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Sampling and Generalized Almost Periodic Extension of Functions

Paulo J. S. G. Ferreira

***Abstract*— This brief explores the connections between nonuniform sampling of a certain function and the almost periodic extension of its Fourier transform. It is shown that the Fourier transform of a band-limited function can be extended (as a weighted sum of translates) as a Stepanoff almost periodic function, to the whole frequency axis. This result leads to a generalized nonuniform sampling theorem which, unlike previous results, does not require the continuity of the Fourier transform of the sampled function, and is valid for finite-energy band-limited functions.**

***Index Terms*— Almost periodic extension, almost periodic functions, nonuniform sampling, Stepanoff almost periodic functions.**

I. Introduction

We use the standard notation L_p for the spaces of complex functions of one real variable such that $\int_{-\infty}^{+\infty} |f(x)|^p\,dx$ exists as a Lebesgue integral. A function $f \in L_p$ $(1 \le p \le 2)$ is bandlimited to σ if its Fourier transform

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)\, e^{-j\omega t}\, dt$$

vanishes almost everywhere for $|\omega| > \sigma$.

The reconstruction of such functions from a knowledge of $f(t_n)$ ($n \in \mathbb{Z}$) is the subject of sampling theory. For an introduction to the topic, see [1]. The historical development of sampling theory is partially discussed in [2] and [3], which also review a number of interesting related results. The review paper by Jerri [4] is an account of the state of the art in sampling as of 1977, whereas Butzer's review [5] concentrates on the results obtained at the Lehrstuhl A für Mathematik, Aachen, Germany. A number of more recent reviews and books are available, such as [6] or [7], the latter which contains an extensive bibliography with more than 1000 entries. Several other developments are discussed in the books by Zayed [8] and Higgins [9], including an expository account of the Feichtinger–Gröchenig theory [10], [11], sampling results associated with Sturm–Liouville problems, the Landau minimum sampling density theorem [12], and much more.

Since this brief addresses the nonuniform or irregular sampling problem, a few words regarding the motivation for the study of such a sampling methodology, along with some of the possible applications, are not devoid of interest. First, nonuniform or irregular sampling measurements do occur naturally in several applications, including optics [13], tomography [14], [15], frequency modulation (FM), and phase, delta, or pulse position demodulation [16], [17]. Second, it is often impossible to uniformly sample certain types of data, such as astronomical or geophysical data, because the signal might not be available for measurement at certain points of the domain. Third, there are several factors that may accidentally lead to irregularly distributed sets of samples: jitter, incorrect samples due to noise or clipping, lost or delayed packets in packet-oriented telecommunication systems, data losses due to channel erasures, and others. Fourth, deliberately randomized sampling might have advantages for some specific applications, including signal analysis at low sampling rates [19]. The proceedings of two recently held workshops [20], [21] mention many theoretical results and engineering applications in reference to nonuniform sampling.

This brief explores the connections between the theory of almost periodic functions and nonuniform sampling. The classical theory of uniform almost periodic functions is due to H. Bohr, and was soon generalized by a number of other mathematicians, including Stepanoff, Wiener, Weyl, Besicovitch, and Schwarz. The reader interested in the theory of almost periodic functions is referred to [22]–[25]. The generalization due to Stepanoff will be especially useful in the context of this brief.

Sampling results have been derived using a diversity of mathematical techniques, including distribution theory, eigenfunction expansions, complex variable methods, reproducing kernel Hilbert spaces, special function theory, abstract harmonic analysis, and more. Given the connections between harmonic analysis and sampling theory, the absence of the theory of almost periodic functions from this list is surprising. It is true that frames [26] and nonharmonic Fourier series are often useful in the context of nonuniform sampling [9], [27]. However, the potential offered by almost periodic function theory seems to have gone unused or unnoticed by many researchers.

The paper by Davis [28] is the only work in the engineering literature of which we are aware that relates or applies the theory of almost periodic functions to sampling. It introduces the idea of an almost periodic extension of a compactly supported function, and cleverly uses it to formulate a nonuniform sampling theorem that contains, as a special case, a previously known result on multichannel sampling. More recently, the mean-periodic continuation method of Katsnelson [29], [30] has lead to new insights regarding sampling for functions with a multiband spectrum with support E, and Riesz bases for $L_2(E)$.

However, the interesting results given in [28] apply only to functions with a continuous Fourier transform. This restriction leaves out band-limited L_2 functions such as $\sin(at)/t$, which play a fundamental role in sampling. Given the importance of finite-energy band-limited functions this is a considerable drawback, already pointed out in [28] as one of the issues worth of further investigation.

It is precisely this matter that we address in this brief. Our results hold true for functions whose Fourier transform is not necessarily continuous, as is the case with L_2 band-limited functions. The sampling theorem and the actual construction of the almost periodic extension upon which it rests remain essentially unchanged. As in [28], the extension method is based on a weighted sum of translates. This brief extends the previously known results to a broader class of functions, and shows that Davis' construction remains valid under less stringent constraints.

Technically, we deal with almost periodic extension of bounded measurable functions, instead of continuous functions as in [28]. This

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leads in a natural way to continuations that are almost periodic in the sense of Stepanoff. This class of functions contains the (continuous) uniformly almost periodic functions used in [28] as a special case. As a consequence of the generalization, we will be able to establish a nonuniform sampling theorem for functions whose Fourier transform is not necessarily continuous, as is the case with L_2 band-limited functions.

II. Generalized Almost Periodic Extension

A function $f\colon \mathbb{R} \to \mathbb{C}$ is uniformly almost periodic (u.a.p.) if given $\epsilon > 0$ there is a real $L > 0$, such that each interval of length L contains at least one τ such that

$$\sup_{x\in\mathbb{R}} |f(x+\tau) - f(x)| < \epsilon.$$

The numbers τ are called ϵ-almost periods of f. A subset of $\mathbb{R}$ is relatively dense with inclusion length L if any interval of length L contains at least one element from the set. Therefore, f is u.a.p. if and only if it is continuous and has a relatively dense set of ϵ-almost periods (for arbitrary $\epsilon > 0$).

Almost periodic sequences (functions from $\mathbb{Z}$ into $\mathbb{C}$) are also useful. A sequence f_i is u.a.p. if, for any $\epsilon > 0$, there is an integer $M > 0$ such that each interval of length M contains at least one integer m such that

$$\sup_{i\in\mathbb{Z}} |f_{i+m} - f_i| < \epsilon.$$

It is known [23] that f_i is u.a.p. if, and only if, there exists a u.a.p. function g such that $f_i = g(i)$.

The Stepanoff S^p norm is defined by

$$\|f\|_{S^p} := \sup_{x\in\mathbb{R}} \left[\frac{1}{\ell}\int_x^{x+\ell} |f(x)|^p\, dx\right]^{1/p}$$

for an arbitrary $\ell > 0$. A function is almost periodic in the sense of Stepanoff (S^p a.p.) if it belongs to the closure in the S^p norm of the set of u.a.p. functions. A study of S^p a.p. functions, which were originally introduced by Stepanoff [31], may be found in [24].

It can be shown that if a function is S^p a.p. for a certain positive ℓ, it is also S^p a.p. for every other $\ell > 0$. If follows from the definition that the set of S^p a.p. functions strictly contains the u.a.p. functions. Every uniformly continuous S^p a.p. function is also a u.a.p. function.

The original idea [28] behind almost periodic extension of a Fourier transform function $\hat{x}$ with support in $[-\sigma, \sigma]$ can be explained as follows. Consider

$$\hat{y}(\omega) := \sum_{i=-\infty}^{+\infty} a_i \hat{x}(\omega - \omega_i) \tag{1}$$

and choose the ω_i and the a_i in such a way that $\hat{y}$ is uniformly almost periodic. The following sufficient condition for uniform almost periodicity of $\hat{y}$ is stated without proof in [28].

Theorem 1: Let $\hat{x}$ be a continuous function with support $[-\sigma, \sigma]$. If a_i is u.a.p., ω_i is uniformly almost linear, and no two translates of $\hat{x}$ overlap, then $\hat{y}$ as defined by (1) is a u.a.p. function.

In the terminology of [28], a sequence α_n of real numbers is uniformly almost linear (u.a.l.) if, and only if, there exists a real constant Ω, such that $n\Omega - \alpha_n$ is a u.a.p. sequence.

The continuity of $\hat{x}$ is unnecessary if $\hat{y}$ is required to be almost periodic in the Stepanoff sense, as we show with the following theorem, which generalizes Theorem 1.

Theorem 2: Let $\hat{x}$ be a bounded measurable function with support contained in $[-\sigma, \sigma]$. If a_i is u.a.p., ω_i is u.a.l., and no two translates of $\hat{x}$ overlap, then $\hat{y}$ defined by (1) is an S^p a.p. function $(p \geq 1)$.

We have to show that, for any $\epsilon > 0$, there is a relatively dense set of ϵ-almost periods of $\hat{y}(\omega)$ in the S^p $(p \geq 1)$ norm, that is, a set of numbers τ such that

$$\sup_{\alpha\in\mathbb{R}} \left[\frac{1}{\ell}\int_\alpha^{\alpha+\ell} |\hat{y}(\omega+\tau) - \hat{y}(\omega)|^p\, d\omega\right]^{1/p} < \epsilon. \tag{2}$$

Since ω_i is u.a.l., there exists $\Omega > 0$ such that $\omega_n = n\Omega + \Delta_n$, Δ_n being a u.a.p. sequence. Both a_i and Δ_i are u.a.p. and, therefore, for any given $\delta > 0$, there exists a relatively dense set P_δ of δ-almost periods of both a_i and Δ_i. The existence of P_δ is not immediately obvious, and is equivalent to the addition theorem for u.a.p. sequences or functions. A clear proof can be found in [22, p. 36], for example.

We will now show that (2) holds for any $\tau = m\Omega$, where $m \in P_\delta$, if δ is sufficiently small.

The precise value of ℓ in (3) is unimportant, and so we may assume without loss of generality that $\ell < 2\sigma$. This implies that every interval $I = [\alpha, \alpha+\ell]$ can be expressed as union of at most three disjoint sets S_1, S_2, and S_3 defined as follows. S_1 is the set of all $\omega \in I$ such that

$$|\omega - \omega_n| < \sigma - \delta \tag{3}$$

for some integer n. S_2 and S_3 are defined similarly, but substituting

$$|\omega - \omega_n| > \sigma + \delta \tag{4}$$

and

$$\sigma - \delta \leq |\omega - \omega_n| \leq \sigma + \delta \tag{5}$$

for (3), respectively. The integral (2) can be expressed as the sum of integrals taken over S_1, S_2, and S_3, which we will now examine separately.

The first case occurs when $\omega \in S_1$. We then have

$$\hat{y}(\omega) = a_n \hat{x}(\omega - \omega_n). \tag{6}$$

Since, for any $m \in P$

$$\begin{aligned} |\omega + m\Omega - \omega_{n+m}| &= |\omega - \omega_n + \Delta_n - \Delta_{n+m}| \\ &\leq |\omega - \omega_n| + |\Delta_n - \Delta_{n+m}| < \sigma \end{aligned}$$

it follows that

$$\begin{aligned} \hat{y}(\omega + m\Omega) &= a_{n+m}\hat{x}(\omega + m\Omega - \omega_{n+m}) \\ &= a_{n+m}\hat{x}(\omega - \omega_n + \theta\delta) \end{aligned} \tag{7}$$

for some real θ such that $|\theta| \leq 1$. Using (6) and (7), we may write

$$\begin{aligned} &|\hat{y}(\omega + m\Omega) - \hat{y}(\omega)|^p \\ &\quad = |a_{n+m}\hat{x}(\omega - \omega_n + \theta\delta) - a_{n+m}\hat{x}(\omega - \omega_n) \\ &\qquad + a_{n+m}\hat{x}(\omega - \omega_n) - a_n\hat{x}(\omega - \omega_n)|^p \\ &\quad = |a_{n+m}[\hat{x}(\omega - \omega_n + \theta\delta) - \hat{x}(\omega - \omega_n)] \\ &\qquad + \hat{x}(\omega - \omega_n)[a_{n+m} - a_n]|^p. \end{aligned}$$

Minkowski's inequality leads to

$$\begin{aligned} &\left[\int_{S_1} |\hat{y}(\omega + m\Omega) - \hat{y}(\omega)|^p\, d\omega\right]^{1/p} \\ &\quad \leq |a_{n+m}| \left[\int_{S_1} |\hat{x}(\omega - \omega_n + \theta\delta) - \hat{x}(\omega - \omega_n)|^p\, d\omega\right]^{1/p} \\ &\qquad + |a_{n+m} - a_n| \left[\int_{S_1} |\hat{x}(\omega - \omega_n)|^p\, d\omega\right]^{1/p} \\ &\quad \leq A\left[\int_{S_1} |\hat{x}(\omega - \omega_n + \theta\delta) - \hat{x}(\omega - \omega_n)|^p\, d\omega\right]^{1/p} + \delta\|\hat{x}\|_p \end{aligned} \tag{8}$$

where

$$A = \sup_k |a_k|.$$

The second case occurs when $\omega \in S_2$, and we then have $\hat{y}(\omega) = 0$. It is easy to see that $\hat{y}(\omega + m\Omega) = 0$ as well, since in any δ-neighborhood of $\omega \in S_2$ we have $\hat{y}(\omega) = 0$, and m is a δ-almost period of Δ_n. Thus

$$\int_{S_2} |\hat{y}(\omega + m\Omega) - \hat{y}(\omega)|^p \, d\omega = 0. \tag{9}$$

To deal with the third and last case, let $\omega \in S_3$. Since $\ell < 2\sigma$, the measure of S_3 does not exceed 4δ. Thus

$$\int_{S_3} |\hat{y}(\omega + m\Omega) - \hat{y}(\omega)|^p \, d\omega \le 4\delta 2^p B \tag{10}$$

where

$$B = \|\hat{x}\|_\infty = \sup_\omega |\hat{x}(\omega)|^p.$$

Using (8)–(10), we have

$$\begin{aligned} &\frac{1}{\ell} \int_\alpha^{\alpha+\ell} |\hat{y}(\omega + m\Omega) - \hat{y}(\omega)|^p \, d\omega \\ &= \left[\int_{S_1} + \int_{S_2} + \int_{S_3} \right] |\hat{y}(\omega + m\Omega) - \hat{y}(\omega)|^p \, d\omega \\ &\le A \left[\int_{S_1} |\hat{x}(\omega - \omega_n + \theta\delta) - \hat{x}(\omega - \omega_n)|^p \, d\omega \right]^{1/p} \\ &\quad + \delta \|\hat{x}\|_p + 4\delta 2^p B \end{aligned}$$

which tends to zero when $\delta \to 0$. This means that δ can always be chosen so small that (2) holds, thus completing the proof.

Note that Theorem 2 reduces to Theorem 1 when $\hat{x}(\omega)$ is continuous, since a S^p a.p. function is u.a.p. if, and only if, it is uniformly continuous [23], [24].

The following sampling theorem is a consequence of Theorem 2, and removes the continuity restriction found in [28].

Theorem 3: Let ω_n and a_n be u.a.l. and u.a.p. sequences, respectively, and let x be an L_2 signal with Fourier transform supported in $[-\sigma, +\sigma]$. If the conditions $a_0 = 1$, $\omega_0 = 0$, $a_{-n} = a_n$, $\omega_{-n} = \omega_n$, and $\omega_n - \omega_{n-1} > 2\sigma$ hold, then:

1) $s(t) = \lim_{n\to\infty} (1/n) \sum_{k=0}^{n-1} a_k e^{jw_k t}$ is nonzero at a countable set $\{t_n\}$ of points only;
2) if $t_n \to \infty$ at least as rapidly as n, then x can be recovered from $s(t_n)x(t_n)$ using

$$x(t) = \lim_{n\to\infty} \sum_{k=-n}^{n} s(t_k)x(t_k)h(t - t_k)$$

where $h(t) = 2\sin(\sigma t)/\Omega t$, and Ω is the constant associated with the u.a.l. sequence ω_n.

The proof can be carried out following [28], noting that Parseval's equation also holds for S^2 a.p. functions [24]. Therefore, the Fourier coefficients b_k of the extension function $\hat{y}$ do satisfy

$$\sum_{k=-\infty}^{+\infty} |b_k|^2 < \infty$$

just as in the u.a.p. case.

This shows that the essential of the construction proposed in [28] remains valid under less stringent constraints and completes our initial goal: to extend the previously known results to a broader class of functions, with possibly discontinuous Fourier transforms, exploring the concept of Stepanoff almost periodicity.

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Synchronizing Hyperchaotic Systems by Observer Design

Giuseppe Grassi and Saverio Mascolo

***Abstract*— In this brief, a technique for synchronizing hyperchaotic systems is presented. The basic idea is to make the driven system a linear observer for the state of the drive system. By developing this approach, a linear time-invariant synchronization error system is obtained, for which a necessary and sufficient condition is given in order to asymptotically stabilize its dynamics at the origin. The suggested tool proves to be effective and systematic in achieving global synchronization. It does not require either the computation of the Lyapunov exponents, or the initial conditions belonging to the same basin of attraction. Moreover, it guarantees synchronization of a wide class of hyperchaotic systems via a *scalar signal*. Finally, the proposed tool is utilized to design a secure communications scheme, which combines conventional cryptographic methods and synchronization of hyperchaotic systems. The utilization of both *cryptography* and hyperchaos seems to make a contribution to the development of communication systems with higher security.**

***Index Terms*—Chaotic encryption, hyperchaotic circuits and systems, synchronization theory.**

I. Introduction

At first thought, chaotic phenomena generated by nonlinear systems would seem singularly unsuited for engineering applications. In reality, the broad-band frequency spectrum makes chaotic signals a natural way of sending and receiving private communications. For this reason, chaotic dynamics, synchronization of coupled dynamic systems, and secure communications have been the topics of many papers over the last few years [1]–[7].

Referring to synchronization, Carroll and Pecora [2] have theoretically and experimentally shown that the dynamics of a drive system and of a driven subsystem (response system) become synchronized if the Lyapunov exponents of the response system are less then zero, assuming that both the systems start in the same basin of attraction. However, most of the methods concern the synchronization of low dimensional systems, characterized by only one positive Lyapunov exponent [2]–[4]. Since this feature limits the complexity of the chaotic dynamics, it is believed that the adoption of higher dimensional chaotic systems, with more than one positive Lyapunov exponent, enhances the security of the communication scheme. Therefore, hyperchaotic systems and hyperchaos synchronization have recently become fields of active research [5]–[7]. In particular, in [5] the synchronization between hyperchaotic systems is achieved by exploiting linear and nonlinear feedback functions, although the attention is not focused on the number of the synchronizing signals. In [6], a linear combination of the original state variables (i.e., a scalar signal) is used to synchronize hyperchaos in Rössler's systems. However, the approach in [6] cannot be considered a systematic technique for synchronization, since the coefficients of the linear combination are somewhat arbitrary. An interesting result has been recently reported in [7], where a parameter control method is proposed to achieve hyperchaos synchronization. In any case, the computation of the Lyapunov exponents is still required in order to verify the synchronization.

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This brief makes a contribution in the context of hyperchaos synchronization. Furthermore, an application to hyperchaos-based cryptography is presented. The key idea is to make the response system a linear observer for the state of the drive system. This approach guarantees synchronization, because an observer has the property that its state converges to the state of the plant; that is, the state of the driven system converges to the state of the drive one. The proposed technique has several advantages over the existing methods. It proves to be simple and rigorous. It does not require either the computation of the Lyapunov exponents or initial conditions belonging to the same basin of attraction. Moreover, global synchronization is achievable in a systematic way for several examples of hyperchaotic systems reported in literature.

The paper is organized as follows. In Section II, a general class of hyperchaotic systems is defined and the well-known concept of linear observer is introduced to formalize the problem of hyperchaos synchronization. Following this approach, a linear time-invariant synchronization error system is derived, along with a necessary and sufficient condition for its asymptotic stabilization. This technique guarantees synchronization of Rössler's system [6], the Matsumoto–Chua–Kobayashi (MCK) circuit [8] and its modified version [9], two oscillators recently reported in literature [10], [11], and a circuit with hysteretic nonlinearity [12]. A major advantage is that all of these systems are synchronized using a *scalar* signal. In order to show the effectiveness of the developed technique, numerical simulations are carried out in Section III, whereas in Section IV, a secure communications scheme is designed, which combines conventional *cryptographic* methods and synchronization of hyperchaotic systems. In Section V, some concluding remarks are given.

II. Hyperchaos Synchronization Using Linear Observer

The goal of synchronization is to design a coupling between two chaotic systems, called drive system and response system, so that their dynamics become identical after a transient time. The coupling is implemented via a synchronizing signal, which is generated by the drive system. In this brief, the attention is focused on the following class of dynamic systems.

Definition 1: A hyperchaotic system belongs to the class C_m if its state and output equations can be written, respectively, as

$$\dot{x}(t) = Ax(t) + Bf(x(t)) + c \tag{1}$$

$$y(t) = h(x(t)) \tag{2}$$

where $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $f = (f_1(x),\, f_2(x),\, \cdots,\, f_m(x))^T \in \Re^{m\times 1}$ with $f_i \neq f_j$ for $i \neq j$, $m \leq n$, $c \in \Re^{n\times 1}$ and $y = (h_1(x),\, h_2(x),\, \cdots,\, h_m(x))^T \in \Re^{m\times 1}$.