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A Comment on the Approximation of Signals by Gaussian Functions

Paulo Jorge S. G. Ferreira

Abstract—**We point out that an approximation property of Gaussian functions, derived in a recent work, is a direct corollary to the work of Wiener on the closure of translations in L_1 and L_2. This observation not only simplifies the proof of the approximation property, but also renders the result applicable, in a more general setting, to other functions (not necessarily Gaussian).**

Index Terms—**Approximation methods, closure of translations, Gaussion functions, nonlinear approximation, nonlinear functions, signal representations, superpositions.**

I. Introduction

The purpose of this note is to comment on certain side aspects of a recent and interesting work [1]. Our remarks in no way compromise the main results and conclusions presented in that paper, which addresses the approximation of finite-energy signals by linear combinations of Gaussian functions:

$$\sum_i a_i g(t - t_i, \sigma_i)$$

where

$$g(t, \sigma_i) = \frac{1}{\sigma_i \sqrt{2\pi}} e^{-t^2/2\sigma_i^2}.$$

It is not our intention to shift attention from the main results and conclusions presented in [1], but simply to address this approximation problem in the light of Wiener's results on the closure of translations which, despite their usefulness and importance, do not seem to be as well known as some of the other works. We hope that our observations might be of use to researchers interested in nonlinear approximation problems such as this, and who remain unaware of Wiener's results.

The purpose of the long Appendix in [1] is to prove that any finite-energy signal which vanishes outside a certain interval can be arbitrarily well approximated by linear combinations of Gaussian functions. This is done very much in the spirit of Lauricella's theorem [2], that is, by showing that any sinusoidal signal $\sin(2\pi kt/T)$, $0 \leq t \leq T$, can be approximated by Gaussian functions.

Our aim is to show that similar, and indeed more general, conclusions follow from the approximation results due to Wiener on the closure of translations in L_1 and L_2—respectively, the Banach and Hilbert spaces of functions f such that $|f|$ and $|f|^2$ is Lebesgue integrable over $(-\infty, +\infty)$. Wiener showed that any function belonging to L_1 can be approximated to any prescribed tolerance, in the L_1 norm, by linear combinations of the translates of a single function $\psi \in L_1$:

$$\sum_{i=1}^{N} a_i \psi(t - t_i)$$

if and only if the Fourier transform of ψ has no zeros. He also showed that a similar result holds in L_2 if and only if the set of zeros of the Fourier transform of ψ has zero measure. Proofs of these results can be found in [3] and [4], for example.

TABLE I
Approximating $\sin(2\pi t)$, $0 \leq t \leq 1$

		Example 1	Example 2
Curve 1	amplitude	-1.21646	-1.17646
	position	-0.25	-0.22779
	width	0.327444	0.315398
Curve 2	amplitude	1.23495	1.26416
	position	0.25	0.255889
	width	0.325604	0.335937
Curve 3	amplitude	-1.23495	-1.26416
	position	0.75	0.744111
	width	0.325604	0.335937
Curve 4	amplitude	1.21646	1.17646
	position	1.25	1.22779
	width	0.327444	0.315398
	Error	1.1e-06	1.3e-10

The Gaussian function $g(t, \sigma)$ clearly belongs to L_1 and L_2, independently of σ, and its Fourier transform certainly has no zeros. Thus, the results obtained by Wiener imply that, for any $f \in L_1$ and $\epsilon > 0$, there is an integer N and constants $(a_i)_{1 \leq i \leq N}$ and $(t_i)_{1 \leq i \leq N}$ such that

$$\int_{-\infty}^{\infty} \left| f(t) - \sum_{i=1}^{N} a_i g(t - t_i, \sigma) \right| dt < \epsilon.$$

A similar result holds for any $f \in L_2$, the approximation now being in the L_2 norm:

$$\int_{-\infty}^{\infty} \left| f(t) - \sum_{i=1}^{N} a_i g(t - t_i, \sigma) \right|^2 dt < \epsilon.$$

These conclusions generalize those obtained, at much greater length, in [1].

The approximation property just discussed holds no matter the value of σ, a somewhat surprising result: the spaces L_1 and L_2 contain very rapidly varying functions, and the results mentioned imply that even very spread-out Gaussian curves can somehow be combined to closely approximate these signals.

Unfortunately, the methods used by Wiener are not constructive, and do not offer any hints on how to pick N, $(a_i)_{1 \leq i \leq N}$ and $(t_i)_{1 \leq i \leq N}$. For example, the approximation of sinusoids as discussed in [1, Appendix] is based on Gaussian curves of fixed width translated to predetermined locations (the extrema of the sinusoid). Much better

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fits are possible if the Gaussians are less constrained, that is, if more of their characteristics (amplitude, position, and width) are used. This is demonstrated in Table I. For the first example, the position of the Gaussians was held at the extrema of $\sin(2\pi t)$, as done in [1], but the remaining parameters were adjusted to minimize the squared error. The result was a very good fit. Using all parameters, as in the second example, reduced the squared error even further, by four orders of magnitude.

As stated in [1], this is a nonlinear least squares curve fitting problem. Hence, we see the practical importance of suboptimal but efficient algorithms like the one proposed in that work.

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A Versatile Algorithm for Two-Dimensional Symmetric Noncausal Modeling

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***Abstract*—In this brief, a novel algorithm is presented for the efficient two-dimensional (2-D) symmetric noncausal finite impulse response (FIR) filtering and autoregressive (AR) modeling. Symmetric filter masks of general boundaries are allowed. The proposed algorithm offers the greatest maneuverability in the 2-D index space in a computationally efficient way. This flexibility can be taken advantage of if the shape of the 2-D mask is not *a priori* known and has to be dynamically configured.**

***Index Terms*—**

Algorithms, filtering, image processing, least mean square error methods, Toeplitz matrices.

I. Introduction

Two-dimensional least squares noncausal modeling is of great importance in a wide range of applications. These include image restoration, image enhancement, image compression, 2-D spectral estimation, detection of changes in image sequences, stochastic texture modeling, edge detection, etc. [1].

Let $x(n_1, n_2)$ be the input of a linear, space invariant 2-D FIR filter. The filter's output $y(n_1, n_2)$ is a linear combination of past input values $x(n_1 - i_1, n_2 - i_2)$ weighted by the *filter coefficients* c_{i_1,i_2} over a support region, or *filter mask* $\mathcal{M}$:

$$y(n_1, n_2) = -\sum_{(i_1,i_2)\in\mathcal{M}} c_{i_1,i_2} x(n_1 - i_1, n_2 - i_2). \tag{1}$$

A fairly general shape for the support region is considered. Thus, $\mathcal{M}$ is allowed to be horizontally convex, i.e., the horizontal line segment joining any two points $(i_1, i_2), (i_1, i_3) \in \mathcal{M}$ lies in $\mathcal{M}$.

The filter is restricted to be linear phase. Thus, the following conditions should be satisfied [1]:

$$\begin{aligned} &\text{mask symmetry} \quad \forall (i_1, i_2) \in \mathcal{M}, \quad \exists(-i_1, -i_2) \in \mathcal{M} \\ &\text{coeff. symmetry} \quad c_{i_1,i_2} = c_{-i_1,-i_2}. \end{aligned}$$

Given an input 2-D signal $x(n_1, n_2)$ and a desired response 2-D signal $z(n_1, n_2)$, the optimal mean-squared error (MSE) 2-D FIR filter is obtained by minimizing the cost function

$$\mathcal{E}[(z(n_1, n_2) - y(n_1, n_2))^2]. \tag{2}$$

$\mathcal{E}[\cdot]$ is the expectation operator. MSE 2-D linear prediction can be handled as a special case of filtering, setting $z(n_1, n_2) = x(n_1, n_2)$ and excluding the origin $\{(0,0)\}$ from the filter mask, i.e., $(i_1, i_2) \in \mathcal{M} - \{(0,0)\}$.

Minimization of (2) with respect to the filter parameters c_{i_1,i_2} leads to a system of linear system of equations, the so-called normal equations. Any well-behaved linear system solver can be applied for the inversion of the 2-D normal equations. However, the special structure of the normal equations gives rise to the development of cost-effective algorithms for the determination of the unknown parameters [2]–[4].

In this paper a new, highly efficient algorithm is developed for the solution of the normal equations in a ***true*** order recursive way [7]. Filter masks of general, horizontally convex shape are allowed. Fast recursions are developed for the updating of lower order filter parameters toward any neighboring point. It can efficiently be applied for the order-recursive estimation of the 2-D MSE FIR filter and system identification, accelerating the exhaustive search procedures required by most of the order determination criteria [8], [9].

II. 2-D Symmetric Support Region

Consider the support region depicted in Fig. 1. More precisely, $\mathcal{M}$ consists of a union of intervals:

$$\mathcal{M} = \bigcup_{i_1=-k_1}^{k_1} \boldsymbol{m}(i_1),$$
$$\boldsymbol{m}(i_1) = \{(i_1, i_2)\colon -k_2(-i_1) \le i_2 \le k_2(i_1)\}.$$

Clearly, $k_1 = \max\{i_1\colon (i_1, i_2) \in \mathcal{M}\}$, $k_2(i_1) = \max\{i_2\colon (i_1, i_2) \in \boldsymbol{m}(i_1)\}$. Then, (1) takes the form

$$y(n_1, n_2) = -\sum_{i_1=-k_1}^{k_1} \sum_{i_2=-k_2(-i_1)}^{k_2(i_1)} c_{i_1,i_2} x(n_1 - i_1, n_2 - i_2).$$

The above equation can be written as a linear regression:

$$y(n_1, n_2) = -\mathcal{X}_{\mathcal{M}}^t(n_1, n_2)\mathcal{C}_{\mathcal{M}} \tag{3}$$

where the regressor (data vector) and the filter coefficients vector are defined by (4), (5), and (5a).

The filter coefficients' symmetry implies that

$$\mathcal{C}_{\mathcal{M}} = \mathcal{J}\mathcal{C}_{\mathcal{M}} \tag{6}$$

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