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Incomplete Sampling Series and the Recovery of Missing Samples from Oversampled Band-Limited Signals

Paulo J. S. G. Ferreira

***Abstract*—It is well known that a band-limited oversampled signal is completely determined even if an arbitrary finite number of samples is lost. We show that an alternative simple proof of this fact carries over to generalized sampling expansions. More precisely, we show that any finite number of missing samples can be recovered from the remaining ones, in the case of generalized Kramer sampling expansions, if an appropriate oversampling constraint is satisfied. The recovery can be accomplished either iteratively or noniteratively.**

I. Introduction

We call a *signal* any function belonging to the space of squared Lebesgue integrable functions defined on the real line, $L_2(-\infty, +\infty)$. A band-limited signal f has a Fourier transform $\hat{f}$ with compact support, and can be written as

$$f(t) = \int_{-\Omega}^{+\Omega} \hat{f}(u) e^{j2\pi ut}\, du \tag{1}$$

where

$$\hat{f}(u) = \int_{-\infty}^{+\infty} f(t) e^{-j2\pi ut}\, dt. \tag{2}$$

The set of all signals for which the representation (1) is valid will be denoted by B_Ω. We define a *sampling set* to be any set of distinct reals $S = \{t_0, t_{\pm 1}, t_{\pm 2}, \cdots\}$ such that the functions

$$K(x, t_n) = e^{j2\pi x t_n} \tag{3}$$

are complete and orthogonal on some interval $(-a, a)$.

Given a band-limited signal $f \in B_\Omega$ and a sampling set $S = \{t_n\}$, we say that S oversamples f if the interval on which the set of functions (3) is complete and orthogonal strictly contains $[-\Omega, +\Omega]$. The usefulness of this terminology will become apparent later.

The sampling set that corresponds to the complex exponential functions (3) is equivalent to $\boldsymbol{Z} = \{0, \pm 1, \pm 2, \cdots\}$. As it is well known from the classical theory of Fourier series, the set of functions (3) with $t_n = n$ is complete and orthogonal on $(-1/2, 1/2)$, and therefore oversamples any band-limited signal in B_Ω with $\Omega < 1/2$.

By the Whittaker–Kotelnikov–Shannon (WKS) sampling theorem [1] a signal $f \in B_\Omega$ is uniquely determined by its equidistant samples taken T seconds apart, provided that $2\Omega \le 1/T$. Following [2], we define the parameter $r = 2\Omega T$, which cannot exceed unity for the WKS theorem to hold. It has been shown [2] by two different methods that when $r < 1$ the samples $\{f(nT)\}$ are redundant, in the sense that any finite number of them can be obtained from the remaining ones by solving a system of linear equations.

In this correspondence, we show that an alternative and simpler proof of the result discussed in [2] can be extended to generalized sampling expansions, in the sense of Kramer, if the concepts of *sampling set* and *oversampling* are adequately formulated.

II. Recovering Lost Samples

Let a signal f, band-limited to $[-\Omega, \Omega]$, be sampled at a rate $1/T$ such that $r = 2\Omega T < 1$. Its sampling series, which relates the value of f at time t with its value at the sampling instants, will be [2]

$$f(t) = r \sum_{k=-\infty}^{+\infty} f(kT)\ \text{sinc}\ [2\Omega(t - kT)] \tag{4}$$

where sinc $(0) = 1$ and sinc $(x) = \sin(\pi x)/\pi x$ whenever $x \neq 0$.

Let $U = \{i_1, i_2, i_3, \cdots, i_n\}$ be a finite set of integers corresponding to the location of the unknown samples of f. Setting $t = mT$ in (4) and concentrating on values of m belonging to U we obtain

$$f(i_j T) = r \sum_{k \notin U} f(kT)\ \text{sinc}\ [r(i_j - k)] + r \sum_{k=1}^{n} f(i_k T)\ \text{sinc}\ [r(i_k - i_j)] \qquad j = 1, 2, \cdots, n. \tag{5}$$

These equations can be written more concisely in matrix form as

$$\boldsymbol{f} = \boldsymbol{h} + \boldsymbol{S}\boldsymbol{f}$$

where $\boldsymbol{f}$ is the $n \times 1$ column vector of unknown samples, $\boldsymbol{h}$ is the $n \times 1$ vector with components given by

$$h_j = r \sum_{k \notin U} f(kT)\ \text{sinc}\ [r(i_j - k)]$$

and which depends only on the known samples, and $\boldsymbol{S}$ denotes the $n \times n$ matrix with entries

$$S_{jk} = r\ \text{sinc}\ [r(i_j - i_k)].$$

The vector $\boldsymbol{f}$ can be found as the solution to the system of equations $(\boldsymbol{I} - \boldsymbol{S})\boldsymbol{f} = \boldsymbol{h}$. We remark that essentially the same proof can be found in a recent work [4].

Contrary to a previous claim [2], $\boldsymbol{A}$ will not be Toeplitz, unless the missing samples are equidistant. This can be readily verified by taking, for example, $U = \{0, 1, 3\}$.

III. Generalizations to Other Sampling Expansions

It is possible to extend the previous result to more general sampling expansions. We will illustrate this by using Kramer's extension [3] of the WKS sampling theorem.

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The author is with the Departamento de Electrónica e Telecomunicações, Universidade de Aveiro, 3800 Aveiro, Portugal, and with INESC, Instituto de Engenharia de Sistemas e Computadores, Aveiro, Portugal.
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Let $K(x, t)$ be a complex-valued function of the real variables x and t such that $K(x, \cdot) \in L_2(I)$ for a certain interval I. The following natural generalization of the concept of sampling set will be useful in the sequel: a *sampling set* is a set of real numbers $\{t_n\}$ that makes the functions $\{K(x, t_i)\}$ complete and orthogonal on some subset of $\boldsymbol{R}$ with finite measure.

Kramer's theorem asserts that, for any function $f \in L_2(-\infty, \infty)$ such that

$$f(t) = \int_I g(x) K(x, t)\, dx$$

we have

$$f(t) = \sum_{n=-\infty}^{+\infty} f(t_n) S(t_n, t)$$

where

$$S(t_n, t) = \frac{\int_I K(x, t) K^*(x, t_n)\, dx}{\int_I |K(x, t_n)|^2\, dx}.$$

Clearly, and since $S(t_n, t_m) = \delta_{nm}$, if no additional restrictions are imposed upon f or $\{t_n\}$ the samples $f(t_n)$ will be independent.

Therefore, we will assume from now on that the sampling set $\{t_n\}$ *oversamples* f, that is, leads to a set of functions $K(x, t_n)$ complete and orthogonal over an interval I that strictly contains the support J of g. The following version of Kramer's theorem holds under this hypothesis.

Theorem 1: Let $K(x, t)$ be a complex-valued function of the real variables x and t such that $K(x, \cdot) \in L_2(I)$. Let $f \in L_2(-\infty, +\infty)$ be a function such that the representation

$$f(t) = \int_J g(x) K(x, t)\, dx, \qquad (J \subset I)$$

is valid, and let $\{t_n\}$ be a sampling set over I which oversamples f. Then, if the integral

$$\int_I |K(x, t)|^2\, dx$$

is uniformly bounded in t, we have

$$f(t) = \sum_{n=-\infty}^{+\infty} f(t_n) S(t_n, t)$$

the convergence being uniform, and where

$$S(t_n, t) = \frac{\int_J K(x, t) K^*(x, t_n)\, dx}{\int_I |K(x, t_n)|^2\, dx}.$$

We expand g on I as

$$g(x) = \sum_{n=-\infty}^{+\infty} c_n K^*(x, t_n). \tag{6}$$

Upon multiplication by $K(x, t_m)$ and termwise integration over I we obtain

$$\int_J g(x) K(x, t_m)\, dx = c_m \int_I |K(x, t_m)|^2\, dx$$

since $g(x)$ vanishes outside $J \subset I$. The coefficients c_n are therefore given by

$$c_n = \frac{f(t_n)}{\int_I |K(x, t_n)|^2\, dx}.$$

Substituting this in (6), multiplying by $K(x, t)$ and integrating over J, we obtain

$$f(t) = \sum_{n=-\infty}^{+\infty} f(t_n) S(t_n, t).$$

The convergence is uniform since f and c_n are uniformly bounded. The interdependence of the samples of f is a consequence of the form of $S(t_n, t)$, and a generalization of the result discussed in Section II can now be easily derived, using the same method. The only difference consists in the substitution of r sinc $[2\Omega(t_n - t_k)]$ by $S(t_n, t_k)$ in matrix $\boldsymbol{A}$. However, in order to recover any finite set of missing samples, it is necessary to show that $\boldsymbol{A} = \boldsymbol{I} - \boldsymbol{S}$ is nonsingular. It follows from the next theorem that this is indeed true, for virtually all known examples of Kramer sampling expansions.

Theorem 2: If the functions $K(x, t_i)$ are orthonormal and analytic, the matrix $\boldsymbol{S}$ with elements $S_{ij} = S(t_i, t_j)$ is positive definite and all its eigenvalues are strictly less than unity.

The quadratic form $\boldsymbol{v}^H \boldsymbol{S} \boldsymbol{v}$ associated with $\boldsymbol{S}$ is given by

$$\sum_{i=1}^{n} \sum_{j=1}^{n} v_i v_j^* \int_J K(x, t_i) K^*(x, t_j)\, dx.$$

This is equivalent to

$$\int_J \left| \sum_{i=1}^{n} v_i K(x, t_i) \right|^2 dx \tag{7}$$

which is, clearly, a nonnegative quantity. To establish its positivity, we argue as follows. Expression (7) is zero if and only if

$$\phi(x) = \sum_{i=1}^{n} v_i K(x, t_i)$$

is the zero function in J. Since ϕ is analytic, it would have to vanish in I also. But this contradicts the linear independence of the $K(x, t_j)$, and, therefore, (7) must never vanish.

To show that the eigenvalues of $\boldsymbol{S}$ cannot exceed unity, we note that, since J is strictly contained in I,

$$\boldsymbol{v}^H \boldsymbol{S} \boldsymbol{v} < \int_I \left| \sum_{i=1}^{n} v_i K(x, t_i) \right|^2 dx = \sum_{i=1}^{n} |v_i|^2.$$

But no eigenvalue of $\boldsymbol{S}$ can exceed the maximum value assumed by the quadratic form $\boldsymbol{v}^H \boldsymbol{S} \boldsymbol{v}$ on the sphere $\|v\| = 1$, and, consequently, the theorem must be true.

It is possible to show that a matrix $\boldsymbol{S}$ with elements given by

$$S_{mn} = \int_J f_m(x) f_n^*(x)\, dx$$

where the f_m are functions orthonormal on I, is not positive definite in general, for arbitrary proper subsets J of I. For this purpose we consider the well-known Haar basis, a set of real, piecewise constant functions $f_i(x)$, complete and orthonormal on [0, 1]. Some of these functions vanish in certain subintervals of [0, 1]. It is this property that it is fundamental here. Let $f_i(x)$ be a function of the Haar basis that vanishes in a certain subinterval J of $I = [0, 1]$. Define a vector $\boldsymbol{v}$ by $\|v\| = 1$ and $v_j = 0$ for all $j \neq i$. We see that $\boldsymbol{v}^H \boldsymbol{S} \boldsymbol{v}$ reduces to $\int_J |f_i(x)|\, dx$, an expression that is equal to zero.

The recovery of missing samples can be accomplished either by iterative or noniterative methods. The noniterative method calls for the solution of the equations $(I - S)x = h$. Since the eigenvalues of the matrix S do not exceed unity, these same equations can also be solved by a variety of linear stationary iterative methods of the first order [5], such as the Jordan, JOR, Gauss-Seidel, SOR methods, and others. The (slow) Jordan method, for example, is based on the iteration

$$x_{n+1} = Sx_n + h.$$

The convergence of the method follows directly from theorem 2. We note that the Papoulis-Gerchberg algorithm leads to a similar equation [2] for the Fourier case. The use of iterative methods with faster convergence rates than the Jordan method might be preferable, particularly because S does not seem to possess a structure that renders the direct computation of the solution especially attractive from a computational perspective, at least in the general case.

IV. Conclusions

We have seen how the loss of any finite number of samples from a signal band limited and oversampled in a generalized sense can be tolerated, and the signal recovered without error. We suggested simple iterative or noniterative procedures, valid for the general case, through which the recovery process can be performed.

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Corrections to "Regularized Fast Recursive Least Squares Algorithms for Adaptive Filtering"

Amrane Houacine

In the above paper,[1] the following corrections must be made.
The corrections to the text are as follows.
In (4.2), replace n by N.
In matrix equations (4.4) and (4.5) replace the vector v_1 by v_i.
Equation (5.11) must be

$$x_{m+1}(i) = \begin{bmatrix} x_m(i) \\ x_{i-m} \end{bmatrix}.$$

In (6.17), (6.18), (6.23), and (6.26) replace the vector w_i by $\overline{w}_i$.
In (6.18) also replace v_i^t by $\overline{v}_i^t$.
In (6.28) replace $\overline{w}$ by $\overline{w}_i$.
The corrections to the tables are as follows.
In Table II, in the fifth equation, r_i must be replaced by r_{i-1}:

$$M_i = \lambda^{-1}[M_{i-1} + w_i(r_{i-1})^{-1}w_i^t].$$

In Table VIII, the fifth equation must be

$$\overline{u}_i = \overline{w}_i/\overline{r}_i$$

and the eighth equation must be as follows:

$$\overline{S}_i = \lambda^{-1/2}\left[\overline{S}_{i-1}\,\Sigma + \left(\overline{g}_i\overline{r}_i/(\overline{r}_{i-1} - \overline{r}_i) + \begin{bmatrix} 0 \\ \overline{k}_{i-1} \end{bmatrix}\right)\overline{u}_i^t\right].$$

In Table IX, the eighth equation must be

$$\overline{S}_i = \lambda^{-1/2}\left[-\overline{S}_{i-1}\,\Sigma + \left(\overline{g}_i\overline{r}_{i-1}/(\overline{r}_{i-1} - \overline{r}_i) - \begin{bmatrix} \overline{k}_i \\ 0 \end{bmatrix}\right)\overline{u}_i^t\right].$$

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The author is with the Institute of Electronics, University of Sciences and Technology of Algiers, Algiers, Algeria.
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[1] A. Houacine, *IEEE Trans. Signal Processing*, vol. 39, no. 4, pp. 860–871, Apr. 1991.

Improved Polarity Coincidence Spectral-Centroid Estimator

A. Laurenti and G. Scarano

***Abstract*—An extension of the polarity coincidence (PC) method for estimating the center frequency of a power spectral density (PSD) is proposed. By applying Price's theorem, the variance of this new estimator for Gaussian processes is evaluated. Its performance is compared and related to those of the averaged output of an ideal tuned FM receiver.**

I. Introduction

Parametric spectrum estimation of wide-sense stationary processes based on spectral moments evaluation is often required in many physical applications.

In particular, the first-order spectral moment, i.e., the centroid of the spectrum, is a very useful parameter to investigate the character of some phenomena. For instance, the mean Doppler velocity of scatterers in radar echoes, the slope of the dispersive attenuation in seismic records, etc., are measured by the centroid displacement.

In a recent paper [3], Gupta has considered in detail three spectrum-centroid estimators and has compared them with the classical correlation method by evaluating their accuracies. Among them the polarity coincidence (PC) estimator was originally proposed by Pickard [6] and analyzed by Pawula [5]. In the complex extension of some of these forms presented in [2], the PC estimator has not been considered.

The aim of this paper is to investigate the performance of an improved version of the PC estimator named the complex polarity coincidence (CPC) estimator. A comparison with the PC estimator

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A. Laurenti is with Contraves Italiana S.p.A., I-00139 Rome, Italy.
G. Scarano was with the C.N.R., Instituto di Acustica "O.M. Corbino," I-00189 Rome, Italy. He is now with the Institute Elettronica, Università di Perugia, I-06100 Perugia, Italy.
IEEE Log Number 9104018.