Nonuniform Sampling of Nonbandlimited Signals

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***Abstract*— We give a nonuniform sampling series for nonbandlimited functions, together with error bounds.**

I. Introduction

SAMPLING theory is a topic with important applications in several fields, which has attracted the attention of many authors. For an introduction to sampling theory, see [1]. The papers in [2]–[4] give a rather complete survey of the field and of its history. For more recent developments and up-to-date bibliographies, see [5] and [6].

Briefly, sampling theory is the study of series of the form

$$f(t) = \sum_{k=-\infty}^{+\infty} f(t_k)\phi_k(t)$$

also called sampling expansions or sampling series. In this context, the t_k are called the sampling points. The WKS theorem, which lies at the core of the theory, applies to bandlimited functions, that is, functions $f \in L_2(\mathbf{R})$ whose Fourier transforms vanish outside some interval $[-w, w]$. The WKS theorem asserts that

$$f(t) = \sum_{k=-\infty}^{+\infty} f\left(\frac{k\pi}{w}\right)\frac{\sin[w(t-k\frac{\pi}{w})]}{w(t-k\frac{\pi}{w})} \tag{1}$$

absolutely and uniformly but has been generalized along several broad lines of research: extension to other spaces (for example, $L_p(\mathbf{R}^n)$ spaces); extension to nonuniform sampling; extension to nonbandlimited functions.

The restriction to bandlimited functions subsists in most generalizations of the theorem, but there are important results concerning the approximation of nonbandlimited functions by sampling series. Generally speaking, they establish that certain classes of functions may be arbitrarily well approximated by sampling expansions if the sampling period π/w is sufficiently small. Among these works, we mention [3], [7], and [8], which, however, consider only equidistant samples.

In this letter, we give a nonuniform sampling expansion for nonbandlimited functions (duration-limited functions of bounded variation). The bound on the approximation error, remarkably, does not depend on the distribution of the sampling points.

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II. Results

We recall that a function f defined in $[a, b]$ is said to be of bounded variation if

$$\sum_{k=1}^{N} |f(x_k) - f(x_{k-1})| \leq C < \infty \tag{2}$$

for every partition $a = x_0 < x_1 < \ldots < x_N = b$ of $[a, b]$. The variation of f on $[a, b]$, which is denoted by V_f, is the least upper bound of (2) over all finite partitions of $[a, b]$. As usual, we write $f(x) = O(g(x))$ when $x \to \infty$ if there is a constant C such that $f(x) \leq C\, g(x)$ when $x \to \infty$.

Theorem: **Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function of bounded variation vanishing outside $[0, 1]$, and let ϵ be an arbitrarily small positive real. Denote by $\{t_i\}$ any N reals such that**

$$\frac{i-1}{N} < t_i < \frac{i}{N}, \tag{3}$$

for $1 \leq i \leq N$, and let f_w denote a low-pass smoothed version of f

$$f_w(t) = \int_0^1 f(\tau)\frac{\sin[w(t-\tau)]}{\pi(t-\tau)}\, d\tau \tag{4}$$

where w is a positive real. Then, for any $t < -\epsilon$ or $t > 1 + \epsilon$

$$\left| f_w(t) - \frac{1}{N}\sum_{k=1}^{N} f(t_k)\frac{\sin[w(t-t_k)]}{\pi(t-t_k)} \right| = \frac{O(w)}{N} \tag{5}$$

whereas, for $0 \leq t \leq 1$

$$\left| f_w(t) - \frac{1}{N}\sum_{k=1}^{N} f(t_k)\frac{\sin[w(t-t_k)]}{\pi(t-t_k)} \right| = \frac{O(w \log w)}{N}. \tag{6}$$

Proof: To simplify the writing, let

$$s_N(t) = \frac{1}{N}\sum_{k=1}^{N} f(t_k)\frac{\sin[w(t-t_k)]}{\pi(t-t_k)}.$$

The mean value theorem, which is applied to (4), asserts the existence of N reals ξ_i satisfying (3), for $1 \leq i \leq N$, and such that

$$f_w(t) = \frac{1}{N}\sum_{k=1}^{N} f(\xi_k)\frac{\sin[w(t-\xi_k)]}{\pi(t-\xi_k)}.$$

This leads to

$$\begin{aligned} &|f_w(t) - s_N(t)| \\ &\leq \frac{1}{N}\sum_{k=1}^{N}\left| f(\xi_k)\frac{\sin[w(t-\xi_k)]}{\pi(t-\xi_k)} - f(t_k)\frac{\sin[w(t-t_k)]}{\pi(t-t_k)} \right| \\ &\leq \frac{V_F(w,t)}{N} \end{aligned} \tag{7}$$

where $V_F(w,t)$ denotes the variation of the function $F(x) = f(x)\frac{\sin w(x-t)}{\pi(x-t)}$, for $0 < x < 1$. However

$$V_F(w,t) \le V_f \|T\|_\infty + V_T \|f\|_\infty = V_f \frac{w}{\pi} + V_T \|f\|_\infty$$

where $\|\cdot\|_\infty$ denotes the L_∞ norm, and V_T denotes the variation of the function $T(x) = \frac{\sin w(x-t)}{\pi(x-t)}$, for $0 < x < 1$, which, in turn, is w/π times the variation V_S of $S(x) = \sin(x)/x$, for $x \in [-wt, w(1-t)]$. Thus

$$V_F(w,t) \le V_f \frac{w}{\pi} + \|f\|_\infty \frac{w}{\pi} V_S. \quad (8)$$

Before trying to estimate V_S, it is convenient to clarify the following. The points x, where S attains a local maximum or minimum, which will play a role in the subsequent discussion, will be denoted by x_n, $n \in \mathbf{Z}$, with the following convention: $x_0 = 0$ is the point where S attains its absolute maximum; x_n, with n positive, denotes the nth relative maximum or minimum to the right of the origin; and x_n, with n negative, has a similar meaning with respect to the left of the origin.

To estimate V_S, we need to distinguish between the following two cases: i) $0 \le t \le 1$, ii) $t < 0$ or $t > 1$.

In the first case, the interval $[-wt, w(1-t)]$ contains the origin, and, consequently, is the union of $[0,a]$ and $[-b,0]$ for some nonnegative a, b (b is zero when $t = 0$, and a is zero when $t = 1$). Since S is even, we may concentrate on the variation of S when $0 < x < a$ only. In this case, we have

$$V_S \le |S(x_0)| + 2\left(\sum_{k=1}^{i-1} |S(x_k)|\right) + |S(x_i)|$$

where x_i denotes the smallest of the $\{x_n\}$ greater than a. Now

$$\begin{aligned} V_S &\le 1 + \frac{2}{\pi}\sum_{k=1}^{i}\frac{1}{k} \\ &\le 1 + \frac{2}{\pi} + \frac{2}{\pi}\sum_{k=2}^{i}\int_{k-1}^{k}\frac{dx}{x} \\ &\le 1 + \frac{2}{\pi} + \frac{2}{\pi}\log i \\ &\le 1 + \frac{2}{\pi} + \frac{2}{\pi}\log\frac{x_i}{\pi} \end{aligned}$$

which shows that the variation of S in $[0,a]$ is $O(\log a)$.

Let us go back to $V_F(w,t)$ and (8) assuming that $\epsilon < t < 1-\epsilon$. The variation V_S of S for $x \in [-wt, w(1-t)]$ equals the sum of the variations in $[0, w|t|]$ and $[0, w|1-t|]$. By the previous result, there are positive constants A, B such that

$$V_F(w,t) \le V_f \frac{w}{\pi} + \|f\|_\infty \frac{w}{\pi}[A\log|wt| + B\log(w|1-t|)]. \quad (9)$$

However

$$A\log|wt| + B\log(w|1-t|) = = (A+B)\log w + A\log|t| + B\log|1-t|$$

remains bounded in t if $\epsilon < t < 1-\epsilon$ (recall that we are excluding the possibilities $t < 0$ or $t > 1$, which will be dealt with in the sequel). This means that $V_F(w,t) = O(w\log w)$, and therefore

$$|f_w(t) - s_N(t)| = \frac{O(w\log w)}{N}$$

which is (6). This still holds when $t = 0$ or $t = 1$. Indeed, when t is zero, the term $A\log|wt|$ in (9) can be omitted (the interval $[0, w|t|]$ need not be considered). Similarly, when $t = 1$, the term $B\log(w|1-t|)$ can be dropped.

To deal with the cases $t < 0$ or $t > 1$, let us again consider (8). We need to estimate V_S, which is the variation of S in $I = [-wt, w(1-t)]$. When $t < 0$ or $t > 1$, the interval I does not contain the origin. Since S is even, we may assume that $t < 0$, which means that I will be of the form $[a,b]$, with $b > a > 0$. The variation of S in such an interval satisfies

$$V_S \le |S(x_i)| + 2\left(\sum_{k=i+1}^{j-1}|S(x_k)|\right) + |S(x_j)|$$

where x_i is the greatest of the $\{x_n\}$ less than a, and x_j is the smallest of the $\{x_n\}$ greater than b. Then

$$V_S \le \frac{2}{\pi}\sum_{k=i}^{j}\frac{1}{k} \le \frac{2}{\pi}\sum_{k=i}^{j}\int_{k-1}^{k}\frac{dx}{x} \le \frac{2}{\pi}\log\frac{j}{i-1} \le \frac{2}{\pi}\log\frac{x_j}{x_{i-2}}.$$

This shows that V_S in $[a,b]$ is $O(\log b/a)$, and therefore, setting $a = w|t|$, $b = w|1-t|$, we have

$$V_F(w,t) \le V_f\frac{w}{\pi} + \|f\|_\infty \frac{w}{\pi}A\log\left|\frac{1-t}{t}\right|$$

where A is a constant (independent of w or t). When $t < -\epsilon$ or $t > 1+\epsilon$, the function $\log(|1-t|/|t|)$ is bounded, and therefore

$$|f_w(t) - s_N(t)| = \frac{O(w)}{N},$$

which is (5), as we wished to show.

III. Remarks

Remark 1: The theorem holds even if f is discontinuous (but still of bounded variation). In this case, the proof of (7) is somewhat longer and requires the second mean value theorem [9]. From that point on, the arguments are similar, and therefore, for brevity, we limited ourselves to the continuous case.

Remark 2: Note that (5) and (6) involve f_w, which is a bandlimited approximation to f and a sampling sum whose coefficients are values of f itself. As w grows, f_w becomes an increasingly better approximation of f. Regardless of the value of w, (5) and (6) can always be made (uniformly) as small as desired by proper choice of N.

REFERENCES

[1] R. J. Marks, II, *Introduction to Shannon Sampling and Interpolation Theory*. Berlin: Springer, 1991.

[2] A. J. Jerri, "The Shannon sampling theorem—Its various extensions and applications: A tutorial review," *Proc. IEEE*, vol. 65, no. 11, pp. 1565–1596, Nov. 1977.

[3] P. L. Butzer, "A survey of the Whittaker–Shannon sampling theorem and some of its extensions," *J. Math. Res. Expos.*, vol. 3, no. 1, pp. 185–212, Jan. 1983.

[4] J. R. Higgins, "Five short stories about the cardinal series", *Bull. Amer. Math. Soc., New Ser.*, vol. 12, no. 1, pp. 45–89, 1985.

[5] R. J. Marks, II, Ed., *Advanced Topics in Shannon Sampling and Interpolation Theory*. Berlin: Springer, 1993.

[6] A. I. Zayed, *Advances in Shannon's Sampling Theory*. Boca Raton: CRC, 1993.

[7] P. L. Butzer and W. Splettstößer, "A sampling theorem for duration-limited functions with error estimates," *Inform. Contr.*, vol. 34, pp. 55–65, 1977.

[8] P. L. Butzer, S. Ries, and R. L. Stens, "Approximation of continuous and discontinuous functions by generalized sampling series," *J. Approx. Theory*, vol. 50, no. 1, pp. 25–39, 1987.

[9] E. C. Titchmarsh, *The Theory of Functions*. Oxford: Oxford University Press, 1986.