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SUPERRESOLUTION, THE RECOVERY OF MISSING SAMPLES, AND VANDERMONDE MATRICES ON THE UNIT CIRCLE

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ABSTRACT

The purpose of this paper is to study the conditioning of complex Vandermonde matrices, in reference to applications such as superresolution and the problem of recovering missing samples in band-limited signals. The results include bounds for the singular values of Vandermonde matrices whose nodes are complex numbers on the unit circle. It is shown that, under certain conditions, such matrices can be quite well-conditioned, contrarily to what happens in the real case.

1. INTRODUCTION

Vandermonde matrices with real nodes are generally believed to be ill-conditioned, a reputation that they certainly deserve, at least when the nodes are real [5]. In fact, the condition number of real Vandermonde matrices has been shown to growth exponentially with the matrix order, at least for positive or symmetric nodes (and there is a conjecture that such node distribution is optimal).

That is not always the case when the nodes of the matrix are allowed to be complex. There are Vandermonde matrices with complex nodes that are unitary, the most important example being the $N \times N$ Fourier matrix F, the elements of which are defined by

$$F_{ab} = \frac{1}{\sqrt{N}} e^{-\mathrm{i}\frac{2\pi}{N}ab}. \qquad (1)$$

This Vandermonde matrix, being unitary, is perfectly conditioned (its spectral condition number is one). However, examples of highly ill-conditioned complex Vandermonde matrices can also be easily found.

The purpose of this paper is to study the conditioning of complex Vandermonde matrices, mentioning applications such as superresolution and the problem of recovering missing samples in band-limited signals. The results include bounds for the singular values (and hence bounds for the spectral condition number) of Vandermonde matrices whose nodes are complex numbers on the unit circle.

Certain classes of complex Vandermonde matrices have been studied before. Reference [1], for example, studies the conditioning of Vandermonde matrices generated by a set of points which is uniformly distributed modulo one on the unit circle (the nodes are given by the Van der Corput sequence, see [9]). Also, certain engineering problems, such as interpolation, superresolution, extrapolation, and the recovery of missing samples in band-limited signals, also lead to problems in which complex Vandermonde matrices do appear. For all those problems, an understanding of the spectrum of the underlying matrices would be highly beneficial.

1.1. Notation

Concerning notation, let $x \in \mathbb{C}^N$ denote a signal, and $\hat{x} \in \mathbb{C}^N$ its Fourier transform, defined by $\hat{x} = Fx$, where F is the $N \times N$ unitary Fourier matrix defined by (1). We denote by E_N the set $\{0, 1, \ldots, N-1\}$, and let S_a and S_b be two subsets of E_N of M elements each. We say that S_a and S_b are *equivalent* if the elements of S_a can be obtained by addition of an integer constant, modulo N, to the elements of S_b. This means that S_a and S_b are related by a circular shift or permutation. We say that a subset of E_N of cardinality $M < N$ is *contiguous* if it is equivalent to $E_M = \{0, 1, \ldots, M-1\}$.

A $n \times n$ Vandermonde matrix with nodes

$$a_0, a_1, \ldots, a_{n-1}$$

has the general form

$$V(a_0, \ldots, a_{n-1}) := \begin{bmatrix} 1 & 1 & \cdots & 1 \\ a_0 & a_1 & \cdots & a_{n-1} \\ \vdots & \vdots & & \vdots \\ a_0^{n-1} & a_1^{n-1} & \cdots & a_{n-1}^{n-1} \end{bmatrix}.$$

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A Vandermonde matrix on the unit circle is a Vandermonde matrix whose nodes are complex numbers of unitary modulus, $|a_i| = 1$.

2. MOTIVATION

A few words are in order to explain the possible motivation for studying Vandermonde matrices on the unit circle. This class of matrices appears in a variety of problems, such as interpolation, extrapolation, superresolution, and band-limited interpolation. In [14], for example, one starts with the N equations $\hat{x} = Fx$, and then one disregards all but the m equations that correspond to the known frequency-domain samples $\hat{x}_i$, that is,

$$\hat{x}_i = \sum_{j=0}^{n-1} F_{ij} x_j, \quad (i \in S_f), \tag{2}$$

where S_f denotes the set of m known Fourier transforms samples. For completeness, the set of known time-domain data samples will be denoted below by S_t, even if, as it happens here, $S_t = \{0, 1, \dots, n-1\}$.

If $m \geq n$ one might hope to solve these equations for the n unknown x_j, $0 \leq j < n$. When the elements of S_f and S_t are not contiguous, this necessary condition is not sufficient (it is easy to exhibit examples of sets S_t and S_f such that the relevant matrix has no inverse). But when either S_f or S_t are contiguous the matrix is Vandermonde, up to a scale factor, and consequently nonsingular.

This discrete, finite-dimensional version of the superresolution problem [14] is a special case of the following problem [3], which is related to band-limited interpolation, and error correction in erasure channels:

Problem 1 *Let S_t and S_f be two proper nonempty subsets of $\{0, 1, \dots, N-1\}$. Given the sets $x(S_t) := \{x_i : i \in S_t\}$ and $\hat{x}(S_f) := \{\hat{x}_i : i \in S_f\}$, find x and $\hat{x}$.*

These remarks outline some of the connections between Vandermonde matrices on the unit circle and interpolation, extrapolation, superresolution, and the recovery of missing samples in band-limited data records.

2.1. Condition number

Rewrite the equation (2) as

$$\hat{x}_{i_j} = \sum_{k=0}^{n-1} F_{i_j k} x_k, \quad (0 \leq j < m).$$

We adhere to the notation $Ax = b$, where

$$A_{jk} := F_{i_j k} = \frac{1}{\sqrt{N}} e^{-\mathrm{i}\frac{2\pi}{N} i_j k},$$

and $b_j = \hat{x}_{i_j}$. Clearly, the $m \times n$ matrix A is related to a Vandermonde matrix with nodes proportional to

$$e^{-\mathrm{i}\frac{2\pi}{N} i_0}, e^{-\mathrm{i}\frac{2\pi}{N} i_1}, \dots, e^{-\mathrm{i}\frac{2\pi}{N} i_{m-1}}.$$

One of the crucial issues regarding linear equations such as $Ax = b$ is the conditioning of A. The condition number with respect to a matrix norm $\|\cdot\|$ of a nonsingular matrix A is defined as

$$\kappa(A) := \|A\| \, \|A^{-1}\|.$$

It is easy to see that $\kappa(A) \geq \|I\|$, in any matrix norm. From the practical point of view $\kappa(A)$ is highly important since it can be shown that errors Δb in the data vector b imply errors Δx in the solution to $Ax = b$, the relative norm of which is bounded by

$$\frac{\|\Delta x\|}{\|x\|} \leq \kappa(A) \frac{\|\Delta b\|}{\|b\|}.$$

Moreover, there are error vectors Δb for which equality is attained.

One of the norms most often used is the spectral norm (the matrix norm induced by the standard Euclidean vector norm). The condition number of a matrix with respect to the spectral norm (which we will refer to as the spectral condition number) is always greater than or equal to unity. If the matrix A is positive definite and Hermitian, then

$$\kappa(A) = \frac{\lambda_{\mathsf{max}}}{\lambda_{\mathsf{min}}},$$

and if A is not Hermitian (even rectangular), then

$$\kappa(A) = \frac{\sigma_{\mathsf{max}}}{\sigma_{\mathsf{min}}}.$$

The λ and σ denote the eigenvalues and singular values of A, respectively.

We have seen that the superresolution problem and in other related problems A is Vandermonde. Nevertheless, the results discussed below show that Vandermonde matrices with *reasonably well distributed nodes* can indeed be quite well conditioned.

3. RESULTS

The matrix AA^H is $m \times m$ and $A^H A$ is $n \times n$. Both matrices have the same (real) nonzero eigenvalues, which are the squares of the nonzero singular values of the $m \times n$ Vandermonde A. It is clear that

$$(AA^H)_{ab} = \sum_{j=0}^{n-1} F_{i_a j} F^*_{j i_b} = \frac{1}{N} \sum_{j=0}^{n-1} w^{(i_a - i_b) j},$$

and

$$(A^H A)_{ab} = \sum_{j=0}^{m-1} F^*_{a i_j} F_{i_j b} = \frac{1}{N} \sum_{j=0}^{m-1} w^{-i_j (a-b)},$$

with

$$w = e^{-\mathrm{i}\frac{2\pi}{N}}.$$

These equations are similar to those that define the matrices of the direct methods studied in [2, 7, 11], an observation that further links the band-limited extrapolation, interpolation and sampling problems with the class of Vandermonde matrices on the unit circle.

The singular values of A are the square roots of the eigenvalues of AA^H or $A^H A$. These eigenvalues have been studied (see for example [4]), and the results can immediately be applied to A as well. However, the bounds given in this work are more general (the distribution of the roots of unity was subject more stringent constraints in [4] than in the present work).

Let D denote the maximum distance between any two of the i_k, d denote the minimum distance, and $\alpha := \pi D/N$. The distances should be taken in the circular sense (modulo N). We have obtained several upper bounds for the largest singular value of A, and lower bounds for the smallest, which are useful if d is reasonably large (in particular, the bounds do not yield anything useful if $d = 1$, which should not cause surprise because when $d = 1$ the matrix A can be very ill-conditioned). Two of the simplest bounds are

$$\begin{aligned} \sigma^2_{\mathsf{max}}(A) &\leq \frac{n-1+\ell}{N} + \frac{\alpha^2 N}{3\ell\sin^2(\alpha)d^2}, \\ \sigma^2_{\mathsf{min}}(A) &\geq \frac{n+1-\ell}{N} - \frac{\alpha^2 N}{3\ell\sin^2(\alpha)d^2}. \end{aligned} \tag{3}$$

Minimizing the first expression (or maximizing the second) with respect to ℓ, considered as a real variable, leads to

$$\ell = \frac{\alpha N}{\sqrt{3}\sin(\alpha)d}.$$

The nearest integer to this real number is the optimum value of ℓ. More specific bounds can be obtained by letting α take appropriate values. For example, if $\alpha = \pi/4$ one has

$$\sigma^2_{\mathsf{max}}(A) \leq \frac{n-1+\ell}{N} + \frac{\pi^2 N}{24\ell d^2}.$$

$$\sigma^2_{\mathsf{min}}(A) \geq \frac{n+1-\ell}{N} - \frac{\pi^2 N}{24\ell d^2},$$

and the optimum value of ℓ is the nearest integer to

$$\ell = \frac{\pi N}{\sqrt{24}d}.$$

We will now explain the method used to obtain the bounds. A matrix of the form AA^H can be written as

$$M = [\, M_{pq}\,]_{p,q=0}^{m-1} = [\, s(i_p - i_q)\,]_{p,q=0}^{m-1},$$

where

$$s(x) = \frac{1}{N}\sum_{k=0}^{n-1} w^{kx}.$$

In addition to $s(k)$ and the matrix M, we will need two additional functions $s^+(k)$ and $s^-(k)$. We impose upon their discrete Fourier transforms the conditions

$$\begin{gathered} \hat{s}^-(k) \in \mathbb{R}, \quad \hat{s}^+(k) \in \mathbb{R}, \\ 0 \leq \hat{s}^-(k) \leq \hat{s}(k) \leq \hat{s}^+(k). \end{gathered} \tag{4}$$

A specific choice for $s^-(k)$ and $s^+(k)$ will be considered shortly, but other choices are possible provided that (4) is satisfied.

The signals $s^+(k)$ and $s^-(k)$ generate the matrices

$$M^+ = [\, M^+_{pq}\,]_{p,q=0}^{m-1} := [\, s^+(i_p - i_q)\,]_{p,q=0}^{m-1},$$

$$M^- = [\, M^-_{pq}\,]_{p,q=0}^{m-1} := [\, s^-(i_p - i_q)\,]_{p,q=0}^{m-1}.$$

It is easy to show using (4) that the three matrices M, M^+ and M^- are positive semidefinite, and that they satisfy the inequalities

$$0 \leq M^- \leq M \leq M^+. \tag{5}$$

Also,

$$\begin{array}{ccccc} \lambda_{\mathsf{max}}(M^-) & \leq & \lambda_{\mathsf{max}}(M) & \leq & \lambda_{\mathsf{max}}(M^+), \\ \lambda_{\mathsf{min}}(M^-) & \leq & \lambda_{\mathsf{min}}(M) & \leq & \lambda_{\mathsf{min}}(M^+). \end{array} \tag{6}$$

The Geršgorin discs [8] associated with the matrix M are the sets

$$D_i := \{z \in \mathbb{C} : |z - M_{ii}| \leq R_i(M)\},$$

where

$$M_{ii} = s(0) = \frac{n}{N},$$

$$R_i(M) := \sum_{\substack{j=0 \\ j\neq i}}^{m-1} |M_{ij}|.$$

We now introduce a family of signals $V(k; a, b, \ell)$, periodic in k with period N, whose discrete Fourier transforms $\hat{V}(k; a, b, \ell)$ are real and trapezoidal. The meaning of the parameters a, b and ℓ is the following: $\hat{V}(k; a, b, \ell)$ is equal to one for $k \in [a, b]$, and decreases linearly from one down to zero in ℓ samples.

The inverse discrete Fourier transform of the trapezoidal function is given by the expression

$$\begin{aligned} V(k; a, b, \ell) &= \frac{1}{\sqrt{N}}\sum_{j=0}^{N-1} \hat{V}(j; a, b, \ell) e^{\mathrm{i}\frac{2\pi}{N}jk} \\ &= e^{\mathrm{i}\pi(a+b)k/N} \frac{\sin(\pi\ell k/N)\sin[\pi(b-a+\ell)k/N]}{\sqrt{N}\ell\sin^2(\pi k/N)}. \end{aligned}$$

Note that $\hat{V}(k; a, b, 1)$ is a rectangular window with $b-a+1$ unitary samples (periodic with period N).

We will avoid the matrix M, which is not diagonally dominant, and we will focus on the Geršgorin discs of M^+ and M^- instead. The first step is to estimate $R_i(M^+)$ and $R_i(M^-)$, for certain M^+ and M^-. The results will be used to bound the eigenvalues of M itself, using (6). The condition number of M is given by

$$\kappa(M) := \frac{\lambda_{\max}(M)}{\lambda_{\min}(M)}.$$

The bounds

$$\lambda_{\max}(M) \le \lambda_{\max}(M^+) \le s^+(0) + \max_i R_i(M^+), \quad (7)$$

$$\lambda_{\min}(M) \ge \lambda_{\min}(M^-) \ge s^-(0) - \max_i R_i(M^-), \quad (8)$$

lead to

$$\kappa(M) \le \frac{\lambda_{\max}(M^+)}{\lambda_{\min}(M^-)} \le \frac{s^+(0) + \max_i R_i(M^+)}{s^-(0) - \max_i R_i(M^-)}. \quad (9)$$

It is now necessary to select two candidate functions s^+ and s^-, and the corresponding matrices M^+ and M^-. The discrete Fourier transforms of the trapezoidal signals mentioned above are among the simplest possibilities for s^+ and s^-. The choice

$$\hat{s}^+(k) := \frac{1}{\sqrt{N}} \hat{V}(k; 0, n-1, \ell), \quad \ell > 1,$$

is in agreement with (4). The discrete Fourier transform is

$$\begin{aligned} s^+(k) &= \frac{1}{\sqrt{N}} V(k; 0, n-1, \ell) \\ &= e^{\mathrm{i}\pi(n-1)k/N} \frac{\sin(\pi \ell k/N) \sin[\pi(n-1+\ell)k/N]}{N\ell \sin^2(\pi k/N)}, \end{aligned}$$

and it should be noted that

$$s^+(0) = \frac{n-1+\ell}{N}.$$

A trapezoidal $\hat{s}^-$ compatible with (4) can be obtained from

$$\hat{s}^-(k) := \frac{1}{\sqrt{N}} \hat{V}(k; \ell-1, n-\ell, \ell),$$

and consequently

$$\begin{aligned} s^-(k) &= \frac{1}{\sqrt{N}} V(k; \ell-1, n-\ell, \ell) \\ &= e^{\mathrm{i}\pi(n-1)k/N} \frac{\sin(\pi \ell k/N) \sin[\pi(n-\ell+1)k/N]}{N\ell \sin^2(\pi k/N)}, \end{aligned}$$

with

$$s^-(0) = \frac{n-\ell+1}{N}.$$

These equations and the previously discussed results can now be easily used to estimate the eigenvalues. Note that in general there are free parameters (in this example, ℓ). For best results, ℓ can be adjusted to minimize (maximize) the estimate for the largest (smallest) eigenvalue. It can be shown that

$$\sum_{\substack{p=0 \\ p \ne q}}^{m-1} \frac{1}{\sqrt{N}} |V(i_p - i_q; a, b, \ell)| \le \frac{\alpha^2 N}{3\ell \sin^2(\alpha) d^2}.$$

Use has been made of

$$\sin x \ge \frac{\sin \alpha}{\alpha} x, \quad x \in [0, \alpha], \quad 0 \le \alpha \le \pi.$$

If D denotes the maximum distance between any two error positions, then one can take $\alpha = \pi D/N$.

The bounds for the radii of the Geršgorin discs for the matrices M^+ and M^- are direct corollaries of the inequalities.

For general α, one has

$$\lambda_{\max}(M) \le \frac{n-1+\ell}{N} + \frac{\alpha^2 N}{3\ell \sin^2(\alpha) d^2},$$

$$\lambda_{\min}(M) \ge \frac{n+1-\ell}{N} - \frac{\alpha^2 N}{3\ell \sin^2(\alpha) d^2},$$

which lead to (3) (the eigenvalues of $M = AA^H$ are the squares of the singular values of A). Minimizing the first expression (or maximizing the second) with respect to ℓ, considered as a real variable, leads to

$$\ell = \frac{\alpha N}{\sqrt{3} \sin(\alpha) d}.$$

The nearest integer to this real number is the optimum value of ℓ, denoted by ℓ_{opt}. More specific bounds can be obtained by giving α appropriate values.

4. FINAL REMARKS

These bounds are not intended to be used when $d = 1$ (that is, when there are long gaps in the data, or when contiguous errors occur). It is well known that in such circumstances severe ill-conditioning will occur (for data with a contiguous nonzero spectrum, such as low-pass data).

It is known [13] that the value of the determinant Δ of the complex Vandermonde that appears in the superresolution or band-limited extrapolation problems (in which $d = 1$) satisfies

$$N^{n/2} |\Delta| \le 2^{n(n-1)/2} C(n) \sin\left(\frac{\pi}{N}\right)^{n(n-1)/2}$$

and this is asymptotically correct. One has

$$C(n) = 1!\,2!\,3!\,4! \cdots (n-1)!,$$

and estimating $C(n)$ using the Euler-MacLaurin summation formula shows that

$$C(n)^{1/n} \approx \exp(\frac{n}{2}\log(n) - \frac{3n}{4} + 1).$$

The smallest eigenvalue of the matrix therefore satisfies

$$|\lambda_{\mathsf{min}}| \leq \frac{C(n)^{1/n}}{\sqrt{N}} \left[2\sin\left(\frac{\pi}{N}\right)\right]^{(n-1)/2}.$$

Usually, n is much smaller than N, and it follows that λ_{min} is bounded by a quantity $O(\sqrt{n}e^{-3n/4})$ times $O[(\pi\sqrt{n}/N)^n]$. The matrix may become singular to machine precision even for moderate values of n and N (say, $n = 10$ and $N = 100$).

As for the singular values, the asymptotic equivalence of the matrix and the prolate matrix can be combined with results that go back to Slepian [10] and Szego [6], quantifying the ill-conditioning of such matrices, which have even been proposed as good test matrices for numerical algorithms [12].

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