isomorphisms between spaces of functions and spaces of sequences

The isomorphisms described in sections 2.1 and 2.2 between the Hilbert spaces introduced there are depicted in figure 2.1. It would be interesting, for reasons that will become apparent in the next chapters, to construct an explicit isomorphism between $\mathcal{H}^2(\mathbb{E})$ and $\mathcal{H}^2(\mathbb{C}^+)$, or between $L^2(\mathbb{T})$ and $L^2(\mathsf{i}\mathbb{R})$. As it happens, it is easy to construct one family of such isomorphisms using the bilinear transformation. In the engineering literature, to the best of our knowledge, this transformation was used in this context for the first time by Steiglitz [1965a]. We have not yet attempted to trace its utilization, in a similar context, in the mathematical literature, but it is almost certain that mathematicians have used it well before Steiglitz did (see, e.g., [Hoffman, 1988]).

The bilinear transformation we are interested in is actually a conformal mapping (a change of variables) that maps the interior (exterior) of the unit circle into the left (right) half of the complex plane. The reason for the choice of this conformal mapping is primarily related to the guaranteed domains of analyticity of the functions belonging respectively to $\mathcal{H}^2(\mathbb{E})$ and to $\mathcal{H}^2(\mathbb{C}^+)$, since the first (domain) is transformed into the second by this conformal mapping. This change of variables is given by

$$s = a\frac{z-1}{z+1}, \qquad ds = \frac{2a}{(z+1)^2}\,dz, \tag{2.17}$$

and the corresponding inverse change of variables is given by

$$z = \frac{a+s}{a-s}, \qquad dz = \frac{2a}{(a-s)^2}\,ds. \tag{2.18}$$

The parameter a is an arbitrary strictly positive real number that is normally replaced by 1. We will keep this degree of freedom in the specification of the bilinear transformation, since this will be useful later on. The conformal mapping induced by (2.17) is depicted in figure 2.2.

To avoid confusion, in this section we will represent functions belonging to $\mathcal{H}^2(\mathbb{E})$ by normal capital letters, such as for example F, and functions belonging to $\mathcal{H}^2(\mathbb{C}^+)$ by capital letters in a sans-serif font, such as for example F. To establish an isomorphism between $\mathcal{H}^2(\mathbb{E})$ and $\mathcal{H}^2(\mathbb{C}^+)$ we will apply the change of variables given by (2.18) to the alternative form for the inner product of $\mathcal{H}^2(\mathbb{E})$, given by the middle expression in (2.8). Our goal is to obtain a final expression that is in the form of the middle expression in (2.16), which is an alternative form for the inner product of $\mathcal{H}^2(\mathbb{C}^+)$. Furthermore, we will require that each function of $\mathcal{H}^2(\mathbb{E})$ be transformed into one and only one function of $\mathcal{H}^2(\mathbb{C}^+)$ and vice versa.

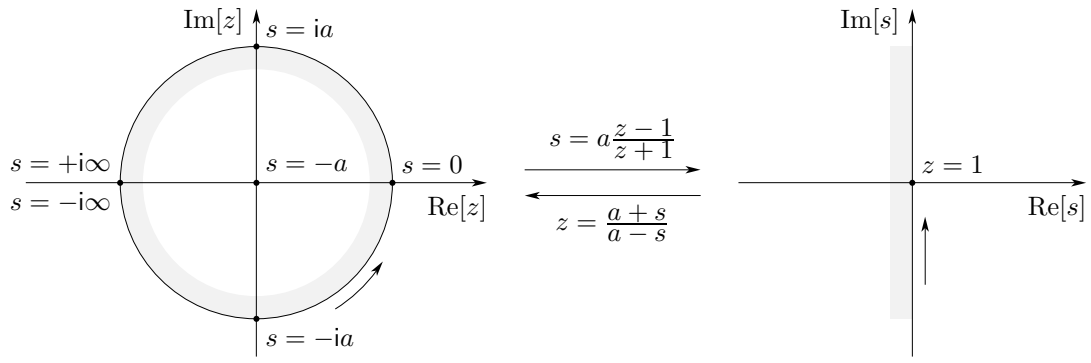

Figure 2.2: Conformal mapping of the bilinear transformation (2.17). The points (marked with dots) placed in each complex plane correspond to the image of the points of the other plane with the given coordinates. Note that any contour integral over the boundary of the unit circle is transformed into a line integral over the imaginary axis in the s plane. Note also that the unit circle $\{z \,:\, |z| < 1\}$ is transformed into the left half plane $\{s \,:\, \mathrm{Re}[s] < 0\}$, and that $\mathbb{E}$ is transformed into $\mathbb{C}^+$.

As soon as both these tasks are done we will have obtained an isomorphism between $\mathcal{H}^2(\mathbb{E})$ and $\mathcal{H}^2(\mathbb{C}^+)$.[29] The result of the change of variables mentioned above is

$$\frac{1}{2\pi\mathsf{i}} \oint_{\mathbb{T}} F(z)G^*(1/z^*)\,\frac{dz}{z} = \frac{1}{2\pi\mathsf{i}} \int_{\mathsf{i}\mathbb{R}} F\Big(\frac{a+s}{a-s}\Big)\; G^*\Big(\frac{a-s^*}{a+s^*}\Big)\; \frac{2a}{(a-s)(a+s)}\, ds. \tag{2.19}$$

Apart from the term $\frac{2a}{(a-s)(a+s)}$ the rest of the expression is already in the required form. This extra term, that was introduced by the change of the measure of integration, can, fortunately, be factored as follows

$$\frac{2a}{(a-s)(a+s)} = \alpha(s)\,\alpha^*(-s^*), \qquad \alpha(s) = A(s)\frac{\sqrt{2a}}{a+s}$$

where $A(s)$ is an all-pass function, i.e., a function that satisfies $A(s)A^*(-s^*) = 1$. These functions satisfy $|A(\mathsf{i}\omega)| = 1$ for all real ω, and so are unimodular on the imaginary axis. With the transformation

$$H(z) \mapsto \mathsf{H}(s) = \alpha(s)\, H\Big(\frac{a+s}{a-s}\Big) \tag{2.20}$$

equation (2.19) can be rewritten as

$$\frac{1}{2\pi\mathsf{i}} \oint_{\mathbb{T}} F(z)G^*(1/z^*)\,\frac{dz}{z} = \frac{1}{2\pi\mathsf{i}} \int_{\mathsf{i}\mathbb{R}} \mathsf{F}(s)\mathsf{G}^*(-s^*)\, ds$$

which is exactly the form of the inner product of $\mathcal{H}^2(\mathbb{C}^+)$. The problem now lies in the choice of the all-pass function $A(s)$ such that each function of $\mathcal{H}^2(\mathbb{E})$ is transformed into one and

[29] Because the inner products of $\mathcal{H}^2(\mathbb{E})$ and of $L^2(\mathbb{T})$ are given by exactly the same formula, and because the same happens between the inner products of $\mathcal{H}^2(\mathbb{C}^+)$ and of $L^2(\mathsf{i}\mathbb{R})$, the isomorphism we are about to obtain between $\mathcal{H}^2(\mathbb{E})$ and $\mathcal{H}^2(\mathbb{C}^+)$ will be equally valid for the spaces $L^2(\mathbb{T})$ and $L^2(\mathsf{i}\mathbb{R})$. The difficult part in the derivation we are about to undertake is to make sure that each function of $\mathcal{H}^2(\mathbb{E})$ in transformed into a function of $\mathcal{H}^2(\mathbb{C}^+)$ and vice versa. That restriction is not necessary for an isomorphism between $L^2(\mathbb{T})$ and $L^2(\mathsf{i}\mathbb{R})$.

Figure 2.3: Diagram of the isomorphisms between the Hilbert spaces described in this report. Each number inside curly brackets represents the equation number that defines the corresponding isomorphism.

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only one function of $\mathcal{H}^2(\mathbb{C}^+)$ and vice versa. We will show that the simplest possible choice for $A(s)$, viz. $A(s) = 1$, satisfies this requirement, with the added advantage that rational functions are transformed into rational functions.

Setting $A(s) = 1$ the transformation (2.20) becomes

$$H(z) \mapsto \mathsf{H}(s) = \frac{\sqrt{2a}}{a+s} H\left(\frac{a+s}{a-s}\right). \tag{2.21}$$

Any element $F(z)$ of $\mathcal{H}^2(\mathbb{E})$ can be written in the form[30]

$$F(z) = \sum_{i=0}^{\infty} c_i z^{-i} \tag{2.22}$$

with

$$\sum_{i=0}^{\infty} |c_i|^2 < \infty$$

Applying (2.21) to this $F(z)$ yields

$$\mathsf{F}(s) = \sum_{i=0}^{\infty} c_i \frac{\sqrt{2a}}{a+s} \left(\frac{a-s}{a+s}\right)^i. \tag{2.23}$$

But since the functions $\frac{\sqrt{2a}}{a+s}\left(\frac{a-s}{a+s}\right)^i$ form a complete orthonormal set in $\mathcal{H}^2(\mathbb{C}^+)$ (cf. section 2.3), by the Riesz-Fisher theorem $\mathsf{F}(s)$ belongs to $\mathcal{H}^2(\mathbb{C}^+)$. From this argument it is clear that to each $F(z)$ corresponds one and only one $\mathsf{F}(s)$ and vice versa. The isomorphism is then established.

The inverse of (2.21) is given by

$$\mathsf{H}(s) \mapsto H(z) = \frac{\sqrt{2a}\, z}{z+1} \mathsf{H}\left(a\frac{z-1}{z+1}\right). \tag{2.24}$$

As expected, applying this transformation to (2.23) gives (2.22). In this way, a Laguerre series expansion (in the Laplace transform domain) can be transformed into a power series, a result that was noticed by Schetzen [1970] using a more direct approach.

To conclude this section we present in figure 2.3 an updated diagram of the isomorphisms described so far.

[30] Recall that $\{1, z^{-1}, z^{-2}, \ldots\}$ is the standard basis of $\mathcal{H}^2(\mathbb{E})$.

2.5 The supremum norm and related function spaces

The purpose of this section is to describe briefly two Hardy spaces that are related to another kind of approximations, in the so-called supremum or $\mathcal{H}^\infty$ norm. The spaces in question are $\mathcal{H}^\infty(\mathbb{E})$, the Hardy space of functions analytic on $\mathbb{E}$ that are essentially bounded on $\mathbb{T}$, and $\mathcal{H}^\infty(\mathbb{C}^+)$, the Hardy space of functions analytic on $\mathbb{C}^+$ vanishing at "infinity" that are essentially bounded on $\mathrm{i}\mathbb{R}$, equipped respectively with the norms[31]

$$\|F\|_\infty = \operatorname*{ess\,sup}_{z\in\mathbb{T}} |F(z)|$$

and

$$\|F\|_\infty = \operatorname*{ess\,sup}_{s\in\mathrm{i}\mathbb{R}} |F(s)|.$$

A function with a finite essential supremum is said to be essentially bounded. The main property of these two Hardy spaces is that they are closed, i.e., all Cauchy sequences in one of those spaces converge to an element of that space. In fact, they are Banach spaces, which are precisely closed normed spaces.

The reason why we have introduced these two Hardy spaces is because some bases of $\mathcal{H}^2(\mathbb{E})$ and $\mathcal{H}^2(\mathbb{C}^+)$ exhibit the desirable property, in terms of $\mathcal{H}^\infty$ approximations, of being absolutely bounded. This means that there is a common bound for the $\mathcal{H}^\infty$ norm of all elements of one of those basis.

We will not describe here the advantages of the absolute ($\mathcal{H}^\infty$) boundedness of a Hilbert space basis. In this report we are merely interested in verifying if some specific bases that will be introduced in the following chapter are absolutely bounded or not, and in providing conditions that assure that they are.

It is interesting to verify that the raw bilinear transformation, defined by (2.17) or by (2.18), defines one isometry between $\mathcal{H}^\infty(\mathbb{E})$ and $\mathcal{H}^\infty(\mathbb{C}^+)$. This is so because $\mathbb{T}$ is mapped (smoothly) into $\mathrm{i}\mathbb{R}$ and vice versa, and it is in precisely these two sets that the essential supremum is evaluated.

[31] The essential supremum of a Lebesgue measurable function, denoted by ess sup, is the smallest possible (i.e., the infimum) value of the supremum of all functions that differ from the given one in a set of null measure. Contrary to the normal supremum, in the essential supremum the value of a function in a set of null measure is irrelevant.

Chapter 3

Rational orthonormal functions on the unit circle and on the imaginary axis

In this chapter we address the problem of the construction of orthonormal sets of rational functions in the Hardy spaces $\mathcal{H}^2(\mathbb{E})$ and $\mathcal{H}^2(\mathbb{C}^+)$. The principal reason for our interest in this problem stems from the fact that finite-dimensional systems have transfer functions that are rational (and proper). The utilization of a rational set of orthonormal functions to approximate an element, which may not be rational, of, say, $\mathcal{H}^2(\mathbb{E})$, gives rise to approximations (truncated orthonormal expansions) that are rational. In terms of transfer functions this is tantamount to approximate a transfer function of a possibly infinite-dimensional system by the transfer function of a finite-dimensional system. Therefore, this approximation can be realized and used very easily in practice.[1]

This approximation problem can also be formulated in the time domain, since we have already built an isomorphism between $\mathcal{H}^2(\mathbb{E})$ and $\ell^2(\mathbb{N}_0)$, and between $\mathcal{H}^2(\mathbb{C}^+)$ and $L^2(\mathbb{R}^+)$. There, the approximation problem becomes one of approximating the unit pulse response ($\ell^2(\mathbb{N}_0)$) or unit impulse response ($L^2(\mathbb{R}^+)$) of a given system by a linear combination of orthonormalized generalized exponentials. To give a more appealing picture of the problem we are about to address we will start our presentation of the rational orthonormal functions on the unit circle and on the imaginary axis precisely in the time domain.

3.1 Rational orthonormal functions on the unit circle

Consider the sequence $(z_i)_{i=0}^{\infty}$ of complex numbers, not necessarily distinct, that belong to $\mathbb{D}$. Let ν_i be the number of times that the value of z_i appears *before* in this sequence, i.e., let $\nu_i = \sum_{j=0}^{i-1} \delta_{z_i,z_j}$. For each $i \in \mathbb{N}_0$ let[2]

$$f_i(k) = C_{\nu_i}^k z_i^{k-\nu_i}, \qquad k \in \mathbb{N}_0.$$

Note that by definition $C_{\nu_i}^k$ vanishes for $k < \nu_i$, and that $C_{\nu_i}^k$ is a polynomial in k of degree ν_i for $k \geq \nu_i$. Since $|z_i| < 1$ it is obvious that $f_i(k) \in \ell^2(\mathbb{N}_0)$. For $\nu_i = 0$ the sequences $f_i(k)$ are

[1] We remark that the rational orthonormal functions are exponentially stable. The approximation is therefore exponentially stable as well.

[2] By convention $0^0 = 1$. This case occurs if $z_i = 0$. (The corresponding $f_i(k)$ is then equal to δ_{k,ν_i}.)

discrete exponentials, i.e., of the form z_i^k, and for $\nu_i > 0$ they are delayed generalized discrete exponentials, i.e., delayed discrete exponentials multiplied by a polynomial in k. It is a well know fact that these sequences are linearly independent.[3]

Consider now the (monic) polynomials

$$D_{n+1}(z) = \prod_{i=0}^{n}(z - z_i) = \sum_{i=0}^{n+1} a_{n+1,j}\, z^i, \qquad n \in \mathbb{N}_0,$$

of degree $n+1$. It is a well know fact from the theory of ordinary linear difference equations with constant coefficients that all solutions of the homogeneous difference equation

$$\sum_{i=0}^{n+1} a_{n+1,i} h(k+i) = 0, \qquad k \in \mathbb{N}_0, \tag{3.1}$$

are of the form

$$h_n(k) = \sum_{i=0}^{n} b_i f_i(k) \tag{3.2}$$

where the values of the coefficients b_i can be uniquely determined from the values of $h(k)$ for $k = 0, 1, \ldots, n$ [Jury, 1964]. Equation (3.1) is the hallmark of a time-invariant finite-dimensional linear system with degree $n+1$ and with poles $z_0, \ldots, z_n$.[4] In fact, the unit pulse response of all such systems have the form (3.2).[5]

From the previous paragraph we know that $h_n(k)$, which is a linear combination of the first $n+1$ sequences $f_i(k)$ [cf. (3.2)], is the unit pulse response of a finite-dimensional linear system of degree $n+1$. This makes the approximation of a given sequence $h(k)$ of $\ell^2(\mathbb{N}_0)$ by $h_n(k)$ very interesting from the view point of system approximation, since the impulse responses of all stable and causal linear systems belong to that Hilbert space. Unfortunately, except in the special case $z_i = 0$ for all $i \in \mathbb{N}_0$, the sequences $f_i(k)$ are not orthonormal, and this makes the approximation problem more difficult that necessary. To simplify this approximation problem, we are interested in the orthonormalization of these sequences using the Gram-Schmidt algorithm with normalization.[6] It turns out that this orthonormalization

[3] If $z_i \neq 0$ for all $i \in \mathbb{N}_0$ this linear independence can be established as follows. For each $n \in \mathbb{N}_0$ construct an $n+1 \times n+1$ matrix with elements $[f_i(k+n)]_{k,i=0}^{n}$. The elements of the i-th column of this matrix have a common non-zero factor, which is $z_i^{n-\nu_i}$. After removing these factors each column of the resulting matrix is either a discrete exponential (if $\nu_i = 0$), or a discrete exponential multiplied by a polynomial in k (if $\nu_i > 0$). But then, by simple column operations this matrix can be put in the form of a (possibly confluent) Vandermonde matrix, which is know to be regular [Davis, 1975]. This proves the linear independence of the first $n+1$ sequences for the case discussed here. The case $z_i = 0$ for one or more values of i requires a more complex proof, and is left as an exercise to the interested reader.

[4] Depending on the initial conditions, some of these poles may be canceled. That's why we have not talked about the McMillan degree of the system.

[5] Note that the standard state-space realization of a strictly proper, in z, finite-dimensional discrete-time system gives rise to an equation similar to (3.1) but valid only for $k \in \mathbb{N}$ and with $h(0) = 0$. Our discussion is for a strictly proper, in z^{-1}, finite-dimensional system.

[6] The Gram-Schmidt algorithm with normalization replaces each sequence $f_i(k)$ by a sequence $g_i(k)$ of the form

$$g_i(k) = \sum_{j=0}^{i} \alpha_{i,j}\, f_j(k) = \sum_{j=0}^{i-1} \beta_{i,j}\, g_j(k) + \alpha_{i,i}\, f_i(k), \qquad \alpha_{i,i} \neq 0,$$

such that $g_i(k)$ is normal and orthogonal to $f_j(k)$ (or, equivalently, to $g_j(k)$) for $j = 0, \ldots, i-1$. In our case this is always possible because the sequences $f_i(k)$ are linearly independent. It is easy to verify that the sequences $g_i(k)$ and $f_i(k)$ are related by a lower-triangular non-singular linear transformation.

can be carried out much more easily in the Hardy space $\mathcal{H}^2(\mathbb{E})$, where its solution has a very simple and elegant form. Note that it is possible to translate this problem to $\mathcal{H}^2(\mathbb{E})$ using the isomorphism (the z transform) between $\ell^2(\mathbb{N}_0)$ and $\mathcal{H}^2(\mathbb{E})$ described in the previous chapter.

The z transforms of the sequences $f_i(k)$ defined at the beginning of this section are given by[7]

$$F_i(z) = \frac{z}{(z - z_i)^{1+\nu_i}}.$$

Our task is to construct the functions

$$G_i(z) = \sum_{j=0}^{i} \alpha_{i,j} F_j(z) \tag{3.3}$$

such that $\langle G_i, G_j \rangle = \delta_{ij}$. Note that $G_i(z)$ has the form

$$G_i(z) = \frac{z P_i(z)}{\prod_{j=0}^{i}(z - z_j)}$$

for some polynomial $P_i(z)$ of formal degree i. This can be seen easily by reducing (3.3) to a common denominator. We begin with the explicit determination of the first two functions, and then jump to the general case.

Consider first the determination of $G_0(z)$. In particular, consider the determination of $\alpha_{0,0}$ such that $\langle G_0, G_0 \rangle = 1$. An elementary calculation using the residue theorem shows that

$$\langle G_0, G_0 \rangle = \frac{1}{2\pi \mathsf{i}} \oint_{\mathbb{T}} G_0(z) G_0^*(1/z^*) \frac{dz}{z} = \frac{1}{2\pi \mathsf{i}} \oint_{\mathbb{T}} \frac{|\alpha_{0,0}|^2}{(z - z_0)(1 - z_0^* z)} \, dz = \frac{|\alpha_{0,0}|^2}{1 - |z_0|^2}.$$

The normalization condition then implies that $\alpha_{0,0} = \kappa_0 \sqrt{1 - |z_0|^2}$, where κ_0 is an arbitrary unimodular complex number (i.e., of modulus one). $G_0(z)$ is then given by

$$G_0(z) = \kappa_0 \sqrt{1 - |z_0|^2} \, \frac{z}{z - z_0}.$$

Consider now the determination of $G_1(z)$. First of all, $G_1(z)$ must be orthogonal of $G_0(z)$. But

$$\langle G_0, G_1 \rangle = \frac{1}{2\pi \mathsf{i}} \oint_{\mathbb{T}} G_0(z) G_1^*(1/z^*) \frac{dz}{z} = 0$$

implies that $G_1^*(1/z^*)$ must vanish for $z = z_0$. Therefore

$$P_1(z) = \gamma_1 (1 - z_0^* z)$$

for some γ_1 yet to be determined. A careful consideration shows that this formula is also valid for $z_0 = 0$. The remaining condition, $\langle G_1, G_1 \rangle = 1$, determines the value of γ_1. An elementary calculation shows that

$$\begin{aligned}
\langle G_1, G_1 \rangle &= \frac{1}{2\pi \mathsf{i}} \oint_{\mathbb{T}} \frac{\gamma_1 (1 - z_0^* z)}{(z - z_0)(z - z_1)} \frac{\gamma_1^* (z - z_0)}{(1 - z_0^* z)(1 - z_1^* z)} \, dz \\
&= \frac{1}{2\pi \mathsf{i}} \oint_{\mathbb{T}} \frac{|\gamma_1|^2}{(z - z_1)(1 - z_1^* z)} \, dz \\
&= \frac{|\gamma_1|^2}{1 - |z_1|^2}.
\end{aligned}$$

[7]The reason why we defined the sequences $f_i(k)$ to include the term $C_{\nu_i}^k$ instead of just k^{ν_i} and delayed then by ν_i samples was just to make $F_i(z)$ very simple. There is no loss of generality in doing so.

The normalization condition then implies that $\gamma_1 = \kappa_1\sqrt{1-|z_1|^2}$, where κ_1 is an arbitrary unimodular complex number. $G_1(z)$ is then given by

$$G_1(z) = \kappa_1\sqrt{1-|z_1|^2}\,\frac{z(1-z_0^*z)}{(z-z_0)(z-z_1)}.$$

The determination of $G_1(z)$ gives sufficient clues about how to proceed in the general case. Consider then the determination of $G_i(z)$, for $i \in \mathbb{N}_0$. (Although the cases $i = 0$ and $i = 1$ have already been treated, we include them here since the argument we are about to present is also applicable to them.) If $i > 0$ the condition $\langle G_0, G_i\rangle = 0$ forces $P_i(1/z^*)$ to vanish for $z = z_0$. But then, if $i > 1$ the condition $\langle G_1, G_i\rangle = 0$ forces $P_i(1/z^*)$ to vanish also for $z = z_1$.[8] Using this argument for $\langle G_j, G_i\rangle = 0$, $j = 2, \ldots, i-1$, we conclude that $P_i(z)$ has the form

$$P_i(z) = \gamma_i \prod_{j=0}^{i-1}(1-z_j^*z)$$

for some γ_i to be determined from the condition $\langle G_i, G_i\rangle = 1$. An elementary calculation shows that

$$\langle G_i, G_i\rangle = \frac{1}{2\pi\mathsf{i}}\oint_{\mathbb{T}} \frac{|\gamma_i|^2}{(z-z_i)(1-z_i^*z)}\,dz = \frac{|\gamma_i|^2}{1-|z_i|^2}.$$

The normalization condition then implies that $\gamma_i = \kappa_i\sqrt{1-|z_i|^2}$, where κ_i is an arbitrary unimodular complex number. $G_i(z)$ is then given by

$$G_i(z) = \kappa_i\sqrt{1-|z_i|^2}\,\frac{z\prod_{j=0}^{i-1}(1-z_j^*z)}{\prod_{j=0}^{i}(z-z_j)},$$

which can be rewritten in the form

$$G_i(z) = \kappa_i A_{i-1}(z) L_i(z) \tag{3.4a}$$

with

$$A_{i-1}(z) = \prod_{j=0}^{i-1}\left(\frac{1-z_j^*z}{z-z_j}\right) \tag{3.4b}$$

and

$$L_i(z) = \sqrt{1-|z_i|^2}\,\frac{z}{z-z_i}. \tag{3.4c}$$

Note that $A_{i-1}(z)$ is an all-pass function since $A_{i-1}(z)A_{i-1}^*(1/z^*) = 1$, which implies that $|A_{i-1}(e^{\mathsf{i}\omega})| = 1$ for $\omega \in [-\pi, \pi]$. Note also that $L_i(z)$ is a first order low-pass function. Therefore, $G_i(z)$ is a cascade of i first order all-pass sections followed by a first order low-pass section. Since $G_i(z)$ and $G_{i-1}(z)$ share the same all-pass section $A_{i-2}(z)$, the whole set of orthonormal functions can be generated by a cascade of first order all-pass sections, tapered by first order low-pass sections, as shown in figure 3.1.

[8] Note that in this case the pole of $G_1(z)$ at $z = z_0$ is canceled by the zero of $P_i(1/z^*)$ also at $z = z_0$, and so the contour integral will only have one pole (z_1) inside the unit circle. This argument is only strictly valid if $z_0 \neq 0$. A careful analysis shows that its conclusion remains valid for this special case.

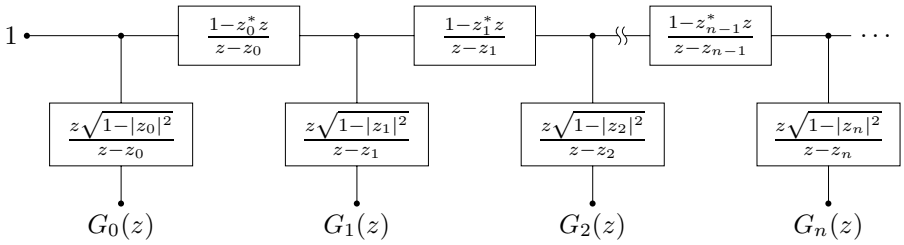

Figure 3.1: Block diagram of the generation of the orthonormal functions $G_i(z)$ given by (3.4).

As far as we know the functions (3.4) were discovered almost simultaneously in 1925 by Malmquist [1926] (see also [Walsh, 1969]), and by Takenaka [1925],[9] and were later rediscovered by Young and Huggins [1962a], and by Broome [1965] (and by [Ninness and Gustafsson, 1994]!). The first publications about this problem in the engineering literature appear to be the already mentioned conference paper of Young and Huggins [1962a] and the (also already mentioned) paper of Broome [1965]. These two papers are based on the equivalent problem of the construction of rational orthonormal functions in $\mathcal{H}^2(\mathbb{C}^+)$ that had appeared around ten years previously in the engineering literature [Kautz, 1954]. A fairly complete account of this equivalent problem will be given in the next section.

The special case in which the sequence $(z_i)_{i=0}^{\infty}$ is periodic is of considerable importance and was suggested as far back as [Walsh, 1969]. Its importance is related to the fact that it is relatively easy to obtain asymptotic results for orthonormal expansions in this case. For models based on this kind of functions the same is true, as was shown in [Wahlberg, 1991], [Wahlberg, 1994], and [Van den Hof, Heuberger and Bokor, 1995], respectively for the cases of sequences of period one (Laguerre functions), period two ("two-parameter" Kautz functions), and arbitrary period (the system based orthonormal functions of Heuberger *et al.*).

The orthonormal functions given by (3.4), although perfectly satisfactory to a mathematician, are not very satisfactory to an engineer. The problem is that some of then do not have real inverse z transforms when their poles are real and/or form complex conjugate pairs. This is particularly annoying if they (the inverse z transforms) are used to approximate a real sequence, since then to form the approximation one has to use complex numbers.[10] Fortunately, there is a simple way to avoid this problem [Young and Huggins, 1962a; Broome, 1965], and this will be the theme for the rest of this already long section.

The z transform $F(z)$ of a real sequence $f(k)$ satisfies the condition $F(z) = F^*(z^*)$. We will say that the functions satisfying this condition are z-real functions. In particular, if $F(z)$ is an irreducible z-real rational function then the coefficients of its numerator and denominator polynomials are real.

Consider first the case of a real z_i. In this case $G_i(z)$ will only be z-real if κ_i is real and if the poles of $A_{i-1}(z)$ are either real or form complex conjugate pairs.[11] In terms of the sequence $(z_j)_{j=0}^{\infty}$ this forces z_i to be preceded by either real numbers or by pairs of complex conjugate numbers.

[9] In the mathematical literature, the poles of $G_i(z)$ are usually placed in $\mathbb{E}$ and not in $\mathbb{D}$. Mathematicians usually prefer to approximate functions analytic in $\mathbb{D}$ and not in $\mathbb{E}$ as we do here. Clearly, the two problems are equivalent.

[10] The approximation will be real valued if the poles are real and/or form complex conjugate pairs.

[11] Since the zeros of $A_{i-1}(z)$ are the inversion with respect to the unit circle of its poles, it is enough to specify the poles.

Consider now the case of a complex z_i. In this case $G_i(z)$ will not be z-real no matter how we select κ_i and the poles of $A_{i-1}(z)$. As we will see, one way to obtain a function that is z-real is to form a linear combination of $G_i(z)$ and $G_{i+1}(z)$. Note that this forces z_{i+1} to be the complex conjugate of z_i, since the denominator of the resulting function must be a polynomial with real coefficients. Note also that $G_{i+1}(z)$ will not be z-real, and so we actually need two different linear combinations of $G_i(z)$ and $G_{i+1}(z)$ to form two new orthonormal z-real functions.

Any linear combination of $G_i(z)$ and $G_{i+1}(z)$ has the form

$$\begin{aligned}\alpha G_i(z) + \beta G_{i+1}(z) &= A_{i-1}(z)\frac{z[\alpha(z - z_{i+1}) + \beta(1 - z_i^* z)]}{(z - z_i)(z - z_{i+1})} \\ &= A_{i-1}(z)\frac{z[z(\alpha - \beta z_i^*) + (\beta - \alpha z_{i+1})]}{(z - z_i)(z - z_{i+1})},\end{aligned}$$

which can be made z-real if $z_{i+1} = z_i^*$; it is already z-real if both z_i and z_{i+1} are real. In fact, it is possible to construct two z-real orthonormal functions of the form

$$G_{r,i}(z) = \gamma_{r,i} A_{i-1}(z)\frac{z(z - a)}{(z - z_i)(z - z_{i+1})}$$

$$G_{r,i+1}(z) = \gamma_{r,i+1} A_{i-1}(z)\frac{z(z - b)}{(z - z_i)(z - z_{i+1})}$$

with (obviously) real $\gamma_{r,i}$, $\gamma_{r,i+1}$, a, and b. Without loss of generality we will make $\gamma_{r,i} > 0$ and $\gamma_{r,i+1} > 0$. Note that a and b are not independent. In fact, one determines the other, since

$$\langle G_{r,i}, G_{r,i+1}\rangle = \gamma_{r,i}\gamma_{r+1,i}\,\frac{(1 + ab)(1 + z_i z_{i+1}) - (a + b)(z_i + z_{i+1})}{(1 - z_i z_{i+1})(1 - z_i^2)(1 - z_{i+1}^2)} \tag{3.5}$$

can only vanish if

$$b = \frac{a(z_i + z_{i+1}) - (1 + z_i z_{i+1})}{a(1 + z_i z_{i+1}) - (z_i + z_{i+1})}.$$

Notice that b is obtained from a via a bilinear transformation. For $z_{i+1} = z_i^*$ and $|z_i| < 1$ we have $\left|\frac{z_i + z_{i+1}}{1 + z_i z_{i+1}}\right| < 1$. In this case, if $|a| < 1$ then $|b| > 1$ (and also $|a| = 1$ gives $|b| = 1$, and $|a| > 1$ gives $|b| < 1$).

Obviously, a determines $\gamma_{r,i}$ and b determines $\gamma_{r,i+1}$. Setting $b = a$ in (3.5), so that this formula becomes the square of the $\mathcal{H}^2(\mathbb{E})$ norm of $G_{r,i}(z)$, and applying the normality condition yields

$$\gamma_{r,i} = \sqrt{\frac{(1 - z_i z_{i+1})(1 - z_i^2)(1 - z_{i+1}^2)}{(1 + a^2)(1 + z_i z_{i+1}) - 2a(z_i + z_{i+1})}}.$$

Of course, a similar formula exists for $\gamma_{r,i+1}$.

As an alternative to the above construction of z-real orthonormal functions, it is also possible to use a $G_{r,i+1}(z)$ of the form

$$G_{r,i+1}(z) = \gamma_{r,i+1} A_{i-1}(z)\frac{z(1 - bz)}{(z - z_i)(z - z_{i+1})}.$$

In this case we have

$$\langle G_{r,i}, G_{r,i+1}\rangle = \gamma_{r,i}\gamma_{r+1,i}\,\frac{(1 + ab)(z_i + z_{i+1}) - (a + b)(1 + z_i z_{i+1})}{(1 - z_i z_{i+1})(1 - z_i^2)(1 - z_{i+1}^2)},$$

which vanishes if

$$b = \frac{a(1 + z_i z_{i+1}) - (z_i + z_{i+1})}{a(z_i + z_{i+1}) - (1 + z_i z_{i+1})}$$

or if

$$a = \frac{b(1 + z_i z_{i+1}) - (z_i + z_{i+1})}{b(z_i + z_{i+1}) - (1 + z_i z_{i+1})}.$$

Again, b is obtained from a via a bilinear transformation. However, in this case if $|a| < 1$ then $|b| < 1$, when $z_{i+1} = z_i^*$ and $|z_i| < 1$. The formula for $\gamma_{r,i+1}$ is similar to the one for $\gamma_{r,i}$, because $G_{r,i+1}(z)$ can be put in the form of $G_{r,i}(z)$ if it is multiplied by the first order all-pass factor $(z - b)/(1 - bz)$, which does not change its $L^2(\mathbb{T})$ norm.

Without doubt, the simplest possible choice of a and b that forces (3.5) to vanish is $a = \pm 1$ and $b = \mp 1$ [Broome, 1965]. Notice that these two equivalent choices for a and b are independent of z_i and z_{i+1}.[12] The choice $a = 1$ and $b = -1$ gives the z-real orthonormal functions

$$G_{r,i}(z) = \sqrt{(1 + z_i)(1 + z_{i+1})(1 - z_i z_{i+1})/2}\, A_{i-1}(z)\, \frac{z(z - 1)}{(z - z_i)(z - z_{i+1})}$$

and

$$G_{r,i+1}(z) = \sqrt{(1 - z_i)(1 - z_{i+1})(1 - z_i z_{i+1})/2}\, A_{i-1}(z)\, \frac{z(z + 1)}{(z - z_i)(z - z_{i+1})},$$

which can be rewritten in the form

$$G_{r,i}(z) = \sqrt{(1 - k_1)/2}\, \sqrt{1 - k_2^2}\, A_{i-1}(z)\, \frac{z(z - 1)}{z^2 + k_1(1 + k_2)z + k_2} \tag{3.6a}$$

and

$$G_{r,i+1}(z) = \sqrt{(1 + k_1)/2}\, \sqrt{1 - k_2^2}\, A_{i-1}(z)\, \frac{z(z + 1)}{z^2 + k_1(1 + k_2)z + k_2} \tag{3.6b}$$

with $k_1 = -(z_i + z_{i+1})/(1 + z_i z_{i+1})$ and $k_2 = z_i z_{i+1}$. It is possible to prove (Schur-Cohn test) that z_i and z_{i+1} will be in $\mathbb{D}$ if and only if both k_1 and k_2 have modulus smaller than one. Moreover, if k_1 and k_2 are real then z_1 and z_2 are either real or form a complex conjugate pair. Another interesting choice for a and b consists of setting $a = 0$, which makes $b = \frac{1+z_i z_{i+1}}{z_i + z_{i+1}}$. The two z-real orthonormal functions are in this case given by

$$G_{r,i}(z) = \sqrt{1 - k_1^2}\, \sqrt{1 - k_2^2}\, A_{i-1}(z)\, \frac{z^2}{z^2 + k_1(1 + k_2)z + k_2} \tag{3.7a}$$

and

$$G_{r,i+1}(z) = \sqrt{1 - k_2^2}\, A_{i-1}(z)\, \frac{z(k_1 z + 1)}{z^2 + k_1(1 + k_2)z + k_2}. \tag{3.7b}$$

Note that $k_1 = -1/b$. A final interesting case, based on the alternative construction of the z-real orthonormal functions applied to the case $z_i = u + \mathrm{i}v$ and $z_{i+1} = u - \mathrm{i}v$, is

$$G_{r,i}(z) = \sqrt{\frac{(1-u^2-v^2)(1-2u+u^2+v^2)(1+2u+u^2+v^2)}{(1+u^2)(1+u^2+v^2)-4u^2}}\, A_{i-1}(z)\, \frac{z(z - u)}{z^2 - (2u)z + (u^2 + v^2)} \tag{3.8a}$$

[12] In fact this situation is more general. Let $f(k) \in \ell^2(\mathbb{Z})$ be a *real* signal. Then $f(k+1) + f(k)$ is orthogonal to $f(k + 1) - f(k)$. (The same is true for *real* wide-sense stationary signals.) In terms of z transforms this implies that $(z+1)F(z)$ is orthogonal to $(z-1)F(z)$ for all $F(z)$ with finite energy. This fact was pointed out to the author by Dr. Ir. Bert den Brinker and by Harm Belt.

and

$$G_{r,i+1}(z) = (1-u^2+v^2)\sqrt{\tfrac{1-u^2-v^2}{(1+u^2)(1+u^2+v^2)-4u^2}}\, A_{i-1}(z)\, \tfrac{z\left(1-u\frac{1-u^2-v^2}{1-u^2+v^2}z\right)}{z^2-(2u)z+(u^2+v^2)}. \tag{3.8b}$$

The usefulness of these last two formulas lies in the fact that when $v = 0$ we have $G_{r,i} = G_i(z)$ and $G_{r,i+1}(z) = G_{i+1}(z)$.

3.2 Rational orthonormal functions on the imaginary axis

The structure of this section is very similar to the structure of the previous one. This is due to the isomorphism between $\mathcal{H}^2(\mathbb{E})$ and $\mathcal{H}^2(\mathbb{C}^+)$. We will discuss the similarities between these two sections in the next one.

Consider the sequence $(s_i)_{i=0}^{\infty}$ of complex numbers, not necessarily distinct, that belong to $\mathbb{C}^+$. Let ν_i be the number of times that the value of s_i appears *before* in this sequence, i.e., let $\nu_i = \sum_{j=0}^{i-1} \delta_{s_i,s_j}$. For each $i \in \mathbb{N}_0$ let

$$f_i(t) = \frac{t^{\nu_i}}{\nu_i!}\, e^{-s_i t}, \qquad t \in \mathbb{R}^+.$$

Since $\mathrm{Re}[s_i] > 0$ it is obvious that $f_i(t) \in L^2(\mathbb{R}^+)$. For $\nu_i = 0$ the functions $f_i(t)$ are exponentials, and for $\nu_i > 0$ they are generalized exponentials, i.e., exponentials multiplied by a polynomial in t. It is a well know fact that these sequences are linearly independent.[13]

Consider now the (monic) polynomials

$$D_{n+1}(s) = \prod_{i=0}^{n}(s+s_i) = \sum_{i=0}^{n+1} a_{n+1,j}\, s^i, \qquad n \in \mathbb{N}_0,$$

of degree $n+1$. It is a well know fact from the theory of ordinary linear differential equations with constant coefficients that all solutions of the homogeneous differential equation

$$\sum_{i=0}^{n+1} a_{n+1,i} \frac{d^i h(t)}{dt^i} = 0, \qquad t \in \mathbb{R}^+, \tag{3.9}$$

are of the form

$$h_n(t) = \sum_{i=0}^{n} b_i f_i(t) \tag{3.10}$$

where the values of the coefficients b_i can be uniquely determined from the values of $h^{(i)}(0^+)$ for $i = 0, 1, \ldots, n$. Equation (3.9) is the hallmark of a strictly proper time-invariant finite-dimensional linear system with degree $n+1$ and with poles $-s_0, \ldots, -s_n$.[14] In fact, the unit pulse response of all such systems have the form (3.10).

[13] This can be established as follows. Sample the functions at equidistant intervals. The result of that sampling will be discrete exponentials or discrete generalized exponentials. If the sampling period is T then the discrete (generalized) exponentials will have a z_i given by $z_i = e^{-s_i T}$. Note that $|z_i| < 1$ and $z_i \neq 0$. Note also that if $T < \pi/\max_{i=0,\ldots,n} |\mathrm{Im}[s_i]|$, then for $0 \leq i, j \leq n$ we have $z_i \neq z_j$ if $s_i \neq s_j$. Since as we have seen in the last section the discrete (generalized) exponentials are linearly independent, so are the (generalized) exponentials discussed here.

[14] Depending on the initial conditions, some of these poles may be canceled. That's why we have not talked about the McMillan degree of the system.

From the previous paragraph we know that $h_n(t)$, which is a linear combination of the first $n+1$ functions $f_i(t)$ [cf. (3.10)], is the impulse response of a finite-dimensional linear system of degree $n+1$. This makes the approximation of a given function $h(t)$ of $L^2(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$ by $h_n(t)$ very interesting from the view point of system approximation, since the impulse responses with finite energy of stable and causal linear systems belong to this set. Unfortunately, the sequences $f_i(t)$ are not orthonormal, and this makes the approximation problem more difficult that necessary. To simplify this approximation problem, we are interested in the orthonormalization of these sequences using the Gram-Schmidt algorithm with normalization. It turns out that this orthonormalization can be carried out much more easily in the Hardy space $\mathcal{H}^2(\mathbb{C}^+)$, where its solution has a very simple and elegant form. Note that it is possible to translate this problem to $\mathcal{H}^2(\mathbb{C}^+)$ using the isomorphism (the Laplace transform) between $L^2(\mathbb{R}^+)$ and $\mathcal{H}^2(\mathbb{C}^+)$ described in the previous chapter.

The Laplace transforms of the functions $f_i(t)$ defined at the beginning of this section are given by[15]

$$F_i(s) = \frac{1}{(s+s_i)^{1+\nu_i}}.$$

Our task is to construct the functions

$$G_i(s) = \sum_{j=0}^{i} \alpha_{i,j} F_j(s) \tag{3.11}$$

such that $\langle G_i, G_j \rangle = \delta_{ij}$. Note that $G_i(s)$ has the form

$$G_i(s) = \frac{P_i(s)}{\prod_{j=0}^{i}(s+s_j)}$$

for some polynomial $P_i(s)$ of formal degree i. This can be seen easily by reducing (3.11) to a common denominator. We begin with the explicit determination of the first two functions, and then jump to the general case.

Consider first the determination of $G_0(s)$. In particular, consider the determination of $\alpha_{0,0}$ such that $\langle G_0, G_0 \rangle = 1$. An elementary calculation using the residue theorem shows that[16]

$$\langle G_0, G_0 \rangle = \frac{1}{2\pi \mathsf{i}} \int_{\mathsf{i}\mathbb{R}} G_0(s) G_0^*(-s^*)\, ds = \frac{1}{2\pi \mathsf{i}} \int_{\mathsf{i}\mathbb{R}} \frac{|\alpha_{0,0}|^2}{(s+s_0)(-s+s_0^*)}\, ds = \frac{|\alpha_{0,0}|^2}{s_0+s_0^*}.$$

The normalization condition then implies that $\alpha_{0,0} = \kappa_0 \sqrt{2\mathrm{Re}[s_0]}$, where κ_0 is an arbitrary unimodular complex number. $G_0(s)$ is then given by

$$G_0(s) = \kappa_0 \sqrt{2\mathrm{Re}[s_0]}\, \frac{1}{s+s_0}.$$

Consider now the determination of $G_1(s)$. First of all, $G_1(s)$ must be orthogonal of $G_0(s)$. But

$$\langle G_0, G_1 \rangle = \frac{1}{2\pi \mathsf{i}} \int_{\mathsf{i}\mathbb{R}} G_0(s) G_1^*(-s^*)\, ds = 0$$

[15] The reason why we defined the functions $f_i(t)$ to include the term $1/\nu_i!$ was just to make $F_i(s)$ very simple.

[16] Remember that the integration on the imaginary axis can be performed easily using contour integration.

implies that $G_1^*(-s^*)$ must vanish for $s = -s_0$. Therefore

$$P_1(s) = \gamma_1(s - s_0^*)$$

for some γ_1 yet to be determined. The remaining condition, $\langle G_1, G_1\rangle = 1$, determines the value of γ_1. An elementary calculation shows that

$$\begin{aligned}\langle G_1, G_1\rangle &= \frac{1}{2\pi\mathsf{i}}\int_{\mathsf{i}\mathbb{R}} \frac{\gamma_1(s-s_0^*)}{(s+s_0)(s+s_1)}\frac{\gamma_1^*(-s-s_0)}{(-s+s_0^*)(-s+s_1^*)}\,ds \\ &= \frac{1}{2\pi\mathsf{i}}\oint_{\mathbb{T}} \frac{|\gamma_1|^2}{(s+s_1)(-s+s_1^*)}\,ds \\ &= \frac{|\gamma_1|^2}{s_1+s_1^*}.\end{aligned}$$

The normalization condition then implies that $\gamma_1 = \kappa_1\sqrt{2\mathrm{Re}[s_1]}$, where κ_1 is an arbitrary unimodular complex number. $G_1(s)$ is then given by

$$G_1(s) = \kappa_1\sqrt{2\mathrm{Re}[s_1]}\,\frac{s-s_0^*}{(s+s_0)(s+s_1)}.$$

The determination of $G_1(s)$ gives sufficient clues about how to proceed in the general case. Consider then the determination of $G_i(s)$, for $i \in \mathbb{N}_0$. (Although the cases $i = 0$ and $i = 1$ have already been treated, we include them here since the argument we are about to present is also applicable to them.) If $i > 0$ the condition $\langle G_0, G_i\rangle = 0$ forces $P_i(-s^*)$ to vanish for $s = -s_0$. But then, if $i > 1$ the condition $\langle G_1, G_i\rangle = 0$ forces $P_i(-s^*)$ to vanish also for $s = -s_1$.[17] Using this argument for $\langle G_j, G_i\rangle = 0$, $j = 2, \ldots, i-1$, we conclude that $P_i(s)$ has the form

$$P_i(s) = \gamma_i\prod_{j=0}^{i-1}(s - s_j^*)$$

for some γ_i to be determined from the condition $\langle G_i, G_i\rangle = 1$. An elementary calculation shows that

$$\langle G_i, G_i\rangle = \frac{1}{2\pi\mathsf{i}}\int_{\mathsf{i}\mathbb{R}} \frac{|\gamma_i|^2}{(s+s_i)(-s+s_i^*)}\,ds = \frac{|\gamma_i|^2}{s_i+s_i^*}.$$

The normalization condition then implies that $\gamma_i = \kappa_i\sqrt{2\mathrm{Re}[s_i]}$, where κ_i is an arbitrary unimodular complex number. $G_i(s)$ is then given by

$$G_i(s) = \kappa_i\sqrt{2\mathrm{Re}[s_i]}\,\frac{\prod_{j=0}^{i-1}(s-s_j^*)}{\prod_{j=0}^{i}(s+s_j)},$$

which can be rewritten in the form

$$G_i(s) = \kappa_i A_{i-1}(s)L_i(s) \tag{3.12a}$$

with

$$A_{i-1}(s) = \prod_{j=0}^{i-1}\left(\frac{s-s_j^*}{s+s_j}\right) \tag{3.12b}$$

[17] Note that in this case the pole of $G_1(s)$ at $s = -s_0$ is canceled by the zero of $P_i(-s^*)$ also at $s = -s_0$, and so the contour integral will only have one pole ($-s_1$) on the left half plane.

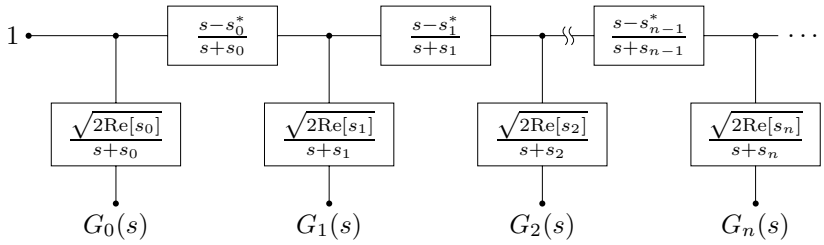

Figure 3.2: Block diagram of the generation of the orthonormal functions $G_i(s)$ given by (3.12).

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and

$$L_i(s) = \sqrt{2\mathrm{Re}[s_i]}\,\frac{1}{s+s_i}. \tag{3.12c}$$

Note that $A_{i-1}(s)$ is an all-pass function since $A_{i-1}(s)A_{i-1}^*(-s^*) = 1$ which implies that $|A_{i-1}(\mathrm{i}\omega)| = 1$ for $\omega \in \mathbb{R}$. Note also that $L_i(s)$ is a first order low-pass function. Therefore, $G_i(s)$ is a cascade of i first order all-pass sections followed by a first order low-pass section. Since $G_i(s)$ and $G_{i-1}(s)$ share the same all-pass section $A_{i-2}(s)$, the whole set of orthonormal functions can be generated by a cascade of first order all-pass sections, tapered by first order low-pass sections, as shown in figure 3.2.

According to Lee [1960], this structure of the functions $G_i(s)$ was discovered by his students, following some work about this subject by Wiener and himself. Apparently, these results were first divulged to a broad audience by Kautz, first in a technical report [Kautz, 1952], and afterwards in a well known paper [Kautz, 1954]. A very good explanation of the construction of the orthonormal functions can be found in chapter 17 of [Lee, 1960], and also in [Mendel, 1966].

The orthonormal functions given by (3.12), although perfectly satisfactory to a mathematician, are not very satisfactory to an engineer. The problem is that some of then do not have real inverse Laplace transforms when their poles are real and/or form complex conjugate pairs. This is particularly annoying if they (the inverse Laplace transforms) are used to approximate a real function, since then to form the approximation one has to use complex numbers.[18] Fortunately, there is a simple way to avoid this problem [Kautz, 1952; Kautz, 1954], and this will be the theme for the rest of this already long section.

The Laplace transform $F(s)$ of a real function $f(t)$ satisfies the condition $F(s) = F^*(s^*)$. We will say that the functions satisfying this condition are s-real functions. In particular, if $F(s)$ is an irreducible s-real rational function then the coefficients of its numerator and denominator polynomials are real.

Consider first the case of a real s_i. In this case $G_i(s)$ will only be s-real if κ_i is real and if the poles of $A_{i-1}(s)$ are either real or form complex conjugate pairs.[19] In terms of the sequence $(s_j)_{j=0}^{\infty}$ this forces s_i to be preceded by either real numbers or by pairs of complex conjugate numbers.

Consider now the case of a complex s_i. In this case $G_i(s)$ will not be s-real no matter how we select κ_i and the poles of $A_{i-1}(s)$. As we will see, one way to obtain a function that

[18] The approximation will be real valued if the poles are real and/or form complex conjugate pairs.

[19] Since the zeros of $A_{i-1}(s)$ are the mirror images with respect to the imaginary axis of its poles, it is enough to specify the poles.

is s-real is to form a linear combination of $G_i(s)$ and $G_{i+1}(s)$. Note that this forces s_{i+1} to be the complex conjugate of s_i, since the denominator of the resulting function must be a polynomial with real coefficients. Note also that $G_{i+1}(s)$ will not be s-real, and so we actually need two different linear combinations of $G_i(s)$ and $G_{i+1}(s)$ to form two new orthonormal s-real functions.

Any linear combination of $G_i(s)$ and $G_{i+1}(s)$ has the form

$$\begin{aligned}\alpha G_i(s) + \beta G_{i+1}(s) &= A_{i-1}(s)\frac{\alpha(s+s_{i+1}) + \beta(s-s_i^*)}{(s+s_i)(s+s_{i+1})} \\ &= A_{i-1}(s)\frac{s(\alpha+\beta) + (\alpha s_{i+1} - \beta s_i^*)}{(s+s_i)(s+s_{i+1})},\end{aligned}$$

which can be made s-real if $s_{i+1} = s_i^*$ or if both s_i and s_{i+1} are real.[20] In fact, it is possible to construct two s-real orthonormal functions of the form

$$G_{r,i}(s) = \gamma_{r,i} A_{i-1}(s)\frac{s-a}{(s+s_i)(s+s_{i+1})}$$

$$G_{r,i+1}(s) = \gamma_{r,i+1} A_{i-1}(s)\frac{s-b}{(s+s_i)(s+s_{i+1})}$$

with (obviously) real $\gamma_{r,i}$, $\gamma_{r,i+1}$, a, and b. Without loss of generality we will make $\gamma_{r,i} > 0$ and $\gamma_{r,i+1} > 0$. Note that a and b are not independent. In fact, one determines the other, since

$$\langle G_{r,i}, G_{r,i+1}\rangle = \gamma_{r,i}\gamma_{r+1,i}\,\frac{ab + s_i s_{i+1}}{2(s_i + s_{i+1})s_i s_{i+1}} \tag{3.13}$$

can only vanish if

$$b = -\frac{s_i s_{i+1}}{a}.$$

Obviously, a determines $\gamma_{r,i}$ and b determines $\gamma_{r,i+1}$. Setting $b = a$ in (3.13), so that this formula becomes the square of the $\mathcal{H}^2(\mathbb{C}^+)$ norm of $G_{r,i}(s)$, and applying the normality condition yields

$$\gamma_{r,i} = \sqrt{\frac{2(s_i + s_{i+1})s_i s_{i+1}}{a^2 + s_i s_{i+1}}}.$$

Of course, a similar formula exists for $\gamma_{r,i+1}$.

As an alternative to the above construction of s-real orthonormal functions, it is also possible to use a $G_{r,i+1}(s)$ of the form

$$G_{r,i+1}(s) = \gamma_{r,i+1} A_{i-1}(s)\frac{1-bs}{(s+s_i)(s+s_{i+1})}.$$

In this case we have

$$\langle G_{r,i}, G_{r,i+1}\rangle = -\gamma_{r,i}\gamma_{r+1,i}\,\frac{a + b\,s_i s_{i+1}}{2(s_i + s_{i+1})s_i s_{i+1}}$$

which vanishes if

$$b = -\frac{a}{s_i s_{i+1}}.$$

[20] This second case is included here just for curiosity, since in that case both $G_i(s)$ and $G_{i+1}(s)$ are already s-real.

The value of $\gamma_{r,i+1}$ is also easy to compute, being given by

$$\gamma_{r,i+1} = \sqrt{\frac{2(s_i + s_{i+1})s_i s_{i+1}}{1 + b^2\, s_i s_{i+1}}}.$$

One of the simple ways of selecting a and b such that (3.13) vanishes is $a = \pm\sqrt{s_i s_{i+1}}$ and $b = \mp\sqrt{s_i s_{i+1}}$ [Kautz, 1954]. The choice $a = \sqrt{s_i s_{i+1}}$ and $b = -a$ gives the s-real orthonormal functions

$$G_{r,i}(s) = \sqrt{u_i}\, A_{i-1}(s)\, \frac{s - \sqrt{v_i}}{s^2 + u_i s + v_i} \tag{3.14a}$$

and

$$G_{r,i+1}(s) = \sqrt{u_i}\, A_{i-1}(s)\, \frac{s + \sqrt{v_i}}{s^2 + u_i s + v_i} \tag{3.14b}$$

with $u_i = s_i + s_{i+1}$ and $v_i = s_i s_{i+1}$. Another interesting choice, which uses the alternative construction of the s-real orthonormal functions, consists of setting $a = 0$ [Ross, 1964], which also makes $b = 0$. The two s-real orthonormal functions are in this case given by

$$G_{r,i}(s) = \sqrt{2u_i}\, A_{i-1}(s)\, \frac{s}{s^2 + u_i s + v_i} \tag{3.15a}$$

and

$$G_{r,i+1}(s) = \sqrt{2u_i v_i}\, A_{i-1}(s)\, \frac{1}{s^2 + u_i s + v_i}. \tag{3.15b}$$

3.3 Links between rational orthonormal functions on the unit circle and those on the imaginary axis

As we stated in the beginning of section 3.2, the results presented in that section are quite similar to the results presented in section 3.1. The culprit of this similarity is the isomorphism between $\mathcal{H}^2(\mathbb{E})$ and $\mathcal{H}^2(\mathbb{C}^+)$ induced by the bilinear transformation (with a normalization term to preserve inner products) described in section 2.4. To avoid confusion, in this section we will represent functions belonging to $\mathcal{H}^2(\mathbb{E})$ by normal capital letters, such as for example G, and functions belonging to $\mathcal{H}^2(\mathbb{C}^+)$ by capital letters in a sans-serif font, such as for example G. This is precisely the notation already used in section 2.4.

Suppose we want to apply the transformation (2.21) to a function of the form $F(z)G(z)$. The result of this transformation is

$$F(z)G(z) \mapsto \frac{\sqrt{2a}}{a+s} F\left(\frac{a+s}{a-s}\right) G\left(\frac{a+s}{a-s}\right).$$

This transformation can also be seen to be the transformation of $F(z)$ by (2.21) multiplied by the transformation of $G(z)$ by (2.18).[21]

Consider the transformation of $(1 - z_j^* z)/(z - z_j)$, i.e., of the j-th term of $A_{i-1}(z)$ as given by (3.4b), by the raw bilinear transformation. Some simple algebra yields

$$\frac{1 - z_j^* z}{z - z_j} \mapsto \left(-\frac{1 + z_j^*}{1 + z_j}\right)\left(\frac{s - s_j^*}{s + s_j}\right)$$

[21] This shows that transfer functions should be transformed using the raw bilinear transformation, while signals should be transformed using the bilinear transformation with the normalization term.

with[22]

$$s_j = -a\frac{z_j - 1}{z_j + 1}. \tag{3.16}$$

This is the j-th term of $\mathsf{A}_{i-1}(s)$ as given by (3.12c) multiplied by $-(1 + z_j^*)/(1 + z_j)$, which is a unimodular complex number.

Consider now the transformation of $L_i(z)$ as given by (3.4c) by the bilinear transformation with normalization. Some simple but tedious algebra yields

$$\sqrt{1 - |z_i|^2}\,\frac{z}{z - z_i} \mapsto \sqrt{\frac{a + s_i}{a + s_i^*}}\,\sqrt{2\mathrm{Re}[s_i]}\,\frac{1}{s + s_i}$$

again with s_i given by (3.16). This result is no more no less that $\mathsf{L}_i(s)$, as given by (3.12b), multiplied by $\sqrt{\frac{a+s_i}{a+s_i^*}}$, which is a unimodular complex number.

The preceding results show that $G_i(z)$, as given by (3.4), is transformed into $\mathsf{G}_i(s)$, as given by (3.4), times a well defined unimodular constant. There is then a one to one mapping between these two sets of rational orthonormal functions. The interested (and patient) reader may also try to relate in a similar way (3.6) with (3.14).

3.4 Completeness considerations

Under what conditions is the set $\{G_i(z)\}_{i=0}^{\infty}$ with $G_i(z)$ given by (3.4) complete? Under what conditions is the set $\{\mathsf{G}_i(s)\}_{i=0}^{\infty}$ with $\mathsf{G}_i(s)$ given by (3.12) complete? These are the questions (for which the answers are well known) that will be answered in this section.

We start with a theorem of Szász [Paley and Wiener, 1934; Clement, 1963]. Let $L^2[0,1]$ denote the Hilbert space of square integrable functions on the interval $[0,1]$ equipped with the usual inner product. Then, the set $\{x^{\lambda_i}\}_{i=0}^{\infty}$, with $\mathrm{Re}[\lambda_i] > -1/2$ and $\lambda_i \neq \lambda_j$ if $i \neq j$, is complete in $L^2[0,1]$ if and only if

$$\sum_{i=0}^{\infty} \frac{\mathrm{Re}[\lambda_i + 1/2]}{1 + |\lambda_i|^2} = \infty.$$

We will not attempt to prove this theorem here. The reader may consult [Paley and Wiener, 1934] for a proof. Note that the functions x^{λ_i} are square integrable on $[0,1]$ if and only if $\mathrm{Re}[\lambda_i] > -1/2$.

Now, let $x = e^{-t}$. Then

$$\int_0^1 f(x)g^*(x)\,dx = \int_0^{\infty} \left[e^{-t/2}f(e^{-t})\right]\left[e^{-t/2}g^*(e^{-t})\right]\,dt.$$

It is then clear that the transformation

$$f(x) \mapsto e^{-t/2}f(e^{-t})$$

defines one isomorphism between $L^2[0,1]$ and $L^2(\mathbb{R}^+)$. This isomorphism maps the functions x^{λ_i} into

$$x^{\lambda_i} \mapsto e^{-(\lambda_i+1/2)t} = e^{-s_i t}$$

[22] Note that z_j is transformed into s_j precisely by (2.17). Remember that the poles of $\mathsf{G}_i(s)$ are $-s_0, \ldots, -s_i$.

with $s_i = \lambda_i + 1/2$. It is then clear that the set $\{e^{-s_i t}\}_{i=0}^{\infty}$, with $\mathrm{Re}[s_i] > 0$ and $s_i \neq s_j$ if $i \neq j$, is complete in $L^2(\mathbb{R}^+)$ if and only if [Huggins, 1956; Clement, 1963; Young, 1968]

$$\sum_{i=0}^{\infty} \frac{\mathrm{Re}[s_i]}{1 + |s_i - 1/2|^2} = \infty. \tag{3.17}$$

This result[23] gives a partial answer to the second question posed at the beginning of this section. As we have seen in section 3.2 the Laplace transforms of the Gram-Schmidt orthonormalization of the functions $e^{-s_i t}$ are precisely the functions $\mathsf{G}_i(s)$ of $\mathcal{H}^2(\mathbb{C}^+)$ given by (3.12). Therefore (3.17) is also a necessary and sufficient condition for the completeness of these functions in $\mathcal{H}^2(\mathbb{C}^+)$, provided $s_i \neq s_j$ if $i \neq j$. For example, the Legendre functions, for which $s_i = (2i+1)p$ with $p > 0$ [Lee, 1960], are complete in $\mathcal{H}^2(\mathbb{C}^+)$.

As described in the previous section the bilinear transformation with the normalization term maps the functions $\mathsf{G}_i(s)$ into the functions $G_i(z)$. These last functions will then form a complete set in $\mathcal{H}^2(\mathbb{E})$ if (3.17) is satisfied with $s_i = -a(z_i - 1)/(z_i + 1)$ [cf. (3.16)] and if $z_i \neq z_j$ when $i \neq j$. To obtain a completeness condition directly in terms of the sequence $(z_i)_{i=0}^{\infty}$ we replace s_i by $a(1 - z_i)/(1 + z_i)$ in (3.17) to obtain

$$\sum_{i=0}^{n} \frac{\mathrm{Re}[s_i]}{1 + |s_i - 1/2|^2} = \sum_{i=0}^{n} \frac{4a(1 + |z_i|)(1 - |z_i|)}{4|1 + z_i|^2 + |(1 + 2a)z_i + (1 - 2a)|^2}.$$

Setting $a = 1/2$, which we are entitled to do since a is arbitrary,[24] it is possible to verify that

$$\frac{1}{5} < \frac{1}{2} \frac{1 + |z_i|}{|1 + z_i|^2 + |z_i|^2} \leq \frac{1}{\sqrt{10} - 3}$$

for any z_i with modulus smaller that one.[25] But then

$$\frac{1}{5} \sum_{i=0}^{n} (1 - |z_i|) < \sum_{i=0}^{n} \frac{\mathrm{Re}[s_i]}{1 + |s_i - 1/2|^2} \leq \frac{1}{\sqrt{10} - 3} \sum_{i=0}^{n} (1 - |z_i|) \tag{3.18}$$

with the consequence that the two series $\sum_{i=0}^{\infty} \frac{\mathrm{Re}[s_i]}{1+|s_i-1/2|^2}$ and $\sum_{i=0}^{\infty}(1 - |z_i|)$ will converge or diverge together.[26] Therefore [Szász, 1953]

$$\sum_{i=0}^{\infty} (1 - |z_i|) = \infty \tag{3.19}$$

[23] Let $s_i = u_i + \mathrm{i}v_i$, with $u_i > 0$. Since

$$\frac{4}{5} \frac{u_i}{1 + u_i^2 + v_i^2} < \frac{u_i}{1 + (u_i - 1/2)^2 + v_i^2} \leq \frac{17 + \sqrt{17}}{17 - \sqrt{17}} \frac{u_i}{1 + u_i^2 + v_i^2},$$

it is clear that condition (3.17) is equivalent to the condition

$$\sum_{i=0}^{\infty} \frac{\mathrm{Re}[s_i]}{1 + |s_i|^2} = \infty.$$

The left hand side of the inequality is approached with $u_i \to 0$, and the equality sign of the right hand side is attained for $u_i = (1 + \sqrt{17})/4$, both with $v_i = 0$.

[24] Any other positive value of a could also have been used, at the expense of somewhat more labor to arrive at the same final result.

[25] The infimum is attained for $z = 1$ and the maximum for $z = 1 - \sqrt{5/2} \approx -0.581$.

[26] Note that both series have positive terms only.

is a necessary and sufficient condition for the completeness in $\mathcal{H}^2(\mathbb{E})$ of the functions $G_i(z)$, given by (3.4), provided $z_i \neq z_j$ for $i \neq j$. This result was already known to Takenaka [1925].

So far we have addressed the completeness problem only in the case where the poles of the functions $G_i(z)$ and $\mathsf{G}_i(s)$ are simple. As we will see shortly the completeness conditions that we have presented for these cases remain valid if we allow poles with arbitrary multiplicities. The method of proof will be to show that if (3.19) holds then it is possible to approximate arbitrarily well the z transform of any discrete exponential by a linear combination of the functions $G_i(z)$, with no restriction on their pole locations except that they must be in $\mathbb{D}$. Since it is very easy to construct a complete set with the z transforms of these discrete exponentials (that was, after all, the subject of the first part of this section), this proves the completeness in the more general case.[27] The equivalent problem for $\mathcal{H}^2(\mathbb{C}^+)$ can then be dealt with with (3.18) and the results of the last section.

Let $F(z) = z/(z-u)$ and let $F_n(z)$ be the best approximation to $F(z)$ by a linear combination of the first $n+1$ functions $G_i(z)$ given by (3.4), i.e.,

$$F_n(z) = \sum_{i=0}^{n} c_i \, G_i(z)$$

with

$$c_i = \langle F, G_i \rangle = \frac{\sqrt{1-|z_i|^2}}{1-z_i^* u} \prod_{j=0}^{i-1} \left(\frac{u-z_j}{1-z_j^* u} \right).$$

Let

$$\xi_n = \langle F - F_n, F - F_n \rangle = \frac{1}{1-|u|^2} - \sum_{i=0}^{n} \left[\frac{1-|z_i|^2}{(1-z_i^* u)(1-z_i u^*)} \prod_{j=0}^{i-1} \left| \frac{u-z_j}{1-z_j^* u} \right|^2 \right].$$

Using the identity

$$\frac{1}{1-|u|^2} - \frac{1-|z_i|^2}{(1-z_i^* u)(1-z_i u^*)} = \frac{1}{1-|u|^2} \left| \frac{u-z_i}{1-z_i^* u} \right|^2$$

it is easy to prove by induction that ξ_n is also given by

$$\xi_n = \frac{1}{1-|u|^2} \prod_{i=0}^{n} \left| \frac{u-z_i}{1-z_i^* u} \right|^2. \tag{3.20}$$

We want to find under which conditions ξ_n goes to zero when n goes to infinity and for all $u \in \mathbb{D}$.[28] The condition $\lim_{n\to\infty} \xi_n = 0$ is equivalent to the condition

$$\lim_{n\to\infty} \sum_{i=0}^{n} \log \left| \frac{u-z_i}{1-z_i^* u} \right|^2 = -\infty.$$

[27] Here we use a result of Lauricella [Davis, 1975]: "Let X be a normed linear space and let $\{x_n\}$ be a closed system (in X). Then a second system $\{y_n\}$ is closed in X if and only if it is closed in $\{x_n\}$. By this we mean that each x_n can be approximated arbitrarily closely by linear combinations of the y_n."

[28] The knowledgeable reader may notice that (3.20) is, if we exclude the $1/(1-|u|^2)$ factor, the square of the modulus of a Blaschke product [Garnett, 1981] evaluated at an arbitrary point of $\mathbb{D}$. In this light the result we are about to obtain is well known. Note, however, that zero or more poles of the Blaschke product may have infinite multiplicities, and this may occur even if the pole is at the origin of the z plane. An informal proof of the completeness of the rational orthonormal functions under consideration here, which is based entirely on some properties of Blaschke products, can be found in [Ninness and Gustafsson, 1994].

After some simple algebraic manipulations it becomes

$$\lim_{n\to\infty}\sum_{i=0}^{n}\log[1-x_i(1-|z_i|)]=-\infty \tag{3.21}$$

with

$$x_i=\frac{(1-|u|^2)(1+|z_i|)}{(1-z_i^*u)(1-z_iu^*)}.$$

Since this condition will be obviously satisfied if $u=z_i$ for some $i\in\mathbb{N}_0$ we will assume henceforth that $u\neq z_i$ for all $i\in\mathbb{N}_0$. In that case it is clear that $0<x_i(1-|z_i|)<1$ because $0<|(u-z_i)/(1-z_i^*u)|<1$ for all u with modulus smaller that one and satisfying the restriction just stated, and for all z_i with modulus smaller that one. After some tedious algebra it is also possible to verify that[29]

$$2\frac{1-|u|}{1+|u|}<x_i<2\frac{1+|u|}{1-|u|}.$$

Consider first the sufficiency of (3.19). Since $-\log(1-x)>x$ for $0<x<1$ it is clear that

$$-\sum_{i=0}^{n}\log[1-x_i(1-|z_i|)]>\sum_{i=0}^{n}x_i(1-|z_i|)>2\frac{1-|u|}{1+|u|}\sum_{i=0}^{n}(1-|z_i|).$$

Therefore the divergence of (3.21) implies the divergence of (3.19).

Consider now the necessity of (3.19). Assume that (3.19) is not satisfied. Then, for every $\epsilon>0$ there exists a $n_\epsilon\in\mathbb{N}_0$ such that $1-|z_i|<\epsilon$ for $i\geq n_\epsilon$. Since $-\log(1-x)<\frac{x}{1-x}$ for $0<x<1$, it is possible to verify that, for $\epsilon<\frac{1}{2}\frac{1-|u|}{1+|u|}$,

$$-\sum_{i=n_\epsilon}^{+\infty}\log[1-x_i(1-|z_i|)]<\sum_{i=n_\epsilon}^{+\infty}\frac{x_i}{1-x_i(1-|z_i|)}(1-|z_i|)<\frac{2\frac{1+|u|}{1-|u|}}{1-2\epsilon\frac{1+|u|}{1-|u|}}\sum_{i=n_\epsilon}^{+\infty}(1-|z_i|).$$

Since by assumption the summation in (3.19) converges, setting ϵ to a value smaller than $\frac{1}{2}\frac{1-|u|}{1+|u|}$ leads to the conclusion that the summation in (3.21) will also converge.

From previous arguments it is clear that (3.17) is both necessary and sufficient for the completeness in $\mathcal{H}^2(\mathbb{C}^+)$ of the functions $\mathsf{G}_i(s)$ given by (3.12) without restrictions on the multiplicity of their poles.[30] For example, the Laguerre functions introduced in section 2.3, for which $s_i=p$ for all i and with $p>0$, are complete in $\mathcal{H}^2(\mathbb{C}^+)$.

3.5 Uniform boundedness considerations

Under what conditions is the set $\{G_i(z)\}_{i=0}^{\infty}$ with $G_i(z)$ given by (3.4) uniformly bounded? Under what conditions is the set $\{\mathsf{G}_i(s)\}_{i=0}^{\infty}$ with $\mathsf{G}_i(s)$ given by (3.12) uniformly bounded? These are the questions that will be answered in this section.

It is clear from the definition of the $\mathcal{H}^\infty$ norm that $\|FG\|_\infty=\|F\|_\infty\|G\|_\infty$. The $\mathcal{H}^\infty$ norm of the functions $G_i(z)$, defined by (3.4), is then given by

$$\|G_i\|_\infty=\sqrt{1-|z_i|^2}\max_{z\in\mathbb{T}}\frac{1}{|z-z_i|}$$

[29] The infimum occurs for $z=-u/|u|$ and the supremum for $z=u/|u|$.

[30] It is worth mentioning that this fact was proven directly by Dzhrbashyan [1979].

because $zA_{i-1}(z)$ is an all-pass function and so $\|zA_{i-1}(z)\|_\infty = 1$. A simple calculation yields[31]

$$\|G_i\|_\infty = \sqrt{\frac{1+|z_i|}{1-|z_i|}}.$$

If the functions $G_i(z)$ are to be uniformly bounded then $\|G_i\|_\infty \leq M$ for all $i \in \mathbb{N}_0$. Clearly, M must be no smaller than 1. Another simple calculation yields

$$|z_i| \leq \frac{M^2-1}{M^2+1}, \qquad i \in \mathbb{N}_0. \tag{3.22}$$

Note that this implies that the poles of the functions $G_i(z)$ cannot be arbitrarily close to the unit circle. Because of this, and for the functions discussed here, uniform boundedness in $\mathcal{H}^\infty(\mathbb{E})$ implies also completeness in $\mathcal{H}^2(\mathbb{E})$.

The $\mathcal{H}^\infty$ norm of the functions $\mathsf{G}_i(z)$, defined by (3.12), is given by

$$\|\mathsf{G}_i\|_\infty = \sqrt{2\mathrm{Re}[s_i]} \max_{s \in \mathrm{i}\mathbb{R}} \frac{1}{|s+s_i|}$$

because $\mathsf{A}_{i-1}(s)$ is an all-pass function and so $\|\mathsf{A}_{i-1}(s)\|_\infty = 1$. A simple calculation yields[32]

$$\|\mathsf{G}_i\|_\infty = \sqrt{\frac{2}{\mathrm{Re}[s_i]}}.$$

If the functions $\mathsf{G}_i(z)$ are to be uniformly bounded then $\|\mathsf{G}_i\|_\infty \leq M$ for all $i \in \mathbb{N}_0$. Another simple calculation yields

$$\mathrm{Re}[s_i] \geq \frac{2}{M^2}, \qquad i \in \mathbb{N}_0. \tag{3.23}$$

Note that this implies that the poles of the functions $\mathsf{G}_i(s)$ cannot be arbitrarily close to the imaginary axis. Note also that in this case, and for the functions discussed here, uniform boundedness in $\mathcal{H}^\infty(\mathbb{C}^+)$ does *not* imply completeness in $\mathcal{H}^2(\mathbb{C}^+)$.

In the next chapter we will construct rational orthonormal functions that can be obtained from the ones described in this chapter via a unitary transformation. The uniform boundedness results described here cannot be applied directly to these other rational orthonormal functions because they are based of the specific form of the functions $G_i(z)$ and $\mathsf{G}_i(s)$. However, they can be adapted easily to these other rational orthonormal functions. The following argument will use the functions $G_i(z)$. It remains valid for the functions $\mathsf{G}_i(s)$.

Let $\mathbf{g} = [\, G_0(z) \quad G_1(z) \quad \cdots \,]^T$ be a vector (with an infinite number of elements) holding the rational orthonormal functions described in section 3.1. Let $\mathbf{h} = [\, H_0(z) \quad H_1(z) \quad \cdots \,]^T$ be another vector (again with an infinite number of elements) holding the rational orthonormal functions obtained by applying a unitary transformation to the functions $G_i(z)$. Mathematically, this unitary transformation is described by a unitary matrix $\mathbf{T}$ (with infinite dimensions) and $\mathbf{g}$ and $\mathbf{h}$ are related by $\mathbf{h} = \mathbf{T}\mathbf{g}$. In particular,

$$H_i(z) = \sum_{j=0}^{\infty} t_{ij} G_j(z), \qquad i \in \mathbb{N}_0.$$

[31] The maximum is attained for $z = z_i/|z_i|$.

[32] The maximum is attained for $s = -\mathrm{i}\,\mathrm{Im}[s_i]$.

The $\mathcal{H}^\infty$ norm of $H_i(z)$ is then bounded by

$$\|H_i\|_\infty \leq \sum_{j=0}^{\infty} |t_{ij}| \, \|G_j\|_\infty .$$

Assuming that $\|G_i\|_\infty \leq M$ for all i produces the bound

$$\|H_i\|_\infty \leq M \sum_{j=0}^{\infty} |t_{ij}|.$$

This shows that the functions $H_i(z)$ are uniformly bounded if both the functions $G_i(z)$ and the ℓ^1 norm of the rows of $\mathbf{T}$ are also uniformly bounded. In particular, if each row of $\mathbf{T}$ has only a bounded number of non-zero elements then the ℓ^1 norm of the rows of $\mathbf{T}$ is uniformly bounded. This is precisely the situation that will be encountered in the next chapter.

3.6 Orthonormal expansions and interpolation

In this section we will describe a very interesting and old connection between $\|\cdot\|_2$ approximations and interpolation [Walsh, 1969]. In what is to follow we will need what is usually called the Cauchy formula for functions of $\mathcal{H}^2(\mathbb{E})$, which is

$$\left\langle \frac{z}{z-u}, F(z) \right\rangle = F^*(1/u^*), \qquad |u| < 1, \quad F(z) \in \mathcal{H}^2(\mathbb{E}).$$

Note that $F(z)$ need not be analytic on $\mathbb{T}$, and that $F(z)$ is evaluated at a point where it is analytic. Note also that this formula is in agreement with the contour integral in (2.8). This formula can be generalized in the obvious way to any rational function with poles in $\mathbb{D}$ appearing in the left hand side of the inner product. We just evaluate the inner product using the residue theorem just as if $H(z)$ were analytic on $\mathbb{T}$ [Riesz and Sz.-Nagy, 1990]. The corresponding formula for $\mathcal{H}^2(\mathbb{C}^+)$ functions is

$$\left\langle \frac{1}{s+u}, F(s) \right\rangle = F^*(u^*), \qquad \mathrm{Re}[u] > 0, \quad H(s) \in \mathcal{H}^2(\mathbb{C}^+).$$

The same kind of remarks apply to this formula as well.

Consider the approximation of the function $F(z)$ by the best linear combination of the functions $G_0(z), \ldots, G_n(z)$, as usual given by (3.4). Denote the approximation by $F_n(z)$ and its error by $E_n(z)$. Note that $E_n(z)$ is itself a function of $\mathcal{H}^2(\mathbb{E})$ and so it is analytic on $\mathbb{E}$. The principle of orthogonality of best approximations in Hilbert spaces gives the equations

$$\langle G_i, E_n \rangle = 0, \qquad i = 0, \ldots, n.$$

But applying Cauchy's formula to $\langle G_0, E_n \rangle = 0$ implies that $E_n(1/z_0^*) = 0$, and so $E_n(z)$ is of the form $E_n(z) = (1 - z_0^* z) E_{n,1}(z)$ with $E_{n,1}(z)$ analytic on $\mathbb{E}$. A second application of Cauchy's (generalized) formula this time to $\langle G_1, E_n \rangle = 0$ implies that $E_{n,1}(1/z_1^*) = 0$, and so $E_{n,1}(z)$ is of the form $E_{n,1}(z) = (1 - z_1^* z) E_{n,2}(z)$ with $E_{n,2}(z)$ analytic on $\mathbb{E}$.[33] Successive

[33] As the reader may have noticed, this argument is very similar to the one used in section 3.1 to determine, up to a constant, the numerator of the functions $G_i(z)$.

application of this argument leads to the conclusion that $E_n(z)$ is of the form[34]

$$E_n(z) = E_{n,n}(z) \prod_{i=0}^{n} (1 - z_i^* z) \tag{3.24}$$

where $E_{n,n}(z)$ is an analytic function on $\mathbb{E}$. The most important consequence of (3.24) is that

$$E_n^{(\nu_i)}(1/z_i^*) = 0, \qquad i = 0, \ldots, n,$$

i.e., the error of the approximation vanishes at the points $z = 1/z_i^*$. If two or more points coincide ($\nu_i > 0$ for some i) then its derivatives up to the number of times the point is repeated minus one also vanish. This is precisely the characteristics of a Hermite interpolation [Davis, 1975]. In other words, the truncated orthonormal expansion of $F(z)$ satisfies the conditions (Hermite interpolation)

$$F^{(\nu_i)}(1/z_i^*) = F_n^{(\nu_i)}(1/z_i^*), \qquad i = 0, \ldots, n. \tag{3.25}$$

Note that $F(z)$ is analytic at $z = 1/z_i^*$, even if $z_i = 0$.

Consider now the approximation of the function $\mathsf{F}(s)$ by the best linear combination of the functions $\mathsf{G}_0(s), \ldots, \mathsf{G}_n(s)$, as usual given by (3.12). Denote the approximation by $\mathsf{F}_n(s)$ and its error by $\mathsf{E}_n(s)$. Using an argument in everything similar to the one used for $\mathcal{H}^2(\mathbb{E})$ functions yields

$$\mathsf{E}_n(s) = E_{n,n}(s) \prod_{i=0}^{n} (s - s_i^*),$$

and so the error of the approximation vanishes at the points $s = s_i^*$. If two or more points coincide ($\nu_i > 0$ for some i) then its derivatives up to the number of times the point is repeated minus one also vanish. Again, this are the characteristics of a Hermite interpolation, and we have

$$\mathsf{F}^{(\nu_i)}(s_i^*) = \mathsf{F}_n^{(\nu_i)}(s_i^*), \qquad i = 0, \ldots, n. \tag{3.26}$$

Note that $\mathsf{F}(s)$ is analytic at $s = s_i^*$.

3.7 Optimality conditions for the poles of the rational orthonormal functions

Consider the problem of the approximation of a function $\mathsf{f}(t) \in L^2(\mathbb{R}^+)$ by a linear combination of the functions $e^{-s_0 t}, \ldots, e^{-s_n t}$, with $s_i \neq s_j$ for $i \neq j$. Let $\mathsf{f}_n(t)$ denote the best approximation in the $L^2(\mathbb{R}^+)$ norm, given by

$$\mathsf{f}_n(t) = \sum_{i=0}^{n} w_{n,i}\, e^{-s_i t},$$

[34] An alternative way to obtain this result is to use a complete set to perform the orthonormal expansion of $F(z)$. In this case it is clear that the error term must have a factor of the form $A_n(z)$, which is in accord with the result presented here.

and let $\mathsf{e}_n(t) = \mathsf{f}(t) - \mathsf{f}_n(t)$ denote the error of this best approximation.[35] Then, from the orthogonality principle we have

$$\int_0^\infty \mathsf{e}_n(t) e^{-s_i^* t}\, dt = 0, \qquad i = 0, \ldots, n.$$

This integral is no more no less than the Laplace transform of $\mathsf{e}_n(t)$ evaluated at $s = s_i^*$, $i = 0, \ldots, n$, and so we have[36]

$$\mathsf{F}(s_i^*) = \mathsf{F}_n(s_i^*), \qquad i = 0, \ldots, n. \tag{3.27}$$

This system of linear equations specify the parameters $w_{n,0}, \ldots, w_{n,n}$, given the parameters $s_0, \ldots, s_n$.

Let ξ_n denote the squared error of the approximation to $\mathsf{f}(t)$ by $\mathsf{f}_n(t)$, i.e., $\xi_n = \|\mathsf{e}_n(t)\|_2^2$. This squared error can be optimized with respect to the exponents $s_i = u_i + \mathsf{i} v_i$. A necessary condition for the existence of a stationary point of ξ_n is[37]

$$\frac{\partial \xi_n}{\partial u_i} = \frac{\partial \xi_n}{\partial v_i} = 0, \qquad i = 0, \ldots, n.$$

Clearly, the global minimum of ξ_n will be among the solutions of this nonlinear system of equations. These so-called stationarity equations can be rewritten in the form

$$\frac{\partial \xi_n}{\partial u_i} = \left\langle \mathsf{e}_n(t), -t\, w_{n,i}\, e^{-s_i t} \right\rangle + \left\langle -t\, w_{n,i}\, e^{-s_i t}, \mathsf{e}_n(t) \right\rangle = 0$$

$$\frac{\partial \xi_n}{\partial v_i} = \left\langle \mathsf{e}_n(t), -\mathsf{i} t\, w_{n,i}\, e^{-s_i t} \right\rangle + \left\langle -\mathsf{i} t\, w_{n,i}\, e^{-s_i t}, \mathsf{e}_n(t) \right\rangle = 0$$

for $i = 0, \ldots, n$. After some simple algebra it is possible to verify that the only solutions of these equations satisfy[38]

$$\left\langle \mathsf{e}_n(t), t\, w_{n,i}\, e^{-s_i t} \right\rangle = 0, \qquad i = 0, \ldots, n.$$

In terms of Laplace transforms this is equivalent to[39]

$$w_{n,i}\, \mathsf{E}_n'(s_i^*) = 0, \qquad i = 0, \ldots, n.$$

In two papers about this subject, [Aigrain and Williams, 1949a; Kammler, 1976], the authors implicitly assumed that the $w_{n,i}$'s were different from zero, with the result that these parameters were omitted from this second set of stationarity conditions. It is clear that if one or

[35] Clearly, this approximation problem is equivalent to the one studied in the last section, except that we impose here (for simplicity of presentation) the restriction that $\nu_i = 0$ for all i.

[36] Note that this result was also obtained in the last section, but in a more general way.

[37] Since this turns out to be a nonlinear problem we have to accept the fact that a first order local analysis of the function ξ_n produces a set of points that correspond to local minima, saddle points, and local maxima (local maxima do not exist if the weights are optimal) of that function (the stationary points). From this (first order) local analysis it is virtually impossible to obtain conditions that are satisfied *only* by the global minimum of ξ_n. The term "optimality conditions" that we will use later on, will therefore refer to the necessary (but not sufficient) conditions that the global minimum will have to satisfy.

[38] Another way to obtain this result uses the interesting trick presented in [Brandwood, 1983; Calvez, Vilbe and Glouannec, 1985; van den Bos, 1994] to avoid differentiating $\langle \mathsf{e}_n(t), \mathsf{e}_n(t) \rangle$ with respect to the real and imaginary parts of s_i.

[39] A prime denotes here differentiation with respect to s.

more of the $w_{n,i}$'s are equal to zero then the approximation is composed of a smaller number of exponentials. It stands to reason that these special cases cannot correspond to the global minimum of ξ_n. Excluding the uninteresting cases in which one of more $w_{n,i}$ vanish, we obtain the equations

$$\mathsf{F}'(s_i^*) = \mathsf{F}_n'(s_i^*), \qquad i = 0, \ldots, n. \tag{3.28}$$

Together with (3.27) these nonlinear equations are usually known by the name of Aigrain-Williams equations [Aigrain and Williams, 1949a]. Several solutions may exist for the parameters $w_{n,i}$ and s_i.

These equations were generalized by Kammler [1976] to the case of repeated s_i's, in which case some exponentials are replaced by exponentials multiplied by powers of t. The result of this generalization is

$$\begin{array}{l} \mathsf{F}^{(2\nu_i)}(s_i^*) = \mathsf{F}_n^{(2\nu_i)}(s_i^*) \\ \mathsf{F}^{(2\nu_i+1)}(s_i^*) = \mathsf{F}_n^{(2\nu_i+1)}(s_i^*) \end{array}, \qquad i = 0, \ldots, n. \tag{3.29}$$

If $\nu_i = 0$ for $i = 0, \ldots, n$ these equations become exactly (3.27) and (3.28). On the light of the results of the last section, we observe that for the optimal values of the exponents $-s_i$ (poles of $\mathsf{G}_i(s)$) the Hermite interpolation property of the approximation is much stronger than the one that exists for non-optimal exponents.

About ways of solving the Aigrain-Williams equations the reader may consult one or more of the following papers: [McBride, Schaefgen and Steiglitz, 1966], [McDonough and Huggins, 1968], [Marzollo, 1969], [Miller, 1970], [Miller, 1973], and [Connell, 1974]. Some useful and interesting remarks concerning the inversion of the so-called Hilbert matrix (which is notoriously ill conditioned) that appears in some of these methods can be found in [Miller, 1969].

Consider now the problem of the approximation of a sequence $f(k) \in \ell^2(\mathbb{N}_0)$ by a linear combination of the sequences $z_0^k, \ldots, z_n^k$, with $z_i \neq z_j$ for $i \neq j$. Let $f_n(k)$ denote the best approximation on the ℓ^2 norm, given by

$$f_n(k) = \sum_{i=0}^{n} w_{n,i}\, z_i^k,$$

and let $e_n(k) = f(k) - f_n(k)$ denote the error of this best approximation.[40] Then, from the orthogonality principle we have

$$\sum_{k=0}^{\infty} e_n(k)\, (z_i^*)^k = 0, \qquad i = 0, \ldots, n.$$

This summation is no more no less than the z transform of $e_n(k)$ evaluated at $z = 1/z_i^*$, $i = 0, \ldots, n$, and so we have[41]

$$F(1/z_i^*) = F_n(1/z_i^*), \qquad i = 0, \ldots, n. \tag{3.30}$$

This system of linear equations specify the parameters $w_{n,0}, \ldots, w_{n,n}$, given the parameters $z_0, \ldots, z_n$.

[40] Clearly, this approximation problem is equivalent to the one studied in the last section, except that we impose here (for simplicity of presentation) the restriction that $\nu_i = 0$ for all i.

[41] Note that this result was also obtained in the last section, but in a more general way.

Let ξ_n denote the squared error of the approximation to $f(k)$ by $f_n(k)$, i.e., $\xi_n = \|e_n(k)\|_2^2$. This squared error can be optimized with respect to the parameters $z_i = u_i + \mathrm{i}\, v_i$. A necessary condition for the existence of a stationary point of ξ_n is

$$\frac{\partial \xi_n}{\partial u_i} = \frac{\partial \xi_n}{\partial v_i} = 0, \qquad i = 0, \ldots, n.$$

These so-called stationarity equations can be rewritten in the form

$$\frac{\partial \xi_n}{\partial u_i} = \left\langle e_n(k), k\, w_{n,i}\, z_i^{k-1} \right\rangle + \left\langle k\, w_{n,i}\, z_i^{k-1}, e_n(k) \right\rangle = 0$$

$$\frac{\partial \xi_n}{\partial v_i} = \left\langle e_n(k), \mathrm{i} k\, w_{n,i}\, z_i^{k-1} \right\rangle + \left\langle \mathrm{i} k\, w_{n,i}\, z_i^{k-1}, e_n(k) \right\rangle = 0$$

for $i = 0, \ldots, n$. After some simple algebra it is possible to verify that the only solutions of these equations satisfy

$$\left\langle e_n(k), k\, w_{n,i}\, z_i^{k-1} \right\rangle = 0, \qquad i = 0, \ldots, n.$$

In terms of z transforms this is equivalent to[42]

$$w_{n,i}\, E_n'(1/z_i^*) = 0, \qquad i = 0, \ldots, n.$$

In [Miller, 1973], the author implicitly assumed that the $w_{n,i}$'s were different from zero, with the result that these parameters were omitted from this second set of stationarity conditions. It is clear that if one or more of the $w_{n,i}$'s are equal to zero then the approximation is composed of a smaller number of discrete exponentials. It stands to reason that these special cases cannot correspond to the global minimum of ξ_n. Excluding the uninteresting cases in which one of more $w_{n,i}$ vanish, we obtain the equations

$$F'(1/z_i^*) = F_n'(1/z_i^*), \qquad i = 0, \ldots, n. \tag{3.31}$$

Together with (3.30) these equations define the parameters $w_{n,i}$ and z_i. Several solutions are possible.

Using the results of section 2.4, the connection between (3.27) and (3.30), and between (3.28) and (3.31), is clear. In a similar manner we can connect (3.29) with

$$\begin{array}{l} F^{(2\nu_i)}(1/z_i^*) = F_n^{(2\nu_i)}(1/z_i^*) \\ F^{(2\nu_i+1)}(1/z_i^*) = F_n^{(2\nu_i+1)}(1/z_i^*) \end{array}, \qquad i = 0, \ldots, n, \tag{3.32}$$

which is more general than (3.30) and (3.31). Again, these formulas can be interpreted as a Hermite interpolation.

[42] A prime denotes here differentiation with respect to z.

Chapter 4

A state-space approach to the realization of the rational orthonormal functions on the unit circle and on the imaginary axis

In this chapter we describe state-space realizations of the rational orthonormal functions described in the previous chapter. We will see that a balanced realization of a all-pass transfer function gives rise naturally to a finite-dimensional set of rational orthonormal functions. Even more interesting is the fact that a cascade of balanced realizations of all-pass transfer functions is also a balanced realization of a all-pass transfer function. This observation leads in a natural way to the rational orthonormal functions described in sections 3.1 and 3.2.

4.1 State-space descriptions of finite-dimensional discrete- and continuous-time systems

The state-space description of a multi-input multi-output discrete-time system of order n is

$$\left[\begin{array}{c} \mathbf{x}(k+1) \\ \hdashline \mathbf{y}(k) \end{array}\right] = \left[\begin{array}{c:c} \mathbf{A} & \mathbf{B} \\ \hdashline \mathbf{C} & \mathbf{D} \end{array}\right] \left[\begin{array}{c} \mathbf{x}(k) \\ \hdashline \mathbf{u}(k) \end{array}\right]. \tag{4.1}$$

If this system has m inputs and l outputs then the dimensions of the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are respectively $n \times n$, $n \times m$, $l \times n$, and $l \times m$. Using z transforms and assuming zero initial conditions it is easy to verify that

$$\mathbf{X}(z) = (z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(z)$$

and that

$$\mathbf{Y}(z) = [\mathbf{D} + \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}]\mathbf{U}(z).$$

The transfer function of this system is then given by

$$\mathbf{H}(z) = \mathbf{D} + \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

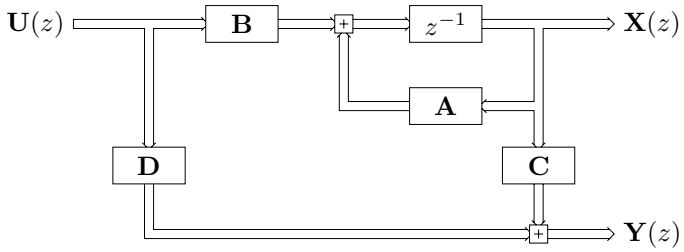

Figure 4.1: Standard state-space block diagram (z transform version).

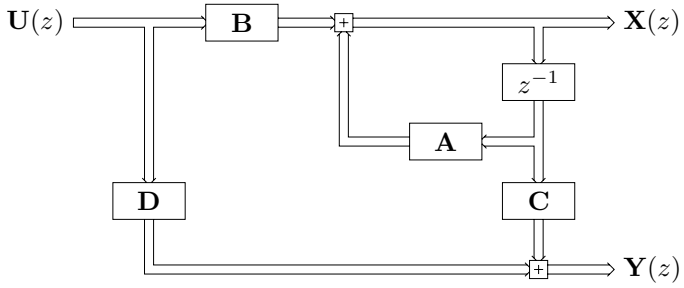

Figure 4.2: Modified state-space block diagram (z transform version).

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and its inverse z transform (the unit pulse response) can be seen to be given by

$$\mathbf{h}(k) = \begin{cases} \mathbf{D} & \text{if } k = 0, \\ \mathbf{C}\mathbf{A}^{k-1}\mathbf{B} & \text{if } k > 0. \end{cases} \tag{4.2}$$

From (4.1) it is clear that there is a delay between the inputs and the states. For reasons that will become apparent later on, we do not want this to happen. A simple solution to this "problem" is to change the definition of the states in such a way that the new definition corresponds to the old one advanced by one sample. The resulting equations are

$$\left[\begin{array}{c} \mathbf{x}(k) \\ \hdashline \mathbf{y}(k) \end{array}\right] = \left[\begin{array}{c:c} \mathbf{A} & \mathbf{B} \\ \hdashline \mathbf{C} & \mathbf{D} \end{array}\right] \left[\begin{array}{c} \mathbf{x}(k-1) \\ \hdashline \mathbf{u}(k) \end{array}\right], \tag{4.3}$$

$$\mathbf{X}(z) = z(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(z),$$

and

$$\mathbf{Y}(z) = [\mathbf{D} + \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}]\mathbf{U}(z).$$

Obviously, the external properties of these two state-space descriptions are equal. The only difference between them lies in their definition of the states. This can be seen easily in figures 4.1 and 4.2 (note the placement of the delay z^{-1}). Most of the results known for the standard state-space model are also valid for the modified one with little or no alterations.

Two important matrices related to state-space descriptions are the so-called controllability matrix[1] $\mathbf{M}_c$, given by

$$\mathbf{M}_c = \left[\begin{array}{cccc} \mathbf{B} & \mathbf{A}\mathbf{B} & \mathbf{A}^2\mathbf{B} & \cdots \end{array}\right],$$

[1]For discrete-time systems this matrix is better known as the reachability matrix. We will use the name controllability instead of reachability to treat simultaneously, in some parts of the text, the continuous- and the discrete-time cases.

and the so-called observability matrix $\mathbf{M}_o$, given by

$$\mathbf{M}_o = \left[\begin{array}{cccc} \mathbf{C}^H & (\mathbf{CA})^H & (\mathbf{CA}^2)^H & \cdots \end{array} \right]^H.$$

Another two important matrices related to the two just introduced are the so-called controllability Gramian $\mathbf{P}$, given by

$$\mathbf{P} = \mathbf{M}_c\mathbf{M}_c^H = \sum_{i=0}^{\infty} \mathbf{A}^i\mathbf{B}\mathbf{B}^H(\mathbf{A}^H)^i = \sum_{i=0}^{\infty} (\mathbf{A}^i\mathbf{B})(\mathbf{A}^i\mathbf{B})^H,$$

and the so-called observability Gramian $\mathbf{Q}$, given by

$$\mathbf{Q} = \mathbf{M}_o^H\mathbf{M}_o = \sum_{i=0}^{\infty} (\mathbf{A}^H)^i\mathbf{C}^H\mathbf{C}\mathbf{A}^i = \sum_{i=0}^{\infty} (\mathbf{C}\mathbf{A}^i)^H(\mathbf{C}\mathbf{A}^i).$$

It is not difficult to verify that if the system is stable these two matrices satisfy the Lyapunov equations

$$\mathbf{A}\mathbf{P}\mathbf{A}^H + \mathbf{B}\mathbf{B}^H = \mathbf{P} \tag{4.4}$$

and

$$\mathbf{A}^H\mathbf{Q}\mathbf{A} + \mathbf{C}^H\mathbf{C} = \mathbf{Q}. \tag{4.5}$$

A very important observation that will be heavily used later on is that for $m = 1$ (one input signal) and $U(z) = 1$ (the input signal is a unit pulse at $k = 0$) we have

$$\mathbf{P} = \sum_{i=0}^{\infty} \mathbf{x}(k)\mathbf{x}^H(k),$$

i.e., the element of the i-th line and j-th column of $\mathbf{P}$ is the $\ell^2(\mathbb{N}_0)$ inner product between $x_i(k)$ and $x_j(k)$.

The state-space description of a multi-input multi-output continuous-time system of order n is

$$\left[\begin{array}{c} \dot{\mathbf{x}}(t) \\ \hdashline \mathbf{y}(t) \end{array} \right] = \left[\begin{array}{c:c} \mathbf{A} & \mathbf{B} \\ \hdashline \mathbf{C} & \mathbf{D} \end{array} \right] \left[\begin{array}{c} \mathbf{x}(t) \\ \hdashline \mathbf{u}(t) \end{array} \right].$$

If this system has m inputs and l outputs then the dimensions of the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are respectively $n\times n$, $n\times m$, $l\times n$, and $l\times m$. Using Laplace transforms and assuming zero initial conditions it is easy to verify that

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)$$

and that

$$\mathbf{Y}(s) = [\mathbf{D} + \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}]\mathbf{U}(s).$$

The transfer function of this system is then given by

$$\mathbf{H}(s) = \mathbf{D} + \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

and its inverse Laplace transform (the impulse response) can be seen to be given by

$$\mathbf{h}(t) = \mathbf{D}\delta(t) + \mathbf{C}e^{\mathbf{A}t}\mathbf{B}.$$

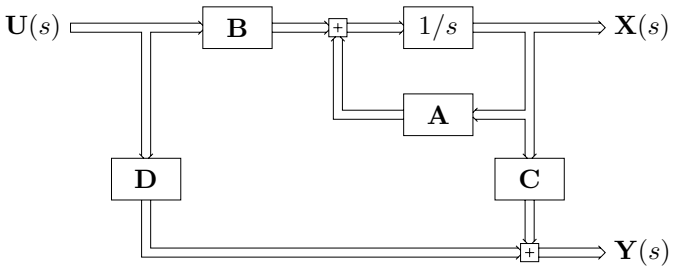

Figure 4.3: State space block diagram (Laplace transform version).

The block diagram of a finite-dimensional continuous-time system is presented in figure 4.3.

The controllability and observability matrices $\mathbf{M}_c$ and $\mathbf{M}_o$ are defined in exactly the same way as in discrete-time state space models. Two important matrices related to continuous-time state-space models are the so-called controllability Gramian $\mathbf{P}$, given by

$$\mathbf{P} = \int_0^\infty e^{\mathbf{A}t}\mathbf{B}\mathbf{B}^H e^{\mathbf{A}^H t}\,dt = \int_0^\infty (e^{\mathbf{A}t}\mathbf{B})(e^{\mathbf{A}t}\mathbf{B})^H\,dt,$$

and the so-called observability Gramian $\mathbf{Q}$, given by

$$\mathbf{Q} = \int_0^\infty e^{\mathbf{A}^H t}\mathbf{C}^H\mathbf{C}e^{\mathbf{A}t}\,dt = \int_0^\infty (\mathbf{C}e^{\mathbf{A}t})^H(\mathbf{C}e^{\mathbf{A}t})\,dt.$$

It is not difficult to verify that if the system is stable these two matrices satisfy the Lyapunov equations

$$\mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^H + \mathbf{B}\mathbf{B}^H = \mathbf{0} \tag{4.6}$$

and

$$\mathbf{A}^H\mathbf{Q} + \mathbf{Q}\mathbf{A} + \mathbf{C}^H\mathbf{C} = \mathbf{0}. \tag{4.7}$$

A very important observation that will be heavily used later on is that for $m = 1$ (one input signal) and $U(s) = 1$ (the input signal is a unit impulse at $t = 0$) we have

$$\mathbf{P} = \int_0^\infty \mathbf{x}(t)\mathbf{x}^H(t)\,dt,$$

i.e., the element of the i-th line and j-th column of $\mathbf{P}$ is the $L^2(\mathbb{R}^+)$ inner product between $x_i(t)$ and $x_j(t)$.

A comment is in order here. A matrix $\mathbf{X}$ whose elements are given by $\langle x_i, x_j\rangle$, $1 \leq i, j \leq n$, is called a Gramian matrix [Davis, 1975]. This matrix is clearly Hermitian and semi-positive definite. If the $x_1, \ldots, x_n$ are linearly independent then this matrix is positive definite. In this light the controllability Gramian of a realization of a single-input system collects the inner products of the impulse responses from the input to the states of that realization. The interpretation of the observability Gramian is somewhat more complicated and requires the introduction of a new set of input signals, which are directly added to the states (one new input signal for each state). These extra input signals can represent, e.g., quantization noise introduced by working with finite-precision arithmetic in discrete-time systems or noise present in an analog implementation of a continuous-time system. For single-output systems the observability Gramian collects the inner products of the impulse responses from these additional input signals to the output [Mullis and Roberts, 1976b; Mullis and Roberts, 1976c].

4.2 Balanced state-space realizations of finite-dimensional systems

Let[2] $\mathbf{x} = \mathbf{T}\mathbf{x}'$ where $\mathbf{T}$ is a $n \times n$ non-singular matrix. This linear transformation of the state variables gives rise to an equivalent description of the same finite-dimensional linear system. The new description can be obtained from the old via the following relations, which are valid for the three state-space descriptions introduced in section 4.1,

$$\mathbf{A}' = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}, \qquad \mathbf{B}' = \mathbf{T}^{-1}\mathbf{B}, \qquad \mathbf{C}' = \mathbf{C}\mathbf{T}, \qquad \mathbf{D}' = \mathbf{D}.$$

Note that

$$\left[\begin{array}{c|c} \mathbf{A}' & \mathbf{B}' \\ \hline \mathbf{C}' & \mathbf{D}' \end{array}\right] = \left[\begin{array}{c|c} \mathbf{T}^{-1} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{I} \end{array}\right] \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right] \left[\begin{array}{c|c} \mathbf{T} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{I} \end{array}\right].$$

The controllability and observability Gramians are transformed according to

$$\mathbf{P}' = \mathbf{T}^{-1}\mathbf{P}\mathbf{T}^{-H}, \qquad \mathbf{Q}' = \mathbf{T}^{H}\mathbf{Q}\mathbf{T}.$$

Note that $\mathbf{P}'\mathbf{Q}' = \mathbf{T}^{-1}\mathbf{P}\mathbf{Q}\mathbf{T}$, i.e., $\mathbf{P}'\mathbf{Q}'$ is similar to $\mathbf{P}\mathbf{Q}$.

From their definitions the controllability and observability Gramians are at least semi-positive definite Hermitian matrices. If $\mathbf{P}$ is positive definite the system is said to be controllable, otherwise it is said to be uncontrollable. If $\mathbf{Q}$ is positive definite the system is said to be observable, otherwise it is said to be unobservable. If the following we will deal exclusively with controllable and observable systems, i.e., with minimal realizations of systems.

Since by assumption $\mathbf{P}$ and $\mathbf{Q}$ are positive definite Hermitian matrices they can be factored in the forms

$$\mathbf{P} = \mathbf{R}_p\mathbf{R}_p^H, \qquad \text{and} \qquad \mathbf{Q} = \mathbf{R}_q\mathbf{R}_q^H.$$

There are many possible ways to accomplish these factorizations. One that is very convenient for computational purposes is the Cholesky factorization [Golub and van Loan, 1989]. In that specific case $\mathbf{R}_p$ and $\mathbf{R}_q$ are lower-triangular matrices. Another interesting possibility is to compute $\mathbf{R}_p$ and $\mathbf{R}_q$ from the singular value decompositions[3] of $\mathbf{P}$ and $\mathbf{Q}$.

The matrix

$$\mathbf{R}_q^H\mathbf{R}_p\mathbf{R}_p^H\mathbf{R}_q = (\mathbf{R}_q^H\mathbf{R}_p)(\mathbf{R}_q^H\mathbf{R}_p)^H$$

is also a positive definite Hermitian matrix, and so it has the singular value decomposition

$$\mathbf{R}_q^H\mathbf{R}_p\mathbf{R}_p^H\mathbf{R}_q = \mathbf{U}\mathbf{\Sigma}^2\mathbf{U}^H$$

where $\mathbf{U}$ is a unitary matrix, i.e., $\mathbf{U}\mathbf{U}^H = \mathbf{U}^H\mathbf{U} = \mathbf{I}$, and where $\mathbf{\Sigma}$ is a diagonal matrix with positive diagonal elements.[4]

Consider the linear transformation of the state vector defined by

$$\mathbf{T} = \mathbf{R}_q^{-H}\mathbf{U}\mathbf{\Sigma}^{\alpha}.$$

[2] Here a prime does not denote a derivative. It denotes another set of variables.

[3] Or eigenvalue decompositions...

[4] The diagonal elements of $\mathbf{\Sigma}$ are called the Hankel singular values of the system. In the early signal processing literature they where called the second order modes of the system [Mullis and Roberts, 1976b; Mullis and Roberts, 1976c].

Simple computations give

$$\mathbf{P}' = \mathbf{\Sigma}^{2(1-\alpha)} \qquad \text{and} \qquad \mathbf{Q}' = \mathbf{\Sigma}^{2\alpha}.$$

Note that both Gramians of the new state space realization are diagonal. Three cases of this transformation are worth of mention [Moore, 1981]. With $\alpha = 1$, we get

$$\mathbf{T}_{\text{input}} = \mathbf{R}_q^{-H}\mathbf{U}\mathbf{\Sigma}, \qquad \mathbf{P}_{\text{input}} = \mathbf{I}, \qquad \text{and} \qquad \mathbf{Q}_{\text{input}} = \mathbf{\Sigma}^2.$$

In this case the controllability Gramian is the identity matrix. The corresponding realization is said to be input balanced. With $\alpha = 0$ we get

$$\mathbf{T}_{\text{output}} = \mathbf{R}_q^{-H}\mathbf{U}, \qquad \mathbf{P}_{\text{output}} = \mathbf{\Sigma}^2, \qquad \text{and} \qquad \mathbf{Q}_{\text{output}} = \mathbf{I}.$$

In this case the observability Gramian is the identity matrix. The corresponding realization is said to be output balanced. Finally, with $\alpha = 1/2$ we get

$$\mathbf{T}_{\text{internal}} = \mathbf{R}_q^{-H}\mathbf{U}\mathbf{\Sigma}^{1/2}, \qquad \mathbf{P}_{\text{internal}} = \mathbf{\Sigma}, \qquad \text{and} \qquad \mathbf{Q}_{\text{internal}} = \mathbf{\Sigma}.$$

In this case the two Gramians are equal (and diagonal). The corresponding realization is said to be internally balanced.

Balanced realizations of finite dimensional systems have many applications and play an important role in modern linear systems theory, automatic control theory, and digital signal processing theory. Among its many applications are balanced model order reduction [Moore, 1981; Pernebo and Silverman, 1982; Fernando and Nicholson, 1983; Al-Saggaf and Franklin, 1988], and the reduction of roundoff-noise in digital filters [Markel and Gray, 1975; Mullis and Roberts, 1976b; Mullis and Roberts, 1976c; Roberts and Mullis, 1987]. They also play an important part in optimal Hankel norm approximations [Glover, 1984].[5] In our context (rational orthonormal functions) input balanced realizations play an important part. This is so because since the controllability Gramian of these realizations is the identity matrix, the (rational) transfer functions from the input to the states are orthonormal (see, e.g., [Bodin and Wahlberg, 1994]). It turns out that all-pass transfer functions appear quite naturally in this context.

4.3 Balanced realizations of all-pass transfer functions

All-pass transfer functions have many interesting properties [Roberts and Mullis, 1987; Regalia, Mitra and Vaidyanathan, 1988]. Among then is the fact that the singular values of rational all-pass transfer functions are all equal to one. This in turn implies that each rational stable all-pass transfer function has balanced minimal realizations for which both the controllability and observability Gramians are equal to the identity matrix (of appropriate dimensions).

To prove that all singular values of a discrete-time all-pass transfer function are equal to one we will use a method similar to one described in [Roberts and Mullis, 1987] (see

[5] The remarkable solution to the problem of finding the optimal approximation in the so-called Hankel norm was discovered by three famous Russian mathematicians: Adamjan, Arov and Kreĭn [1971] (see also [Adamjan, Arov and Kreĭn, 1978]). It is not very often that a (then) recently discovered mathematical result had such a quick and direct impact in the engineering field (in this case mainly automatic control theory).

also [Vaidyanathan, 1985]). A square, stable, rational all-pass transfer function satisfies the equation

$$\mathbf{H}(z)\mathbf{H}^H(1/z^*) = \mathbf{H}^H(z)\mathbf{H}(1/z^*) = \mathbf{I}.$$

Since

$$\mathbf{H}(z) = \mathbf{D} + \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = \mathbf{D} + \sum_{i=1}^{\infty} \mathbf{C}\mathbf{A}^{i-1}\mathbf{B}z^{-i}$$

this implies that

$$\left[\mathbf{D} + \sum_{i=1}^{\infty} \mathbf{C}\mathbf{A}^{i-1}\mathbf{B}z^{-i}\right]\left[\mathbf{D}^H + \sum_{j=1}^{\infty} \mathbf{B}^H(\mathbf{A}^H)^{j-1}\mathbf{C}^H z^{j}\right] = \mathbf{I}.$$

Equating the terms in z^{-k} in both sides of this equation yields, after some trivial simplifications,

$$\mathbf{D}\mathbf{D}^H + \mathbf{C}\mathbf{P}\mathbf{C}^H = \mathbf{I} \tag{4.8}$$

for $k = 0$, and

$$\mathbf{C}\mathbf{A}^{k-1}[\mathbf{B}\mathbf{D}^H + \mathbf{A}\mathbf{P}\mathbf{C}^H] = \mathbf{0} \tag{4.9}$$

for $k > 0$ [$k < 0$ yields the Hermitian transpose of (4.9)]. Since by assumption the realization $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ of $\mathbf{H}(z)$ is minimal the matrix $\mathbf{M}_o$ has full rank and so (4.9) for $k > 0$ implies

$$\mathbf{B}\mathbf{D}^H + \mathbf{A}\mathbf{P}\mathbf{C}^H = \mathbf{0}. \tag{4.10}$$

In a similar way the equation $\mathbf{H}^H(z)\mathbf{H}(1/z^*) = \mathbf{I}$ gives rise to the equations

$$\mathbf{D}^H\mathbf{D} + \mathbf{B}^H\mathbf{Q}\mathbf{B} = \mathbf{I} \tag{4.11}$$

and

$$\mathbf{C}^H\mathbf{D} + \mathbf{A}^H\mathbf{Q}\mathbf{B} = \mathbf{0}. \tag{4.12}$$

For the reader's convenience we repeat here the Lyapunov equations for the Gramians $\mathbf{P}$ and $\mathbf{Q}$:

$$\mathbf{A}\mathbf{P}\mathbf{A}^H + \mathbf{B}\mathbf{B}^H = \mathbf{P} \tag{4.13}$$

$$\mathbf{A}^H\mathbf{Q}\mathbf{A} + \mathbf{C}^H\mathbf{C} = \mathbf{Q}. \tag{4.14}$$

Provided $\mathbf{H}(z)$ is stable these two equations have only one possible solution. If in addition its state space realization in minimal then both $\mathbf{P}$ and $\mathbf{Q}$ are positive definite and there is at least one realization that is input balanced, i.e., with $\mathbf{P} = \mathbf{I}$. For this realization (4.13), (4.10), and (4.8) imply that

$$\left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right]\left[\begin{array}{c|c} \mathbf{A}^H & \mathbf{C}^H \\ \hline \mathbf{B}^H & \mathbf{D}^H \end{array}\right] = \left[\begin{array}{c|c} \mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{I} \end{array}\right], \tag{4.15}$$

and so

$$\left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right] \tag{4.16}$$

is a unitary matrix. Multiplying the two matrices in (4.15) in the reversed order yields

$$\left[\begin{array}{c|c} \mathbf{A}^H & \mathbf{C}^H \\ \hline \mathbf{B}^H & \mathbf{D}^H \end{array}\right]\left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right] = \left[\begin{array}{c|c} \mathbf{A}^H\mathbf{A} + \mathbf{C}^H\mathbf{C} & \mathbf{A}^H\mathbf{B} + \mathbf{C}^H\mathbf{D} \\ \hline \mathbf{B}^H\mathbf{A} + \mathbf{D}^H\mathbf{C} & \mathbf{B}^H\mathbf{B} + \mathbf{D}^H\mathbf{D} \end{array}\right] = \left[\begin{array}{c|c} \mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{I} \end{array}\right].$$

Note that this is compatible with (4.14), (4.12), and (4.11) provided $\mathbf{Q} = \mathbf{I}$. In fact, since the Lyapunov equation has a unique solution $\mathbf{Q}$ must be the identity matrix. So, $\mathbf{PQ}$ is in this case equal to the identity matrix. This implies that the singular values of $\mathbf{H}(z)$ are all equal to one.

To prove that all singular values of a continuous-time all-pass transfer function are equal to one we will use a method described in [Glover, 1984]. A square, stable, rational all-pass transfer function satisfies the equation

$$\mathbf{H}(s)\mathbf{H}^H(-s^*) = \mathbf{H}^H(s)\mathbf{H}(-s^*) = \mathbf{I}$$

and so $\mathbf{H}(s) = \mathbf{H}^{-H}(-s^*)$. Applying the matrix inversion lemma[6] to

$$\mathbf{H}^H(-s^*) = \mathbf{D}^H + \mathbf{B}^H(-s\mathbf{I} - \mathbf{A}^H)^{-1}\mathbf{C}^H$$

yields

$$\mathbf{H}^{-H}(-s^*) = \mathbf{D}^{-H} + \mathbf{D}^{-H}\mathbf{B}^H[s\mathbf{I} - (-\mathbf{A}^H + \mathbf{C}^H\mathbf{D}^{-H}\mathbf{B}^H)]^{-1}\mathbf{C}^H\mathbf{D}^{-H}.$$

Therefore

$$(-\mathbf{A}^H + \mathbf{C}^H\mathbf{D}^{-H}\mathbf{B}^H, \mathbf{C}^H\mathbf{D}^{-H}, \mathbf{D}^{-H}\mathbf{B}^H, \mathbf{D}^{-H})$$

is another realization of $\mathbf{H}(s)$. Assuming that both realizations are minimal implies that there exists a unique $\mathbf{T}$ such that

$$\begin{aligned}
\mathbf{T}^{-1}\mathbf{A}\mathbf{T} &= -\mathbf{A}^H + \mathbf{C}^H\mathbf{D}^{-H}\mathbf{B}^H && \text{(4.17a)}\\
\mathbf{T}^{-1}\mathbf{B} &= \mathbf{C}^H\mathbf{D}^{-H} && \text{(4.17b)}\\
\mathbf{C}\mathbf{T} &= \mathbf{D}^{-H}\mathbf{B}^H && \text{(4.17c)}\\
\mathbf{D} &= \mathbf{D}^{-H}. && \text{(4.17d)}
\end{aligned}$$

Using (4.17b) in (4.17a) and multiplying the resulting equation by $\mathbf{T}$ from the left yields

$$\mathbf{A}\mathbf{T} + \mathbf{T}\mathbf{A}^H - \mathbf{B}\mathbf{B}^H = \mathbf{0}.$$

Since the solution of the Lyapunov equation (4.6) is unique we conclude that for the realization $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ of $\mathbf{H}(s)$ we have $\mathbf{P} = -\mathbf{T}$. Using (4.17c) in (4.17a) and multiplying the resulting equation by $\mathbf{T}^{-1}$ from the right yields

$$\mathbf{T}^{-1}\mathbf{A} + \mathbf{A}^H\mathbf{T}^{-1} - \mathbf{C}^H\mathbf{C} = \mathbf{0}.$$

Since the solution of the Lyapunov equation (4.7) is unique we conclude that for the realization $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ of $\mathbf{H}(s)$ we have $\mathbf{Q} = -\mathbf{T}^{-1}$. But this implies that $\mathbf{PQ} = \mathbf{I}$, and so the (non-zero) singular values of $\mathbf{H}(s)$ are all equal to one. We note that for a balanced realization of $\mathbf{H}(s)$ we have $\mathbf{T} = -\mathbf{I}$. In this case equations (4.17) and the Lyapunov equations become

$$\begin{aligned}
\mathbf{0} &= \mathbf{A} + \mathbf{A}^H + \mathbf{B}\mathbf{B}^H && \text{(4.18a)}\\
\mathbf{0} &= \mathbf{A} + \mathbf{A}^H + \mathbf{C}^H\mathbf{C} && \text{(4.18b)}\\
\mathbf{0} &= \mathbf{B} + \mathbf{C}^H\mathbf{D} && \text{(4.18c)}\\
\mathbf{0} &= \mathbf{C} + \mathbf{D}\mathbf{B}^H && \text{(4.18d)}\\
\mathbf{I} &= \mathbf{D}\mathbf{D}^H. && \text{(4.18e)}
\end{aligned}$$

In particular, for scalar all-pass transfer functions we may use $\mathbf{D} = 1$ and $\mathbf{C} = -\mathbf{B}^H$.

[6] The matrix inversion lemma is

$$(\mathbf{A} + \mathbf{B}\mathbf{C}\mathbf{D})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}(\mathbf{C}^{-1} + \mathbf{D}\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{D}\mathbf{A}^{-1}$$

for matrices with appropriate dimensions and provided all relevant inverses exist.

To terminate this section we present some examples of balanced realizations of first and second order all-pass transfer functions. Since any minimal balanced realization of an all-pass transfer function is also input balanced all-pass transfer functions can also be used to generate rational orthonormal functions. In fact, as we will see in the next section, they are the natural building blocks of the families of rational orthonormal functions discussed in section 3.

• Balanced realization of $(1-u^*z)/(z-u)$:

$$\left[\begin{array}{c|c}\mathbf{A}&\mathbf{B}\\\hline\mathbf{C}&\mathbf{D}\end{array}\right]=\left[\begin{array}{c|c}u&\sqrt{1-|u|^2}\\\hline\sqrt{1-|u|^2}&-u^*\end{array}\right]. \tag{4.19}$$

• Balanced realization of $(s-p^*)/(s+p)$:

$$\left[\begin{array}{c|c}\mathbf{A}&\mathbf{B}\\\hline\mathbf{C}&\mathbf{D}\end{array}\right]=\left[\begin{array}{c|c}-p&\sqrt{2\mathrm{Re}[p]}\\\hline-\sqrt{2\mathrm{Re}[p]}&1\end{array}\right]. \tag{4.20}$$

• Balanced realization of the last second order section of (3.6):

$$\left[\begin{array}{c|c}\mathbf{A}&\mathbf{B}\\\hline\mathbf{C}&\mathbf{D}\end{array}\right]=\left[\begin{array}{cc|c}[k_2(1-k_1)-(1+k_1)]/2&-\sqrt{1-k_1^2}\,(1+k_2)/2&\sqrt{(1-k_1)(1-k_2^2)/2}\\\sqrt{1-k_1^2}\,(1+k_2)/2&[(1-k_1)-k_2(1+k_1)]/2&\sqrt{(1+k_1)(1-k_2^2)/2}\\\hline-\sqrt{(1-k_1)(1-k_2^2)/2}&\sqrt{(1+k_1)(1-k_2^2)/2}&k_2\end{array}\right]. \tag{4.21}$$

• Balanced realization of the last second order section of (3.7):

$$\left[\begin{array}{c|c}\mathbf{A}&\mathbf{B}\\\hline\mathbf{C}&\mathbf{D}\end{array}\right]=\left[\begin{array}{cc|c}-k_1&-k_2\sqrt{1-k_1^2}&\sqrt{(1-k_1^2)(1-k_2^2)}\\\sqrt{1-k_1^2}&-k_1k_2&k_1\sqrt{1-k_2^2}\\\hline0&\sqrt{1-k_2^2}&k_2\end{array}\right]. \tag{4.22}$$

• Balanced realization of the last second order section of (3.14):

$$\left[\begin{array}{c|c}\mathbf{A}&\mathbf{B}\\\hline\mathbf{C}&\mathbf{D}\end{array}\right]=\left[\begin{array}{cc|c}-u/2&-u/2-\sqrt{v}&\sqrt{u}\\-u/2+\sqrt{v}&-u/2&\sqrt{u}\\\hline-\sqrt{u}&-\sqrt{u}&1\end{array}\right]. \tag{4.23}$$

• Balanced realization of the last second order section of (3.15):

$$\left[\begin{array}{c|c}\mathbf{A}&\mathbf{B}\\\hline\mathbf{C}&\mathbf{D}\end{array}\right]=\left[\begin{array}{cc|c}-u&-\sqrt{v}&\sqrt{2u}\\\sqrt{v}&0&0\\\hline-\sqrt{2u}&0&1\end{array}\right]. \tag{4.24}$$

4.4 Cascade of all-pass sections

One nice property of all-pass transfer functions is that the cascade of two (or more) balanced realizations of all-pass transfer functions is also all-pass and balanced. This will be proven

shortly. This property allows the construction of families of rational orthonormal functions: one just cascades several balanced realizations of all-pass transfer functions and the rational orthonormal functions are just the transfer functions from the input to the states of each all-pass section of the cascade. All rational orthonormal functions discussed in section 3 can be obtained in this way.

Remark. In fact, there are other ways of interconnecting two or more balanced realizations of arbitrary all-pass transfer functions for which the resulting transfer functions remains stable, all-pass, and balanced [Roberts and Mullis, 1987]. In particular, arbitrary connections between outputs and inputs of these transfer functions, including feedback connections, are allowed (in the discrete-time case feedback connections may originate delay-free loops that must be disposed off). A particularly interesting example that involves feedback is the normalized lattice structure of Gray and Markel [1975] (see also [Roberts and Mullis, 1987]), which realizes in a very simple way any rational all-pass transfer function. In fact, we have already encountered one example of this structure, viz. (4.22).

Let $(\mathbf{A}_1, \mathbf{B}_1, \mathbf{C}_1, \mathbf{D}_1)$ and $(\mathbf{A}_2, \mathbf{B}_2, \mathbf{C}_2, \mathbf{D}_2)$ be the state-space realizations of two finite dimensional linear systems, say $\mathbf{H}_1(z)$ and $\mathbf{H}_2(z)$. (We will discuss only the discrete-time case. The continuous-time case is identical except for an obvious change of notation.) The cascade or series connection of these two systems is obtained by forcing the output of the first system to be the input of the second (the number of outputs of the first system and the number of inputs of the second must then agree). In state space terms this means that

$$\left[\begin{array}{c} \mathbf{x}_1(k) \\ \hdashline \mathbf{y}_1(k) \end{array}\right] = \left[\begin{array}{c:c} \mathbf{A}_1 & \mathbf{B}_1 \\ \hdashline \mathbf{C}_1 & \mathbf{D}_1 \end{array}\right] \left[\begin{array}{c} \mathbf{x}_1(k-1) \\ \hdashline \mathbf{u}_1(k) \end{array}\right], \qquad \left[\begin{array}{c} \mathbf{x}_2(k) \\ \hdashline \mathbf{y}_2(k) \end{array}\right] = \left[\begin{array}{c:c} \mathbf{A}_2 & \mathbf{B}_2 \\ \hdashline \mathbf{C}_2 & \mathbf{D}_2 \end{array}\right] \left[\begin{array}{c} \mathbf{x}_2(k-1) \\ \hdashline \mathbf{u}_2(k) \end{array}\right]$$

with $\mathbf{u}_2(k) = \mathbf{y}_1(k)$. One state space description of the cascade connection of these two systems is then given by

$$\begin{aligned} \left[\begin{array}{c} \mathbf{x}_1(k) \\ \mathbf{x}_2(k) \\ \hdashline \mathbf{y}_2(k) \end{array}\right] &= \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 & \mathbf{B}_2 \\ \mathbf{0} & \mathbf{C}_2 & \mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1(k) \\ \mathbf{x}_2(k-1) \\ \mathbf{y}_1(k) \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 & \mathbf{B}_2 \\ \mathbf{0} & \mathbf{C}_2 & \mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{A}_1 & \mathbf{0} & \mathbf{B}_1 \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{C}_1 & \mathbf{0} & \mathbf{D}_1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1(k-1) \\ \mathbf{x}_2(k-1) \\ \mathbf{u}_1(k) \end{bmatrix} \\ &= \left[\begin{array}{cc:c} \mathbf{A}_1 & \mathbf{0} & \mathbf{B}_1 \\ \mathbf{B}_2\mathbf{C}_1 & \mathbf{A}_2 & \mathbf{B}_2\mathbf{D}_1 \\ \hdashline \mathbf{D}_2\mathbf{C}_1 & \mathbf{C}_2 & \mathbf{D}_2\mathbf{D}_1 \end{array}\right] \left[\begin{array}{c} \mathbf{x}_1(k-1) \\ \mathbf{x}_2(k-1) \\ \hdashline \mathbf{u}_1(k) \end{array}\right]. \end{aligned}$$

Obviously, for this case the overall transfer function is $\mathbf{H}(z) = \mathbf{H}_2(z)\mathbf{H}_1(z)$, and the states are given by $\mathbf{X}_1(z) = z(z\mathbf{I} - \mathbf{A}_1)^{-1}\mathbf{B}_1\,\mathbf{U}_1(z)$ and by $\mathbf{X}_2(z) = z(z\mathbf{I} - \mathbf{A}_2)^{-1}\mathbf{B}_2\,\mathbf{H}_1(z)\,\mathbf{U}_1(z)$.

For balanced realizations of an all-pass transfer function $\mathbf{H}(z) = \mathbf{D} + \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$ the matrix

$$\left[\begin{array}{c:c} \mathbf{A} & \mathbf{B} \\ \hdashline \mathbf{C} & \mathbf{D} \end{array}\right]$$

is unitary (see last section). For the cascade of two balanced realizations of two all-pass transfer functions it is clear that

$$\left[\begin{array}{cc:c} \mathbf{A}_1 & \mathbf{0} & \mathbf{B}_1 \\ \mathbf{B}_2\mathbf{C}_1 & \mathbf{A}_2 & \mathbf{B}_2\mathbf{D}_1 \\ \hdashline \mathbf{D}_2\mathbf{C}_1 & \mathbf{C}_2 & \mathbf{D}_2\mathbf{D}_1 \end{array}\right] = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 & \mathbf{B}_2 \\ \mathbf{0} & \mathbf{C}_2 & \mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{A}_1 & \mathbf{0} & \mathbf{B}_1 \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{C}_1 & \mathbf{0} & \mathbf{D}_1 \end{bmatrix}$$

is also unitary, since it is the product of two unitary matrices. The realization

$$\left(\begin{bmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{B}_2\mathbf{C}_1 & \mathbf{A}_2 \end{bmatrix}, \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2\mathbf{D}_1 \end{bmatrix}, \begin{bmatrix} \mathbf{D}_2\mathbf{C}_1 & \mathbf{C}_2 \end{bmatrix}, \mathbf{D}_2\mathbf{D}_1 \right) \tag{4.25}$$

of $\mathbf{H}_2(z)\mathbf{H}_1(z)$ is therefore also balanced.

For the continuous-time case precisely the same conclusion can be reached for the realization (4.25) via the simplification of the appropriate Lyapunov equations with equations (4.18).

For the reader's convenience we present here the state-space equations of the cascade of the realizations of 4 transfer functions (not necessarily all-pass), $\mathbf{H}_1(z), \ldots, \mathbf{H}_4(z)$, with compatible dimensions. Here they are:

$$\left[\begin{array}{c} \mathbf{x}_1(k) \\ \mathbf{x}_2(k) \\ \mathbf{x}_3(k) \\ \mathbf{x}_4(k) \\ \hdashline y_4(k) \end{array}\right] = \left[\begin{array}{cccc:c} \mathbf{A}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_1 \\ \mathbf{B}_2\mathbf{C}_1 & \mathbf{A}_2 & \mathbf{0} & \mathbf{0} & \mathbf{B}_2\mathbf{D}_1 \\ \mathbf{B}_3\mathbf{D}_2\mathbf{C}_1 & \mathbf{B}_3\mathbf{C}_2 & \mathbf{A}_3 & \mathbf{0} & \mathbf{B}_3\mathbf{D}_2\mathbf{D}_1 \\ \mathbf{B}_4\mathbf{D}_3\mathbf{D}_2\mathbf{C}_1 & \mathbf{B}_4\mathbf{D}_3\mathbf{C}_2 & \mathbf{B}_4\mathbf{C}_3 & \mathbf{A}_4 & \mathbf{B}_4\mathbf{D}_3\mathbf{D}_2\mathbf{D}_1 \\ \hdashline \mathbf{D}_4\mathbf{D}_3\mathbf{D}_2\mathbf{C}_1 & \mathbf{D}_4\mathbf{D}_3\mathbf{C}_2 & \mathbf{D}_4\mathbf{C}_3 & \mathbf{C}_4 & \mathbf{D}_4\mathbf{D}_3\mathbf{D}_2\mathbf{D}_1 \end{array}\right] \left[\begin{array}{c} \mathbf{x}_1(k-1) \\ \mathbf{x}_2(k-1) \\ \mathbf{x}_3(k-1) \\ \mathbf{x}_4(k-1) \\ \hdashline \mathbf{u}_1(k) \end{array}\right]. \tag{4.26}$$

From these equations the reader should be able to infer easily the form of the state-space equations for an arbitrary number of sections.

4.5 Links between discrete- and continuous- time state-space realizations of all-pass transfer functions

As in section 3.3, the bilinear transformation described in section 2.4 provides a link between discrete-time and continuous-time state space realizations of all-pass transfer functions. In fact, a clever use of the bilinear transformation (with and without normalization) allows the construction of a continuous-time state-space realization which has exactly the same controllability and observability Gramians as a discrete-time state-space realization of a given stable transfer function (not necessarily all-pass) [Glover, 1984]. In the following a subscript of d (c) will denote a discrete-time (continuous-time) variable or signal.

The state space description of a finite-dimensional discrete-time linear system is [cf. section 4.1]

$$\begin{aligned} \mathbf{X}_d(z) &= \mathbf{A}_d\, z^{-1}\mathbf{X}_d(z) + \mathbf{B}_d\, \mathbf{U}_d(z) \\ \mathbf{Y}_d(z) &= \mathbf{C}_d\, z^{-1}\mathbf{X}_d(z) + \mathbf{D}_d\, \mathbf{U}_d(z). \end{aligned}$$

Replacing z by $(a+s)/(a-s)$ [cf. (2.18)] this state space realization can be put in the form

$$\begin{aligned} s\mathbf{X}_d(\cdot) &= a(\mathbf{A}_d+\mathbf{I})^{-1}(\mathbf{A}_d-\mathbf{I})\,\mathbf{X}_d(\cdot) + (\mathbf{A}_d+\mathbf{I})^{-1}\mathbf{B}_d\,(a+s)\mathbf{U}_d(\cdot) \\ (a+s)\mathbf{Y}_d(\cdot) &= 2a\mathbf{C}_d(\mathbf{A}_d+\mathbf{I})^{-1}\,\mathbf{X}_d(\cdot) + [\mathbf{D}_d - \mathbf{C}_d(\mathbf{A}_d+\mathbf{I})^{-1}\mathbf{B}_d]\,(a+s)\mathbf{U}_d(\cdot) \end{aligned}$$

with $(\cdot)$ denoting $(\frac{a+s}{a-s})$. Note that $(\mathbf{A}_d + \mathbf{I})^{-1}$ is always well defined since the module of the eigenvalues of $\mathbf{A}_d$ are strictly smaller that 1 (for stable systems). Multiplying the first equation by $\sqrt{2a}/(a+s)$ and dividing the second by $a+s$ we get

$$\begin{aligned} s\mathbf{X}_c(s) &= a(\mathbf{A}_d + \mathbf{I})^{-1}(\mathbf{A}_d - \mathbf{I})\,\mathbf{X}_c(s) + \sqrt{2a}\,(\mathbf{A}_d + \mathbf{I})^{-1}\mathbf{B}_d\,\mathbf{U}_c(s) && \text{(4.27a)} \\ \mathbf{Y}_c(s) &= \sqrt{2a}\,\mathbf{C}_d(\mathbf{A}_d + \mathbf{I})^{-1}\,\mathbf{X}_c(s) + [\mathbf{D}_d - \mathbf{C}_d(\mathbf{A}_d + \mathbf{I})^{-1}\mathbf{B}_d]\,\mathbf{U}_c(s) && \text{(4.27b)} \end{aligned}$$

with [cf. (2.21)]

$$\mathbf{X}_c(s) = \frac{\sqrt{2a}}{a+s}\mathbf{X}_d\left(\frac{a+s}{a-s}\right)$$

and

$$\mathbf{U}_c(s) = \mathbf{U}_d\left(\frac{a+s}{a-s}\right), \qquad \mathbf{Y}_c(s) = \mathbf{Y}_d\left(\frac{a+s}{a-s}\right).$$

Note that $\mathbf{X}_d(z)$ is transformed with the normalization factor and that $\mathbf{U}_d(z)$ and $\mathbf{Y}_d(z)$ are transformed without the normalization factor. The reason for these choices is two-fold: the inner products between state variables are preserved; and a unit pulse in the input is transformed into a Dirac impulse (their z and Laplace transforms, respectively, are both equal to 1). Therefore, the controllability Gramian is preserved. In fact, as we will see shortly, both Gramians are preserved [Glover, 1984]. Also, the transfer function of the original system is transformed without normalization. All-pass systems are then transformed into all-pass systems.

From (4.27) we have[7]

$$\begin{aligned} \mathbf{A}_c &= a(\mathbf{A}_d + \mathbf{I})^{-1}(\mathbf{A}_d - \mathbf{I}), && \text{(4.28a)} \\ \mathbf{B}_c &= \sqrt{2a}\,(\mathbf{A}_d + \mathbf{I})^{-1}\mathbf{B}_d, && \text{(4.28b)} \\ \mathbf{C}_c &= \sqrt{2a}\,\mathbf{C}_d(\mathbf{A}_d + \mathbf{I})^{-1}, && \text{(4.28c)} \\ \mathbf{D}_c &= \mathbf{D}_d - \mathbf{C}_d(\mathbf{A}_d + \mathbf{I})^{-1}\mathbf{B}_d. && \text{(4.28d)} \end{aligned}$$

Note that $(\mathbf{A}_d + \mathbf{I})^{-1}$ and $(\mathbf{A}_d - \mathbf{I})$ commute and so

$$\mathbf{A}_c = a(\mathbf{A}_d + \mathbf{I})^{-1}(\mathbf{A}_d - \mathbf{I}) = a(\mathbf{A}_d - \mathbf{I})(\mathbf{A}_d + \mathbf{I})^{-1} = a[\mathbf{I} - 2(\mathbf{A}_d + \mathbf{I})^{-1}].$$

From equations (4.28) it is easy to verify that

$$\begin{aligned} \mathbf{A}_d &= (a\mathbf{I} - \mathbf{A}_c)^{-1}(a\mathbf{I} + \mathbf{A}_c), && \text{(4.29a)} \\ \mathbf{B}_d &= \sqrt{2a}\,(a\mathbf{I} - \mathbf{A}_c)^{-1}\mathbf{B}_c, && \text{(4.29b)} \\ \mathbf{C}_d &= \sqrt{2a}\,\mathbf{C}_c(a\mathbf{I} - \mathbf{A}_c)^{-1}, && \text{(4.29c)} \\ \mathbf{D}_d &= \mathbf{D}_c + \mathbf{C}_c(a\mathbf{I} - \mathbf{A}_c)^{-1}\mathbf{B}_c. && \text{(4.29d)} \end{aligned}$$

Note that because $a > 0$ and because for stable systems the eigenvalues of $\mathbf{A}_c$ have negative real parts $(a\mathbf{I} - \mathbf{A}_c)$ is always a regular matrix.

Using (4.28) in the Lyapunov equation [cf. (4.6)]

$$\mathbf{A}_c\mathbf{P}_c + \mathbf{P}_c\mathbf{A}_c^H + \mathbf{B}_c\mathbf{B}_c^H = \mathbf{0}$$

[7] See [Jonckheere and Li, 1987] for formulas similar to these ones but for a general bilinear mapping, of the form $z = (as+b)/(cs+d)$, $ad - bc \neq 0$. See also [Ober and Montgomery-Smith, 1990].

yields

$$a(\mathbf{A}_d + \mathbf{I})^{-1}[(\mathbf{A}_d - \mathbf{I})\mathbf{P}_c(\mathbf{A}_d^H + \mathbf{I}) + (\mathbf{A}_d + \mathbf{I})\mathbf{P}_c(\mathbf{A}_d^H - \mathbf{I}) + 2\mathbf{B}_d\mathbf{B}_d^H](\mathbf{A}_d^H + \mathbf{I})^{-1} = \mathbf{0}.$$

Since $\mathbf{A}_d + \mathbf{I}$ is always a regular matrix it is easy to show that this equation is equivalent to

$$\mathbf{A}_d\mathbf{P}_c\mathbf{A}_d^H + \mathbf{B}_d\mathbf{B}_d^H = \mathbf{P}_c,$$

which is just the corresponding Lyapunov equation for the discrete-time system [cf. (4.4)]. Since this Lyapunov equation has a unique solution $\mathbf{P}_c = \mathbf{P}_d$, as expected.

Using (4.28) in the Lyapunov equation [cf. (4.7)]

$$\mathbf{A}_c^H\mathbf{Q}_c + \mathbf{Q}_c\mathbf{A}_c + \mathbf{C}_c^H\mathbf{C}_c = \mathbf{0}$$

yields

$$a(\mathbf{A}_d + \mathbf{I})^{-1}[(\mathbf{A}_d^H - \mathbf{I})\mathbf{Q}_c(\mathbf{A}_d + \mathbf{I}) + (\mathbf{A}_d^H + \mathbf{I})\mathbf{Q}_c(\mathbf{A}_d - \mathbf{I}) + 2\mathbf{C}_d^H\mathbf{C}_d](\mathbf{A}_d + \mathbf{I})^{-1} = \mathbf{0},$$

where we have used the fact that $(\mathbf{A}_d + \mathbf{I})^{-1}$ and $(\mathbf{A}_d - \mathbf{I})$ commute. Since $\mathbf{A}_d + \mathbf{I}$ is always a regular matrix it is easy to show that this equation is equivalent to

$$\mathbf{A}_d^H\mathbf{Q}_c\mathbf{A}_d + \mathbf{C}_d^H\mathbf{C}_d = \mathbf{Q}_c$$

which is just the corresponding Lyapunov equation for the discrete-time system [cf. (4.5)]. Since this Lyapunov equation has a unique solution $\mathbf{Q}_c = \mathbf{Q}_d$.

As we have shown in one direction[8] the transformations (4.28) and (4.29) preserve the controllability and observability Gramians. In particular, balanced realizations of all-pass transfer functions are transformed into balanced realizations of all-pass transfer functions. The corresponding rational orthonormal functions extracted from these balanced realizations are thus transformed ones into the others, just like in section 3.3. This was expected, since the state variables were transformed using the bilinear transformation with normalization.

4.6 The generalized orthonormal basis functions of Heuberger *et al.*

We have shown in section 4.4 that the cascade of balanced realizations of all-pass transfer functions gives rise to a family of rational orthonormal functions. Perhaps the most interesting members of this family are the ones obtained by cascading the same realization of a given all-pass transfer function. This kind of construction gives rise of a less general family of rational orthonormal functions that was introduced by Heuberger in his interesting thesis [Heuberger, 1991] and explored in subsequent papers [Heuberger and Bosgra, 1990; Heuberger, Van den Hof and Bosgra, 1993a; Heuberger, Van den Hof and Bosgra, 1993b; Heuberger, Van den Hof and Bosgra, 1995; Van den Hof et al., 1995]. For the discrete-time case the functions of the standard basis of $\mathcal{H}^2(\mathbb{E})$ belong to this more restrict family. The Laguerre functions are another interesting example, both for the discrete- and the continuous-time cases. Following the terminology of Heuberger *et al.* we will refer to the rational orthonormal functions obtained by cascading the same balanced realization of a scalar all-pass transfer function as

[8] The other direction is left as an exercise for the reader.

the generalized orthonormal basis functions (GOBFs). Because of the repetitive structure of this construction the GOBFs have some very interesting properties. We will just describe the ones that are more directly related with the contents of this report. We will discuss first the discrete-time case and afterwards the continuous-time case.

Let $(\mathbf{A}, \mathbf{B}, \mathbf{C}, D)$ be one balanced realization of a scalar (and stable) all-pass transfer function $G(z) = D + \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$ with McMillan degree m. The m rational orthonormal functions obtained from this realization are given by the elements of the vector[9]

$$\mathbf{V}(z) = z(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}.$$

The inverse z transform of $\mathbf{V}(z)$ is

$$\mathbf{v}(k) = \mathbf{A}^k\,\mathbf{B}, \qquad k \geq 0. \tag{4.30}$$

The j-th element of $\mathbf{V}(z)$ will be denoted by $V_j(z)$. Denote by $z_0, \ldots, z_{m-1}$ the m poles of $G(z)$. It is easy to check that the functions $V_1(z), \ldots, V_m(z)$ can be obtained by applying a unitary transformation (in this case multiplication by an $m{\times}m$ unitary matrix) to the functions $G_0(z), \ldots, G_{m-1}(z)$ described in section 3.1.

The cascade of n identical realizations of $G(z)$ gives rise to the orthonormal functions $V_j(z)G^i(z)$, with $i = 0, \ldots, n-1$ and $j = 1, \ldots, m$. These functions can also be obtained by applying a unitary transformation (in this case multiplication by a block diagonal $mn{\times}mn$ unitary matrix with blocks of size $m{\times}m$ and identical diagonal blocks) to the functions $G_0(z), \ldots, G_{nm-1}(z)$, provided $z_{i+mk} = z_i$ for $k > 0$ and $0 \leq i < m$.[10] Using the results of sections 3.4 and 3.5 it is then trivial to verify that the GOBFs form an orthonormal basis of $\mathcal{H}^2(\mathbb{E})$, and that they are also uniformly bounded in $\mathcal{H}^\infty(\mathbb{E})$. These orthonormal functions are naturally partitioned in n vectors with m elements each, one vector per all-pass section. These vectors are given by the simple formula

$$\mathbf{V}_i(z) = G^i(z)\mathbf{V}(z), \qquad i = 0, \ldots, n-1.$$

Using the structure of the obvious generalization of (4.26) to an arbitrary number of cascaded transfer functions it is easy to verify that

$$z\mathbf{V}_i(z) = \mathbf{A}\mathbf{V}_i(z) + \sum_{j=1}^{i} \mathbf{B}D^{j-1}\mathbf{C}\,\mathbf{V}_{i-j}(z) + \mathbf{B}D^i\,z$$

since in our case $\mathbf{A}_i = \mathbf{A}$, $\mathbf{B}_i = \mathbf{B}$, $\mathbf{C}_i = \mathbf{C}$, and $\mathbf{D}_i = D$ for all i.

Because the functions $V_j(z)G^i(z)$, $1 \leq j \leq m$ and $i \in \mathbb{N}_0$, form an orthonormal basis of $\mathcal{H}^2(\mathbb{E})$ any function belonging to this Hardy space can be expressed in the form

$$F(z) = \sum_{i=0}^{\infty}\sum_{j=1}^{m} c_{ji}\,G^i(z)V_j(z) = \sum_{j=1}^{m} F_j(z) \tag{4.31}$$

[9]Note that we are using the state space description (4.3).

[10]It is quite interesting to observe that Walsh, in his important book [Walsh, 1969, section 9.5], suggested the utilization of a sequence of poles for the rational orthonormal functions that is precisely of this form (i.e., periodic), because in that case it is not difficult to derive asymptotic results for the approximations based on truncated series expansions. In a different context this is confirmed in the recent work of Van den Hof et al. [1995].

with

$$F_j(z) = V_j(z) \sum_{i=0}^{\infty} c_{ji}\, G^i(z), \qquad 1 \le j \le m,$$

and

$$c_{ji} = \left\langle F(z), G^i(z) V_j(z) \right\rangle, \qquad i \in \mathbb{N}_0, 1 \le j \le m.$$

By construction, the functions $F_j(z)$ are orthogonal. Each $F_j(z)$ belongs to a subspace of $\mathcal{H}^2(\mathbb{E})$, which we will denote by $H_j^2(\mathbb{E})$, that only depends on the balanced realization of $G(z)$. It is clear that $\mathcal{H}^2(\mathbb{E})$ is the direct sum of $\mathcal{H}_1^2(\mathbb{E}), \ldots, \mathcal{H}_m^2(\mathbb{E})$, and that each $H_j^2(\mathbb{E})$ is itself a Hilbert space.

Suppose we want to approximate a function $F(z)$ by a truncated version of (4.31), namely

$$F(z;n) = \sum_{i=0}^{n} \sum_{j=1}^{m} c_{ji}\, G^i(z) V_j(z) = \sum_{j=1}^{m} F_j(z;n)$$

with

$$F_j(z;n) = V_j(z) \sum_{i=0}^{n} c_{ji}\, G^i(z), \qquad 1 \le j \le m.$$

Note that in this manner the original approximation problem can be broken in m other problems of the same kind. It will pay off to recast each one of these "smaller" problems in $\mathcal{H}^2(\mathbb{E})$. This is easily done by constructing an isomorphism between $\mathcal{H}_j^2(\mathbb{E})$ and $\mathcal{H}^2(\mathbb{E})$.

The set $\{G^i(z)V_j(z)\}_{i\in\mathbb{N}_0}$ is an orthonormal basis of $\mathcal{H}_j^2(\mathbb{E})$. In order to construct an isomorphism between this Hilbert space and $\mathcal{H}^2(\mathbb{E})$ it is enough to map these functions into the functions w^{-i} of the standard basis of $\mathcal{H}^2(\mathbb{E})$. The function $F_j(z)$ of $\mathcal{H}_j^2(\mathbb{E})$ is then mapped into the function

$$\tilde{F}_j(w) = \sum_{i=0}^{\infty} c_{ji}\, w^{-i}$$

of $\mathcal{H}^2(\mathbb{E})$. Because of the simple structure of the orthonormal basis of $\mathcal{H}_j^2(\mathbb{E})$ that we are using, $F_j(z)$ and $\tilde{F}_j(w)$ are related by the interesting formula

$$F_j(z) = V_j(z)\, \tilde{F}_j(1/G(z)). \tag{4.32}$$

In part of this formula w is replaced by $1/G(z)$. It is easy to verify that $|z| < 1$ corresponds to $|w| < 1$, $|z| = 1$ to $|w| = 1$ and $|z| > 1$ to $|w| > 1$. Note that both $F_j(z)$ and $\tilde{F}_j(w)$ are analytic on $\mathbb{E}$.

The mapping defined by $w = 1/G(z)$ is m to 1, i.e., there are m values of z that are mapped into the same value of w. Therefore $\mathbb{E}$ is mapped into m versions of itself (the same happens to $\mathbb{T}$ and to $\mathbb{D}$). The corresponding inverse mapping has m branches. Each one of these branches maps $\mathbb{E}$ into a region of $\mathbb{E}$. (We can imagine $\mathbb{E}$ as being divided in m regions, each one of which the image of $\mathbb{E}$ under a branch of this inverse mapping.)

Since by definition the isomorphism between $\mathcal{H}_j^2(\mathbb{E})$ and $\mathcal{H}^2(\mathbb{E})$ preserves inner products we have

$$\frac{1}{2\pi \mathsf{i}} \oint_{\mathbb{T}} \tilde{X}_j(w) \tilde{Y}_j^*(1/w^*)\, \frac{dw}{w} = \frac{1}{2\pi \mathsf{i}} \oint_{\mathbb{T}} X_j(z) Y_j^*(1/z^*)\, \frac{dz}{z}.$$

Using (4.32) in this formula gives the result (remember that $G(z)G^*(1/z^*) = 1$)

$$\frac{1}{2\pi \mathsf{i}} \oint_{\mathbb{T}} \tilde{X}_j(w) \tilde{Y}_j^*(1/w^*)\, \frac{dw}{w} = \frac{1}{2\pi \mathsf{i}} \oint_{\mathbb{T}} \tilde{X}_j(1/G(z)) \tilde{Y}_j^*(G^*(z))\, V_j(z) V_j^*(1/z^*) \frac{dz}{z}. \tag{4.33}$$

This very interesting formula suggests once more the change of variables $w = 1/G(z)$. The factor $V_j(z)V_j^*(1/z^*)$ appearing in the integral of the right hand side of this formula would then be just a normalization factor due to the change of the measure of integration. This is actually the case when $w = 1/G(z)$ is a one to one mapping, i.e., when $m = 1$. For $m > 1$ this interpretation of (4.33) is not at all clear since the relevant mapping is m to 1.

Equation (4.32) gives a way to obtain $F_j(z)$ from $\tilde{F}_j(w)$. But, how can we obtain $\tilde{F}_j(w)$ from $F(z)$? This problem was addressed in [Van den Hof et al., 1995] for the case where $F(z)$ is a transfer function. We will address here two different cases: $F(z)$ is the z transform of a $\ell^2(\mathbb{N}_0)$ signal; and $F(z)$ is the transfer function of a causal and stable system. These two cases are slightly different. This is akin to the situation encountered while dealing with the bilinear transformation of section 2.4 when applied to signals and when applied to transfer functions (cf. footnote 21 on page 29).

Instead of finding a formula for $\tilde{F}_j(w)$ it is easier to obtain a formula for the function

$$\tilde{\mathbf{f}}(w) = \sum_{i=0}^{\infty} \mathbf{c}_i \, w^{-i}$$

where

$$F(z) = \sum_{i=0}^{\infty} \mathbf{c}_i^H \, \mathbf{V}_i(z)$$

and

$$\mathbf{c}_i = \langle \mathbf{V}_i(z), F(z) \rangle = \frac{1}{2\pi \mathsf{i}} \oint_{\mathbb{T}} \mathbf{V}_i(z) F^*(1/z^*) \, \frac{dz}{z}.$$

Note that the j-th element of $\tilde{\mathbf{f}}(w)$ is equal to $\tilde{F}_j^*(w^*)$.[11] Following [Van den Hof et al., 1995] we will refer to $\tilde{\mathbf{f}}(w)$ as the Hambo transform of $F(z)$. It turns out that we need to differentiate between the Hambo transform of signals and the Hambo transform of systems, since they are slightly different. The Hambo transform of signals will be denoted by a tilde (eg., $\tilde{\mathbf{f}}(w)$) while the Hambo transform of systems will be denoted by an overbar (eg., $\bar{\mathbf{f}}(w)$).

Consider the function

$$F_\alpha(z) = z^{-\alpha} F(z), \qquad \alpha \in \mathbb{N}_0.$$

For this function we have, for $\alpha > 0$,

$$\begin{aligned}
\mathbf{c}_{\alpha,i} &= \langle \mathbf{V}_i(z), z^{-\alpha} F(z) \rangle \\
&= \left\langle z\mathbf{V}_i(z), z^{1-\alpha} F(z) \right\rangle \\
&= \left\langle \mathbf{A}\mathbf{V}_i(z) + \sum_{j=1}^{i} \mathbf{B} D^{j-1} \mathbf{C} \, \mathbf{V}_{i-j}(z) + \mathbf{B} D^i \, z, z^{1-\alpha} F(z) \right\rangle \\
&= \mathbf{A}\mathbf{c}_{\alpha-1,i} + \sum_{j=1}^{i} \mathbf{B} D^{j-1} \mathbf{C} \, \mathbf{c}_{\alpha-1,i-j} + \left\langle \mathbf{B} D^i, z^{-\alpha} F(z) \right\rangle \\
&= \mathbf{A}\mathbf{c}_{\alpha-1,i} + \sum_{j=1}^{i} \mathbf{B} D^{j-1} \mathbf{C} \, \mathbf{c}_{\alpha-1,i-j} + \mathbf{0}.
\end{aligned}$$

[11] The awkwardness of the relation between the elements of $\tilde{\mathbf{f}}(w)$ and the functions $\tilde{F}_j(w)$ is the price we have to pay for a cleaner formula for $\tilde{\mathbf{f}}(w)$. In essence, this awkwardness is originated by the fact that we have to conjugate one of the arguments of inner products while this does not happen for convolutions. Of course, if we are dealing with real signals and systems (in the time domain), there is no awkwardness at all.

It is not difficult to recognize that this last expression is the convolution of the sequence

$$(\mathbf{A}, \mathbf{BC}, \mathbf{B}D\mathbf{C}, \mathbf{B}D^2\mathbf{C}, \ldots).$$

with the sequence

$$(\mathbf{c}_{\alpha-1,0}, \mathbf{c}_{\alpha-1,1}, \mathbf{c}_{\alpha-1,2}, \mathbf{c}_{\alpha-1,3}, \ldots)$$

The first of the two sequences is just the impulse response of

$$N(w) = \mathbf{A} + \mathbf{B}(w - D)^{-1}\mathbf{C},$$

and so its (signal) Hambo transform is $N(w)$. Note that since $G(z)$ is stable we have $|D| < 1$, and so $N(w)$ is stable. Just like for the z transform, the convolution of the coefficients of two expansions corresponds to the multiplication of their respective Hambo transforms. This yields first

$$\tilde{\mathbf{f}}_\alpha(w) = [\mathbf{A} + \mathbf{B}(w - D)^{-1}\mathbf{C}]\, \tilde{\mathbf{f}}_{\alpha-1}(w).$$

and then

$$\tilde{\mathbf{f}}_\alpha(w) = N^\alpha(w)\tilde{\mathbf{f}}(w). \tag{4.34}$$

Using this formula and the linearity of the Hambo transform it is clear that the (signal) Hambo transform of the function

$$X(z) = \sum_{i=0}^{\infty} x(i)\, z^{-i}$$

is given by

$$\tilde{\mathbf{x}}(w) = \left[\sum_{i=0}^{\infty} x(i)\, N^i(w)\right] \tilde{\mathbf{u}}(w)$$

where $\tilde{\mathbf{u}}(w)$ is the (signal) Hambo transform of the function $U(z) = 1$. To compute $\tilde{\mathbf{u}}(w)$ note that

$$U(z) = 1 = \sum_{i=0}^{\infty} \mathbf{c}_{u,i}^H\, \mathbf{V}_i(z)$$

with

$$\mathbf{c}_{u,i} = \langle \mathbf{V}_i(z), 1 \rangle = \langle \mathbf{v}_i(k), \delta(k) \rangle = \mathbf{v}_i(0).$$

But from (4.30) and from the general structure of (4.26) it is easy to infer that $\mathbf{v}_i(0) = \mathbf{B}D^i$, and so

$$\tilde{\mathbf{u}}(w) = \sum_{i=0}^{\infty} \mathbf{B}D^i\, w^{-i} = w\mathbf{B}(w - D)^{-1}.$$

This leads to the conclusion that

$$\tilde{\mathbf{x}}(w) = \left[\sum_{i=0}^{\infty} x(i)\, N^i(w)\right] w\mathbf{B}(w - D)^{-1}.$$

Note that the realization $(D, \mathbf{C}, \mathbf{B}, \mathbf{A})$ of $N(w)$ is stable and all-pass. Equally interesting is the fact that the transfer function from a perturbation signal directly added to the state of the standard state space realization of $N(w)$[12] to the outputs of that realization is given by $w\mathbf{B}(w - D)^{-1}$.

[12]Recall the interpretation of the observability Gramian.

The Hambo transform for systems can be obtained in the following way. Since (4.32) defines a one to one mapping it is clear that the Hambo transform of the product of two functions is the product of their Hambo transforms. This shows that the Hambo transform inherits (from the z transform) the property that convolutions in the time domain are transformed into products in the transform domain. Let $X(z)$ [the input signal] belong to $\mathcal{H}^2(\mathbb{E})$, and let $H(z)$ [the stable and causal transfer function] belong to $\mathcal{H}^\infty(\mathbb{E})$. Let $Y(z) = H(z)X(z)$ [the output signal]. Then, $Y(z)$ also belongs to $\mathcal{H}^2(\mathbb{E})$ and its signal Hambo transform is given by

$$\tilde{\mathbf{y}}(w) = \left[\sum_{i=0}^{\infty} h(i)\, N^i(w)\right] \left[\sum_{i=0}^{\infty} x(i)\, N^i(w)\right] w\mathbf{B}(w-D)^{-1} = \bar{\mathbf{h}}(w)\tilde{\mathbf{x}}(w).$$

This clearly shows that

$$\bar{\mathbf{h}}(w) = \sum_{i=0}^{\infty} h(i)\, N^i(w).$$

This result could also have been inferred directly from (4.34) and from the linearity of the Hambo transform. Note that for a general signal $F(z)$ we have

$$\tilde{\mathbf{f}}(w) = \bar{\mathbf{f}}(w)\, w\mathbf{B}(w-D)^{-1}.$$

It is now time to discuss the continuous-time GOBFs. Let $(\mathbf{A}, \mathbf{B}, \mathbf{C}, D)$ be one balanced realization of a scalar (and stable) all-pass transfer function $G(s) = D + \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$ with McMillan degree m. The m rational orthonormal functions obtained from this realization are given by the elements of the vector

$$\mathbf{V}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}.$$

The j-th element of $\mathbf{V}(s)$ will be denoted by $V_j(s)$. Denote by $s_0, \ldots, s_{m-1}$ the m poles of $G(s)$. It is easy to check that the functions $V_1(s), \ldots, V_m(s)$ can be obtained by applying a unitary transformation (in this case multiplication by an $m{\times}m$ unitary matrix) to the functions $G_0(s), \ldots, G_{m-1}(s)$ described in section 3.2.

The cascade of n identical realizations of $G(s)$ gives rise to the orthonormal functions $V_j(s)G^i(s)$, with $i = 0, \ldots, n-1$ and $j = 1, \ldots, m$. These functions can also be obtained by applying a unitary transformation (in this case multiplication by a block diagonal $mn{\times}mn$ unitary matrix with blocks of size $m{\times}m$ and identical diagonal blocks) to the functions $G_0(s), \ldots, G_{nm-1}(s)$, provided $s_{i+mk} = s_i$ for $k > 0$ and $0 \le i < m$. Using the results of sections 3.4 and 3.5 it is then trivial to verify that the GOBFs form an orthonormal basis of $\mathcal{H}^2(\mathbb{C}^+)$, and that they are also uniformly bounded in $\mathcal{H}^\infty(\mathbb{C}^+)$. These orthonormal functions are naturally partitioned in n vectors with m elements each, one vector per all-pass section. These vectors are given by the simple formula

$$\mathbf{V}_i(s) = G^i(s)\mathbf{V}(s), \qquad i = 0, \ldots, n-1.$$

Because the functions $V_j(s)G^i(s)$, $1 \le j \le m$ and $i \in \mathbb{N}_0$, form an orthonormal basis of $\mathcal{H}^2(\mathbb{C}^+)$ any function belonging to this Hardy space can be expressed in the form

$$F(s) = \sum_{i=0}^{\infty}\sum_{j=1}^{m} c_{ji}\, G^i(s)V_j(s) = \sum_{j=1}^{m} F_j(s) \tag{4.35}$$

with

$$F_j(s) = V_j(s) \sum_{i=0}^{\infty} c_{ji}\, G^i(s), \qquad 1 \le j \le m,$$

and

$$c_{ji} = \left\langle F(s), G^i(s) V_j(s) \right\rangle, \qquad i \in \mathbb{N}_0, 1 \le j \le m.$$

By construction, the functions $F_j(s)$ are orthogonal. Each $F_j(s)$ belongs to a subspace of $\mathcal{H}^2(\mathbb{C}^+)$, which we will denote by $H_j^2(\mathbb{C}^+)$, that only depends on the balanced realization of $G(s)$. It is clear that $\mathcal{H}^2(\mathbb{C}^+)$ is the direct sum of $\mathcal{H}_1^2(\mathbb{C}^+), \ldots, \mathcal{H}_m^2(\mathbb{C}^+)$, and that each $H_j^2(\mathbb{C}^+)$ is itself a Hilbert space.

Suppose we want to approximate a function $F(s)$ by a truncated version of (4.35), namely

$$F(s;n) = \sum_{i=0}^{n} \sum_{j=1}^{m} c_{ji}\, G^i(s) V_j(s) = \sum_{j=1}^{m} F_j(s;n)$$

with

$$F_j(s;n) = V_j(s) \sum_{i=0}^{n} c_{ji}\, G^i(s), \qquad 1 \le j \le m.$$

Note that in this manner the original approximation problem can be broken in m other problems of the same kind. It will pay off to recast each one of these "smaller" problems in $\mathcal{H}^2(\mathbb{E})$. This is easily done by constructing an isomorphism between $\mathcal{H}_j^2(\mathbb{C}^+)$ and $\mathcal{H}^2(\mathbb{E})$.

The set $\{G^i(s)V_j(s)\}_{i\in\mathbb{N}_0}$ is an orthonormal basis of $\mathcal{H}_j^2(\mathbb{C}^+)$. In order to construct an isomorphism between this Hilbert space and $\mathcal{H}^2(\mathbb{E})$ it is enough to map these functions into the functions w^{-i} of the standard basis of $\mathcal{H}^2(\mathbb{E})$. The function $F_j(s)$ of $\mathcal{H}_j^2(\mathbb{C}^+)$ is then mapped into the function

$$\tilde{F}_j(w) = \sum_{i=0}^{\infty} c_{ji}\, w^{-i}$$

of $\mathcal{H}^2(\mathbb{E})$. Because of the simple structure of the orthonormal basis of $\mathcal{H}_j^2(\mathbb{C}^+)$ that we are using, $F_j(s)$ and $\tilde{F}_j(w)$ are related by the interesting formula

$$F_j(s) = V_j(s)\, \tilde{F}_j(1/G(s)). \tag{4.36}$$

In part of this formula w is replaced by $1/G(s)$. It is easy to verify that $\mathrm{Re}[s] < 0$ corresponds to $|w| < 1$, $\mathrm{Re}[s] = 0$ to $|w| = 1$ and $\mathrm{Re}[s] > 0$ to $|w| > 1$. Note that $F_j(s)$ is analytic on $\mathbb{C}^+$ and $\tilde{F}_j(w)$ is analytic on $\mathbb{E}$.

The mapping defined by $w = 1/G(s)$ is m to 1, i.e., there are m values of s that are mapped into the same value of w. Therefore $\mathbb{C}^+$ is mapped into m versions of $\mathbb{E}$ (the same happens between $\mathbb{T}$ and $\mathrm{i}\mathbb{R}$ and between $\mathbb{C}^-$ and $\mathbb{D}$). The corresponding inverse mapping has m branches. Each one of these branches maps $\mathbb{E}$ into a region of $\mathbb{C}^+$. (We can imagine $\mathbb{C}^+$ as being divided in m regions, each one of which the image of $\mathbb{E}$ under a branch of this inverse mapping.)

Since by definition the isomorphism between $\mathcal{H}_j^2(\mathbb{C}^+)$ and $\mathcal{H}^2(\mathbb{E})$ preserves inner products we have

$$\frac{1}{2\pi\mathrm{i}} \oint_{\mathbb{T}} \tilde{X}_j(w) \tilde{Y}_j^*(1/w^*)\, \frac{dw}{w} = \frac{1}{2\pi\mathrm{i}} \int_{\mathrm{i}\mathbb{R}} X_j(s) Y_j^*(-s^*)\, ds.$$

Using (4.36) in this formula gives the result (remember that $G(s)G^*(-s^*) = 1$)

$$\frac{1}{2\pi \mathsf{i}} \oint_{\mathbb{T}} \tilde{X}_j(w)\tilde{Y}_j^*(1/w^*)\,\frac{dw}{w} = \frac{1}{2\pi \mathsf{i}} \int_{\mathsf{i}\mathbb{R}} \tilde{X}_j(1/G(s))\tilde{Y}_j^*(G^*(s))\,V_j(s)V_j^*(-s^*)ds. \tag{4.37}$$

This very interesting formula suggests once more the change of variables $w = 1/G(s)$. The factor $V_j(s)V_j^*(-s^*)$ appearing in the integral of the right hand side of this formula would then be just a normalization factor due to the change of the measure of integration. This is actually the case when $w = 1/G(s)$ is a one to one mapping, i.e., when $m = 1$, and corresponds to the bilinear transformation described in detail in section 2.4 and in [Steiglitz, 1965a]. For $m > 1$ this interpretation of (4.33) is not at all clear since the relevant mapping is m to 1.

Unlike the discrete-time case, it is not clear how to obtain $\tilde{F}_j(w)$ from $F(s)$. The method used to deduce the Hambo transforms for discrete-time signals and systems appears to fail in the continuous-time case (D is equal to 1, and the representation of a continuous-time system by its Markov parameters is not general enough...).

Chapter 5

System identification models based on rational orthonormal functions

In this final chapter we describe a general system identification model based on the rational orthonormal functions described in the two previous chapters. Special cases of these models include the FIR model [Ljung, 1987; Haykin, 1991], the Laguerre model [King and Paraskevopoulos, 1977; Wahlberg, 1991; Oliveira e Silva, 1995a], the "two-parameter" Kautz model [Wahlberg, 1994], and the GOBFs model [Heuberger et al., 1995]. All these special cases correspond to the utilization of a periodic sequence of poles in the construction of the rational orthonormal functions.

We will describe the main properties of the most general model, without assuming that the sequence of poles is periodic.[1] Our analysis of this general model will be made in a stochastic framework. Therefore, we start with a small introduction to wide-sense stationary stochastic processes, which is followed by the description of the promised system identification model and by the presentation of its most relevant properties. Contrary to what we have done in previous chapters we will only address the discrete-time case. The development of the continuous-time case is in its most significant aspects similar and so is left for the interested reader. The relevant references for this case will be cited along their discrete-time counterparts.

5.1 Wide-sense stationary stochastic processes

Let X and Y be two complex random variables with zero mean and finite variance.[2] The inner product between these two variables is defined as being their covariance, i.e.,

$$\langle X, Y \rangle = \mathrm{E}[XY^*]$$

where the symbol $\mathrm{E}[\cdot]$ denotes the mathematical expectation of its argument. Since the variances of X and Y are finite $\langle X, Y \rangle$ is a well defined complex number. Note that $\langle X, X \rangle = \mathrm{Var}[X]$ is the variance of X. The set of all complex random variables with zero mean and

[1] However, some of the more specialized results, in particular the asymptotic results for finite data records (which we will not describe), are only known for periodic sequences of poles, since it appears that these cases are much more tractable than the general case.

[2] These random variables are of course members of a probability space. For more information concerning this detail we refer the reader to [Porat, 1994].

finite variance (belonging to a given probability space) equipped with the inner product just described is a Hilbert space [Porat, 1994].

Loosely speaking, a discrete-time stochastic process is a sequence of random variables whose statistical properties may (or may not) be inter-dependent. Let $x(k)$ and $y(k)$ be two (discrete-time) stochastic processes. The cross-correlation function between these two stochastic processes is defined by

$$R_{xy}(k_1, k_2) = \mathrm{E}[x(k_1)y^*(k_2)] = \langle x(k_1), y(k_2) \rangle \,.$$

The auto-correlation of the stochastic process $x(k)$ is defined in the same manner, with $y(k)$ replaced by $x(k)$, and will be denoted by $R_{xx}(k_1, k_2)$.

Loosely speaking, a stationary process is one whose probabilistic law it time-invariant. In the same vein, a wide-sense stationary process (up to second order) is one whose first and second order moments (i.e., its mean and auto-correlation function) are time-invariant. In this chapter we will work exclusively with wide-sense stationary stochastic processes that have zero mean and finite variance.

Let $x(k)$ and $y(k)$ be two wide-sense stationary stochastic processes. Then their cross-correlation function can also be given by

$$R_{xy}(\tau) = \mathrm{E}[x(k+\tau)y^*(k)] = \langle x(k+\tau), y(k) \rangle \,,$$

i.e., it is independent of k. The auto-correlation of $x(k)$ is defined in the same manner. The cross-power spectral density between $x(k)$ and $y(z)$ is defined as the Fourier transform of $R_{xy}(\tau)$ and is given by

$$\Phi_{xy}(z) = \sum_{\tau=-\infty}^{\infty} R_{xy}(\tau)\, z^{-\tau}, \qquad z = e^{\mathrm{i}\omega}.$$

Note that $\Phi_{xy}(z)$ converges almost everywhere on $\mathbb{T}$ since $R_{xy}(\tau)$ is a bounded sequence. The power spectral density of $x(k)$ is defined in the same manner and will be denoted by $\Phi_{xx}(z)$. To avoid some mathematical difficulties related with the convergence to $\Phi_{xy}(z)$, in the sequel we will assume that $R_{xy}(\tau)$ is absolutely summable. This assumption makes $\Phi_{xy}(e^{\mathrm{i}\omega})$ a continuous and bounded function for $\omega \in [-\pi, \pi]$.[3] This allows us to use the results of chapter 2 without any technical problems. In particular, under this restriction it is possible to recover $R_{xy}(\tau)$ from $\Phi_{xy}(z)$ with the formula [cf. (2.4)]

$$R_{xy}(\tau) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_{xy}(e^{\mathrm{i}\omega})\, e^{\mathrm{i}\omega\tau}\, d\omega.$$

In the general case this integral would have to be interpreted with care, since $\Phi_{xy}(e^{\mathrm{i}\omega})$ could contain some distributions (e.g., Dirac impulses). With the imposed restriction this will never happen. It is important to note that the variance of $x(k)$, which is equal to $R_{xx}(0)$, is given by

$$\mathrm{Var}[x(k)] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_{xx}(e^{\mathrm{i}\omega})\, d\omega.$$

In a sense, this formula justifies the name "power spectral density" given to $\Phi_{xx}(z)$.

[3] This restriction can be relaxed. See footnote 15 on page 67 for more details.

Let $y(k)$ be the output of a stable system with transfer function $H(z) = \sum_{i=0}^{\infty} h(i)z^{-i}$, excited by a wide-sense stationary stochastic process $x(k)$ with zero-mean and finite variance, and also with an absolutely summable auto-correlation function. Since the process $x(k)$ has zero-mean so will $y(k)$. Furthermore, since $H(z)$ is stable, i.e., $h(k)$ is an absolutely summable sequence, it is a simple exercise to verify that the variance of $y(k)$ is finite,[4] and so $y(k)$ is also a stochastic process with zero-mean and finite variance. The power spectral density of $y(k)$ is given by

$$\begin{aligned}\Phi_{yy}(z) &= \sum_{\tau=-\infty}^{\infty} \left(\sum_{i,j=0}^{\infty} \mathrm{E}[h(i)x(k+\tau-i)h^*(j)x^*(k-j)] \right) z^{-\tau} \\ &= \sum_{i,j=0}^{\infty} h(i)h^*(j) \sum_{\tau=-\infty}^{\infty} R_{xx}(\tau - i + j)\, z^{-\tau} \\ &= H(z)H^*(1/z^*)\Phi_{xx}(z).\end{aligned}$$

In particular, we have [Oppenheim and Schafer, 1975]

$$\Phi_{yy}(e^{\mathrm{i}\omega}) = |H(e^{\mathrm{i}\omega})|^2 \Phi_{xx}(e^{\mathrm{i}\omega})$$

and

$$\mathrm{Var}[y(k)] = \frac{1}{2\pi} \int_{-\pi}^{\pi} |H(e^{\mathrm{i}\omega})|^2 \Phi_{xx}(e^{\mathrm{i}\omega})\, d\omega.$$

The following result will be useful later on. Let $F(z)$ and $G(z)$ be the transfer functions of two stable systems, excited respectively by the stochastic processes $x(k)$ and $y(k)$ (satisfying the usual conditions), and whose outputs are respectively the stochastic processes $u(k)$ and $v(k)$. Then, using a line of reasoning similar to the one used in the last paragraph it is easy to verify that

$$\Phi_{uv}(z) = F(z)G^*(1/z^*)\Phi_{xy}(z),$$

and that

$$\langle u(k), v(k) \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{\mathrm{i}\omega})G^*(e^{\mathrm{i}\omega})\Phi_{xy}(e^{\mathrm{i}\omega})\, d\omega. \tag{5.1}$$

Remark. The following fact is useful to adapt some of the results of this and the next sections to the case of deterministic signals. Suppose that $x(k)$ and $y(k)$ are deterministic signals with z transforms $X(z)$ and $Y(z)$, respectively. Then, the inner product formula (5.1) remains valid if we make $\Phi_{xy}(z) = X(z)Y^*(1/z^*)$. Of course in this case this inner product will be the one of $\ell^2(\mathbb{N}_0)$, computed in the z transform domain.

5.2 System identification models based on rational orthonormal functions

The rational orthonormal functions (on the unit circle) developed in the previous two chapters can be used to approximate arbitrarily well any function of $\mathcal{H}^2(\mathbb{E})$.[5] In particular, they can

[4] Note that $\mathrm{Var}[y(k)] = \sum_{i,j=0}^{\infty} h(i)\Phi_{xx}(j-i)h^*(j) < \|h(k)\|_1^2 \, \|\Phi_{xx}(\tau)\|_1$.

[5] Here, "arbitrarily well" means convergence in the mean, i.e., in the norm of $\mathcal{H}^2(\mathbb{E})$.

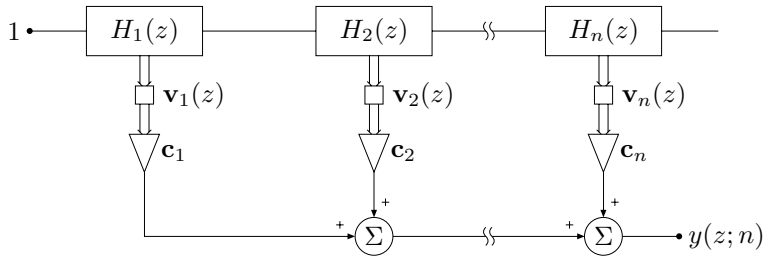

Figure 5.1: Model of the approximation of the transfer function of a stable system by a truncated series of rational orthonormal functions. Note that the input of this block diagram is the z transform of the unit pulse signal, and that the coefficients of the (truncated) series expansion are independent of the number of terms used in the expansion. The transfer functions $H_i(z)$ are scalar and all-pass, and the state variables of the cascade of one of their balanced realizations are denoted by $\mathbf{v}_i(z)$.

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be used to approximate the transfer function of any stable and causal (discrete-time) system, since the transfer functions of all such systems belong to this Hardy space.[6]

Let $H(z)$ be the transfer function we wish to approximate and let the approximation have the form

$$H(z;n) = \sum_{i=0}^{n} c_i \, G_i(z) \tag{5.2}$$

where we have used the rational orthonormal functions $G_i(z)$ of section 3.1 to build the approximation. The coefficients c_i in this case are independent of n and are given by $c_i = \langle H(z), G_i(z) \rangle$. In practice, it is usually easier to approximate $H(z)$ by the rational orthonormal functions built from a cascade of balanced realizations of all-pass transfer functions, as described in chapter 4, specially if all signals are real. Denoting by $\mathbf{v}_i(z)$ the state-variables of the i-th section of the cascade, and assuming we are using n sections, the approximation to $H(z)$ can then be written as[7]

$$H(z;n) = \sum_{i=1}^{n} \mathbf{c}_i^H \mathbf{v}_i(z) \tag{5.3}$$

with

$$\mathbf{c}_i = \langle \mathbf{v}_i(z), H(z) \rangle = \frac{1}{2\pi \mathrm{i}} \oint_{\mathbb{T}} \mathbf{v}_i(z) H^*(1/z^*) \, \frac{dz}{z}.$$

This approximation is depicted schematically in figure 5.1. It is to be noted that (5.2) and (5.3) are two equivalent ways (if n is interpreted in the appropriate way) of describing the same approximation. Note also that in (5.3) the vectors $\mathbf{c}_i$ may have different numbers of elements.

In practical situations it is more often than not difficult to obtain the unit pulse response of an unknown system directly. In such cases it is necessary to infer this unit pulse response from the available input and output signals of the system.[8] This constitutes the so-called

[6] For continuous-time systems it is necessary to impose the additional condition that the impulse response of the stable system has to be square integrable (finite energy).

[7] Note the different meanings of n in this formula and in (5.2), except when all all-pass sections are of first order.

[8] To simplify the discussion we will only address the case of single-input single-output systems.

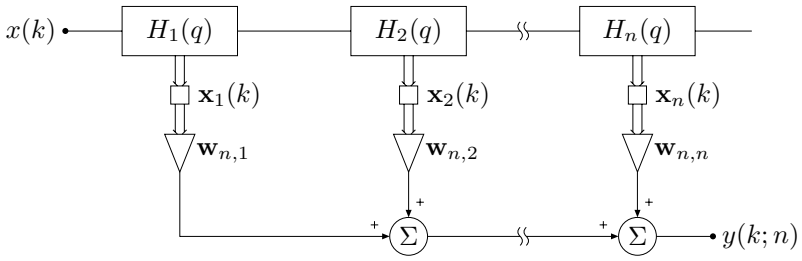

Figure 5.2: Approximation of the output of a stable system by a model based on rational orthonormal functions. The objective of the approximation is the minimization of the mean squared error between the output of this model and the output of the system being identified, when both are excited by the same (arbitrary) input signal. Note that the weights $\mathbf{w}_{n,i}$ of the model depend on the number of all-pass sections used.

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system identification problem. In general, both the input and output signals are corrupted by noise, which is usually assumed to be additive and independent of these signals. In a linear least-squares setup it is current practice to disregard the noise in the input signal. In this case the mismatch observed between the system's output signal (the so-called desired output signal) and the model's output signal, i.e., the error signal, is assumed to be the combined effects of some possible modeling errors and some possible residual uncorrelated noise in the output signal.[9] If there is no noise at all then any mismatch will be due only to modeling errors.

Assume that both the input and output signals are not corrupted by noise. Let $x(k)$ be the input signal of the system to be identified and also of our model. Assume further that this signal is a wide-sense stationary stochastic process with a known auto-correlation function.[10,11] The output of the system will in this case be given by

$$y(k) = \sum_{i=0}^{\infty} h(i)\, x(k-i) = \left[\sum_{i=0}^{\infty} h(i)\, q^{-i}\right] x(k) = H(q)x(k)$$

where q is the advance operator and q^{-1} is its inverse (the delay operator). Our model for the system will be exactly the same as the one in figure 5.1 except that now the input signal will be $x(k)$ instead of a unit pulse. This is depicted in figure 5.2. Note that now the signals $\mathbf{x}_i(k) = \mathbf{v}_i(q)x(k)$ may not be orthonormal, and so the coefficients of the linear combination of these signals may depend on n. This is stressed by changing the name of the coefficients from $\mathbf{c}_i$ to $\mathbf{w}_{n,i}$. Following the terminology of signal processing we will call them weights. The output of the model will then be given by

$$y(k;n) = \sum_{i=1}^{n} \mathbf{w}_{n,i}^{H}\, \mathbf{x}_i(k).$$

[9] It is to be noted that the Total Least Squares technique [Golub and van Loan, 1989; van Huffel and Zha, 1991; Cadzow, 1994] assumes that some noise is present in both signals: input and output. This more natural assumption leads to potentially superior estimation algorithms. This interesting technique is starting to be used in digital signal processing and automatic control problems.

[10] If the process is ergodic this auto-correlation function can be computed as the limit of appropriate time averages of one of its realizations.

[11] The results to be presented shortly can be adapted trivially to a deterministic input signal if the remark at the end of the previous section is used.

A system identification model of this form (but with the classical Kautz functions) was proposed as far back as 1960 by Kitamori [1960]. Recently, it has also been suggested by Williamson [1993] and by Fahmy, Haweel, Elraheem and Gharieb [1993], using this time the discrete Kautz functions of Broome [1965]. The GOBFs model of Heuberger et al. [1995] is another recent (restricted) example of this kind of models.

We will only consider the case in which all all-pass sections of the model are fixed, i.e., they are known a priori and are not allowed to change during the identification process. In this way the identification process is linear and standard least mean squares techniques can be used. If, on the other hand, the poles of the all-pass sections were to be optimized as well our model would be just one particular realization of an ARMA model, and the identification process, which in this case uses an output error configuration, would be nonlinear and vulnerable to local minima.

The analysis to come of this system identification model uses the exact auto- and cross-correlation functions of the input and output signals directly. Therefore, it assumes infinite data records (in the time domain). In this case, it does not make much sense to add an uncorrelated noise (with respect to the input signal) to the output signal of the system (or of the model), since that will not change the analysis of the problem. For finite data records this is not so, and in fact this spurious noise will affect the asymptotic variance of the estimates of the weights and consequently of the transfer function. It will, however, not affect the asymptotic bias of the weights (there is none) as long as this noise is uncorrelated with the input signal [Ljung, 1987]. The asymptotic analysis of this system identification model in its most general form under finite data records appears to be difficult and is an open problem. This analysis has only been performed for the special case in which all the sections of the model are equal. In particular, it has been solved for the FIR case [Ljung and Yuan, 1985], for the Laguerre case [Wahlberg, 1991], for the "two-parameter" Kautz case [Wahlberg, 1994], and, in general, for the GOBF case [Van den Hof et al., 1995].

For a given number of sections of our model, the weights that minimize the variance of the error signal can be computed easily via a linear system of equations, the so-called normal equations, that expresses the orthogonality of the (best) error signal with respect to the signals composing the approximation (cf. the projection theorem in section 2.3). Let

$$e(k;n) = y(k) - y(k;n) = y(k) - \sum_{i=1}^{n} \mathbf{w}_{n,i}^H \, \mathbf{x}_i(k)$$

be the error signal of the model and $\xi_n = \langle e(k;n), e(k;n) \rangle$ its variance. The orthogonality principle can then be expressed as $\langle e(k;n), y(k;n) \rangle = 0$ or, e.g., as

$$\langle \mathbf{x}_i(k), e(k;n) \rangle = \mathbf{0}, \qquad i = 1, \ldots, n. \tag{5.4}$$

This group of n conditions defines a system of linear equations of the form

$$\mathbf{R}_n \, \mathbf{w}_n = \mathbf{p}_n \tag{5.5}$$

where $\mathbf{R}_n$ is an $n \times n$ block matrix with (possibly rectangular) blocks given by

$$\mathbf{r}_{n,ij} = \langle \mathbf{x}_i(k), \mathbf{x}_j(k) \rangle = \mathrm{E}[\mathbf{x}_i(k) \mathbf{x}_j^H(k)] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbf{v}_i(e^{\mathrm{i}\omega}) \mathbf{v}_j^H(e^{\mathrm{i}\omega}) \Phi_{xx}(e^{\mathrm{i}\omega}) \, d\omega, \qquad 1 \le i, j \le n,$$

where $\mathbf{w}_n = [\, \mathbf{w}_{n,1}^H \quad \cdots \quad \mathbf{w}_{n,n}^H \,]^H$, and where $\mathbf{p}_n$ is a partitioned vector with blocks

$$\mathbf{p}_{n,i} = \langle \mathbf{x}_i(k), y(k) \rangle = \mathrm{E}[\mathbf{x}_i(k) y^*(k)] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbf{v}_i(e^{\mathrm{i}\omega}) \Phi_{xy}(e^{\mathrm{i}\omega}) \, d\omega, \qquad 1 \le i \le n.$$

The weights that minimize the variance of $e(k;n)$ are given by [Haykin, 1991]

$$\mathbf{w}_n = \mathbf{R}_n^+ \mathbf{p}_n \tag{5.6}$$

where $\mathbf{R}_n^+$ is the pseudo-inverse of $\mathbf{R}_n$.[12] For these optimal weights the variance of the error signal is given by

$$\begin{aligned}
\xi_n &= \langle y(k) - y(k;n), e(k;n)\rangle = \langle y(k), e(k;n)\rangle \\
&= \langle y(k), y(k)\rangle - \sum_{i=1}^{n} \langle y(k), \mathbf{x}_i\rangle \, \mathbf{w}_{n,i} \\
&= \langle y(k), y(k)\rangle - \sum_{i=1}^{n} \mathbf{p}_{n,i}^H \, \mathbf{w}_{n,i}.
\end{aligned}$$

The condition number[13] of $\mathbf{R}_n$, usually denoted by $\kappa(\mathbf{R}_n)$, plays an important role in the numerical resolution of (5.5), specially if the standard inverse of $\mathbf{R}_n$ is used. A theoretical upper bound for this condition number has been found in Grenander and Szegö [1984] (see also [Gray, 1972]) for the FIR model, in the course of investigations related to Toeplitz matrices.[14] It turns out that this upper bound remains valid for our $\mathbf{R}_n$.

Let $\lambda_{\min}(\mathbf{R}_n)$ and $\lambda_{\max}(\mathbf{R}_n)$ denote respectively the smallest and largest eigenvalues of $\mathbf{R}_n$. Due to the interlacing property of the eigenvalues of nested Hermitian matrices [Golub and van Loan, 1989] it is a simple matter to verify that, for an arbitrary n,

$$\lambda_{\min}(\mathbf{R}_1) \geq \lambda_{\min}(\mathbf{R}_2) \geq \cdots \geq \lambda_{\min}(\mathbf{R}_n),$$

and that

$$\lambda_{\max}(\mathbf{R}_1) \leq \lambda_{\max}(\mathbf{R}_2) \leq \cdots \leq \lambda_{\max}(\mathbf{R}_n).$$

As a consequence, it is always true that

$$\kappa(\mathbf{R}_1) \leq \kappa(\mathbf{R}_2) \leq \cdots \leq \kappa(\mathbf{R}_n)$$

[recall that $\kappa(\mathbf{R}_n) = \sqrt{\lambda_{\max}(\mathbf{R}_n)/\lambda_{\min}(\mathbf{R}_n)}$]. The condition numbers of the matrices $\mathbf{R}_n$, $n = 1, 2, \ldots$, then form a non-decreasing sequence. The question here is: is this sequence bounded? If it is, then the underlying least-mean squares problem cannot become arbitrarily ill-conditioned, no matter how large n is.

Fortunately, following the ideas of Grenander and Szegö [1984], it is easy to obtain bounds for the eigenvalues of $\mathbf{R}_n$. In order to avoid some technical problems we will assume that the auto-correlation function of $x(k)$ is absolutely summable.[15] This implies that its Fourier transform (i.e., its power spectral density) is bounded and continuous on the unit circle.

[12] This pseudo-inverse is equal to the normal inverse when $\mathbf{R}_n$ is non-singular. Either way, let $\mathbf{R}_n = \mathbf{U}_n \mathbf{\Sigma}_n \mathbf{V}_n^H$ be the singular decomposition of $\mathbf{R}_n$ [Golub and van Loan, 1989; Haykin, 1991]. Then $\mathbf{R}_n^+ = \mathbf{V}_n \mathbf{\Sigma}_n^{-1} \mathbf{U}_n^H$. Only the non-zero elements of the diagonal matrix $\mathbf{\Sigma}_n$ are actually inverted in $\mathbf{\Sigma}_n^{-1}$. In our case the pseudo-inverse can also be computed from the eigenvalue decomposition of $\mathbf{R}_n$ since this matrix is Hermitian and non-negative definite.

[13] The condition number of a square and non-negative definite Hermitian matrix is the square root of the ratio between its largest and its smallest eigenvalues.

[14] For the FIR model $\mathbf{R}_n$ is a Toeplitz matrix. Curiously, for the GOBF model $\mathbf{R}_n$ is a block Toeplitz matrix.

[15] To satisfy the reader's curiosity we mention that the results we are about to obtain can be extended to any regular (also called purely indeterministic) stochastic process $x(k)$. Basically, the power spectral density of these processes is absolutely continuous (it does not contain, e.g., Dirac delta impulses, sometimes also called

It is well known that the eigenvalues of a Hermitian matrix are tightly bounded by the extremal values of the so-called Rayleigh quotient

$$\frac{\mathbf{w}_n^H \mathbf{R}_n \mathbf{w}_n}{\mathbf{w}_n^H \mathbf{w}_n}, \qquad \mathbf{w}_n \neq \mathbf{0}.$$

(Here, $\mathbf{w}_n$ is an arbitrary non-zero vector with an appropriate number of elements.) In particular, it is possible to prove that [Golub and van Loan, 1989]

$$\lambda_{\min}(\mathbf{R}_n) = \min_{\mathbf{w}_n \neq \mathbf{0}} \frac{\mathbf{w}_n^H \mathbf{R}_n \mathbf{w}_n}{\mathbf{w}_n^H \mathbf{w}_n}, \qquad \text{and} \qquad \lambda_{\max}(\mathbf{R}_n) = \max_{\mathbf{w}_n \neq \mathbf{0}} \frac{\mathbf{w}_n^H \mathbf{R}_n \mathbf{w}_n}{\mathbf{w}_n^H \mathbf{w}_n}. \tag{5.7}$$

Now,

$$\begin{aligned}
\mathbf{w}_n^H \mathbf{R}_n \mathbf{w}_n &= \sum_{i,j=1}^{n} \mathbf{w}_{n,i}^H \mathbf{r}_{n,ij} \mathbf{w}_{n,j} \\
&= \sum_{i,j=1}^{n} \mathbf{w}_{n,i}^H \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbf{v}_i(e^{\mathrm{i}\omega}) \mathbf{v}_j^H(e^{\mathrm{i}\omega}) \Phi_{xx}(e^{\mathrm{i}\omega})\, d\omega \right] \mathbf{w}_{n,j} \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\sum_{i=1}^{n} \mathbf{w}_{n,i}^H \mathbf{v}_i(e^{\mathrm{i}\omega}) \right] \left[\sum_{j=1}^{n} \mathbf{w}_{n,j}^H \mathbf{v}_j(e^{\mathrm{i}\omega}) \right]^H \Phi_{xx}(e^{\mathrm{i}\omega})\, d\omega.
\end{aligned}$$

Since by assumption $\Phi_{xx}(e^{\mathrm{i}\omega})$ is a continuous function it is easy to obtain simple lower and upper bounds for this last integral: one just replaces $\Phi_{xx}(e^{\mathrm{i}\omega})$ (which is a non-negative function) by its minimum and maximum values in the interval $[-\pi, \pi]$, respectively. This produces the bounds

$$\mathbf{w}_n^H \mathbf{R}_n \mathbf{w}_n \geq m \sum_{i,j=1}^{n} \mathbf{w}_{n,i}^H \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbf{v}_i(e^{\mathrm{i}\omega}) \mathbf{v}_j^H(e^{\mathrm{i}\omega})\, d\omega \right] \mathbf{w}_{n,j}$$

and

$$\mathbf{w}_n^H \mathbf{R}_n \mathbf{w}_n \leq M \sum_{i,j=1}^{n} \mathbf{w}_{n,i}^H \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbf{v}_i(e^{\mathrm{i}\omega}) \mathbf{v}_j^H(e^{\mathrm{i}\omega})\, d\omega \right] \mathbf{w}_{n,j}$$

where

$$m = \min_{\omega \in [-\pi, \pi]} \Phi_{xx}(e^{\mathrm{i}\omega}), \qquad \text{and} \qquad M = \max_{\omega \in [-\pi, \pi]} \Phi_{xx}(e^{\mathrm{i}\omega}).$$

In these expressions the term inside square brackets is easily recognized to be the $\mathcal{H}^2(\mathbb{E})$ inner product between $\mathbf{v}_i(z)$ and $\mathbf{v}_j(z)$ (part of the controllability Gramian!). Since these functions are orthonormal it is easy to verify that

$$\sum_{i,j=1}^{n} \mathbf{w}_{n,i}^H \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbf{v}_i(e^{\mathrm{i}\omega}) \mathbf{v}_j^H(e^{\mathrm{i}\omega})\, d\omega \right] \mathbf{w}_{n,j} = \sum_{i=1}^{n} \mathbf{w}_{n,i}^H \mathbf{w}_{n,i} = \mathbf{w}_n^H \mathbf{w}_n.$$

---

point masses) and satisfies the famous Szegö condition

$$\int_{-\pi}^{\pi} \log[\Phi_{xx}(e^{\mathrm{i}\omega})]\, d\omega > -\infty.$$

Alternatively, we may say that both $\Phi_{xx}(z)$ and $\log[\Phi_{xx}(z)]$ belong to $L^1(\mathbb{T})$. To adapt our results to the more general case it is necessary to replace the minimum and maximum of the power spectral density by the essential infimum and essential supremum, respectively.

Plugging this result in the above two bounds yields

$$m\,\mathbf{w}_n^H\,\mathbf{w}_n \leq \mathbf{w}_n^H\,\mathbf{R}_n\,\mathbf{w}_n \leq M\,\mathbf{w}_n^H\,\mathbf{w}_n.$$

Applying these bounds to (5.7) finally yields

$$m \leq \lambda_{\min}(\mathbf{R}_n) \leq \lambda_{\max}(\mathbf{R}_n) \leq M. \tag{5.8}$$

If $m > 0$ we have

$$\kappa(\mathbf{R}_n) \leq \sqrt{\frac{M}{m}} = \sqrt{\frac{\max_{\omega\in[-\pi,\pi]}\Phi_{xx}(e^{\mathrm{i}\omega})}{\min_{\omega\in[-\pi,\pi]}\Phi_{xx}(e^{\mathrm{i}\omega})}}. \tag{5.9}$$

Note that this result is also valid for the FIR models (for that particular case Grenander and Szegö [1984] obtained this same result in a slightly more general form[16]). For small values of n these bounds may be somewhat conservative.

The interesting thing about these bounds is that they are independent of n. They also shed some light on which characteristics of the input signal are important for system identification with this kind of models. In particular, the power spectral density of the input signal should be bounded away from zero ($m > 0$), a condition known as unconditional persistence of excitation. Also, the dynamic range of this power spectral density should be as small as possible. The independence on n of these bounds has a useful consequence: we can increase the order of the model without making the normal equations "too much" ill-conditioned.

Lets now look at the variance of the error signal. To start with, it is easy to verify that this variance is given by

$$\xi_n(\mathbf{w}_n) = \mathrm{Var}[e(k;n)] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|H(e^{\mathrm{i}\omega}) - H(e^{\mathrm{i}\omega};n)\right|^2\Phi_{xx}(e^{\mathrm{i}\omega})\,d\omega$$

for any value of $\mathbf{w}_n$ (not necessarily optimal). On the other hand, it is also easy to verify that the $\mathcal{H}^2(\mathbb{E})$ norm of the error in the transfer function itself is given by

$$\zeta_n(\mathbf{w}_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|H(e^{\mathrm{i}\omega}) - H(e^{\mathrm{i}\omega};n)\right|^2\,d\omega.$$

Note that the only difference between these two definitions is the presence in the first one of the factor $\Phi_{xx}(e^{\mathrm{i}\omega})$ inside the integral, which is missing in the second one. Assuming again that the auto-correlation function of $x(k)$ is an absolutely summable sequence allows us to write the following trivial inequalities

$$m\,\zeta_n(\mathbf{w}_n) \leq \xi_n(\mathbf{w}_n) \leq M\,\zeta_n(\mathbf{w}_n) \tag{5.10}$$

which are valid for all $\mathbf{w}_n$. These inequalities give us immediately (possibly conservative) error bounds for the $\mathcal{H}^2(\mathbb{E})$ error of the transfer function of the identified system, viz.

$$\frac{\xi_n(\mathbf{w}_n)}{M} \leq \zeta_n(\mathbf{w}_n) \leq \frac{\xi_n(\mathbf{w}_n)}{m}. \tag{5.11}$$

[16]The "slightly more general" here is related to the less restrictive assumption on the spectral density of the input signal made in [Grenander and Szegö, 1984]. See previous footnote.

Of course, we have assumed that $m > 0$ (and also that there is no uncorrelated noise in one of the output signals...). Otherwise the normal equations can become arbitrarily ill-conditioned. Note that these bounds become tighter and tighter the more the ratio M/m approaches 1.[17]

For the value of $\mathbf{w}_n$ that minimizes $\xi_n(\mathbf{w}_n)$ the value of $\zeta_n(\mathbf{w}_n)$ is not, in general, the smallest one possible. This observation and (5.10) leads to the inequality

$$m \min_{\mathbf{w}_n} \zeta_n(\mathbf{w}_n) \leq \min_{\mathbf{w}_n} \xi_n(\mathbf{w}_n).$$

On the other hand, for the value of $\mathbf{w}_n$ that minimizes $\zeta_n(\mathbf{w}_n)$ the value of $\xi_n(\mathbf{w}_n)$ is not, in general, the smallest one possible. This observation and (5.10) leads in turn to the inequality

$$\min_{\mathbf{w}_n} \xi_n(\mathbf{w}_n) \leq M \min_{\mathbf{w}_n} \zeta_n(\mathbf{w}_n).$$

Combining these two inequalities together yields

$$m \min_{\mathbf{w}_n} \zeta_n(\mathbf{w}_n) \leq \min_{\mathbf{w}_n} \xi_n(\mathbf{w}_n) \leq M \min_{\mathbf{w}_n} \zeta_n(\mathbf{w}_n), \tag{5.12}$$

that relates the best approximation errors for the two cases. The beauty of this result is that if the model is based on a *complete* set of rational orthonormal functions, then $\min_{\mathbf{w}_n} \zeta_n(\mathbf{w}_n)$ will decay to zero as n goes to infinity and this forces $\min_{\mathbf{w}_n} \xi_n(\mathbf{w}_n)$ also to decay to zero. Furthermore, if $m > 0$ then the two rates of decay will be the same! Note also that in this case the $\mathcal{H}^2(\mathbb{E})$ error of the transfer function estimate obtained via the minimization of $\xi_n(\mathbf{w}_n)$ will also decay to zero [cf. (5.11)], despite the fact that the value of $\mathbf{w}_n$ obtained in this way will not, in general, minimize $\zeta_n(\mathbf{w}_n)$. It goes without saying (but we say it anyway) that these results are valid for auto-correlation functions that are absolutely summable[18] and only if $m > 0$.

Another interesting feature of the proposed model is the existence of an LMS-like algorithm to adapt on-line its weights.[19] In order to develop this adaptation algorithm we need to compute the partial derivatives of $\xi_n(\mathbf{w}_n)$ with respect to the real and imaginary parts of the elements of $\mathbf{w}_n$.

Let $w_{ij} = \alpha_{ij} + \mathrm{i}\beta_{ij}$ be the j-th element of $\mathbf{w}_{n,i}$ and let $x_{ij}(k)$ be the corresponding j-th signal of the i-th section of our model. It is then easy to verify that

$$\frac{\partial e(k;n)}{\partial \alpha_{ij}} = \frac{\partial}{\partial \alpha_{ij}} \left(y(k) - \sum_{l=1}^{n} \mathbf{w}_{n,l}^H \mathbf{x}_l(k) \right) = -x_{ij}(k)$$

and that

$$\frac{\partial e(k;n)}{\partial \beta_{ij}} = \frac{\partial}{\partial \beta_{ij}} \left(y(k) - \sum_{l=1}^{n} \mathbf{w}_{n,l}^H \mathbf{x}_l(k) \right) = \mathrm{i}\, x_{ij}(k).$$

[17] Note that $M = m$ implies that $x(k)$ is some sort of white noise.

[18] As mentioned before this restriction can be somewhat relaxed.

[19] The problem of the adaptation of the poles of the all-pass sections is much more difficult, because the error surface is in general multi-modal with respect to them. We will not address this interesting (and open) problem here. However, we will describe a simple off-line method (it can, perhaps, be made on-line if block processing is used) based on the balanced model order reduction of a model to which some more sections were added for just this specific purpose.

Since in general for a *real* variable x one has

$$\frac{\partial \langle f, g \rangle}{\partial x} = \left\langle \frac{\partial f}{\partial x}, g \right\rangle + \left\langle f, \frac{\partial g}{\partial x} \right\rangle$$

and because $\xi_n(\mathbf{w}_n) = \langle e(k;n), e(k;n) \rangle$ it is not difficult to show that

$$\frac{\partial \xi_n(\mathbf{w}_n)}{\partial \alpha_{ij}} = -\langle x_{ij}(k), e(k;n) \rangle - \langle e(k;n), x_{ij}(k) \rangle$$

and that

$$\frac{\partial \xi_n(\mathbf{w}_n)}{\partial \beta_{ij}} = \mathsf{i}\, \langle x_{ij}(k), e(k;n) \rangle - \mathsf{i}\, \langle e(k;n), x_{ij}(k) \rangle .$$

It is easy of prove that equating these two partial derivatives to zero for all i and j yields precisely (5.4). It is also interesting to note that

$$\frac{\partial \xi_n(\mathbf{w}_n)}{\partial \alpha_{ij}} + \mathsf{i} \frac{\partial \xi_n(\mathbf{w}_n)}{\partial \beta_{ij}} = -2 \langle x_{ij}(k), e(k;n) \rangle . \tag{5.13}$$

This same formula could also have been obtained using the trick described in [Brandwood, 1983; Calvez et al., 1985; van den Bos, 1994]. We could also have used an equivalent trick described in [Haykin, 1991] (see also the interesting discussion in [Huang and Chen, 1989; Morgan, 1991; Picinbono, 1991]).

Instead of computing the optimal weight vector $\mathbf{w}_n$ in one step using (5.6) suppose we use a steepest descent algorithm to improve an estimate of that optimal weight vector. Due to (5.13) this steepest descent algorithm updates the vector $\mathbf{w}_n$ according to the rule

$$\mathbf{w}_n^{[\eta+1]} = \mathbf{w}_n^{[\eta]} + 2\mu^{[\eta+1]} \langle \mathbf{x}(k;n), e(k;n) \rangle \tag{5.14}$$

where $\mu^{[\eta+1]} > 0$, $\mathbf{x}(k;n) = [\, \mathbf{x}_1^H \quad \cdots \quad \mathbf{x}_n^H \,]^H$, and where η is the iteration number. A careful analysis of this update formula reveals that it can also be written in the form[20]

$$\mathbf{w}_n^{[\eta+1]} = \mathbf{w}_n^{[\eta]} - 2\mu^{[\eta+1]}(\mathbf{R}_n \mathbf{w}_n^{[\eta]} - \mathbf{p}_n). \tag{5.15}$$

Since both $\mathbf{R}_n$ and $\mathbf{p}_n$ are fixed, this way of solving (5.5) is only efficient if $\kappa(\mathbf{R}_n)$ is close to 1 (and if $\mu^{[\eta]}$ is selected properly!). Otherwise (5.6) is more efficient.

Let $\mathbf{w}_n^o = \mathbf{R}_n^{-1}\mathbf{p}_n$ be the optimal weight vector of our model, and let $\mathbf{w}_n^{[\eta]} = \mathbf{w}_n^o + \Delta\mathbf{w}_n^{[\eta]}$. Then, it is easy to verify that (5.15) can be put in the form

$$\Delta\mathbf{w}_n^{[\eta+1]} = [\mathbf{I} - 2\mu^{[\eta+1]}\mathbf{R}_n]\Delta\mathbf{w}_n^{[\eta]}.$$

Let $\mathbf{U}_n\mathbf{\Gamma}_n\mathbf{U}_n^H$ be the eigenvalue decomposition of $\mathbf{R}_n$. It is then true that $[\mathbf{I} - 2\mu^{[\eta+1]}\mathbf{R}_n]$ is equal to $\mathbf{U}_n[\mathbf{I} - 2\mu^{[\eta+1]}\mathbf{\Gamma}_n]\mathbf{U}_n^H$. The vector $\mathbf{w}_n^{[\eta]}$ will then converge to $\mathbf{w}_n^o$ starting from any initial vector $\mathbf{w}_n^{[0]}$ if and only if

$$\prod_{\eta=1}^{\infty} (1 - 2\mu^{[\eta]}\lambda_i) \tag{5.16}$$

[20] It is not difficult to verify that $\langle x_{ij}(k), e(k;n) \rangle$ is equal to the j-th element of the i-th block of the vector $\mathbf{p}_n - \mathbf{R}_n\mathbf{w}_n$.

converges to zero for each eigenvalue λ_i of $\mathbf{R}_n$. In fact, it is enough to analyze the convergence of this infinite product for the extremal eigenvalues of $\mathbf{R}_n$, here denoted by $\lambda_{\min}$ and $\lambda_{\max}$. Note that from our previous analysis we know that

$$m \leq \lambda_{\min} \leq \lambda_{\max} \leq M.$$

As usual, we assume that $m > 0$.

To simplify the study of the convergence of (5.16) assume that $\mu^{[\eta]} = \mu$ for all η, i.e., that the adaptation step size is constant. Then, the steepest descent algorithm will converge unconditionally if

$$|1 - 2\mu\lambda_{\max}| < 1,$$

that is, if

$$\mu < \frac{1}{\lambda_{\max}} \leq \frac{1}{M}.$$

(Note that $\mu = 1/M - \epsilon$, where ϵ is a small positive number, is a safe value for the adaptation step size. This step size will probably be conservative if n is small and/or if $M/m \gg 1$.) However, the fastest adaptation possible[21] will occur for a μ that satisfies

$$1 - 2\mu\lambda_{\min} = -[1 - 2\mu\lambda_{\max}],$$

that is, for

$$\mu = \frac{1}{\lambda_{\max} + \lambda_{\min}}.$$

(Note that $\mu = 1/(M+m)$ will be an estimate of this optimal step size, that as the additional advantage of assuring convergence.) For this optimal step size the convergence rate for the weights of the model is given by

$$\left|1 - 2\frac{1}{\lambda_{\max} + \lambda_{\min}}\,\lambda_{\min}\right| = \left|1 - 2\frac{1}{\lambda_{\max} + \lambda_{\min}}\,\lambda_{\max}\right| = \frac{\lambda_{\max} - \lambda_{\min}}{\lambda_{\max} + \lambda_{\min}}.$$

Note that if $\lambda_{\max}/\lambda_{\min} \gg 1$ this best convergence rate will be close (but smaller) to 1, and so the steepest descent algorithm will perform poorly (slowly). The ideal situation is then to have $\lambda_{\max}/\lambda_{\min}$ as close to 1 as possible, as is well know.[22] This is ensured *for all* n if M/m is as close to 1 as possible.

The steepest descent algorithm just studied requires the knowledge of (the theoretical value of) $\mathbf{R}_n$ and of $\mathbf{p}_n$. In an adaptive filter the goal is to update its internal parameters (i.e., weights) whenever new samples[23] of the input and desired output signals become available, in an attempt to minimize the mismatch between the desired signal and the output of the filter up to that instant in time. (Of course, in adaptive system identification the desired signal will be the output of the system to be identified and instead of a filter we may speak of an adaptive model.) This implies that in an adaptive situation we will have, at most, estimates of $\mathbf{R}_n$ and of $\mathbf{p}_n$, which are based only on the past (and present) of all the signals involved in the adaptation process. If these estimates have a fading memory, i.e., if the value of the

[21] Note that this adaptation speed is for the weights of the model and not for its squared error.

[22] Of course, the best possible situation is to have $\lambda_{\max}/\lambda_{\min} = 1$, which corresponds to a white noise input signal.

[23] Or blocks of samples in block-wise adaptation.

signals in the remote past do not affect significantly the current estimates of these quantities, this adaptation process can cope with sufficiently slow changes in the system being identified.

The (radical) approach taken in the LMS (Least Mean Square) algorithm is to do without an estimate of $\mathbf{R}_n$ and to estimate $\mathbf{p}_n$ using only the current time instant. In order for this approach to work, the objective to the adaptation is to adjust the weights of the model in an attempt to reduce the a posteriori value of $e(k;n)$. This is done by replacing, in the update formula (5.14) of the steepest descent algorithm described earlier, $\langle \mathbf{x}(k;n), e(k;n) \rangle$ by just $\mathbf{x}(k;n)e(k;n)$, in fact removing the expectation operation implicit in the inner product. As a result, the adaptation rule becomes, for a constant adaptation step size,

$$\mathbf{w}_n(k+1) = \mathbf{w}_n(k) + 2\mu\, \mathbf{x}(k;n)e(k;n). \tag{5.17}$$

Note that we have replaced the iteration number by the time variable. As mentioned before, this is a somewhat radical approach, and can be justified by saying that in using $\mathbf{x}(k;n)e(k;n)$ instead of $\langle \mathbf{x}(k;n), e(k;n) \rangle$ we are actually using a local (in time) and noisy estimate of the gradient of $\xi_n(\mathbf{w}_n)$, which is "low-pass filtered" by the repetitive use of (5.17). In a stationary environment, the expectation (across realizations) of this noisy estimate of the gradient coincides with the one of $\xi_n(\mathbf{w}_n)$. This explains why the LMS algorithm is classified as a stochastic gradient algorithm.

The standard stability analysis of the LMS algorithm [Solo, 1989; Haykin, 1991] remains valid for our more general model. This is so because the so-called fundamental assumption [Haykin, 1991] does not use the specific relation (delays) between the states of the transversal filter.[24] Although this analysis it not applicable to the transversal filter, and to our model, it has been verified in practice that it yields a reasonable approximation to the behaviour of the LMS algorithm. From this (approximate) analysis it follows that in order for the LMS algorithm be convergent in the mean and in the mean-square, the adaptation step size should satisfy the conditions [Haykin, 1991]

$$\mu < \frac{1}{\lambda_{\max}}$$

and

$$\frac{1}{2}\sum_{i=1}^{n} \frac{\mu\lambda_i}{1-\mu\lambda_i} < 1,$$

where n is the total number of states of our model and λ_i are the eigenvalues of $\mathbf{R}_n$. We stress that these conditions are only approximations to the truth.

To terminate this extraordinary long section we are going to describe a very simple and elegant method to select a "good" position for the poles of our model. This method is a "trivial" application of the so-called balanced model order reduction, developed by Moore [1981] (see also [Pernebo and Silverman, 1982; Fernando and Nicholson, 1983; Al-Saggaf and Franklin, 1988]).[25] To make the presentation self contained, before presenting our pole

[24] In fact, among other things, it makes the unrealistic assumption that the state vectors $\mathbf{x}(k;n)$ are statistically independent [Haykin, 1991]. This is clearly not true in the transversal filter (and in our model), since there is a well defined relation between the states of the filter. This problem has been circumvented recently for the transversal filter [Butterweck, 1995].

[25] Clearly, we may also use other model order reduction techniques, such as, e.g., a best Hankel norm approximation. Since this is slightly more computationally involved, we will describe only the balanced model order reduction technique.

selection method we will summarize the balanced model order reduction technique of [Moore, 1981].

Let $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ be an internally (minimal) balanced realization of the stable transfer function $H(z)$ (cf. section 4.2). Let m be the number of inputs of this system, n its McMillan degree and l its number of outputs. (The dimensions of the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are then, respectively, $n \times n$, $n \times m$, $l \times n$, and $l \times m$. Recall that in an internally balanced realization both the controllability and the observability Gramians are equal to the diagonal matrix

$$\mathbf{\Sigma} = \mathrm{diag}(\sigma_1, \ldots, \sigma_n).$$

It is standard practice to order the Hankel singular values in decreasing order:

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0.$$

The last inequality is a consequence of our assumption that the realization is minimal. Now, suppose that for some $r < n$ it is true that $\sigma_r \gg \sigma_{r+1}$. In terms of the controllability Gramian (of the internally balanced realization) this has the consequence that the input(s) are loosely coupled to the states $x_{r+1}(k), \ldots, x_n(k)$ (when compared to the other states). In terms of the observability Gramian a similar thing happens: when compared to the other states, the states $x_{r+1}(k), \ldots, x_n(k)$ are loosely coupled to the output signal(s). To perform the (balanced) model order reduction, one just drops these "least influential" states from the state-space representation. Mathematically, one replaces the full state-space realization

$$\left(\left[\begin{array}{c:c} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \hdashline \mathbf{A}_{21} & \mathbf{A}_{22} \end{array} \right], \left[\begin{array}{c} \mathbf{B}_1 \\ \hdashline \mathbf{B}_2 \end{array} \right], \left[\begin{array}{c:c} \mathbf{C}_1 & \mathbf{C}_2 \end{array} \right], \mathbf{D} \right),$$

by the reduced realization $(\mathbf{A}_{11}, \mathbf{B}_1, \mathbf{C}_1, \mathbf{D})$. Of course $\mathbf{A}_{11}$ is a $r \times r$ matrix, and the other matrices have compatible dimensions. Note that there is no need for σ_r to be much larger than σ_{r+1}. The reduction process in just less effective when this is not true. It is possible to prove that the reduced model is always stable [Moore, 1981]. If $\sigma_r \gg \sigma_{r+1}$ then the reduced model will be a very good approximation of the original one, and it is expected that its poles will be "close" to the poles of the best $\mathcal{H}^2(\mathbb{E})$ model. This may not be true for the opposite case, i.e., if $\sigma_r \approx \sigma_{r+1}$, specially if we are underestimating the true order of the system being identified (or if this system has infinite dimension).

So, how can we use this balanced model order reduction to select "good" poles for our own model? We just add to it some more all-pass sections (more about this later), use this extended model to identify (or approximate, if you like) the unknown system (using the LMS algorithm, for example), and then, (what else?), perform the balanced model order reduction to the extended model to get a model with just the number of poles that we really want.[26] Let us elaborate on some details of this procedure.

First of all, adding more all-pass sections does not deteriorate "very much" the condition number of the correlation matrix of the states of the model. This follows from (5.8) and from (5.9). Of course, if the LMS algorithm is used in the identification process there will be a larger excess of mean squared error than what would otherwise be the case. That is the price we have to pay for the better approximation capabilities of our model. It also shows that we should not add too many extra sections (poles).

[26] Of course, we can repeat this procedure several times, until we get satisfactory results, or until we give up!

In the second place, the state space representation of our (normal or extended) model is very easy to find. We just replace the output signal of the cascade of the all-pass sections (which we don't need) by a linear combination of the states. For example, for four sections we get the state space representation [cf.(4.26)]

$$\left[\begin{array}{c} \mathbf{x}_1(k) \\ \mathbf{x}_2(k) \\ \mathbf{x}_3(k) \\ \mathbf{x}_4(k) \\ \hdashline y(k;4) \end{array}\right] = \left[\begin{array}{cccc:c} \mathbf{A}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_1 \\ \mathbf{B}_2\mathbf{C}_1 & \mathbf{A}_2 & \mathbf{0} & \mathbf{0} & \mathbf{B}_2\mathbf{D}_1 \\ \mathbf{B}_3\mathbf{D}_2\mathbf{C}_1 & \mathbf{B}_3\mathbf{C}_2 & \mathbf{A}_3 & \mathbf{0} & \mathbf{B}_3\mathbf{D}_2\mathbf{D}_1 \\ \hdashline \mathbf{B}_4\mathbf{D}_3\mathbf{D}_2\mathbf{C}_1 & \mathbf{B}_4\mathbf{D}_3\mathbf{C}_2 & \mathbf{B}_4\mathbf{C}_3 & \mathbf{A}_4 & \mathbf{B}_4\mathbf{D}_3\mathbf{D}_2\mathbf{D}_1 \\ \hdashline \mathbf{c}_{4,1} & \mathbf{c}_{4,2} & \mathbf{c}_{4,3} & \mathbf{c}_{4,4} & d_4 \end{array}\right] \left[\begin{array}{c} \mathbf{x}_1(k-1) \\ \mathbf{x}_2(k-1) \\ \mathbf{x}_3(k-1) \\ \mathbf{x}_4(k-1) \\ \hdashline x(k) \end{array}\right] \tag{5.18}$$

with

$$\left[\begin{array}{cccc:c} \mathbf{c}_{4,1} & \mathbf{c}_{4,2} & \mathbf{c}_{4,3} & \mathbf{c}_{4,4} & d_4 \end{array}\right] = \left[\begin{array}{cccc} \mathbf{w}_{4,1}^H & \mathbf{w}_{4,2}^H & \mathbf{w}_{4,3}^H & \mathbf{w}_{4,4}^H \end{array}\right] \left[\begin{array}{cccc:c} \mathbf{A}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_1 \\ \mathbf{B}_2\mathbf{C}_1 & \mathbf{A}_2 & \mathbf{0} & \mathbf{0} & \mathbf{B}_2\mathbf{D}_1 \\ \mathbf{B}_3\mathbf{D}_2\mathbf{C}_1 & \mathbf{B}_3\mathbf{C}_2 & \mathbf{A}_3 & \mathbf{0} & \mathbf{B}_3\mathbf{D}_2\mathbf{D}_1 \\ \mathbf{B}_4\mathbf{D}_3\mathbf{D}_2\mathbf{C}_1 & \mathbf{B}_4\mathbf{D}_3\mathbf{C}_2 & \mathbf{B}_4\mathbf{C}_3 & \mathbf{A}_4 & \mathbf{B}_4\mathbf{D}_3\mathbf{D}_2\mathbf{D}_1 \end{array}\right].$$

In order to be able to use the formulas of section 4.2 to compute the corresponding balanced realization it is also necessary to compute the observability Gramian of (5.18). Robust algorithms to do this can be found in [Golub and van Loan, 1989] and references therein. (This is a linear problem.) Note that the controllability Gramian is the identity matrix [Bodin and Wahlberg, 1994]. This can be used to simplify the formulas of section 4.2. More precisely, if $\mathbf{Q} = \mathbf{U}\mathbf{\Sigma}^2\mathbf{U}^H$ is the eigenvalue decomposition of $\mathbf{Q}$, $\mathbf{T}_{\text{internal}} = \mathbf{U}\mathbf{\Sigma}^{-1/2}$ is one possible internally balancing transformation for the system.

Finally, it is necessary to decide how many sections to add to the model and to choose their poles. This is not an easy problem. We can, for example, add a few extra sections with poles scattered randomly in the interior of the unit circle. Another alternative, which we find quite appealing, is to add another copy of the whole model. The extended model then becomes a GOBF model with just two big sections.[27] It is worth mentioning that in this second (and last) case the optimal pole positions for the unextended model is reflected in a very easy way in the extended model: for the optimal set of poles all weights of the second GOBF section must vanish simultaneously. For a white noise input, this beautiful result is a consequence of the results of section 3.7 (check also [McDonough and Huggins, 1968; Marzollo, 1969]). For a general colored input this result is more difficult to prove [Oliveira e Silva, 1997], but remains true. This result can be used to fine-tune the pole positions, using for example an LMS-like adaptation algorithm for the poles themselves, but only when we are confident that we are near the global minimum of the error surface! (Hopefully, the model order reduction step will place them nearby...)

That's all, folks!

[27] The balanced model order reduction method was suggested in [Heuberger et al., 1993b] to find a good set of poles for the GOBF model with an arbitrary number of sections.

Appendix A

Conventions and lists of symbols

The following convention for sums will be used whenever necessary

$$\sum_{k=i}^{j}(\cdot) = 0 \qquad \text{if } i > j,$$

as will be the following convention for products

$$\prod_{k=i}^{j}(\cdot) = 1 \qquad \text{if } i > j.$$

We will say that the z transform of a real sequence is a z-real function and that the Laplace transform of a real function is a s-real function. For z transforms this implies that $F(z) = F^*(z^*)$ and for Laplace transforms this implies that $F(s) = F^*(s^*)$. We will use the letter k to represent the discrete-time variable and the letter t to represent the continuous-time variable.

The following is a list of the principal symbols used in this report, together with their respective descriptions and/or definitions.

General symbols

δ_{ij}	The Kronecker symbol, which is equal to one if $i = j$ and equal to zero otherwise.
$\delta(t)$	The Dirac Delta impulse: $f(t_0) = \int_{-\infty}^{\infty} f(t)\delta(t - t_0)\, dt$.
$\dot{x}(t)$	Derivative with respect to t: $\dot{x}(t) = dx(t)/dt$.
C_j^i	The binomial symbol. It is equal to $\frac{i!}{j!(i-j)!}$ if $0 \leq j \leq i$ and to 0 otherwise.
$\mathrm{E}[\cdot]$	Mathematical expectation.
$\mathrm{Var}[\cdot]$	Variance.
x^+	Denotes the direction of limits. For example, $\lim_{t\to 0^+} f(t) = f(0^+)$ denotes the limit (on the right) of $f(t)$ at $t = 0$, i.e., the limit when t goes to zero by positive values only.
i	The square root of -1. Do not confuse with i.
z^*	Complex conjugate of the complex number z.
$\mathrm{Re}[z]$	Real part of the complex number z.
$\mathrm{Im}[z]$	Imaginary part of the complex number z.

$|z|$ ····· Modulus of the complex number z.

$\operatorname{ess\,sup}$ ····· The essential supremum of a Lebesgue measurable function.

$\langle f, g \rangle$ ····· Inner product between two elements of the same Hilbert space.

$\|f\|_1$ ····· Norm of an element of one of the Banach spaces $\ell^1(\mathbb{N}_0)$ or $L^1(\mathbb{R}^+)$.

$\|f\|_2$ ····· Norm of an element of a Hilbert space: $\|f\|_2 = \sqrt{\langle f, f \rangle}$.

$\|f\|_\infty$ ····· Norm of an element of one of the Hardy spaces $\mathcal{H}^\infty(\mathbb{E})$ or $\mathcal{H}^\infty(\mathbb{C}^+)$.

Symbols related to sets of numbers

$\mathbb{N}$ ····· The set of the natural numbers: 1, 2, 3, ...

$\mathbb{N}_0$ ····· The set of non-negative integers: 0, 1, 2, ...

$\mathbb{Z}$ ····· The set of the integers: ..., −1, 0, 1, ...

$\mathbb{R}$ ····· The set of the real numbers: $(-\infty, \infty)$.

$\mathbb{R}^+$ ····· The set of the strictly positive real numbers: $(0, \infty)$.

$\mathbb{C}$ ····· The set of the complex numbers.

$\mathbb{C}^-$ ····· The set of the complex numbers with strictly negative real part: $\{z \,:\, \mathrm{Re}[z] < 0\}$.

$\mathsf{i}\mathbb{R}$ ····· The set of the purely imaginary complex numbers (the imaginary axis): $\{x \,:\, \mathrm{Re}[z] = 0\}$.

$\mathbb{C}^+$ ····· The set of the complex numbers with strictly positive real part: $\{z \,:\, \mathrm{Re}[z] > 0\}$.

$\mathbb{D}$ ····· The unit circle: $\{z \,:\, |z| < 1\}$.

$\mathbb{T}$ ····· The boundary of the unit circle: $\{z \,:\, |z| = 1\}$.

$\mathbb{E}$ ····· The complement of the closure of the unit circle plus the point at infinity: $\{z \,:\, |z| > 1\} \cup \{\infty\}$.

Symbols related to sets of functions (Hilbert and Hardy spaces)

$\ell^1(\mathbb{N}_0)$ ····· Banach space of absolutely summable sequences on $\mathbb{N}_0$, equipped with the norm $\|f\|_1 = \sum_{k=0}^{\infty} |f(k)|$.

$L^1(\mathbb{R}^+)$ ····· Banach space of absolutely integrable functions on $\mathbb{R}^+$, equipped with the norm $\|f\|_1 = \int_0^\infty |f(t)|\, dt$.

$\ell^2(\mathbb{Z})$ ····· Hilbert space of square summable sequences on $\mathbb{Z}$, equipped with the inner product $\langle f, g \rangle = \sum_{k=-\infty}^{\infty} f(k) g^*(k)$.

$\ell^2(\mathbb{N}_0)$ ····· Hilbert space of square summable sequences on $\mathbb{N}_0$, equipped with the inner product $\langle f, g \rangle = \sum_{k=0}^{\infty} f(k) g^*(k)$.

$L^2(\mathbb{R})$ ····· Hilbert space of square integrable functions on $\mathbb{R}$, equipped with the inner product $\langle f, g \rangle = \int_{-\infty}^{\infty} f(t) g^*(t)\, dt$.

$L^2(\mathbb{R}^+)$ ····· Hilbert space of square integrable functions on $\mathbb{R}^+$, equipped with the inner product $\langle f, g \rangle = \int_0^\infty f(t) g^*(t)\, dt$.

$L^2(\mathbb{T})$ ····· Hilbert space of square integrable functions on $\mathbb{T}$, equipped with the inner product $\langle F, G \rangle = 1/(2\pi) \int_{-\pi}^{\pi} F(e^{\mathsf{i}\omega}) G^*(e^{\mathsf{i}\omega})\, d\omega$. This inner product can also be given by $\langle F, G \rangle = 1/(2\pi\mathsf{i}) \oint_{\mathbb{T}} F(z) G^*(1/z^*) z^{-1}\, dz$.

$L^2(\mathsf{i}\mathbb{R})$ ····· Hilbert space of square integrable functions on $\mathsf{i}\mathbb{R}$, equipped with the inner product $\langle F, G \rangle = 1/(2\pi) \int_{-\infty}^{\infty} F(\mathsf{i}\omega) G^*(\mathsf{i}\omega)\, d\omega$. This inner product can also be given by $\langle F, G \rangle = 1/(2\pi\mathsf{i}) \int_{\mathsf{i}\mathbb{R}} F(s) G^*(-s^*)\, ds$.

$\mathcal{H}^2(\mathbb{E})$ ····· Hardy space of functions analytic on $\mathbb{E}$ and square integrable on $\mathbb{T}$. The inner product is defined in the same way as the one of $L^2(\mathbb{T})$.

$\mathcal{H}^2(\mathbb{C}^+)$ ····· Hardy space of functions analytic on $\mathbb{C}^+$ that vanish at "infinity", and that are also square integrable on $\mathsf{i}\mathbb{R}$. The inner product is defined in the same way as the one of $L^2(\mathsf{i}\mathbb{R})$.

$\mathcal{H}^\infty(\mathbb{E})$ ····· Hardy space of functions analytic on $\mathbb{E}$ and essentially bounded on $\mathbb{T}$, equipped with the norm $\|F\|_\infty = \operatorname{ess\,sup}_{z\in\mathbb{T}} |F(z)|$.

$\mathcal{H}^\infty(\mathbb{C}^+)$ ····· Hardy space of functions analytic on $\mathbb{C}^+$ that vanish at "infinity", and that are also essentially bounded on $\mathsf{i}\mathbb{R}$, equipped with the norm $\|F\|_\infty = \operatorname{ess\,sup}_{s\in\mathsf{i}\mathbb{R}} |F(s)|$.

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