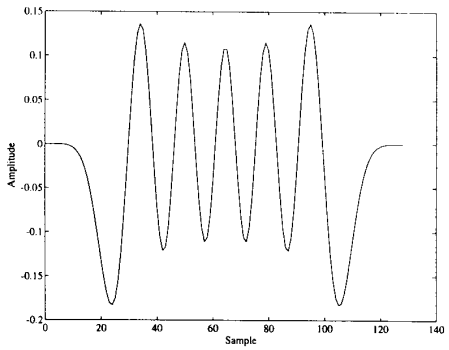

Fig. 3. DPSS $v^{(10)}(N = 128, W = .10)$.

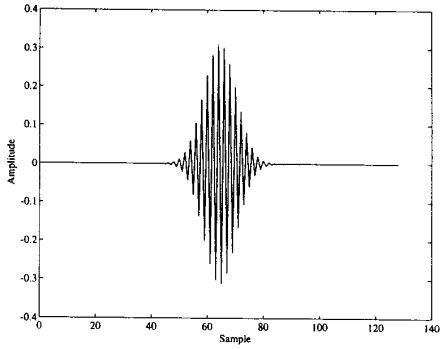

Fig. 4. DPSS $v^{(127)}(N = 128, W = .10)$.

for $i = 1, 2, \ldots$
Solve $(T - \lambda_k I)v_2 = v_1$ for v_2
Normalize v_2 to get a new v_1
end.

The desired DPSS is the last normalized solution, v_1. Even though $(T-\lambda_k I)$ is nearly singular for eigenvalues λ_k of very good accuracy, inverse iteration is known to work well in these cases, many times requiring only one or two iterations [10]. The norm of the residue $r = (T - \lambda_k I)v_1$ may be used to determine the accuracy of the final result. When the known eigenvalue λ_k was accurate to within $\mathbf{TOL} = 1e - 6$ using the bisection method explained previously, it was found that three iterations of the above procedure produced accurate DPSS's. Because matrix $T(N, W)$ is tridiagonal, solving the system equations during each iteration for the latest eigenvector estimate was also simplified (p. 155 of [9]).

III. Some Examples

Figs. 1–5 show some examples of DPSS's generated using the technique described above.

IV. Conclusion

A method for accurate and computationally efficient generation of any single DPSS of large length was presented. The method is easy to implement and should prove useful in situations where only a few DPSS's out of a large set are desired. The availability of long length DPSS's will open the oportunity for their use in applications requiring sequences of long length.

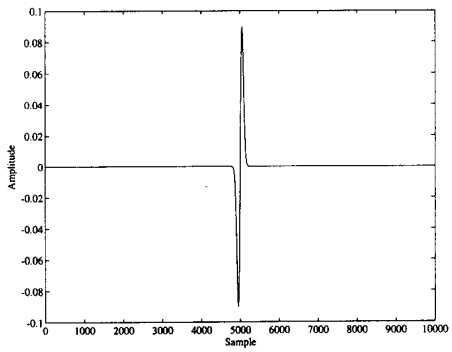

Fig. 5. DPSS $v^{(1)}(N = 10000, W = .10)$.

References

[1] A. D. Wyner, "An analog scrambling scheme which does not expand bandwidth, part I: Discrete time," *IEEE Trans. Inform. Theory*, vol. IT-25, pp. 261–274, May 1979.
[2] ——, "Signal design for PAM data transmission to minimize excess bandwidth," *Bell Syst. Tech. J.*, vol. 57, pp. 3277–3307, Nov. 1978.
[3] A. K. Jain and S. Ranganath, "Extrapolation algorithms for discrete signals with application in spectral estimation," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-29, pp. 830–845, Aug. 1981.
[4] J. D. Mathews, J. K. Breakall, and G. K. Karawas, "The discrete prolate spheroidal filter as a digital signal processing tool," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 33, pp. 1471–1478, Dec. 1985.
[5] D. Slepian and H. O. Pollak, "Prolate spheroidal wave functions, Fourier analysis and uncertainty —I," *Bell Syst. Tech. J.*, vol. 40, pp. 43–64, Jan. 1961.
[6] D. Slepian, "Prolate spheroidal wave functions, Fourier analysis and uncertainty—V: The discrete case," *Bell Syst. Tech. J.*, vol. 57, pp. 1317–1430, May 1978.
[7] F. P. Gantmacher, *Matrix Theory*. New York: Chelsea, 1959, vol. 1.
[8] D. M. Gruenbacher, "Generation of discrete prolate spheroidal sequences for signaling waveforms," M.S. thesis, Kansas State Univ., Manhattan, KS, Oct. 1991.
[9] G. H. Golub and C. F. Van Loan, *Matrix Computations*. Baltimore: Johns Hopkins Univ., 1989, 2nd ed.
[10] G. Peters and J. H. Wilkinson, "Inverse iteration, ill-conditioned equations and Newton's method," *SIAM Rev.*, vol. 21, pp. 339–360, July 1979.

Noniterative and Fast Iterative Methods for Interpolation and Extrapolation

Paulo Jorge S. G. Ferreira

***Abstract*—In this correspondence we study the band-limited interpolation and extrapolation problems for finite-dimensional signals. We show that these problems can be easily reduced to the solution of a set of linear equations with a real symmetric positive-definite matrix S with spectral radius $\rho(\mathbf{S}) < 1$. Thus, the equations can be solved directly or using successive approximation methods. A number of other well known methods which may substantially increase the convergence rate may also be readily applied and are briefly discussed. We state conditions for their convergence, and illustrate their performance through an example.**

Manuscript received May 10, 1994; revised June 6, 1994. The associate editor coordinating the review of this paper and approving it for publication was Prof. Stanley J. Reeves.

The author is with the Departamento de Electrónica e Telecomunicações, Universidade de Aveiro, Instituto de Engenharia de Sistemas e Computadores, 3800 Aveiro, Portugal.

IEEE Log Number 9404788.

I. Introduction

In this correspondence, we study the band-limited interpolation and extrapolation problems for finite-dimensional signals (vectors) in the direct and Fourier transformed domains. In the terminology of [1] this corresponds to the discrete-discrete case. We show that under certain conditions this problem can be easily reduced to the solution of a set of linear equations with a real symmetric positive-definite matrix $\mathbf{S}$ with spectral radius $\rho(\mathbf{S}) < 1$. This fact allows the reconstruction problem to be solved noniteratively or using iterative methods. In either case, the algorithms have minimum dimensionality, that is, the size of the matrix and vectors is equal to the number of missing observations. This stands in contrast with works based on the finite-dimensional Papoulis-Gerchberg algorithm [2], [3], where this size is given by the total number of observations, and which results in singular nonHermitian iteration matrices.

As a consequence of the Hermitian positive-definite character of $\mathbf{S}$, a number of well known alternative iterations can be readily applied to the solution of the restoration problem considered. Again, this is not the case with methods which directly fit into the constrained reconstruction framework presented in [4], or based on the Papoulis-Gerchberg algorithm [2], [3]. We discuss the relations among these methods, and state conditions for the convergence of some of the alternative iterations, illustrating their performance through an example.

II. Preliminaries

The discrete Fourier transform of $\mathbf{x} \in \mathbb{C}^n$ is denoted by $\hat{\mathbf{x}}$ and defined by $\hat{\mathbf{x}} = \mathbf{F}\mathbf{x}$, where $\mathbf{F}$ is the unitary $n \times n$ Fourier matrix with elements $F_{ik} = \frac{1}{\sqrt{n}} e^{j2\pi ik/n}$. The inverse transformation is given by $\mathbf{x} = \mathbf{F}^H \hat{\mathbf{x}}$. We say that $\mathbf{x}$ is band-limited if

$$\hat{\mathbf{x}} = \mathbf{\Gamma}\hat{\mathbf{x}} \tag{1}$$

where $\mathbf{\Gamma}$ is a diagonal matrix containing only zeros or ones, and with at least one zero on the diagonal. The *bandwidth* of a band-limited signal $\mathbf{x}$ is the quantity $\beta = q/n$, where $q < n$ is the number of nonzero elements of $\mathbf{\Gamma}$.

Let A and B be two subsets of $\{0, 1, \ldots, n-1\}$ with k elements. We say that A and B are *equivalent* if the elements of A can be obtained by addition of an integer constant, modulo n, to the elements of B. This means that A and B are related by a circular shift. We say that a subset of $\{0, 1, \ldots, n-1\}$ of cardinality $k < n$ is *contiguous* if it is equivalent to $\{0, 1, \ldots, k-1\}$.

Equation (1) means that a band-limited signal $\mathbf{x}$ satisfies an equation of the form $\mathbf{x} = \mathbf{B}\mathbf{x}$, where $\mathbf{B}$ is a band-limiting matrix given by $\mathbf{B} = \mathbf{F}^H \mathbf{\Gamma} \mathbf{F}$. It is real, symmetric, circulant, nonnegative definite, idempotent, and has q unitary and $n-q$ zero eigenvalues, q being the number of nonzero elements of $\mathbf{\Gamma}$. We say that the set of zero eigenvalues of $\mathbf{B}$ is contiguous if the corresponding set of subscripts in $\mathbf{\Gamma}$ is contiguous. The following results will be needed in the sequel.

Theorem II.1: Let $\mathbf{C}$ be a $n \times n$ nonnegative definite circulant matrix with k contiguous zero eigenvalues. Then every principal submatrix of $\mathbf{C}$ of order not greater than $n-k$ is positive definite.

Proof: Since $\mathbf{F}$ diagonalizes all circulants, the quadratic form $o(\mathbf{x}) = \mathbf{x}^H \mathbf{C}\mathbf{x}$ can be reduced to $o(\mathbf{x}) = \hat{\mathbf{x}}^H \mathbf{\Gamma}\hat{\mathbf{x}}$, where $\mathbf{\Gamma} = \mathbf{F}\mathbf{C}\mathbf{F}^H$ is diagonal. We are interested in the restriction of $o(\mathbf{x})$ to subspaces of dimension not greater than $n-k$, characterized by $x_i = 0$ (for all i belonging to a given set S_t of at least k elements).

If the theorem was false, there would exist a nonzero vector $\mathbf{v}$ belonging to one such subspace and such that $o(\mathbf{v}) = 0$. This would imply that $\hat{v}_i = 0$ for all i belonging to a certain contiguous set S_f with $n-k$ elements. That is, $\mathbf{v}$ would have to satisfy

$$\sum_{i \notin S_t} F_{ki} v_i = 0 \text{ for all } k \in S_f. \tag{2}$$

This is a set of $n-k$ equations for no more than $n-k$ unknowns v_i. Its matrix has linear independent columns when S_f is contiguous (if it is square it will be Vandermonde). It follows that $v_i = 0$ for all $i \notin S_t$, meaning that $\mathbf{v}$ is the zero vector, a contradiction. □

Note that the contiguity of S_f (or, by duality, S_t), is not necessary for the linear independence of the columns of the matrix. It is easy to construct a counterexample using the results in [5].

Lemma II.1: If $\mathbf{A}$ is a Hermitian $n \times n$ nonnegative definite matrix with eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$, then

$$n\lambda_1 \le \sum_{i=1}^{n} A_{ii} \le n\lambda_n.$$

Proof: The trace of $\mathbf{A}$ is the sum of its eigenvalues, and these are real and nonnegative. The inequality is a direct consequence of this. □

III. Results

Let $\mathbf{x}$ be a band-limited vector, satisfying $\mathbf{x} = \mathbf{B}\mathbf{x}$. Assume that a subset of the x_i is unknown. Our purpose is to state conditions for its recovery from the remaining x_i.

Let U be the set of subscripts of the k unknown x_i. The equations $\mathbf{x} = \mathbf{B}\mathbf{x}$ imply that

$$x_i = \sum_{k \in U} B_{ik} x_k + \sum_{k \notin U} B_{ik} x_k$$

for $i \in U$. This set of k equations can be written in matrix form as $\mathbf{u} = \mathbf{S}\mathbf{u} + \mathbf{h}$, where $\mathbf{u}$ is the k-dimensional vector of unknown samples, $\mathbf{S}$ is a $k \times k$ principal submatrix of $\mathbf{B}$, and $\mathbf{h}$ is a known k-dimensional vector. A similar approach has been used in a different context [6]. The following theorem states conditions under which these equations will have a solution.

Theorem III.1: Let $\mathbf{B}$ be a band-limiting matrix with k unity contiguous eigenvalues, and let $\mathbf{x}$ be a vector such that $\mathbf{x} = \mathbf{B}\mathbf{x}$. Let U be a subset of $\{0, 1, \ldots, n-1\}$ with $p \le n-k$ elements. Denote by $\mathbf{S}$ the $p \times p$ matrix obtained by deleting from $\mathbf{B}$ the rows and corresponding columns whose subscripts do not belong to U, and let $\mathbf{h}$ be a p-dimensional vector given by

$$h_i = \sum_{j \notin U} B_{ij} x_j \qquad (i \in U).$$

Then, the p-dimensional vector $\mathbf{u}$ containing the x_i $(i \in U)$ can be found noniteratively from $\mathbf{u} = (\mathbf{I} - \mathbf{S})^{-1}\mathbf{h}$, or iteratively from $\mathbf{u} = \lim_{i \to \infty} \mathbf{u}^{(i)}$, where the $\mathbf{u}^{(i)}$ are defined by $\mathbf{u}^{(i+1)} = \mathbf{S}\mathbf{u}^{(i)} + \mathbf{h}$, and the limit is independent of $\mathbf{u}^{(0)}$.

Proof: The matrix $\mathbf{I} - \mathbf{B}$ is a circulant nonnegative definite matrix with k contiguous zero eigenvalues ($\mathbf{B} = \mathbf{F}^H \mathbf{\Lambda} \mathbf{F}$ is equivalent to $\mathbf{I} - \mathbf{B} = \mathbf{F}^H (\mathbf{I} - \mathbf{\Lambda})\mathbf{F}$). By Theorem II.1 every principal submatrix of $\mathbf{I} - \mathbf{B}$ of order not greater than $n-k$ is positive definite. Hence, $\mathbf{I} - \mathbf{S}$ has an inverse.

In order to justify the convergence of the iterative algorithm we have to show that $\rho(\mathbf{S}) < 1$. Since $\mathbf{I} - \mathbf{S}$ is a principal submatrix of a nonnegative definite idempotent matrix, its eigenvalues must all lie in the interval $[0, 1]$. But since $\mathbf{I} - \mathbf{S}$ is nonsingular, $\mathbf{S}$ may have no eigenvalues equal to 1. □

The identity $\mathbf{u} = \mu(\mathbf{S}\mathbf{u} + \mathbf{h}) + (1 - \mu)\mathbf{u}$, for $\mu \in \mathbb{R}$, suggests the iteration

$$\mathbf{u}^{(i+1)} = [\mu\mathbf{S} + (1 - \mu)\mathbf{I}]\mathbf{u}^{(i)} + \mu\mathbf{h}. \tag{3}$$

If λ is an eigenvalue of $\mathbf{S}$, $\mu\lambda + (1-\mu)$ will be an eigenvalue of $\mu\mathbf{S}+(1-\mu)\mathbf{I}$. It is easy to see that iteration (3) will always converge if $\mu \in (0,2)$, and that proper choice of $\mu \in (1,2)$ may lead to faster convergence.

A variety of more sophisticated iterative methods can be applied to the solution of the interpolation and extrapolation problems. The iteration matrices and vectors of the methods which we will use are summarized in Table I. For details see, for example, [7] and [8]. We will now state conditions under which convergence of these methods is ensured, starting with Jacobi's iteration.

Lemma III.1: The vector $\mathbf{v}$ *is an eigenvector of* $\mathbf{S}$ *pertaining to the eigenvalue* λ *if and only if it is also an eigenvector of the corresponding Jacobi matrix, pertaining to the eigenvalue* $f(\lambda) = (\lambda - \beta)/(1-\beta)$.

Proof: Recall that β is the bandwidth of the matrix $\mathbf{B}$ of which $\mathbf{S}$ is a principal submatrix. The diagonal of $\mathbf{B}$ is equal to the diagonal of $\beta\mathbf{I}$. The Jacobi matrix can be written as $\frac{1}{1-\beta}(\mathbf{S} - \beta\mathbf{I})$ (see Table I). Clearly, $\mathbf{Sv} = \lambda\mathbf{v}$ implies

$$\frac{1}{1-\beta}(\mathbf{S} - \beta\mathbf{I})\mathbf{v} = \frac{\lambda - \beta}{1-\beta}\mathbf{v}.$$

Conversely, if

$$\frac{1}{1-\beta}(\mathbf{S} - \beta\mathbf{I})\mathbf{v} = \mu\mathbf{v}$$

then $\mathbf{Sv} = [(1-\beta)\mu + \beta]\mathbf{v}$, and inversion of $g(\mu) = (1-\beta)\mu + \beta$ gives $f(\lambda)$. □

The following theorem gives a necessary and sufficient condition for the convergence of Jacobi's method under the conditions of Theorem III.1.

Lemma III.2: Let $\mathbf{B}$ *be a band-limiting matrix with bandwidth* β *and let* $\mathbf{S}$ *be a principal submatrix of* $\mathbf{B}$ *with smallest and largest eigenvalues* $0 < \lambda_{\min} < \lambda_{\max} < 1$. *The spectral radius of Jacobi's matrix* $\mathbf{J}_1 = \frac{1}{1-\beta}(\mathbf{S} - \beta\mathbf{I})$ *is*

$$\rho(\mathbf{J}_1) = \max\left\{\frac{\beta - \lambda_{\min}}{1-\beta}, \frac{\lambda_{\max} - \beta}{1-\beta}\right\} \tag{4}$$

and Jacobi's method converges if and only if $\beta < (\lambda_{\min} + 1)/2$.

Proof: Let $\lambda < 1$ be an eigenvalue of $\mathbf{S}$. Then $f(\lambda)$ will be an eigenvalue of $\mathbf{J}_1$, and Jacobi's method will converge if and only if $|f(\lambda)| < 1$ for all eigenvalues λ. Note that $|f|$ is zero for $\lambda = \beta$ and increases linearly as $\lambda \to 0$ and $\lambda \to 1$.

Since the diagonal elements S_{ii} of $\mathbf{S}$ are all equal to β, it follows from Lemma II.1 that $\lambda_{\min} \le \beta$ and $\lambda_{\max} \ge \beta$, with equality if and only if $\mathbf{S}$ is diagonal. This justifies (4). It can be seen using (4) that $\rho(\mathbf{J}_1) < 1$ is equivalent to $\lambda_{\min} > 2\beta - 1$. □

The following result relates the convergence rates of the iterative method of Theorem III.1 and of Jacobi's method, as a function of the bandwidth β.

Theorem III.2: Let $\mathbf{B}$ *be a band-limiting matrix with bandwidth* β *and let* $\mathbf{S}$ *be a principal submatrix of* $\mathbf{B}$ *with smallest and largest eigenvalues* $0 < \lambda_{\min} < \lambda_{\max} < 1$. *Jacobi's method converges with better asymptotic convergence rate than the iteration of Theorem III.1 if and only if*

$$\beta < \frac{\lambda_{\max} + \lambda_{\min}}{1 + \lambda_{\max}}.$$

Proof: We need to show that $\rho(\mathbf{J}_1)$, as given by (4), is less than $\rho(\mathbf{S}) = \lambda_{\max}$ under the stated conditions. In the first place, note that

$$\lambda_{\max} - \frac{\lambda_{\max} - \beta}{1-\beta} = \frac{\beta}{1-\beta}(1 - \lambda_{\max}) > 0$$

TABLE I
ITERATION MATRICES AND VECTORS FOR THE JACOBI, JOR, GAUSS-SEIDEL AND SOR METHODS, APPLIED TO THE SOLUTION OF $\mathbf{Mx} = \mathbf{h}$. THE MATRICES $\mathbf{M}$, $\mathbf{D}$, $\mathbf{L}$ AND $\mathbf{U}$ ARE RELATED BY $\mathbf{M} = \mathbf{D} - \mathbf{A} = \mathbf{D} - \mathbf{L} - \mathbf{U}$. MOREOVER, $\mathbf{D} = \mathrm{diag}(\mathbf{M})$, $\mathbf{U}$ IS STRICTLY UPPER TRIANGULAR, $\mathbf{L}$ IS STRICTLY LOWER TRIANGULAR, AND μ IS A REAL PARAMETER.

Method	*Iteration matrix*	*Vector*
Jacobi	$\mathbf{J}_1 = \mathbf{D}^{-1}\mathbf{A}$	$\mathbf{a}_1 = \mathbf{D}^{-1}\mathbf{h}$
JOR	$\mathbf{J}_\mu = (1-\mu)\mathbf{I} + \mu\mathbf{D}^{-1}\mathbf{A}$	$\mathbf{a}_\mu = \mu\mathbf{D}^{-1}\mathbf{h}$
Gauss-Seidel	$\mathbf{G}_1 = (\mathbf{D} - \mathbf{L})^{-1}\mathbf{U}$	$\mathbf{b}_1 = (\mathbf{D} - \mathbf{L})^{-1}\mathbf{h}$
SOR	$\mathbf{G}_\mu = (\mathbf{D} - \mu\mathbf{L})^{-1}[(1-\mu)\mathbf{D} + \mu\mathbf{U}]$	$\mathbf{b}_\mu = \mu(\mathbf{D} - \mu\mathbf{L})^{-1}\mathbf{h}$

and so $(\lambda_{\max} - \beta)/(1-\beta)$ is always less than $\lambda_{\max}$. On the other hand

$$\lambda_{\max} - \frac{\beta - \lambda_{\min}}{1-\beta} = (\lambda_{\max} + \lambda_{\min}) - \beta(1 + \lambda_{\max})$$

which is positive if and only if $\beta < (\lambda_{\max} + \lambda_{\min})/(1 + \lambda_{\max})$. □

This condition is necessary in the sense that it is not implied by the convergence condition $\beta < (1 + \lambda_{\min})/2$. Note that

$$\frac{\lambda_{\max} + \lambda_{\min}}{1 + \lambda_{\max}} < \frac{\lambda_{\min} + \lambda_{\max}}{2} < \frac{1 + \lambda_{\min}}{2}.$$

Introducing a relaxation parameter in Jacobi's method, we arrive at what Young [8] calls the JOR method. The eigenvalues of the JOR matrix are easily related to those of $\mathbf{S}$.

Lemma III.3: The vector $\mathbf{v}$ *is an eigenvector of* $\mathbf{S}$ *pertaining to the eigenvalue* λ *if and only if it is also an eigenvector of the corresponding JOR matrix pertaining to the eigenvalue*

$$1 - \mu + \mu\frac{\lambda - \beta}{1-\beta}.$$

Proof: As before, β is the bandwidth of the matrix $\mathbf{B}$ of which $\mathbf{S}$ is a principal submatrix. Note that an eigenvalue λ of the Jacobi matrix is mapped into an eigenvalue $1 - \mu + \mu\lambda$ of the JOR matrix (see Table I). □

The interesting situation occurs when $1 < \mu \le 2$, since it may lead to a faster convergent method. Comparing the eigenvalues of $\mathbf{J}_\mu$ as a function of μ, it is easy to see that this will be the case if and only if $\mathbf{J}_1$ has no eigenvalues less than $(\mu - 1)/(\mu + 1) < 1/3$.

The Gauss-Seidel method is similar to Jacobi's method except that the new values of the components of $\mathbf{u}^{(i)}$ are used whenever available. This enables in-place calculation of the successive approximations and often leads to significantly better convergence rates.

In general, the analysis of the Gauss-Seidel method, or of the more general SOR method, is not simple. However, in our case the matrix $\mathbf{I} - \mathbf{S}$ is Hermitian, and its diagonal elements are all positive. Therefore we may directly apply the Ostrowski-Reich theorem [7], which leads to the following corollary.

Corollary III.1: Let $\mathbf{B}$ *be a band-limiting matrix with bandwidth* β *and let* $\mathbf{S}$ *be a principal submatrix of* $\mathbf{B}$ *such that* $\rho(\mathbf{S}) < 1$. *Then the SOR method applied to the solution of* $\mathbf{x} = \mathbf{Sx} + \mathbf{h}$ *will be convergent if and only if* $0 < \mu < 2$.

For any of these methods, the optimum value of μ can often be found experimentally. It is safe to start with $\mu = 1$ and increase it until the desired effect is obtained.

IV. RELATION WITH OTHER ITERATIVE METHODS

The iterative method given by Theorem III.1 is related to the finite-dimensional analog [2], [3] of the Papoulis-Gerchberg algorithm [9], [10], as we will now see.

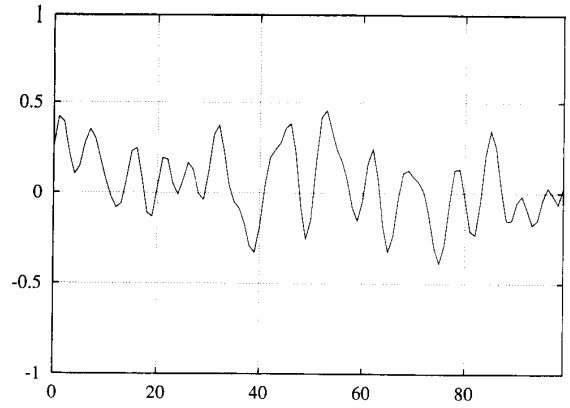

Fig. 1. Band-limited random data vector with dimension 100. The bandwidth is 0.41.

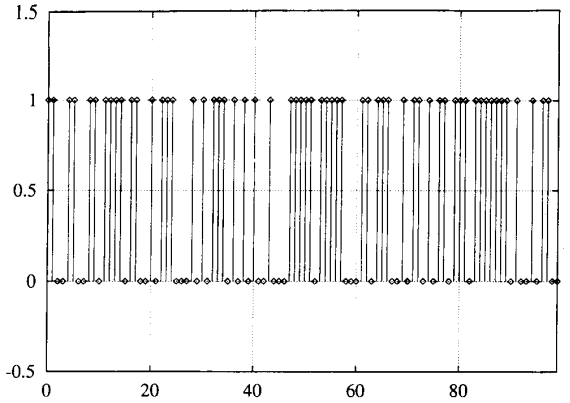

Fig. 2. Sampling sequence used. The density is 0.6, corresponding to a sampling process which introduced a 40% data loss.

Let $\mathbf{x}$ be a band-limited vector, satisfying $\mathbf{x} = \mathbf{B}\mathbf{x}$, and let $\mathbf{D}$ denote the $n \times n$ diagonal matrix defined by $D_{ii} = 0$ if $i \in U$, $D_{ii} = 1$ otherwise. As usual, U is the set of subscripts of k unknown x_i. An algorithm for the recovery of the unknown x_i can be found using several methods. The identity

$$\mathbf{x} = \mathbf{B}(\mathbf{I} - \mathbf{D})\mathbf{x} + \mathbf{B}\mathbf{D}\mathbf{x}$$

for example, immediately suggests the iteration

$$\mathbf{x}^{(i+1)} = \mathbf{B}(\mathbf{I} - \mathbf{D})\mathbf{x}^{(i)} + \mathbf{B}\mathbf{y}$$

where $\mathbf{y} = \mathbf{D}\mathbf{x}$ is a known vector. This method is discussed in [3], and its connection with the Papoulis-Gerchberg algorithm, Youla's alternating projection method [11], or the constrained restoration framework presented in [4] is easily recognized. This iteration can be reduced to the one given in Theorem III.1. In fact, since $y_i = 0$ for $i \in U$, we may rewrite it as

$$x_k^{(i+1)} = \sum_{j \in U} B_{kj} x_j^{(i)} + \sum_{j \notin U} B_{kj} y_j$$

and restriction to $k \in U$ gives an equation of the form $\mathbf{u} = \mathbf{S}\mathbf{u} + \mathbf{h}$, the required result.

V. Examples

A data vector with 100 randomly generated points was filtered with a low-pass filter having bandwidth 0.41. The resulting signal is depicted in Fig. 1.

The process which resulted in lost data was simulated using a randomly generated binary sequence of length 100 and density 0.6, depicted in Fig. 2. It can be thought of as a sampling sequence which introduces a 40% data loss. The zero components of this sequence mark the position of the unknown samples.

The error evolution for the several methods discussed is depicted in Fig. 3 for a few values of the relaxation parameter. The error is, by definition, the Euclidean distance between the reconstructed and original vectors, i.e., the RMS error. The convergence rate of the SOR method is clearly the best, and the effect of μ on the convergence rate is apparent.

VI. Conclusion

A problem commonly found in signal processing is that of recovering k lost samples of a band-limited discrete signal with a total of n samples. The often-studied extrapolation problem, for example, is a special case of this problem. There are a number of approaches to its

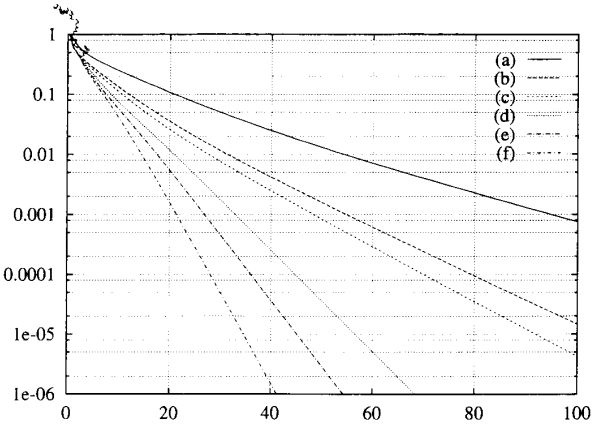

Fig. 3. RMS error versus number of iterations, for the methods: (a) simple iteration; (b) Jacobi; (c) JOR with $\mu = 1.1$; (d) Gauss-Seidel; (e) SOR with $\mu = 1.1$; and (f) SOR with $\mu = 1.2$.

solution, including Papoulis-Gerchberg iteration, but they generally lead to a $n \times n$ singular nonHermitian iteration matrix.

We showed that the finite-dimensional band-limited interpolation and extrapolation problems can be reduced to the solution of a linear set of equations with a symmetric real positive-definite $k \times k$ matrix, with spectral radius less than unity. This immediately gives noniterative and iterative solutions to the problem. The convergence rate of the iterative solution, which, as we have shown, is related to a finite-dimensional alternating projection method, may be improved using relaxation, as expressed by (3). Moreover, a number of well understood and more favorable iterative methods may also be applied, and often allow substantial improvements in the convergence rate. The main advantages of the outlined approach are its simplicity, the fact that it directly leads to a noniterative solution, and the ease with which faster iterations may be applied.

Each iteration of the Papoulis-Gerchberg algorithm requires one multiplication of a circulant $n \times n$ matrix by a vector, that is, a circular convolution of length n, or, equivalently, a direct and an inverse discrete Fourier transform. These can be performed using the FFT algorithm, which requires about $O(n \log_2 n)$ multiplications when n is a power of two. If n is not a highly composite number, each iteration requires $O(n^2)$ multiplications. On the other hand, the methods discussed in this work require $O(k^2)$ multiplications per iteration. If n is not a highly composite number, or if k is

small enough, the reasons why they are computationally attractive are twofold: the more favorable convergence rates, meaning that less iterations are required to obtain usable results; and less computational effort per iteration, meaning that each iteration will complete faster.

REFERENCES

[1] J. L. C. Sanz and T. S. Huang, "Some aspects of band-limited signal extrapolation: Models, discrete approximations, and noise," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-31, no. 6, pp. 1492–1501, Dec. 1983.

[2] M. C. Jones, "The discrete Gerchberg algorithm," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-34, no. 3, pp. 624–626, June 1986.

[3] P. J. S. G. Ferreira, "Interpolation and the discrete Papoulis-Gerchberg algorithm," *IEEE Trans. Signal Processing*, vol. 42, no. 10, pp. xxx-xxx, Oct. 1994.

[4] R. W. Schafer, R. M. Mersereau, and M. A. Richards, "Constrained iterative restoration algorithms," *Proc. IEEE*, vol. 69, no. 4, pp. 432–450, Apr. 1981.

[5] P. J. S. G. Ferreira, "A group of permutations that commute with the discrete Fourier transform," *IEEE Trans. Signal Processing*, vol. 42, no. 2, pp. 444–445, Feb. 1994.

[6] ——, "Incomplete sampling series and the recovery of missing samples from oversampled band-limited signals," *IEEE Trans. Signal Processing*, vol. 40, no. 1, pp. 225–227, Jan. 1992.

[7] R. S. Varga, *Matrix Iterative Analysis*. Englewood Cliffs, NJ: Prentice-Hall, 1962.

[8] D. M. Young, *Iterative Solution of Large Linear Systems*. Orlando, FL: Academic, 1971.

[9] R. W. Gerchberg, "Super resolution through error energy reduction," *Opt. Acta*, vol. 21, no. 9, pp. 709–720, 1974.

[10] A. Papoulis, "A new algorithm in spectral analysis and band-limited extrapolation," *IEEE Trans. Circ. Syst.*, vol. CS-22, no. 9, pp. 735–742, Sept. 1975.

[11] D. C. Youla, "Generalized image restoration by the method of alternating orthogonal projections," *IEEE Trans. Circ. Syst.*, vol. CS-25, no. 9, pp. 694–702, Sept. 1978.

Counting the Degrees of Freedom When Using AIC and MDL to Detect Signals

Douglas B. Williams

***Abstract*—In the well known paper by Wax and Kailath [2], the AIC and MDL criteria for determining the number of signals in a multichannel time-series are presented. An essential element of these criteria is the number of degrees of freedom in the model. We propose a different number for the degrees of freedom and show that the resulting MDL criterion performs noticeably better than the previous criterion. We also show that the same criteria are appropriate for either real or complex data.**

Manuscript received August 22, 1992; revised May 10, 1994. This work was supported by an NSF Research Initiation Award under Contract number MIP-9010790 and by the Office of Naval Research under Contract number N00014-91-J-4129. The associate editor coordinating the review of this paper and approving it for publication was Prof. Daniel Fuhrmann.

The author is with the Georgia Institute of Technology, School of Electrical and Computer Engineering, Digital Signal Processing Laboratory, Atlanta, GA 30332-0250 USA.

IEEE Log Number 9404549.

I. INTRODUCTION

AIC and MDL are model-order determination algorithms that can also be used for determining how many signals are present in vector-valued data. Suppose the $M \times 1$ complex vector $\mathbf{x}(t)$ can be modeled as

$$\mathbf{x}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t) \tag{1}$$

where

- $\mathbf{A}$ is a rank(P) $M \times P$ complex matrix whose columns are determined by the unknown parameters associated with each signal,
- $\mathbf{s}(t)$ is a $P \times 1$ complex vector whose pth element is the waveform of the pth signal, and
- $\mathbf{n}(t)$ is a complex, stationary, and ergodic Gaussian process with zero mean and covariance matrix $E\{\mathbf{n}(t)\mathbf{n}'(t)\} = \sigma_n^2\mathbf{I}$.

The problem is to determine P from N observations of $\mathbf{x}(t)$; i.e., $\mathbf{x}(t_1), \mathbf{x}(t_2), \ldots, \mathbf{x}(t_N)$.

Let $\mathbf{R} = E\{\mathbf{x}(t)\mathbf{x}'(t)\}$ be the covariance matrix of the data $\mathbf{x}(t)$, and $\widehat{\mathbf{R}} = \frac{1}{N}\sum_{i=1}^{N}\mathbf{x}(t_i)\mathbf{x}'(t_i)$ be an estimate of $\mathbf{R}$. The matrix $\mathbf{R}$ is completely determined by its eigenvalues λ_i and eigenvectors $\mathbf{v}_i$. Furthermore, if P uncorrelated signals are present, the $M-P$ smallest eigenvalues of $\mathbf{R}$ are all equal to the noise power σ_n^2, and the vector of parameters $\Theta^{(P)}$ specifying $\mathbf{R}$ can be written as

$$\Theta^{(P)} = \left[\lambda_1, \ldots, \lambda_P, \sigma_n^2, \mathbf{v}_1^T, \ldots, \mathbf{v}_P^T\right]^T.$$

The number of signals are determined from the estimated covariance matrix $\widehat{\mathbf{R}}$. If $l_1, \ldots, l_M$ are the eigenvalues of $\widehat{\mathbf{R}}$ in decreasing order then

$$\mathrm{AIC}(\widehat{P}) = -2(M-\widehat{P})N\ln\left(\frac{\prod_{i=\widehat{P}+1}^{M} l_i^{1/(M-\widehat{P})}}{\frac{1}{M-\widehat{P}}\sum_{i=\widehat{P}+1}^{M} l_i}\right) + 2k$$

and

$$\mathrm{MDL}(\widehat{P}) = -(M-\widehat{P})N\ln\left(\frac{\prod_{i=\widehat{P}+1}^{M} l_i^{1/(M-\widehat{P})}}{\frac{1}{M-\widehat{P}}\sum_{i=\widehat{P}+1}^{M} l_i}\right) + \frac{1}{2}k\ln N$$

where k is the number of degrees of freedom in the model of $\mathbf{R}$ assuming $\widehat{P}$ signals. The estimated value of P is the value from $\widehat{P} = 0, 1, \ldots, M-1$ that minimizes the chosen criterion. Counting the degrees of freedom is where the criteria in this paper differ from those of Wax and Kailath [2]. There are, of course, numerous ways to revise the penalty terms in AIC and MDL. In the following section we present two intuitive arguments for changing the degrees of freedom that are used in these penalty terms. Simulations are then used to demonstrate the resulting improvement in performance.

II. DETERMINING THE NUMBER OF DEGREES OF FREEDOM

Insight into the number of degrees of freedom for this problem can be gained from the classical theory of hypothesis testing. The probability density function of $l_1, \ldots, l_M$ given $\lambda_1, \ldots, \lambda_P$ and $\lambda_{P+1} = \cdots = \lambda_M = \sigma_n^2$ can be approximated by

$$\begin{aligned} f_P(l_1, \ldots, l_M \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_P \geq \lambda_{P+1} = \cdots \\ = \lambda_M = \sigma_n^2) \\ = h(M, N, P, \sigma_n^2, l_i, \lambda_i) \\ \times N^{MN-\frac{1}{2}P(2M-P-1)} \prod_{i=1}^{P} \frac{l_i^{N-M}}{\lambda_i^N} \prod_{i=P+1}^{M} \frac{l_i^{N-M}}{(\sigma_n^2)^N}. \end{aligned}$$