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The existence and uniqueness of the minimum norm solution to certain linear and nonlinear problems

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Abstract

It is shown that the set of fixed points of a nonexpansive operator is either empty or closed and convex. Under rather general conditions this shows that the minimum norm solution of an operator equation of the form $x = Tx$ exists and is unique, provided that T is nonexpansive. This holds in any strictly convex Banach space, a class of spaces that includes Hilbert spaces as particular case, and has consequences in signal and image reconstruction, as well as in other engineering applications.

Key words: Minimum norm solutions, convexity, linear equations, nonexpansive operators, nonlinear equations, fixed points.

1 Introduction

Briefly speaking, a fixed point of a linear or nonlinear operator T is a solution of the equation $x = Tx$. The basic fixed point theorems are Banach's theorem [1, 2], for contractive mappings, Brouwer's theorem [3–6] for continuous mappings in a finite-dimensional space, and Schauder's generalization [4] of Brouwer's theorem to infinite-dimensional Banach spaces. Many other results are discussed in [7, 8].

These fixed point theorems are tools of great importance in signal and image reconstruction, tomography, telecommunications, interpolation, extrapolation, signal enhancement, filter design, among many others [9–16]. A quick glance through [9], for example, should convince any reader of the practical interest of the subject: many interesting problems can be recast as fixed point problems.

For example, let x be a signal of interest, and let y be a distorted version of x. Assume further that y and x are related by an operator equation $y = Dx$

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(y might be the signal measured at the receiving end of a transmission system D, and x the transmitted signal). The problem is how to estimate x given y and the model D of the distortion that x underwent.

If x satisfies a constraint equation $x = Cx$, then

$$x = Cx + \mu(y - DCx),$$

an identity of the form $x = Tx$. Under rather general conditions, the solution to this equation will be the unknown signal x. This is the key to many iterative constrained restoration algorithms

$$x_{i+1} = Cx_i + \mu(y - DCx_i) = Tx_i,$$

as described in [9]. Their analysis is easy if T is a contraction (Banach's theorem applies and guarantees the existence and uniqueness of the solution).

If T is merely nonexpansive [2], as it often is, the analysis of the problem is more complex. A nonexpansive operator may have any number of fixed points, a fact that suggests the study of the minimum norm solutions of $x = Tx$.

Brouwer's theorem and Schauder's theorem are among the results that ensure the existence of fixed points of nonexpansive operators. In this paper we show that the set of fixed points of a nonexpansive operator is either empty or closed and convex. This turns out to be true in all strictly convex Banach spaces, such as L_p, $(1 < p < \infty)$, and in particular in all Hilbert spaces, such as L_2. This easily shows the existence and uniqueness of the minimum norm solution of an operator equation of the form $x = Tx$, if T is nonexpansive.

2 Convexity of the set of fixed points

The set of fixed points of a given nonexpansive mapping may contain any number of elements. For example, the translation $T : f \to f + g$ has no fixed points at all. If $g = 0$ this changes drastically: T reduces to the identity mapping, which is clearly nonexpansive and for which every point in the domain is also a fixed point.

In spite of this wide range of possibilities, there is an useful property that the

[2] A linear or nonlinear operator T defined in a normed space X with norm $\|\cdot\|$ is nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in X$.

set of fixed points of a nonexpansive operator retains.

Theorem 1 *Let X be a strictly convex normed space with norm $\|\cdot\|$. The set of fixed points of a nonexpansive mapping $T : X \to X$ is either empty or closed and convex.*

PROOF. The example $T : f \to f+g$ shows that the set of fixed points can be empty. If there is only one fixed point there is nothing to show. Consequently, let x and y be two fixed points, and let

$$z = \alpha x + (1-\alpha)y.$$

We wish to show that z is also a fixed point. Consider the inequalities

$$\|Tz - x\| = \|Tz - Tx\| \leq \|z - x\| = (1-\alpha)\|x - y\|,$$

$$\|Tz - y\| = \|Tz - Ty\| \leq \|z - y\| = \alpha\|x - y\|.$$

Adding them together leads to

$$\|Tz - x\| + \|Tz - y\| \leq \|x - y\|.$$

But

$$\|Tz - x\| + \|Tz - y\| \geq \|x - y\|,$$

by the triangle inequality, which shows that

$$\|Tz - x\| + \|Tz - y\| = \|x - y\|.$$

Let $a = x - Tz$ and $b = Tz - y$. Then this is $\|a + b\| = \|a\| + \|b\|$, and since the norm is strictly convex, $a = \lambda b$ for some positive constant λ (see [2, p. 336]). This means that Tz is a linear combination of x and y, that is, $Tz = \beta x + (1-\beta)y$ for some real β. The previous results show that

$$\|Tz - x\| = \|z - x\| = (1-\alpha)\|x - y\|, \tag{1}$$

$$\|Tz - y\| = \|z - y\| = \alpha\|x - y\|,$$

and consequently $\beta = \alpha$, and $Tz = z$. □

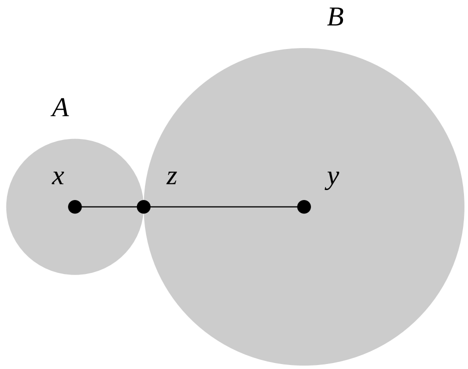

Fig. 1. Geometrical interpretation of theorem 1.

A geometrical interpretation of this result is given in figure 1. The sets A and B are defined by

$$A = \{x : \|Tz - x\| \leq (1-\alpha)\|x-y\|\},$$

$$B = \{x : \|Tz - y\| \leq \alpha\|x-y\|\}.$$

The image of z by T must lie in the intersection of A and B, which reduces to the point z. Thus, Tz must coincide with z.

Note that the theorem ceases to be true if the balls A and B are not strictly convex. Thus, the theorem does not hold in Banach spaces such as L_1, that are not strictly convex. It does hold in L_p, for $1 < p < \infty$, including L_2, as well as any other Hilbert space.

If the normed space X is a Hilbert space, its norm satisfies the parallelogram identity [2]

$$\|a-b\|^2 + \|a+b\|^2 = 2\|a\|^2 + 2\|b\|^2,$$

and the proof becomes simpler. Taking $a = Tz - x$ and $b = z - x$ leads to

$$\|Tz-z\|^2 + \|Tz-x+z-x\|^2 = 2\|Tz-x\|^2 + 2\|z-x\|^2,$$

and using both (1) and the fact that T is nonexpansive

$$\begin{aligned}\|Tz-z\|^2 &\leq 4\|z-x\|^2 - \|Tz-Tx+z-x\|^2 \\ &\leq 4\|z-x\|^2 - (\|Tz-Tx\| + \|z-x\|)^2 \\ &\leq 4\|z-x\|^2 - (\|z-x\| + \|z-x\|)^2,\end{aligned}$$

which means that $\|Tz - z\| = 0$ and consequently $z = Tz$.

Theorem 2 *Any nonempty closed convex subset S of a strictly convex Banach* *space contains a unique element of smallest norm.*

This is a well-known result and we omit the proof. In a Hilbert space, the proof again depends on the parallelogram identity. In a strictly convex Banach space, it is a corollary a the uniqueness of best approximations. See, for example, [2, 17, 18]. The presentation in [18] is based on the (equivalent) concept of a strictly normalized space.

Corollary 3 *Let S be a compact and convex subset of a strictly convex Banach space, and let T be a nonexpansive mapping which carries S into itself. Then, the minimum-norm solution of $x = Tx$ exists and is unique.*

PROOF. By theorem 1, T has a nonempty and convex set of fixed points (Schauder's theorem shows that the set of fixed points is not empty). By theorem 2, this set contains a unique element with minimum norm. □

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