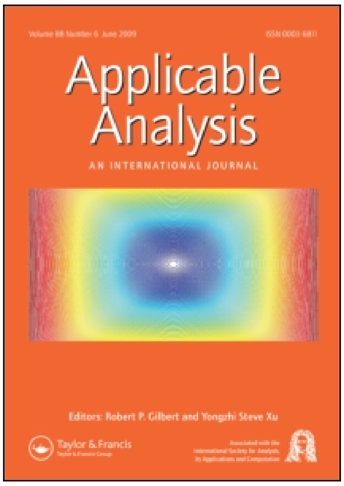


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P. L. Butzer[a]; P. J. S. G. Ferreira[b]; J. R. Higgins[c]; G. Schmeisser[d]; R. L. Stens[a]
[a] Lehrstuhl A für Mathematik, RWTH Aachen University, 52056 Aachen, Germany [b] IEETA/DETI, Universidade de Aveiro, 3810-193 Aveiro, Portugal [c] I.H.P., Montclar, 11250 France [d] Department of Mathematics, University of Erlangen-Nürnberg, 91054 Erlangen, Germany

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# The sampling theorem, Poisson's summation formula, general Parseval formula, reproducing kernel formula and the Paley–Wiener theorem for bandlimited signals – their interconnections

P.L. Butzer[a], P.J.S.G. Ferreira[b], J.R. Higgins[c], G. Schmeisser[d] and R.L. Stens[a*]

[a]*Lehrstuhl A für Mathematik, RWTH Aachen University, 52056 Aachen, Germany;* [b]*IEETA/DETI, Universidade de Aveiro, 3810-193 Aveiro, Portugal;* [c]*I.H.P., 4 rue du Bary, Montclar, 11250France;* [d]*Department of Mathematics, University of Erlangen-Nürnberg, 91054 Erlangen, Germany*

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It is shown that the Whittaker–Kotel'nikov–Shannon sampling theorem of signal analysis, which plays the central role in this article, as well as (a particular case) of Poisson's summation formula, the general Parseval formula and the reproducing kernel formula, are all equivalent to one another in the case of bandlimited functions. Here equivalent is meant in the sense that each is a corollary of the other. Further, the sampling theorem is equivalent to the Valiron–Tschakaloff sampling formula as well as to the Paley–Wiener theorem of Fourier analysis. An independent proof of the Valiron formula is provided. Many of the equivalences mentioned are new results. Although the above theorems are equivalent amongst themselves, it turns out that not only the sampling theorem but also Poisson's formula are in a certain sense the 'strongest' assertions of the six well-known, basic theorems under discussion.

**Keywords:** sampling theorem; bandlimited signals; functions of exponential type; Poisson's summation formula; reproducing kernel formula; Paley–Wiener's theorem

**AMS Subject Classifications:** 30D10; 94A20; 42C15; 46E22

## 1. Introduction

### 1.1. *The central equivalence grouping*

The (classical) sampling theorem of signal analysis, so basic in mathematics and its applications, such as communication engineering, control theory, data and image processing, is connected not only with the name of Claude Shannon [1], but also with the names of Edmund Taylor Whittaker [2], Kinnosuke Ogura [3], Vladimir Aleksandrovich Kotel'nikov [4], Isao Someya [5] and many others, such as de la

*Corresponding author. Email: stens@matha.rwth-aachen.de

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Vallée Poussin [6] and Herbert Raabe [7]; see for example [8–11] and the literature cited there. For applications of the sampling theorem see e.g. [12–14].

Denote by $B_\sigma^p$ for $\sigma > 0$, $1 \le p \le \infty$ the Bernstein space of all entire functions $f\colon \mathbb{C} \to \mathbb{C}$ that belong to $L^p(\mathbb{R})$ when restricted to the real axis and satisfy

$$f(z) = \mathcal{O}_f\big(\exp(\sigma|\mathfrak{Im}\, z|)\big) \quad (|z| \to \infty), \tag{1}$$

i.e. $f$ is exponential type $\sigma$.

The sampling theorem now states:
*For $f \in B_\sigma^2$ with some $\sigma > 0$ we have*

$$f(z) = \sum_{k \in \mathbb{Z}} f\left(\frac{k\pi}{\sigma}\right) \operatorname{sinc}\left(\frac{\sigma z}{\pi} - k\right) \quad (z \in \mathbb{C}),$$

*the convergence being absolute and uniform in strips of bounded width parallel to the real line.*

Here the sinc-function is given by $\operatorname{sinc} z := \sin(\pi z)/(\pi z)$ for $z \in \mathbb{C} \setminus \{0\}$ and $\operatorname{sinc} 0 := 1$. It plays an important role not only in the sampling theorem, but also in most of the other theorems investigated in this article. It is easily verified that this sinc-function belongs to $B_\pi^p$ for all $1 < p \le \infty$. Moreover, the translates $\operatorname{sinc}(\cdot + w)$ belong to $B_\pi^p$ for each fixed $w \in \mathbb{C}$, as can be seen from estimates (5) below. For the important role played by sinc-functions see e.g. [15].

There are (at least) two ways of proving the sampling theorem, namely by applying the Poisson summation formula (PSF) of Fourier/numerical analysis for the particular case of functions belonging to $B_\sigma^1$, or the general Parseval formula (GPF) known from the theory of trigonometric Fourier series. Already Someya [5] used Poisson's formula, so also Ralph Boas [16] (see e.g. [17, Section 3.1; 18, p. 50]).

But the use of Poisson's or Parseval's formula heavily depends on the fact that the Fourier transform of a $B_\sigma^2$-function has compact support. This property is known as Paley–Wiener's theorem (PWT), which connects the growth condition (1) with the support of the Fourier transform.

But in the manifold use of the sampling theorem in signal analysis a great danger is involved, especially when employed as a 'side result' in (many) proofs. In fact, the sampling formula itself, classical sampling formula (CSF), will be shown to yield the PWT. Thus the sampling theorem for $B_\sigma^2$ is fully equivalent to PWT for $B_\sigma^2$-functions, in the sense that each is a corollary of the other. This is one of several new and unexpected results of this article.

Another striking and especially applicable formula of mathematical analysis is the fundamental reproducing kernel formula (RKF), which states under the hypothesis $f \in B_\sigma^2$, $\sigma > 0$ (it holds in fact for $f \in B_\sigma^p$ with $p \neq \infty$) that

$$f(z) = \frac{\sigma}{\pi} \int_{\mathbb{R}} f(u) \operatorname{sinc} \frac{\sigma}{\pi}(z - u) \mathrm{d}u \quad (z \in \mathbb{C}),$$

thus $f(z) = \langle f, \frac{\sigma}{\pi} \operatorname{sinc} \frac{\sigma}{\pi}(z - \cdot) \rangle$, $z \in \mathbb{C}$, in a Hilbert space notation; the function $\frac{\sigma}{\pi} \operatorname{sinc} \frac{\sigma}{\pi}(z - \cdot) \in B_\sigma^2$, for each $z \in \mathbb{C}$, is called the reproducing kernel for $B_\sigma^2$, and $B_\sigma^2$ a reproducing kernel space. Concerning reproducing kernel theory see e.g. [19,20].

Another representation of $B_\sigma^2$-functions by an infinite series similar to CSF is that of Valiron/Tschakaloff (3) below. The interconnection of this formula with CSF was already discussed in [21], where it was shown that each of them can easily be deduced from the other.

Let us now state the six formulae which are to be treated in this article.

**Poisson summation formula (PSF):** *For $f \in B^1_\sigma$ with $\sigma > 0$ we have*

$$\int_{\mathbb{R}} f(u)\mathrm{d}u = \frac{2\pi}{\sigma}\sum_{k\in\mathbb{Z}} f\Big(\frac{2k\pi}{\sigma}\Big).$$

This means that in $B^1_\sigma$ the trapezoidal rule with step size $2\pi/\sigma$ is exact for integration over $\mathbb{R}$.

**General Parseval formula (GPF):** *For $f, g \in B^2_\sigma$ with $\sigma > 0$ we have*

$$\int_{\mathbb{R}} f(u)\overline{g(u)}\mathrm{d}u = \frac{\pi}{\sigma}\sum_{k\in\mathbb{Z}} f\Big(\frac{k\pi}{\sigma}\Big)\overline{g\Big(\frac{k\pi}{\sigma}\Big)}.$$

**Classical sampling formula (CSF):** *For $f \in B^2_\sigma$ with $\sigma > 0$ we have*

$$f(z) = \sum_{k\in\mathbb{Z}} f\Big(\frac{k\pi}{\sigma}\Big)\mathrm{sinc}\Big(\frac{\sigma z}{\pi} - k\Big) \quad (z \in \mathbb{C}), \tag{2}$$

*the convergence being absolute and uniform in strips of bounded width parallel to the real line, thus in particular, on compact sets.*

**Reproducing Kernel formula (RKF):** *For $f \in B^2_\sigma$ with $\sigma > 0$ we have*

$$f(z) = \frac{\sigma}{\pi}\int_{\mathbb{R}} f(u)\mathrm{sinc}\frac{\sigma}{\pi}(z-u)\mathrm{d}u \quad (z \in \mathbb{C}).$$

**Valiron's or Tschakaloff's sampling/interpolation formula (VSF):**[1] *For $f \in B^\infty_\sigma$ with $\sigma > 0$ we have for all $z \in \mathbb{C}$*

$$f(z) = f'(0)z\,\mathrm{sinc}\Big(\frac{\sigma z}{\pi}\Big) + f(0)\mathrm{sinc}\Big(\frac{\sigma z}{\pi}\Big) + \sum_{k\in\mathbb{Z}\setminus\{0\}} f\left(\frac{k\pi}{\sigma}\right)\frac{\sigma z}{k\pi}\mathrm{sinc}\Big(\frac{\sigma z}{\pi} - k\Big), \tag{3}$$

*the convergence being absolute and uniform on compact subsets of $\mathbb{C}$.*

**Paley–Wiener theorem (PWT):** *For $f \in B^2_\sigma$ with $\sigma > 0$ we have*

$$\widehat{f}(v) := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(u)e^{-ivu}\,\mathrm{d}u = 0 \quad (\text{a.e. for } |v| > \sigma).$$

Figure 1 presents the interconnections between the first four formulae under discussion treated in Section 4. Figure 2 shows the additional connections with Valiron's formula and the PWT, which are discussed in Sections 5 and 6.

Although we are only interested in the equivalences of the theorems mentioned, we will give a proof of one of them, namely Valiron's formula, in the appendix. Valiron established this formula in a more general frame as a generalization of Lagrange's interpolation formula in 1925 [22]. Tschakaloff presented this result in the course of the solution of a concrete problem posed by Pólya in the section 'Aufgaben und Lösungen' [Problems and Solutions] of the Jahresbericht der

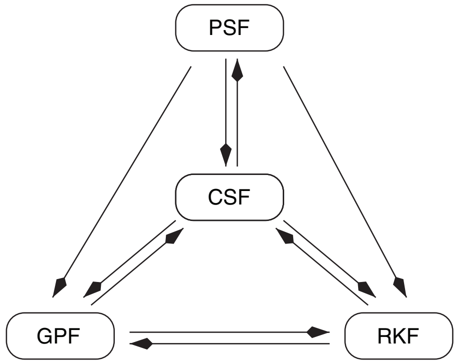


Figure 1. The interconnections between the first four formulae.

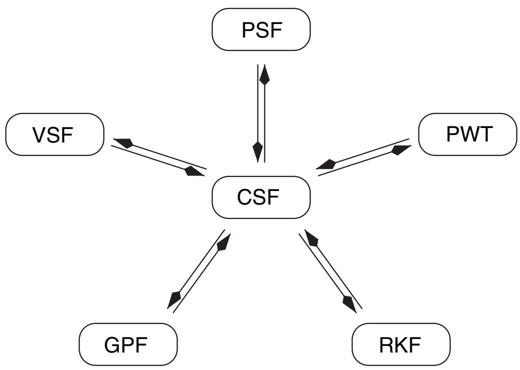


Figure 2. The interconnections with VSF and the PWT.



Deutschen Mathematiker–Vereinigung in 1933 [23]. In fact, it follows immediately from Equation (8) with $\nu = 1$ in this article. In the same volume of the Jahresbericht Pólya commented on Tschakaloff's solution and used it to improve a result by J.M. Whittaker. For the details see [24, Section 1.4].

### 1.2. *Interconnection with further basic theorems*

It is known that the sampling theorem is fully equivalent to the generalized Vandermonde-Chu formula of combinatorial analysis, to the Gauss summation formula of hypergeometric function theory [14], and also to the particular case of Cauchy's integral formula for $f \in B^1_\sigma$ [25], as well as the harmonic sampling formula [26], both of complex analysis.

But there also exists the approximate sampling formula (ASF):

*Let* $f \in L^2(\mathbb{R}) \cap C(\{\mathbb{R})$ *with Fourier transform* $f^\wedge(v) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) e^{-ivu} \mathrm{d}u \in L^1(\mathbb{R})$. *Then*

$$f(t) = \sum_{k \in \mathbb{Z}} f\left(\frac{k}{w}\right) \operatorname{sinc}(wt - k) + (R_w f)(t) \quad (t \in \mathbb{R}),$$

*where the error term*

$$(R_w f)(t) := \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} \left(1 - e^{-it2\pi wn}\right) \int_{(2n-1)\pi w}^{(2n+1)\pi w} f^{\wedge}(v) e^{itv}\, \mathrm{d}v$$

*satisfies the estimate*

$$|(R_w f)(t)| \le \sqrt{\frac{2}{\pi}} \int_{|v|\ge \pi w} |f^{\wedge}(v)| \mathrm{d}v \quad (t \in \mathbb{R}).$$

*In particular, uniformly for* $t \in \mathbb{R}$,

$$f(t) = \lim_{w\to\infty} \sum_{k\in\mathbb{Z}} f\left(\frac{k}{w}\right) \mathrm{sinc}(wt - k).$$

In this respect it is already known that the ASF is equivalent to the general PSF and the Abel–Plana formula of numerical analysis [14,27,28] and even to the famous functional equation of the Riemann zeta function (which is known to be equivalent to the transformation formula of Jacobi's elliptic theta function); see [29]. In view of the recent, unexpected result of Butzer et al. [30] that the CSF is equivalent to the above approximate sampling theorem, the ASF, in fact all of the above theorems have the potential for being equivalent to one another, again in the sense that each is a corollary of the other. This potential would be realized if checks on consistency of hypotheses, side results, etc., are successful across the board, and then the results could be said to be fully equivalent.

But there are also 'exotic' theorems which are equivalent to the different versions of the sampling theorem, such as the Phragmén–Lindelöf principle, the maximum principle and the Cauchy integral formula of complex analysis; see [26]. The PSF and Cauchy's integral theorem are pivotal theorems in many branches of analysis. That they are indeed equivalent to the many results mentioned suggests what pitfalls in proofs must be avoided, what side results cannot be used in order to avoid possible circular reasoning.

As mentioned, ASF is equivalent to CSF. There are more general results of this type; for example, there is an approximate sampling formula in harmonic analysis, as the approximate form of Kluvánek's sampling theorem below[2], and the original form of Kluvánek's sampling theorem, the harmonic analysis version of CSF, is used in its proof in an essential but partial way, in the sense that a square summability condition is also needed.

The converse result is that the approximate form of Kluvánek's sampling theorem implies Kluvánek's theorem itself; this is trivial of course so that one might say that the two are 'close to being equivalent', particularly since the square summability condition 'almost always' holds.

Statements of these results are contained in the following two theorems; for complete details and further references see [31].

Let $G$ be a locally compact abelian group with discrete subgroup $H$, and let the dual group of $G$ be denoted by $\Gamma$. Let $\Gamma$ have a discrete, countable subgroup $\Lambda$ such that $\Gamma/\Lambda$ is compact with measurable transversal $\Omega$ and such that the annihilator $\Lambda^{\perp} = H$ of $\Lambda$ is countable. The usual notations $^{\wedge}$ and $^{\vee}$ for the Fourier and the inverse Fourier transform are used.

Kluvánek's sampling theorem can be stated using an abstract sampling operator $\mathcal{S}_H f$, based on the classical operator (the right-hand side of (2)). For each $f := G \mapsto \mathbb{C}$ write formally

$$(\mathcal{S}_H f)(t) := \frac{1}{m(\Omega)} \sum_{h\in H} f(h)\chi_\Omega^\vee(t-h) = \frac{1}{m(\Omega)} \lim_{n\to\infty} \sum_{j=1}^{n} f(h_j)\chi_\Omega^\vee(t-h_j), \tag{4}$$

where $h_j = 1, 2, \dots$ is an enumeration of $H$ and $m(\Omega)$ denotes the Haar measure of $\Omega$.

The abstract analogue of $B_\sigma^2$ (above) is the Paley–Wiener space

$$PW_\Omega(G) := \{f \in L^2(G) \cap C(G) : f^\wedge(\gamma) = 0 \text{ for almost all } \gamma \notin \Omega\}.$$

Kluvánek's sampling theorem can now be stated:

**Kluvánek's sampling theorem:** *In the notations just established, suppose* $f \in PW_\Omega(G)$. *Then* $m(\Omega) < \infty$ *and* $f$ *has a representation*

$$f(x) = (\mathcal{S}_H f)(x) = \frac{1}{m(\Omega)} \sum_{h\in H} f(h)\chi_\Omega^\vee(x-h),$$

*convergence being absolute, uniform in* $G$ *and also in the* $L^2(G)$*-norm. Furthermore,* $f \in \ell^2(H)$.

The abstract analogue of the classical approximate sampling theorem (cf [30]) can now be stated in terms of the abstract sampling operator $\mathcal{S}_H f$ in (4) and the function class:

$$F^2(G) := \{f \in L^2(G) \cap C(G) : f^\wedge \in L^1(\Gamma)\}.$$

**Approximate form of Kluvánek's sampling theorem:** *Let* $G$, $\Gamma$, $\Lambda$, $\Omega$ *and* $H$ *be as in Kluvánek's theorem. Suppose that* $f \in F^2(G)$ *and* $f \in \ell^2(H)$. *Then*

$$f(t) = (\mathcal{S}_H f)(t) + (\mathcal{R}_H f)(t),$$

*where*

$$|(\mathcal{R}_H f)(t)| \le 2 \int_{\Gamma\setminus\Omega} |f^\wedge(\gamma)| \mathrm{d}m(\gamma).$$

The operator $\mathcal{R}_H$ is a generalization of the error term $\mathcal{R}_w$ in the classical case (above).

It would indeed be of interest to study approximate sampling theorems in a Mellin transform setting on the multiplicative group $\mathbb{R}^+$ as an application of the approximate form of Kluvánek's theorem.

### 1.3. *Characteristics of equivalence groupings: Ideas of Hilbert and Thiele*

We refer to a group of equivalent propositions as an 'equivalence grouping'. Before proceeding we discuss the characteristics that a satisfactory equivalence grouping should possess and some of the pitfalls to be avoided.

The equivalence of a group of propositions means that each proposition is a corollary of the others. The demonstration of the implications often requires a number of auxiliary results, which we call 'side results'. For logical consistency it is necessary to ensure that the side results do not introduce circularity.

Simplicity of proof is an important guideline to bear in mind when considering an equivalence grouping. When the proofs of the implications between propositions are very complex when compared with the proofs of the propositions themselves[3] the grouping seems artificial and of questionable usefulness. Simplicity, on the other hand, conveys the opposite feeling. When the proofs of the implications are all of similar complexity, the equivalent results can be considered to have 'comparable depth' (see also [26, p. 334] and [32]).

To avoid circularity none of the propositions in the equivalence grouping (nor any result implied by any of them) can be used as a side result [33, Section 2]. This suggests another guideline: 'powerful' side results should be avoided as much as possible. However, this may contradict the simplicity guideline, since without powerful side results the complexity of the implication proofs is likely to grow.

What is then a satisfactory equivalence grouping? Roughly speaking, it is a set of propositions linked by implications, the proofs of which are as straightforward as possible, use a minimum of side results and avoid powerful ones. Simplicity, usefulness and power are a matter of taste, and we are left with another question: can (mathematical) criteria for 'simplicity of proof' be given?

In 1900, David Hilbert put forth a collection of problems that he believed could shape the course of mathematics in the twentieth century. In his address to the International Congress of Mathematicians of 1900 he mentioned ten of those problems; the expanded version of his speech, published soon after the congress, contained 23 problems (see [34,35]).

Rüdiger Thiele [36] has discovered in Hilbert's notebook one further problem that Hilbert considered to include in the famous list. The problem asks for the simplest proof of a theorem:

> 'Criteria of simplicity, or proof of the greatest simplicity of certain proofs. Develop a theory of the method of proof in mathematics in general. Under a given set of conditions there can be but one simplest proof. Quite generally, if there are two proofs for a theorem, you must keep going until you have derived each from the other, or until it becomes quite evident what variant conditions (and aids) have been used in the two proofs. Given two routes, it is not right to take either of these two or to look for a third; it is necessary to investigate the area lying between the two routes...'

Part of the interest in equivalence groupings is due to the information that they reveal about the area lying between the equivalent propositions. But can simplicity be measured? Thiele [36] notes that in 1888 Émile Lemoine reduced geometric constructions by rule and compass to five basic operations and quantified the complexity of a construction by counting the number of times that it used the five basic operations. In other similar 'finitary' contexts simplicity can be quantified (see also [32] and the discussion about syzygies in [36]). Proofs are sequences of symbol manipulations, performed according to certain rules or axioms. Each step in a proof consists in the application of an axiom, as in Lemoine's construction. According to this view, assessing simplicity becomes a matter of 'counting beans' [36].

The complexity of a theorem could therefore be defined as the size of the shortest legal sequence of symbol manipulations that produces the theorem. Note the connection with Kolmogorov complexity, according to which the complexity of a sequence of symbols is given by the size of the smallest computer program (or Turing machine) that prints the sequence and halts. The underlying idea is related to a principle in the philosophy of science known as Occam's Razor: 'the simplest explanation is best'. In other words, regular objects have concise descriptions. In this respect see e.g. [37].

We add that the distance between two equivalent propositions could be defined as the length of the shortest equivalence proof. When these distances are short, equivalence groupings play two useful roles: they give prominence to the connections between the set of theorems and they yield simple proofs of each (assuming that the logical value of at least one of the propositions is known).

Simplicity remains important even outside proof theory and foundational issues. In human terms it depends, as Thiele wrote, on the techniques used, one's familiarity with them, the novelty of ideas and so on. We argue that an equivalence grouping is useful when it illustrates the nature of the mathematical landscape lying between the propositions in the group – 'the area between the routes', to use Hilbert's happy choice of words.



### 1.4. *Concerning contents*

Whereas Section 2 is devoted to three auxiliary results, Section 3 contains several corollaries of the CSF, including the Bernstein and Nikol'skiĭ inequalities. Section 4.1 concerns the three implications CSF $\Rightarrow$ RKF, CSF $\Rightarrow$ GPF, CSF $\Rightarrow$ PSF, and Section 4.2 deals with the converse assertions. As to Section 4.3, further implications are treated, thus PSF $\Rightarrow$ GPF, PSF $\Rightarrow$ RKF, GPF $\Rightarrow$ RKF and RKF $\Rightarrow$ GPF.

Sections 5 and 6 deal with the equivalence of CSF with VSF, and CSF with PWT, respectively. Checking over the proofs of this article, the reader will observe that the $L^2(\mathbb{R})$ Fourier transform is used only in Section 6 in conjunction with PWT.

Since all theorems of this article are equivalent among themselves, theoretically it is not clear whether one can speak of a 'stronger' or 'weaker' theorem. However, if the proof of the implication A $\Rightarrow$ B is harder than that of B $\Rightarrow$ A, or uses more difficult side results, then one could say that Theorem B is the more 'difficult' or 'stronger' one.

In fact, the proofs of implications PSF $\Rightarrow$ CSF, PSF $\Rightarrow$ GPF and PSF $\Rightarrow$ RKF (Theorems 4.6, 4.7 and 4.8) are quite elementary, whereas the proof of CSF $\Rightarrow$ PSF (Theorem 4.3) is surely a more difficult one (as it needs, e.g. formula (9), a consequence of CSF). Moreover, there seems to be no direct proof for GPF $\Rightarrow$ PSF or RKF $\Rightarrow$ PSF. In this sense one could say that PSF is a 'stronger' result than the other four.

Concerning CSF $\Rightarrow$ RKF and CSF $\Rightarrow$ GPF (Theorems 4.1 and 4.2), the proofs follow more or less by integrating (a particular case of) CSF. Their respective converse counterparts (Theorems 4.5 and 4.4), however, are more intricate. Here one could say that CSF is stronger than RKF and GPF.

Finally, RKF may be regarded as the 'weakest' formula of the four, since e.g. formula (20), the proof of which is by no means trivial, plays a basic role in the proofs of RKF $\Rightarrow$ CSF and RKF $\Rightarrow$ GPF (Theorems 4.5 and 4.10). For the proof of (20) see the Appendix.

This ranking is confirmed by Figure 1. The formula at the top, PSF, is the 'strongest', the formulae at the bottom, GPF and RKF, are 'weaker' ones.

Concerning VSF and PWT, the former is somewhat 'stronger' than CSF, since it holds for a larger class of functions. But CSF seems in turn to be 'stronger' than PWT.

As will be observed, CSF plays a central role in this article. As can be seen in Figures 1 and 2, it is the only formula which is connected to each of the other five by two implications. One might expect that also Valiron's formula could play this central role, but no attempt has been made in this respect. On the other hand, this seems not possible for any of PSF, RKF, GPF or PWT.

## 2. Some auxiliary results

In the proofs below it will be often convenient to establish the desired result first for functions $f$ belonging to a certain subspace of $B^p_\sigma$ and then to extend this particular case to all of $B^p_\sigma$ by density arguments. In this respect the following lemmas will be useful in the proofs of many theorems.

We first define a suitable subspace of $B^p_\sigma$. For $\sigma > 0$ let

$$\widetilde{B}^1_\sigma := \left\{ f \in B^1_\sigma;\ z^2 f(z) \in B^1_\sigma \right\}.$$

If $f \in \widetilde{B}^1_\sigma$, then also $zf(z) \in B^1_\sigma$ and $f(z) = \mathcal{O}(|z|^{-2})$ for $|z| \to \infty$.[4]

Lemma 2.1 *Let $1 \le p \le \infty$. Then $\widetilde{B}^1_\sigma$ is a subspace of $B^p_\sigma$. Furthermore, there exists for each $f \in B^p_\sigma$ a sequence $(f_n)_{n\in\mathbb{N}} \subset \widetilde{B}^1_\sigma$ such that $\lim_{n\to\infty} f_n(z) = f(z)$ and $\lim_{n\to\infty} f_n'(z) = f'(z)$ for all $z \in \mathbb{C}$. If $p \ne \infty$, then the sequence can be chosen such that in addition $\lim_{n\to\infty} \|f_n - f\|_p = 0$. In particular, $\widetilde{B}^1_\sigma$ is a dense subspace of $B^p_\sigma$ for $1 \le p < \infty$.*

*Proof* The subspace property is obvious, since $f \in \widetilde{B}^1_\sigma$ implies $f(t) = \mathcal{O}(|t|^{-2})$ for $|t| \to \infty$. For the following we first assume $f \in B^p_\pi$ and define for $n \in \mathbb{N}$,

$$f_n(z) := f\left(\left(1 - \frac{1}{n+1}\right)z\right) \operatorname{sinc}^3\left(\frac{z}{3(1+n)}\right) \quad (z \in \mathbb{C}).$$

One easily checks that $f_n \in \widetilde{B}^1_\pi$ and that the assertions concerning pointwise convergence hold. As to the convergence in the norm, let $\varepsilon > 0$. Then there exists $R > 0$ such that

$$\int_{|u| > R/2} |f(u)|^p \mathrm{d}u < \varepsilon,$$

and hence also

$$\int_{|u|>R} |f_n(u)|^p \mathrm{d}u \le \frac{n+1}{n} \int_{|u| > nR/(n+1)} |f(u)|^p \mathrm{d}u \le 2 \int_{|u| \ge R/2} |f(u)|^p \mathrm{d}u < 2\varepsilon.$$

So one obtains

$$\begin{aligned} \|f_n - f\|_p &\le \left\{ \int_{|u|\le R} |f_n(u) - f(u)|^p \mathrm{d}u \right\}^{1/p} + \left\{ \int_{|u|>R} |f_n(u)|^p \mathrm{d}u \right\}^{1/p} + \left\{ \int_{|u|>R} |f(u)|^p \mathrm{d}u \right\}^{1/p} \\ &\le \left\{ \int_{|u|\le R} |f_n(u) - f(u)|^p \mathrm{d}u \right\}^{1/p} + (2\varepsilon)^{1/p} + \varepsilon^{1/p}. \end{aligned}$$

The norm convergence now follows from the fact that $\lim_{n\to\infty} f_n(u) = f(u)$ uniformly on the compact interval $[-R, R]$.

For arbitrary $\sigma > 0$ one has only to note that $f \in B^p_\sigma$ if and only if the function

$$h: \mathbb{C} \to \mathbb{C},\ z \mapsto f\left(\frac{\pi z}{\sigma}\right)$$

belongs to $B^p_\pi$. The results then follow by a linear transformation. ■

LEMMA 2.2 *If $(a_k)_{k\in\mathbb{Z}}$ is a sequence of complex numbers with $\sum_{k\in\mathbb{Z}} |a_k|^p < \infty$ and $1 \le p < \infty$, then for $j = 1, 2$ the series*

$$\sum_{k\in\mathbb{Z}} a_k \operatorname{sinc}(z-k)^j \quad (z \in \mathbb{C})$$

*are absolutely convergent and the convergence is uniform in strips of bounded width parallel to the real line.*

*Proof* Let $|\Im\mathfrak{m}\, z| \le M$ and $N \in \mathbb{N}$. Then one has by Hölder's inequality with $1/p + 1/q = 1$,

$$\sum_{|k|\ge N} |a_k \operatorname{sinc}^j (z-k)| \le \left\{ \sum_{|k|\ge N} |a_k|^p \right\}^{1/p} \left\{ \sum_{|k|\ge N} |\operatorname{sinc}(z-k)|^{jq} \right\}^{1/q}.$$

It is now enough to show that the last series has a bound not depending on $z$.

Indeed, using the estimate

$$|\operatorname{sinc} z| \le \min\left\{ e^{\pi|y|}, \frac{e^{\pi|y|}}{\pi|z|} \right\} \le \frac{2\, e^{\pi|y|}}{1+\pi|z|} \quad (z = x+iy,\ x, y \in \mathbb{R}), \tag{5}$$

we find that

$$\begin{aligned}\left\{ \sum_{|k|\ge N} |\operatorname{sinc}(z-k)|^{jq} \right\}^{1/q} &\le 2^j e^{\pi|y|j} \left\{ \sum_{|k|\ge N} \frac{1}{(1+\pi|x-k|)^{jq}} \right\}^{1/q} \\ &\le 2^j e^{\pi jM} \left\{ \sum_{k\in Z} \frac{1}{(1+\pi|x-k|)^{jq}} \right\}^{1/q} \le 2^j e^{\pi jM} \left\{ 2 + 2\sum_{k=1}^{\infty} \frac{1}{(1+\pi k)^{jq}} \right\}^{1/q}.\end{aligned}$$

Since $jp > 1$, the latter series is convergent with limit independent of $z$. This completes the proof. ■

In the proofs below one of the main problems is the justification of the interchange of summation and integration or the interchange of summation and taking a limit. The following lemma will be helpful in this respect.

LEMMA 2.3 (a) *Let $f \in L^p(\mathbb{R})$, $g \in L^q(\mathbb{R})$ with $1 \le p \le \infty$ and $1/p + 1/q = 1$. Let $(f_n)_{n\in\mathbb{N}} \subset L^p(\mathbb{R})$, $(g_n)_{n\in\mathbb{N}} \subset L^q(\mathbb{R})$ such that $\lim_{n\to\infty} \|f_n - f\|_p = 0$ and $\lim_{n\to\infty} \|g_n - g\|_q = 0$; then*

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n(u)\overline{g_n(u)}\mathrm{d}u = \int_{\mathbb{R}} f(u)\overline{g(u)}\mathrm{d}u.$$

*In particular, there holds*

$$\lim_{n\to\infty}\int_{\mathbb{R}} f_n(u)\overline{g(u)}\mathrm{d}u = \int_{\mathbb{R}} f(u)\overline{g(u)}\mathrm{d}u.$$

(b) *Let* $a := (\alpha_k)_{k\in\mathbb{Z}} \in l^p(\mathbb{Z})$, $b := (\beta_k)_{k\in\mathbb{Z}} \in l^q(\mathbb{Z})$ *and for each* $n\in\mathbb{N}$ *let* $a_n := (\alpha_{n,k})_{k\in\mathbb{Z}} \in l^p(\mathbb{Z})$, $b_n := (\beta_{n,k})_{k\in\mathbb{Z}} \in l^q(\mathbb{Z})$ *with* $\lim_{k\to\infty}\|a_n - a\|_{l^p(\mathbb{Z})} = 0$, $\lim_{n\to\infty}\|b_n - b\|_{l^q(\mathbb{Z})}$; *then*

$$\lim_{n\to\infty}\sum_{k=-\infty}^{\infty}\alpha_{n,k}\overline{\beta_{n,k}} = \sum_{k=-\infty}^{\infty}\alpha_k\overline{\beta_k}$$

*and in particular*,

$$\lim_{n\to\infty}\sum_{k=-\infty}^{\infty}\alpha_{n,k}\overline{\beta_k} = \sum_{k=-\infty}^{\infty}\alpha_k\overline{\beta_k}.$$

*Proof* It follows by Hölder's inequality that

$$\begin{aligned}
&\left|\int_{\mathbb{R}} f_n(u)\overline{g_n(u)}\mathrm{d}u - \int_{\mathbb{R}} f(u)\overline{g(u)}\mathrm{d}u\right| \\
&\quad\le \int_{\mathbb{R}} |f_n(u) - f(u)||g_n(u)|\mathrm{d}u + \int_{\mathbb{R}} |g_n(u) - g(u)||f(u)|\mathrm{d}u \\
&\quad\le \|f_n - f\|_p\,\|g_n\|_q + \|g_n - g\|_p\,\|f\|_q.
\end{aligned}$$

The assumptions upon the sequences $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ imply that the right-hand side vanishes for $n\to\infty$. This proves parts (a) and (b) follows analogously. ■

## 3. The classical sampling formula and its consequences

The CSF enables us to deduce several basic well-known results concerning Bernstein spaces, which will be used in several proofs below.

THEOREM 3.1 *Let* $f\in B^p_\sigma$, *where* $1\le p\le\infty$, $\sigma>0$. *The following statements are consequences of CSF.*

(i) *Boas' differentiation formula*:

$$f'(z) = \frac{4\sigma}{\pi^2}\sum_{k\in\mathbb{Z}}\frac{(-1)^{k+1}}{(2k-1)^2} f\left(z + \frac{(2k-1)\pi}{2\sigma}\right) \quad (z\in\mathbb{C}), \tag{6}$$

*the series converging absolutely and uniformly in strips of bounded width parallel to the real line.*

(ii) *Bernstein's inequality*:

$$\|f'\|_p \le \sigma\|f\|_p. \tag{7}$$

*Moreover*, $B^p_\sigma$ *is invariant under differentiation, i.e. if* $f\in B^p_\sigma$, *then* $f'\in B^p_\sigma$.

(iii) *Nikol'skiĭ's inequality: For* $h>0$ *and* $p\neq\infty$,

$$\left\{h\sum_{k\in\mathbb{Z}}|f(kh)|^p\right\}^{1/p}\leq(1+\sigma h)\|f\|_p. \tag{8}$$

(iv) *Derivative sampling formula: For* $p\neq\infty$ *and all* $z\in\mathbb{C}$,

$$f(z)=\sum_{k\in\mathbb{Z}}\left\{f\Big(\frac{2k\pi}{\sigma}\Big)+\Big(\frac{\sigma z}{2\pi}-k\Big)\frac{2\pi}{\sigma}f'\Big(\frac{2k\pi}{\sigma}\Big)\right\}\operatorname{sinc}^2\Big(\frac{\sigma z}{2\pi}-k\Big)\quad(z\in\mathbb{C}), \tag{9}$$

*where the series converges absolutely and uniformly in strips of bounded width parallel to the real line.*

(v) *Orthogonality of the* sinc-*functions*: *There holds*

$$\int_{\mathbb{R}}\operatorname{sinc}(u-j)\operatorname{sinc}(u-k)\mathrm{d}u=\begin{cases}1, & j=k\\ 0, & j\neq k.\end{cases}\quad(j,k\in\mathbb{Z}). \tag{10}$$

*Proof* As in case of La. 2.1 it suffices to prove the statements for $\sigma=\pi$ only, since the results for general $\sigma$ follow from those special cases by scaling. We will show this in more detail in the proof of (i).

(i) Let $\sigma=\pi$ and assume additionally that $f\in\widetilde{B}^1_\pi$. Then $f\in B^2_\pi$ and CSF yields

$$f(t)=\sum_{k\in\mathbb{Z}}f(k)\operatorname{sinc}(t-k)\quad(t\in\mathbb{R}),$$

where the series is uniformly convergent on $\mathbb{R}$. Moreover, the termwise differentiated series is also uniformly convergent on $\mathbb{R}$ in view of the fact that the additional assumption $f\in\widetilde{B}^1_\pi$ implies $f(t)=\mathcal{O}(|t|^{-2})$ for $t\to\pm\infty$. So we obtain

$$f'(t)=\sum_{k\in\mathbb{Z}}f(k)\operatorname{sinc}'(t-k)\quad(t\in\mathbb{R}).$$

Evaluating this equation at $t=1/2$, we find that

$$f'\Big(\frac{1}{2}\Big)=\frac{4}{\pi}\sum_{k\in\mathbb{Z}}\frac{(-1)^{k+1}}{(2k-1)^2}f(k). \tag{11}$$

But this formula is valid even for *every* $f\in B^p_\pi$. In fact, choose a sequence $(f_n)_{n\in\mathbb{N}}\subset\widetilde{B}^1_\sigma$ as in La. 2.1, apply (11) to each $f_n$ and let $n\to\infty$.

Now it easily follows from the definition of the Bernstein spaces that $f\in B^p_\pi$ implies $g:=f(\cdot+z-1/2)\in B^\infty_\pi$ for any $z\in\mathbb{C}$. Applying now (11) to $g$, we obtain (6) for $\sigma=\pi$.

In order to prove (6) for $f\in B^p_\sigma$ with $\sigma>0$ arbitrary, one applies (6) for $\sigma=\pi$ to $f(\frac{\pi z}{\sigma})\in B^p_\pi$ and replaces $z$ by $\frac{\sigma z}{\pi}$ afterwards. The absolute and uniform convergence of the series follows from the fact that $f$ is bounded in strips of bounded width parallel to the real axis in view of (1).

(ii) For $z=t\in\mathbb{R}$ and $\sigma=\pi$, we deduce from (6) that

$$\left|\frac{f'(t)}{\pi}\right|\leq\frac{4}{\pi^2}\sum_{k\in\mathbb{Z}}\frac{1}{(2k-1)^2}\left|f\Big(t+k-\frac{1}{2}\Big)\right|.$$

Taking the $L^p(\mathbb{R})$-norm on both sides and noting the shift invariance of the norm this yields

$$\frac{1}{\pi}\|f'\|_p \le \frac{4}{\pi^2}\sum_{k\in\mathbb{Z}}\frac{1}{(2k-1)^2}\left\|f\left(\cdot+k-\frac{1}{2}\right)\right\|_p = \frac{4}{\pi^2}\sum_{k\in\mathbb{Z}}\frac{1}{(2k-1)^2}\|f\|_p.$$

Using (6) for calculating the derivative of $\cos\pi z$ at $z=1/2$, we find that

$$\frac{4}{\pi^2}\sum_{k\in\mathbb{Z}}\frac{1}{(2k-1)^2}=1.$$

This proves (7) for $\sigma=\pi$; the general case follows again by a linear transformation.

Finally, assuming that $|f(z)|\le M\exp(\pi|\mathfrak{Im}\,z|)$, we easily deduce from (6) for $\sigma=\pi$ that $|f'(z)|\le \pi M\exp(\pi|\mathfrak{Im}\,z|)$. Hence $f\in B_\pi^p$ implies $f'\in B_\pi^p$.

(iii) The proof of inequality (8) follows by trivial tools from elementary integral calculus together with Bernstein's inequality (7); see [39, p. 123].

(iv) First assume that $f\in \widetilde{B}_\pi^1$ and define

$$F(z) := \begin{cases} \dfrac{\pi[f(2z)-\sum_{k\in\mathbb{Z}}f(2k)\operatorname{sinc}^2(z-k)]}{\sin\pi z}, & z\in\mathbb{C}\setminus\mathbb{Z},\\ (-1)^z 2f'(2z), & z\in\mathbb{Z}. \end{cases} \tag{12}$$

We show that $F\in B_\pi^2$.

The assumption $f\in \widetilde{B}_\pi^1$ implies $\sum_{k\in\mathbb{Z}}|f(2k)|<\infty$. Thus the infinite series in (12) is uniformly convergent on compact sets, implying that $F$ is analytic for $z\in\mathbb{C}\setminus\mathbb{Z}$. By l'Hospital's rule it is easily verified that $F$ is continuous at the integers, i.e. $F$ is an entire function[5].

Next we note that CSF applies to $f$. Representing $f(z)$ by the sampling formula and replacing $z$ by $2z$ afterwards, we deduce from (12) that

$$F(z)=\frac{\pi\left[\sum_{k\in\mathbb{Z}}f(k)\operatorname{sinc}(2z-k)-\sum_{k\in\mathbb{Z}}f(2k)\operatorname{sinc}^2(z-k)\right]}{\sin\pi z},\quad (z\in\mathbb{C}\setminus\mathbb{Z}).$$

Combining the terms of even $k$ in the first series with the terms in the second series, we obtain after a short trigonometric calculation that

$$F(z)=\sum_{k\in\mathbb{Z}}(-1)^k\left[f(2k)\operatorname{sinc}'(z-k)+f(2k+1)\pi\operatorname{sinc}\left(z-k-\frac{1}{2}\right)\right],\quad (z\in\mathbb{C}). \tag{13}$$

Using that $\sum_{k\in\mathbb{Z}}|f(2k)|<\infty$ and $\sum_{k\in\mathbb{Z}}|f(2k+1)|<\infty$ together with the estimates

$$|\operatorname{sinc}'(z-k)|\le c_1 e^{\pi|\mathfrak{Im}\,z|},\qquad \left|\operatorname{sinc}\left(z-k-\frac{1}{2}\right)\right|\le c_2 e^{\pi|\mathfrak{Im}\,z|},\quad (k\in\mathbb{Z}\setminus\{0\},\ z\in\mathbb{C})$$

for two constants $c_1$, $c_2$, we conclude that

$$|F(z)|\le Me^{\pi|\mathfrak{Im}\,z|}. \tag{14}$$

Now the representation (13) gives

$$tF(t)=\sum_{k\in\mathbb{Z}}(-1)^k\left[f(2k)t\operatorname{sinc}'(t-k)+f(2k+1)\pi t\operatorname{sinc}\left(t-k-\frac{1}{2}\right)\right],\quad (t\in\mathbb{R}). \tag{15}$$

It can be shown in an elementary way that there exist constants $d_1$ and $d_2$ such that

$$\left|\frac{t}{k}\operatorname{sinc}'(t-k)\right| \le d_1, \quad \left|\frac{t}{k}\operatorname{sinc}\left(t-k-\frac{1}{2}\right)\right| \le d_2, \quad (k\in\mathbb{Z}\setminus\{0\},\ t\in\mathbb{R}).$$

On the other hand, by the hypotheses $z^2 f(z)\in B_\pi^1$ we have in view of (8) that $\sum_{k\in\mathbb{Z}}|kf(2k)|<\infty$ and $\sum_{k\in\mathbb{Z}}|kf(2k+1)|<\infty$. Hence it follows from the representation (15) that $tF(t)$ is bounded on $\mathbb{R}$, which in turn implies that $F\in L^2(\mathbb{R})$. Observing (14), we end up with $F\in B_\pi^2$.

Now CSF applies to $F$ and we obtain

$$F(z)=\sum_{k\in\mathbb{Z}} F(k)\operatorname{sinc}(z-k) = 2\sum_{k\in\mathbb{Z}}(-1)^k f'(2k)\operatorname{sinc}(z-k), \quad (z\in\mathbb{C}).$$

By the definition of $F$ this can be rewritten as

$$\begin{aligned} f(2z) &= \sum_{k\in\mathbb{Z}} f(2k)\operatorname{sinc}^2(z-k) + \frac{2\sin\pi z}{\pi}\sum_{k\in\mathbb{Z}}(-1)^k f'(2k)\operatorname{sinc}(z-k) \\ &= \sum_{k\in\mathbb{Z}} f(2k)\operatorname{sinc}^2(z-k) + 2\sum_{k\in\mathbb{Z}} f'(2k)(z-k)\operatorname{sinc}^2(z-k), \quad (z\in\mathbb{C}). \end{aligned}$$

This yields (9) for $\sigma=\pi$ in replacing $z$ by $z/2$.

In order to get rid of the more restrictive assumption $f\in \widetilde{B}_\pi^1$, we first show that the series (9) converges absolutely and uniformly in strips of bounded width parallel to the real line for all $f\in B_\pi^p$. Rewriting (9) for $\sigma=\pi$ as

$$f(z)=\sum_{k\in\mathbb{Z}} f(2k)\operatorname{sinc}^2\left(\frac{z}{2}-k\right)+\frac{2}{\pi}\sum_{k\in\mathbb{Z}} f'(2k)\operatorname{sinc}\left(\frac{z}{2}-k\right)\sin\pi\left(\frac{z}{2}-k\right), \tag{16}$$

the first series converges absolutely and uniformly in strips of bounded width parallel to the real line by La. 2.2, since $\sum_{k\in\mathbb{Z}}|f(2k)|^p<\infty$ by (8). Concerning the second series, one has

$$\sum_{k\in\mathbb{Z}}\left|f'(2k)\operatorname{sinc}\left(\frac{z}{2}-k\right)\sin\pi\left(\frac{z}{2}-k\right)\right| \le \left|\sin\frac{\pi z}{2}\right|\sum_{k\in\mathbb{Z}}\left|f'(2k)\operatorname{sinc}\left(\frac{z}{2}-k\right)\right|$$

and the convergence assertion follows again by La. 2.2, in view of $\sum_{k\in\mathbb{Z}}|f'(2k)|^p<\infty$ by (8) and (7).

Finally, one can easily extend the desired result to all of $B_\pi^p$. To this end assume $f\in B_\pi^p$, and let $(f_n)_{n\in\mathbb{N}}$ be a sequence as in La. 2.1. As shown above, (9) is valid for each $f_n$, i.e. (cf (16)),

$$f_n(z)=\sum_{k\in\mathbb{Z}} f_n(2k)\operatorname{sinc}^2\left(\frac{z}{2}-k\right)+\frac{2}{\pi}\sum_{k\in\mathbb{Z}} f_n'(2k)\operatorname{sinc}\left(\frac{z}{2}-k\right)\sin\pi\left(\frac{z}{2}-k\right).$$

Letting $n\to\infty$ it easily follows by La. 2.3 (b) that (9) also holds for the limit function $f$. Indeed, one has only to note that by (8) and (7),

$$\begin{aligned} &\|f_n(2\,\cdot)-f(2\,\cdot)\|_{l^p(\mathbb{Z})} \le c\|f_n-f\|_p = o(1) \quad (n\to\infty),\\ &\|f_n'(2\,\cdot)-f'(2\,\cdot)\|_{l^p(\mathbb{Z})} \le c\|f_n'-f'\|_p \le c\pi\|f_n-f\|_p = o(1) \quad (n\to\infty), \end{aligned}$$

and that $(\operatorname{sinc}(\frac{z}{2}-k))\in l^q(\mathbb{Z})$ as well as $(\operatorname{sinc}(\frac{z}{2}-k)\sin\pi(\frac{z}{2}-k))\in l^q(\mathbb{Z})$.

The above proof was inspired by the proof of the derivative sampling theorem in [26] and will be basic for the proof of Theorem 4.3.

(v) For $s, u \in \mathbb{R}$ the function $\operatorname{sinc}(\cdot + u - s)$ belongs to $B_\pi^2$, and we can apply CSF to obtain

$$\operatorname{sinc}(z + u - s) = \sum_{k \in \mathbb{Z}} \operatorname{sinc}(k + u - s)\operatorname{sinc}(z - k), \quad (z \in \mathbb{C}),$$

and in particular for $z = t - u$,

$$\operatorname{sinc}(t - s) = \sum_{k \in \mathbb{Z}} \operatorname{sinc}(k + u - s)\operatorname{sinc}(t - u - k), \quad (s, t, u \in \mathbb{R}).$$

Next we integrate both sides over $[0, 1]$ with respect to $u$ and note that integration and summation may be interchanged since the series converges uniformly with respect to $u$ on compact sets. Therefore,

$$\begin{aligned}\operatorname{sinc}(t - s) &= \sum_{k \in \mathbb{Z}} \int_0^1 \operatorname{sinc}(k + u - s)\operatorname{sinc}(t - u - k)\mathrm{d}t \\ &= \sum_{k \in \mathbb{Z}} \int_k^{k+1} \operatorname{sinc}(v - s)\operatorname{sinc}(t - v)\mathrm{d}v = \int_{\mathbb{R}} \operatorname{sinc}(s - u)\operatorname{sinc}(t - u)\mathrm{d}u.\end{aligned}$$

This yields the orthogonality (10) by choosing $s$, $t$ as integers. ■

## 4. A first group of equivalences

### 4.1. *The classical sampling formula implies certain assertions*

THEOREM 4.1 *CSF ⇒ RKF.*

*Proof* Let $z \in \mathbb{C}$ and $u \in \mathbb{R}$. Together with $f$, the shifted function $f(\cdot + u)$ also belongs to $B_\pi^2$. Now we apply CSF to $f(\cdot + u)$ and replace $z$ by $z - u$ in the resulting equation. This gives

$$f(z) = \sum_{k \in \mathbb{Z}} f(k + u)\operatorname{sinc}(z - u - k).$$

Integrating both sides over $[0, 1]$ with respect to $u$ and noting that integration and summation may be interchanged since the series converges uniformly with respect to $u$ on compact sets, we obtain as in the proof of Theorem 3.1 (v),

$$\begin{aligned}f(z) &= \sum_{k \in \mathbb{Z}} \int_0^1 f(k + u)\operatorname{sinc}(z - u - k)\mathrm{d}u \\ &= \sum_{k \in \mathbb{Z}} \int_k^{k+1} f(u)\operatorname{sinc}(z - u)\mathrm{d}u = \int_{\mathbb{R}} f(u)\operatorname{sinc}(z - u)\mathrm{d}u.\end{aligned}$$

This shows that RKF holds. ■

THEOREM 4.2 *CSF ⇒ GPF.*

*Proof* First assume $f, g \in \widetilde{B}_\pi^1$. Since in this particular case the sequences $(f(k))_{k\in\mathbb{Z}}$ and $(g(k))_{k\in\mathbb{Z}}$ both belong to $l^1(\mathbb{Z})$, it is easily seen that the series $\sum_{k\in\mathbb{Z}} f(k)\operatorname{sinc}(\cdot - k)$ and $\sum_{k\in\mathbb{Z}} g(k)\operatorname{sinc}(\cdot - k)$ converge in $L^2(\mathbb{R})$-norm towards $f$ and $g$, respectively.

Defining now $S_n f := \sum_{|k|\le n} f(k)\operatorname{sinc}(\cdot - k)$, it follows by La. 2.3 that

$$\begin{aligned}\int_{\mathbb{R}} f(u)\overline{g(u)}\mathrm{d}u &= \lim_{n\to\infty}\int_{\mathbb{R}}(S_n f)(u)\overline{(S_n g)(u)}\mathrm{d}u \\ &= \lim_{n\to\infty}\sum_{|k|\le n}\sum_{|j|\le n} f(j)\overline{g(k)}\int_{\mathbb{R}}\operatorname{sinc}(u-j)\operatorname{sinc}(u-k)\mathrm{d}u.\end{aligned}$$

In view of the orthogonality (10) the latter double sum reduces to $\sum_{|k|\le n} f(k)\overline{g(k)}$, giving GPF for $f, g \in \widetilde{B}^1_\pi$.

To extend the particular case to arbitrary $f, g \in B^2_\pi$, we choose, according to La. 2.1, two sequences $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty}\|f_n - f\|_2 = 0$ and $\lim_{n\to\infty}\|g_n - g\|_2 = 0$. It follows from the first part of the proof that

$$\int_{\mathbb{R}} f_n(u)\overline{g_n(u)}\mathrm{d}u = \sum_{k\in\mathbb{Z}} f_n(k)\overline{g_n(k)}, \quad (n\in\mathbb{N}).$$

For $n\to\infty$ we obtain GPF by La. 2.3 with $p = q = 2$, since $\lim_{n\to\infty}\|f_n - f\|_2 = 0$ and $\lim_{n\to\infty}\|g_n - g\|_2 = 0$ imply $\lim_{n\to\infty}\|f_n(\cdot) - f(\cdot)\|_{l^2(\mathbb{Z})} = 0$ and $\lim_{n\to\infty}\|g_n(\cdot) - g(\cdot)\|_{l^2(\mathbb{Z})} = 0$ in view of (8). ■

THEOREM 4.3 *CSF ⇒ PSF.*

*Proof* The proof is a slight modification of the proof in [26]. If $f \in B^1_{2\pi}$ then $g := f(\frac{\cdot}{2}) \in B^1_\pi$ and we can apply (9) to $g$ and replace $z$ by $2t \in \mathbb{R}$ afterwards. This yields

$$f(t) = \sum_{k\in\mathbb{Z}}\left\{f(k) + (t-k)f'(k)\right\}\operatorname{sinc}^2(t-k), \quad (t\in\mathbb{R}). \tag{17}$$

Now we assume, in addition, that $f \in \widetilde{B}^1_\pi$. An application of (17) to $zf(z)$ then yields

$$tf(t) = \sum_{k\in\mathbb{Z}}\left\{kf(k) + (t-k)[f(k) + kf'(k)]\right\}\operatorname{sinc}^2(t-k), \quad (t\in\mathbb{R}),$$

and after a small rearrangement and dividing by $t$ we obtain,

$$f(t) = \sum_{k\in\mathbb{Z}} f(k)\operatorname{sinc}^2(t-k) + \sum_{k\in\mathbb{Z}}(-1)^k kf'(k)\operatorname{sinc}(t-k)\operatorname{sinc} t, \quad (t\in\mathbb{R}). \tag{18}$$

The advantage of the representation (18) in comparison with (17) is that all terms of the sums in (18) are absolutely integrable over $\mathbb{R}$.

Now we integrate both sides of (18) over $\mathbb{R}$. Since $f \in B^1_\pi$, and hence $(f(k))_{k\in\mathbb{Z}} \in l^1(\mathbb{Z})$, the first series can be integrated term by term.

As regards the second series, we first recall that $zf(z)$ belongs to $B^1_\pi$. By Theorem 3.1 (ii), the derivative $f(z) + zf'(z)$ also belongs to $B^1_\pi$. Since $f \in B^1_\pi$, it follows that $zf'(z)$ belongs to $B^1_\pi$. Now (8) guarantees that $(kf'(k))_{k\in\mathbb{Z}} \in l^1(\mathbb{Z})$, and so the second series can be integrated term by term, too. Hence it follows that

$$\int_{\mathbb{R}} f(u)\mathrm{d}u = \sum_{k\in\mathbb{Z}} f(k)\int_{\mathbb{R}}\operatorname{sinc}^2(u-k)\mathrm{d}u + \sum_{k\in\mathbb{Z}}(-1)^k kf'(k)\int_{\mathbb{R}}\operatorname{sinc}(u-k)\operatorname{sinc} u\,\mathrm{d}u.$$

Recalling (10), we obtain PSF immediately. Finally, the additional assumption $f \in \widetilde{B}_\pi^1$ can be dropped by approximating $f \in B_\pi^1$ by a sequence $(f_n)_{n\in\mathbb{N}} \subset \widetilde{B}_\pi^1$ according to La. 2.1, and using La. 2.3 with $g=1$, $p=1$, $q=\infty$. ■

### 4.2. *Certain assertions imply the classical sampling formula*

Let us observe that GPF, PSF and PWT are real variable statements, but CSF, VSF and RKF are complex variable assertions. In order to prove that GPF implies CSF one needs the following identity principle of complex analysis to obtain the full complex version of CSF. The same holds when proceeding from GPF, PSF or PWT to one of the three CSF, VSF or RKF.

**Weak identity principle:** *If a function from $B_\sigma^2$ vanishes on the real line, then it is identically zero.*

Now the CSF, VSF and RKF easily imply the weak identity principle (WIP). However, it does not seem to be possible to deduce WIP from PSF, GPF or PWT. On the other hand, WIP is covered by the classical identity theorem of complex analysis. Indeed, it suffices to know that functions in a Bernstein space have locally a power series expansion having positive radius of convergence. In fact, if a function vanishes identically on the real line, then all its derivatives vanish, and hence at each point of $\mathbb{R}$ it has a Taylor expansion, all coefficients of which are zero.

If WIP is not admitted as a side result, then one has to restrict the assertions of CSF, VSF and RKF to the real variable frame.

THEOREM 4.4 *GPF + WIP $\Rightarrow$ CSF.*

*Proof* Let $t \in \mathbb{R}$ and note that the sinc-function is real-valued on the real line. Then, applying GPF with $g = \operatorname{sinc}(t-\cdot)$, we have

$$\int_{\mathbb{R}} f(u)\operatorname{sinc}(t-u)\mathrm{d}u = \sum_{k\in\mathbb{Z}} f(k)\operatorname{sinc}(t-k).$$

Next, applying GPF with $f$ replaced by $f(t-\cdot)$ and $g = \operatorname{sinc}$, we deduce

$$\int_{\mathbb{R}} f(t-u)\operatorname{sinc} u\,\mathrm{d}u = \sum_{k\in\mathbb{Z}} f(t-k)\operatorname{sinc} k = f(t).$$

The integrals on the left-hand sides are equal, and so CSF follows for $t \in \mathbb{R}$.

In order to extend CSF to $z \in \mathbb{C}$, we consider the function $h$ defined by

$$h(z) := \sum_{k\in\mathbb{Z}} f(k)\operatorname{sinc}(z-k), \quad (z\in\mathbb{C}). \tag{19}$$

We want to show that $h \in B_\pi^2$ and that $h = f$ on $\mathbb{C}$. As to the convergence of the series (19), applying GPF with $f = g$ yields

$$\sum_{k\in\mathbb{Z}} |f(k)|^2 = \int_{\mathbb{R}} |f(u)|^2\mathrm{d}u,$$

showing by La. 2.2 that the series is absolutely and uniformly convergent in strips of bounded width parallel to the real line and hence defines an entire function.

Next we apply GPF to $f=g=\operatorname{sinc}$, noting that $\operatorname{sinc}(z-\cdot)\in B_\pi^2$ for each fixed $z\in\mathbb{C}$, to obtain

$$\sum_{k\in\mathbb{Z}}|\operatorname{sinc}(z-k)|^2=\int_{\mathbb{R}}|\operatorname{sinc}(z-u)|^2\mathrm{d}u.$$

Now it follows by Cauchy–Schwarz' inequality that

$$\begin{aligned}\left\{\sum_{k\in\mathbb{Z}}|f(k)\operatorname{sinc}(z-k)|\right\}^2 &\le \sum_{k\in\mathbb{Z}}|f(k)|^2\cdot\sum_{k\in\mathbb{Z}}|\operatorname{sinc}(z-k)|^2\\ &=\int_{\mathbb{R}}|f(u)|^2\mathrm{d}u\cdot\int_{\mathbb{R}}|\operatorname{sinc}(z-u)|^2\mathrm{d}u.\end{aligned}$$

Since $|\operatorname{sinc}(z-u)|^2\le\exp(2\pi|y|)$ and also $|\sin\pi(z-u)|^2\le\exp(2\pi|y|)$ for all $z=x+iy\in\mathbb{C}$ and $u\in\mathbb{R}$, the last integral can be estimated by

$$\int_{\mathbb{R}}|\operatorname{sinc}(z-u)|^2\mathrm{d}u=\int_{\mathbb{R}}|\operatorname{sinc}(iy-u)|^2\mathrm{d}u\le\int_{-1}^{1}e^{2\pi|y|}\mathrm{d}u+\int_{|u|>1}\frac{e^{2\pi|y|}}{y^2+u^2}\mathrm{d}u\le 4e^{2\pi|y|}.$$

Hence it follows that

$$|h(z)|\le M\exp(\pi|\mathfrak{Im}\,z|)\quad(z\in\mathbb{C})$$

for some constant $M$.

By the first part of the proof we have that $h(t)=f(t)$ for $t\in\mathbb{R}$, yielding $h\in L^2(\mathbb{R})$. Altogether we have shown $h\in B_\pi^2$. Hence $h$ is a function in $B_\pi^2$, which coincides on the real line with $f\in B_\pi^2$. By WIP $h$ and $f$ coincide on the whole complex plane, which is CSF. ∎

Theorem 4.5 *RKF⇒CSF.*

*Proof* First we prove the sinc summation formula

$$\operatorname{sinc}(\alpha-\beta)=\sum_{k\in\mathbb{Z}}\operatorname{sinc}(\alpha-k)\operatorname{sinc}(\beta-k),\quad(\alpha,\beta\in\mathbb{C}).\tag{20}$$

It follows almost immediately from the cotangent expansion[6]

$$\pi\cot\pi z=\frac{1}{z}+\sum_{k=1}^{\infty}\frac{2z}{z^2-k^2}=\frac{1}{z}+\sum_{k=1}^{\infty}\left(\frac{1}{z+k}+\frac{1}{z-k}\right),\quad(z\in\mathbb{C}\setminus\mathbb{Z}).$$

Indeed, it suffices to deduce (20) for $\alpha,\beta\in\mathbb{C}\setminus\mathbb{Z}$ only. Writing down two copies of the latter expansion, we obtain

$$\pi(\cot\pi\alpha-\cot\pi\beta)=\frac{1}{\alpha}-\frac{1}{\beta}+\sum_{k=1}^{\infty}\frac{2\alpha}{\alpha^2-k^2}-\sum_{k=1}^{\infty}\frac{2\beta}{\beta^2-k^2}=\sum_{k\in\mathbb{Z}}\left(\frac{1}{\alpha-k}-\frac{1}{\beta-k}\right).$$

Hence there follows

$$\frac{\pi}{\alpha-\beta}(\cot\pi\alpha-\cot\pi\beta)=\sum_{k\in\mathbb{Z}}\frac{1}{\alpha-\beta}\left(\frac{1}{\alpha-k}-\frac{1}{\beta-k}\right),$$

so that

$$\frac{\pi \sin \pi(\alpha-\beta)}{(\alpha-\beta)\sin \pi\alpha \, \sin \pi\beta} = \sum_{k\in\mathbb{Z}} \frac{1}{(\alpha-k)(\beta-k)}.$$

A small rearrangement finally gives the sinc formula (20).

Proceeding formally, one substitutes the expansion (20) into the integral of the RKF assertion. An interchange of integration and summation then yields

$$f(z) = \sum_{k\in\mathbb{Z}} \operatorname{sinc}(z-k) \int_{\mathbb{R}} f(u)\operatorname{sinc}(u-k)\mathrm{d}u = \sum_{k\in\mathbb{Z}} f(k)\operatorname{sinc}(z-k) \tag{21}$$

by another application of RKF. This is already the desired result.

Now to the precise proof: We have to justify the interchange of integration and summation in the above formal proof. Using the Hilbert space notation[7] $\langle f, g\rangle := \int_{\mathbb{R}} f(u)\overline{g(u)}\mathrm{d}u$ for the inner product in $L^2(\mathbb{R})$, we can rewrite RKF as

$$f(k) = \big\langle f, \operatorname{sinc}(\cdot - k)\big\rangle, \tag{22}$$

and, in particular, we deduce the orthogonality of the shifted sinc-functions from RKF, namely,

$$\big\langle \operatorname{sinc}(\cdot - j), \operatorname{sinc}(\cdot - k)\big\rangle = \operatorname{sinc}(j-k) = \delta_{j,k}, \quad (j,k\in\mathbb{Z}). \tag{23}$$

Next to the convergence of the sampling series on the right-hand side of (21). In this respect we have,

$$\begin{aligned}
&\Big\| f - \sum_{|k|\le n} f(k)\operatorname{sinc}(\cdot - k)\Big\|_2^2 \\
&= \Big\langle f - \sum_{|j|\le n} f(j)\operatorname{sinc}(\cdot - j), f - \sum_{|k|\le n} f(k)\operatorname{sinc}(\cdot - k)\Big\rangle \\
&= \|f\|_2^2 - \sum_{|k|\le n} \overline{f(k)}\big\langle f, \operatorname{sinc}(\cdot - k)\big\rangle - \sum_{|j|\le n} f(j)\overline{\big\langle f, \operatorname{sinc}(\cdot - j)\big\rangle} \\
&\quad + \sum_{|k|\le n}\sum_{|j|\le n} f(j)\overline{f(k)}\langle \operatorname{sinc}(\cdot - j), \operatorname{sinc}(\cdot - k)\rangle.
\end{aligned}$$

Using (22) and (23), we obtain

$$\|f\|_2^2 - \sum_{|k|\le n} |f(k)|^2 = \Big\| f - \sum_{|k|\le n} f(k)\operatorname{sinc}(\cdot - k)\Big\|_2^2 \ge 0,$$

showing that $\sum_{k\in\mathbb{Z}}|f(k)|^2 \le \|f\|_2 < \infty$.[8] From La. 2.2 it now follows that the sampling series is absolutely and uniformly convergent on strips of bounded width parallel to the real line.

In order to justify the interchange of integration and summation in (21), we consider for fixed $z\in\mathbb{C}$ the sinc summation formula (20) in the form

$$\operatorname{sinc}(z-u) = \sum_{k\in\mathbb{Z}} \operatorname{sinc}(z-k)\operatorname{sinc}(u-k) \quad (u\in\mathbb{R}). \tag{24}$$

For $m,\, n \in \mathbb{N}$ one has by (23),

$$\begin{aligned}
&\Bigg\| \sum_{m \le |k| \le n} \operatorname{sinc}(z-k)\operatorname{sinc}(\cdot - k) \Bigg\|_2^2 \\
&\quad = \Bigg\langle \sum_{m \le |k| \le n} \operatorname{sinc}(z-k)\operatorname{sinc}(\cdot - k),\ \sum_{m \le |j| \le n} \operatorname{sinc}(z-j)\operatorname{sinc}(\cdot - j) \Bigg\rangle \\
&\quad = \sum_{m \le |k| \le n} \operatorname{sinc}(z-k) \sum_{m \le |j| \le n} \operatorname{sinc}(z-j) \langle \operatorname{sinc}(\cdot - j), \operatorname{sinc}(\cdot - k) \rangle \\
&\quad = \sum_{m \le |k| \le n} |\operatorname{sinc}(z-k)|^2.
\end{aligned}$$

Since $\sum_{k \in \mathbb{Z}} |\operatorname{sinc}(z-k)|^2 < \infty$, it follows that

$$\Bigg( \sum_{|k| \le n} \operatorname{sinc}(z-k)\operatorname{sinc}(\cdot - k) \Bigg)_{n \in \mathbb{N}}$$

is a Cauchy sequence in $L^2(\mathbb{R})$, meaning that the series in (24) converges also in $L^2(\mathbb{R})$-norm towards $\operatorname{sinc}(z - \cdot)$. This justifies the interchange of integration and summation in (21) by La. 2.3 and completes the proof. ■

THEOREM 4.6 *PSF + WIP ⇒ CSF.*

*Proof* Let $f \in B^2_\pi$, then for $t \in \mathbb{R}$, the functions $f\operatorname{sinc}(t - \cdot)$ and $f(t - \cdot)\operatorname{sinc}$ both belong to $B^1_{2\pi}$, and so PSF applies. It yields

$$\int_{\mathbb{R}} f(u)\operatorname{sinc}(t-u)\,\mathrm{d}u = \sum_{k \in \mathbb{Z}} f(k)\operatorname{sinc}(t-k)$$

and

$$\int_{\mathbb{R}} f(t-u)\operatorname{sinc} u\,\mathrm{d}u = \sum_{k \in \mathbb{Z}} f(t-k)\operatorname{sinc} k = f(t).$$

The two integrals on the left-hand sides are equal as is seen by a change of variables. Hence CSF holds for $t \in \mathbb{R}$.

Now, if $f \in B^2_\pi$, then the function $z \mapsto f(z)\overline{f(\overline{z})}$, $z \in \mathbb{C}$, belongs to $B^1_{2\pi}$, and so by PSF

$$\int_{\mathbb{R}} |f(u)|^2\mathrm{d}u = \sum_{k \in \mathbb{Z}} |f(k)|^2,$$

yielding, in particular, $\sum_{k \in \mathbb{Z}} |f(k)|^2 < \infty$. Similarly, one has by applying PSF to the function $w \mapsto \operatorname{sinc}(z-w)\overline{\operatorname{sinc}(z-\overline{w})}$, $w \in \mathbb{C}$, which belongs also to $B^1_{2\pi}$,

$$\int_{\mathbb{R}} |\operatorname{sinc}(z-u)|^2\mathrm{d}u = \sum_{k \in \mathbb{Z}} |\operatorname{sinc}(z-k)|^2.$$

Using these two identities, the extension of CSF to $\mathbb{C}$ and the absolute and uniform convergence on strips of bounded width parallel to the real line now follow exactly as in the proof of Theorem 4.4. ■

The implication PSF $\Rightarrow$ CSF is well-known; see e.g. [17, Section 3.1; 26]. Similarly, CSF was known to yield RKF [26], but the converse, RKF $\Rightarrow$ CSF, is a new result of this article.

### 4.3. *Some further implications*

THEOREM 4.7 *PSF $\Rightarrow$ GPF.*

*Proof* Under the hypothesis of GPF for $\sigma = \pi$, the function $h(z) := \overline{g(\overline{z})}$ also belongs to $B^2_\pi$. Since $f \cdot h \in B^1_{2\pi}$, we may apply PSF to $f \cdot h$, which gives GPF. ■

THEOREM 4.8 *PSF + WIP $\Rightarrow$ RKF.*

*Proof* Let $t \in \mathbb{R}$. Again, by a change of variables and an application of PSF to $f(t - \cdot)$sinc, we deduce

$$\int_{\mathbb{R}} f(u)\operatorname{sinc}(t-u)\mathrm{d}u = \int_{\mathbb{R}} f(t-u)\operatorname{sinc} u \,\mathrm{d}u = \sum_{k\in\mathbb{Z}} f(t-k)\operatorname{sinc} k = f(t),$$

and so RKF holds for $t \in \mathbb{R}$. The extension to $z \in \mathbb{C}$ follows by WIP, since $z \mapsto \int_{\mathbb{R}} f(u)\operatorname{sinc}(z-u)\mathrm{d}u$ is obviously an entire function of exponential type $\pi$. ■

THEOREM 4.9 *GPF + WIP $\Rightarrow$ RKF.*

*Proof* Let $t \in \mathbb{R}$. By a change of variables and an application of GPF with $f$ replaced by $f(t - \cdot)$ and $g = \operatorname{sinc}$, we obtain

$$\int_{\mathbb{R}} f(u)\operatorname{sinc}(t-u)\mathrm{d}u = \int_{\mathbb{R}} f(t-u)\operatorname{sinc} u \,\mathrm{d}u = \sum_{k\in\mathbb{Z}} f(t-k)\operatorname{sinc} k = f(t).$$

This is indeed RKF for $t \in \mathbb{R}$; the extension to the complex plane follow as above by WIP. ■

THEOREM 4.10 *RKF $\Rightarrow$ GPF.*

*Proof* As in the proof of RKF $\Rightarrow$ CSF (Theorem 4.5) we make use of the sinc summation formula (20). Proceeding formally, we have

$$\begin{aligned}\int_{\mathbb{R}} \overline{g(u)} f(u)\mathrm{d}u &= \int_{\mathbb{R}} \overline{g(u)} \int_{\mathbb{R}} f(v)\operatorname{sinc}(u-v)\mathrm{d}v\,\mathrm{d}u \\ &= \int_{\mathbb{R}} \overline{g(u)} \left\{ \int_{\mathbb{R}} f(v) \left\{ \sum_{k\in\mathbb{Z}} \operatorname{sinc}(v-k)\operatorname{sinc}(u-k)\mathrm{d}v \right\} \right\} \mathrm{d}u \\ &= \sum_{k\in\mathbb{Z}} \left\{ \int_{\mathbb{R}} \overline{g(u)}\operatorname{sinc}(u-k)\mathrm{d}u \right\} \left\{ \int_{\mathbb{R}} f(v)\operatorname{sinc}(v-k)\mathrm{d}v \right\} \\ &= \sum_{k\in\mathbb{Z}} \overline{g(k)} f(k). \end{aligned} \tag{25}$$

In order to make this proof precise we again have to justify the interchange of summation and integration. The arguments are the same as in the proof of Theorem 4.5. Indeed, there it was shown that RKF implies the orthogonality relation (23) and that this in turn implies the $L^2(\mathbb{R})$-convergence of the sequence $(\sum_{|k|\le n}\operatorname{sinc}(u-k)\operatorname{sinc}(\cdot - k))_{n\in\mathbb{N}}$ towards $\operatorname{sinc}(u - \cdot)$ for each fixed $u \in \mathbb{R}$ as well as $\sum_{k\in\mathbb{Z}} |f(k)|^2 \le \|f\|_2 < \infty$.

On the other hand, one has $m,n\in\mathbb{N}$ by (23),

$$\Big\|\sum_{m\le|k|\le n} f(k)\mathrm{sinc}(\cdot-k)\Big\|_2^2 = \Big\langle \sum_{m\le|k|\le n} f(k)\mathrm{sinc}(\cdot-k), \sum_{m\le|j|\le n}\overline{f(j)}\mathrm{sinc}(\cdot-j)\Big\rangle$$
$$= \sum_{m\le|k|\le n} f(k)\sum_{m\le|j|\le n}\overline{f(j)}\langle \mathrm{sinc}(\cdot-j),\mathrm{sinc}(\cdot-k)\rangle$$
$$= \sum_{m\le|k|\le n} |f(k)|^2$$

and it follows that $(\sum_{|k|\le n} f(k)\mathrm{sinc}(\cdot-k))_{n\in\mathbb{N}}$ is a Cauchy sequence and hence a convergent sequence in $L^2(\mathbb{R})$.

The precise proof is now a consequence of La. 2.3, since in view of RKF and the $L^2(\mathbb{R})$-convergence of the series involved,

$$\begin{aligned}\int_{\mathbb{R}}\overline{g(u)}f(u)\mathrm{d}u &= \int_{\mathbb{R}}\overline{g(u)}\Big\{\int_{\mathbb{R}} f(v)\Big\{\sum_{k\in\mathbb{Z}}\mathrm{sinc}(v-k)\mathrm{sinc}(u-k)\Big\}\mathrm{d}v\Big\}\mathrm{d}u\\ &= \int_{\mathbb{R}}\overline{g(u)}\Big\{\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} f(v)\mathrm{sinc}(v-k)\mathrm{d}v\,\mathrm{sinc}(u-k)\Big\}\mathrm{d}u\\ &= \int_{\mathbb{R}}\overline{g(u)}\Big\{\sum_{k\in\mathbb{Z}} f(k)\mathrm{sinc}(u-k)\Big\}\mathrm{d}u\\ &= \sum_{k\in\mathbb{Z}} f(k)\int_{\mathbb{R}}\overline{g(u)}\mathrm{sinc}(u-k)\mathrm{d}u\\ &= \sum_{k\in\mathbb{Z}} f(k)\overline{g(k)}.\end{aligned}$$

This completes the precise proof of GPF. ■

It should be noted that we have only used the $L^2(\mathbb{R})$-convergence of the sampling series $\sum_{|k|\le n} f(k)\mathrm{sinc}(\cdot-k)$, but not the fact that it converges towards $f$. So we have not used CSF in the proof of RKF $\Rightarrow$ GPF above.

The implication CSF $\Rightarrow$ RKF (Theorem 4.1) as well as CSF $\Rightarrow$ PSF (Theorem 4.3) and its converse PSF $\Rightarrow$ CSF (Theorem 4.6) can also be found in [26]; see also [17, Section 3.1]. Concerning CSF $\Leftrightarrow$ GPF (Theorems 4.2 and 4.4), although the equivalence is known in the frame of orthogonal expansions in Hilbert space, in our approach we did not want to make use of Hilbert space theory as side results. The other five implications of Section 4 are new results of this article, although some of the proofs are quite elementary.

## 5. The classical sampling formula and Valiron's sampling formula

THEOREM 5.1 *CSF $\Rightarrow$ VSF.*

*Proof* If $f\in B_\pi^\infty$, then the function

$$g(z) := \begin{cases}\dfrac{f(z)-f(0)}{z}, & z\ne 0\\ f'(0), & z=0\end{cases}$$

belongs to $B^2_\pi$, and so CSF applies. For $z \neq 0$, it yields

$$\frac{f(z)-f(0)}{z} = f'(0)\operatorname{sinc} z + \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{f(k)-f(0)}{k}\operatorname{sinc}(z-k).$$

In particular, this formula holds for $f = \operatorname{sinc}$, which gives

$$\frac{\operatorname{sinc} z - 1}{z} = -\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{k}\operatorname{sinc}(z-k).$$

Combining the two equations, we obtain

$$f(z) = f'(0)z\operatorname{sinc} z + f(0)\operatorname{sinc} z + \sum_{k\in\mathbb{Z}\setminus\{0\}} f(k)\frac{z}{k}\operatorname{sinc}(z-k),$$

which is (3) for $\sigma = \pi$.

The absolute and uniform convergence of the series on compact subsets of $\mathbb{C}$ follows in view of $\sum_{k\neq 0} |\frac{f(k)}{k}|^2 \leq \|f\|^2_\infty \sum_{k\neq 0}\frac{1}{k^2} < \infty$. ■

THEOREM 5.2 *VSF ⇒ CSF.*

*Proof* First assume $f \in \widetilde{B}^1_\pi$ and apply VSF to $zf(z)$ to obtain,

$$zf(z) = f(0)z\operatorname{sinc} z + \sum_{k\in\mathbb{Z}\setminus\{0\}} f(k)z\operatorname{sinc}(z-k) \quad (z\in\mathbb{C}).$$

Dividing by $z$ yields CSF.

The problem now is to extend this particular case to all of $B^2_\pi$, since we do not know anything about the sequence $(f(k))_{k\in\mathbb{Z}}$. Note that we cannot use inequality (8), since this was proved as a consequence of CSF. On the other hand, one can prove assertions (i), (ii) and (iii) of Theorem 3.1 by starting with VSF instead of CSF. The only difference in the proofs is that (11) now has to be deduced from VSF, which is even easier, because the Valiron series (3) with $\sigma = \pi$ can be differentiated term by term for each $f \in B^\infty_\pi$ in view of the uniform convergence on compact sets. Thus, differentiating (3) with $\sigma = \pi$ and evaluating at $z = \frac{1}{2}$ yields (11) for each $f \in B^\infty_\pi$. The proofs of the assertions (i), (ii) and (iii) of Theorem 3.1 now follow in exactly the same manner as given above. In particular, (8), i.e. $(f(k))_{k\in\mathbb{Z}} \in l^2(\mathbb{Z})$, can also be deduced from VSF.

In order to complete the proof of 'VSF ⇒ CSF', we approximate $f \in B^2_\pi$ by a sequence $(f_n)_{n\in\mathbb{N}}$ according to La. 2.1. Since CSF holds for $f_n$ it also holds for the limit function $f$ by La. 2.3. As a consequence of La. 2.2, the convergence of the series in CSF is uniform in strips of bounded width parallel to the real line. ■

The proof of CSF ⇒ VSF (Theorem 5.1) can already be found in [38, p. 220], see also [21]. The converse direction VSF ⇒ CSF (Theorem 5.2) was proved in [21] using the series expansion

$$f'(t) = \sum_{k\in\mathbb{Z}\setminus\{0\}} f(t+k)\frac{(-1)^{k+1}}{k}, \quad (f\in B^1_\pi, t\in\mathbb{R})$$

to be found in [40, Section 4.2]. The proof of this representation for $f'$, however, was implicitly based on the PWT. Thus it does not fit into our approach.

For the interconnections of Valiron's/Tschakaloff's formula with two basic formulae of Euler see [41].

## 6. The classical sampling formula and the Paley–Wiener theorem

Since the PWT connects the space $B^2_\sigma$ with the Fourier transform, we need some basic facts about the Fourier transform on $L^2(\mathbb{R})$. First we mention that the Fourier transform is an isomorphism from $L^2(\mathbb{R})$ onto itself; see e.g. [42, Chapter 5.2]. Further, we need the transform of the sinc-function.

LEMMA 6.1 *For $\alpha$, $t \in \mathbb{R}$, $\alpha \neq 0$, there holds the formula*

$$\left[\operatorname{sinc}\alpha(\cdot - t)\right]^{\wedge}(v) = \frac{1}{\sqrt{2\pi}\,\alpha}\operatorname{rect}\left(\frac{v}{\alpha}\right)e^{-ivt} \quad a.e. \tag{26}$$

*where*

$$\operatorname{rect}(t) := \begin{cases} 1, & |t| < \pi \\ \frac{1}{2}, & |t| = \pi. \\ 0, & |t| > \pi \end{cases}$$

*Proof* The Fourier transform of the rect-function is given by

$$\left[\frac{1}{\sqrt{2\pi}\,\alpha}\operatorname{rect}\left(\frac{\cdot}{\alpha}\right)e^{i\cdot t}\right]^{\wedge}(v) = \operatorname{sinc}\alpha(v - t) \quad a.e. \tag{27}$$

This is shown by a simple integration. Equation (26) now follows from (27) by the Fourier inversion formula. (For an elementary proof of (26) without using the inversion formula see [43, p. 14].) ∎

We also need a result from trigonometric Fourier series. In fact, if $g \in L^2(-\pi, \pi)$, then the Fourier coefficients of $g$ belong to $l^2(\mathbb{Z})$ and the Fourier series of $g$ converges towards $g$ with respect to $L^2(-\pi, \pi)$-norm. In this respect see [42, Chapter 4.2] or any textbook on Fourier series.

THEOREM 6.2 *$PWT + WIP \Rightarrow CSF$.*

*Proof* Let $f \in B^2_\pi$ and assume that the Fourier transform $f^{\wedge}(v)$ vanishes a.e. outside $[-\pi, \pi]$. Then, noting (26), one has for the partial sums of the sampling series,

$$\begin{aligned}\left[\sum_{|k|\le n} f(k)\operatorname{sinc}(\cdot - k)\right]^{\wedge}(v) &= \sum_{|k|\le n} f(k)[\operatorname{sinc}(\cdot - k)]^{\wedge}(v) \\ &= \left\{\frac{1}{\sqrt{2\pi}}\sum_{|k|\le n} f(k)e^{-ikv}\right\}\operatorname{rect} v, \quad (v \in \mathbb{R}).\end{aligned}$$

Recalling that the Fourier transform in an isometry from $L^2(\mathbb{R})$ onto itself, we deduce

$$\begin{aligned}\left\| f(t) - \sum_{|k|\le n} f(k)\operatorname{sinc}(t - k)\right\|_{L^2(\mathbb{R})} &= \left\| f^{\wedge}(v) - \left\{\frac{1}{\sqrt{2\pi}}\sum_{|k|\le n} f(k)e^{-ikv}\right\}\operatorname{rect} v\right\|_{L^2(\mathbb{R})} \\ &= \left\| f^{\wedge}(v) - \frac{1}{\sqrt{2\pi}}\sum_{|k|\le n} f(k)e^{-ikv}\right\|_{L^2(-\pi,\pi)} \\ &= \left\| f^{\wedge}(v) - \frac{1}{\sqrt{2\pi}}\sum_{|k|\le n} f(-k)e^{ikv}\right\|_{L^2(-\pi,\pi)}.\end{aligned} \tag{28}$$

Furthermore, it follows from the Fourier inversion formula that

$$f(-k) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \widehat{f}(v) e^{-ikv} \mathrm{d}v,$$

i.e. $\sum_{k\in\mathbb{Z}} f(-k)e^{ikv}$ is the trigonometric Fourier series of $f^\wedge \in L^2(-\pi,\pi)$, which is known to converge to $f^\wedge$ in $L^2(-\pi,\pi)$-norm. Hence the right-hand side of (28) vanishes for $n\to\infty$. This, in turn shows that the sampling series converges in $L^2(\mathbb{R})$-norm towards $f$.

As to the pointwise convergence, it follows from the theory of trigonometric Fourier series that the Fourier coefficients of $f^\wedge \in L^2(-\pi, \pi)$ are square summable, giving $\sum_{k\in\mathbb{Z}} |f(-k)|^2 < \infty$. This implies by La. 2.2 that the sampling series of $f$ converges uniformly on $\mathbb{R}$. Since $f$ is continuous, we have that CSF holds for all $t\in\mathbb{R}$. The extension to the complex version can be shown as in the proof of Theorem 4.4, where the uniform convergence of the series in strips of bounded width parallel to the real line follows again by La. 2.2. ■

Theorem 6.3 *CSF ⇒ PWT.*

*Proof* Again we can restrict ourselves to $\sigma = \pi$. Noting that CSF implies the orthogonality relation (10) in view of Theorem 3.1, it follows as in the proof of Theorem 4.10 that the sampling series converges in $L^2(\mathbb{R})$-norm. Moreover, using CSF once more, the $L^2(\mathbb{R})$-limit must be equal to $f$.

Now, the Fourier transform is a bounded linear operator on $L^2(\mathbb{R})$, and we obtain by (26) that

$$f^\wedge(v) = \sum_{k\in\mathbb{Z}} f(k)[\mathrm{sinc}(\cdot - k)]^\wedge(v) = \left\{ \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} f(k) e^{-ikv} \right\} \mathrm{rect}\, v,$$

where the convergence of the infinite series is to be understood in the norm of $L^2(\mathbb{R})$. This shows that $f^\wedge$ vanishes a.e. outside $[-\pi,\pi]$. ■

For a different proof of PWT ⇒ CSF (Theorem 6.2) see [38, p. 220]; the converse, CSF ⇒ PWT (Theorem 6.3), is new. There has been no attempt to establish direct proofs for the equivalence of VSF or PWT with PSF, GPF or RKF.

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## Notes

1. Georges (Jean Marie) Valiron, born 7 September 1884 in Lyon, died March 1955 in Paris. He received his Agrégé de mathématiques in 1908, first taught at the lyceum of Besançon (Doubs) while continuing his studies in complex function theory. At Besançon he was the teacher of Georges Bloch (1894–1948) and his brother André. In 1914, he wrote his dissertation 'Sur les fonctions entières d'ordre nul et d'ordre fini et en particulier sur les fonctions à correspondance régulière' under É. Borel at the Université de Paris. Thereafter he taught as Professor in Valence (Drôme), and in March–June 1921 he presented a two-hour course on 'Dirichlet series and factorial series' at the reorganized University of Strasbourg where he remained until 1931. M. Fréchet was his colleague there from 1921 to 1927. Already in 1923 appeared his well-known 'Lectures on the general theory of integral functions' (220 pp; reprinted Chelsea 1949; latest edition, Iyer Press, March 2007) based on lectures he had presented at the University College of Wales (Aberystwyth). In 1931, he received the Chair for analysis at the Faculté des Sciences at Paris. In 1942, appeared his 'Théorie des fonctions' (real and complex) followed in 1945 by 'Équations fonctionnelles et applications' (reprinted in one volume by Masson 1966, by J. Gabay 1989). His speciality was complex function theory (entire and meromorphic functions). One of his four doctoral students is the Fields Medallist Laurent Schwartz, and according to the Mathematics Genealogy Project (http://genealogy.math.ndsu.nodak.edu/index.php) Valiron has 1860 academic descendants. Schwartz received his doctorate in 1943 at Clermond-Ferrand where the Université Louis Pasteur-Strasbourg was evacuated during WW II. Valiron was the president of the Société Mathématique de France in 1938, and received the Prix Poncelet in 1948.
   Liubomir Nikolov Tschakaloff (Любомир Николов Чакалов, also transliterated as Tschakalov, Chakalov or similarly) was born on 18 February 1886 into the family of an impoverished tailor of Samokov in Bulgaria, one of eleven children. The young Tschakaloff went to school in Samokov, a small town near Sofia, and then completed his schooling in the town of Plovdiv. By 1904, his attachment to mathematics was so strong that it prompted him to walk from Samokov to Sofia to enrol in the University there. He entered the University in the autumn of that year as a mathematics student.
   He graduated with honours in 1908, and in 1909 became an assistant at Sofia University. During the period 1910–1912 he pursued advanced studies at the Universities of Leipzig and Göttingen, coming into contact with some of the most famous mathematicians of the age, particularly Hilbert and Klein; Edmund Landau encouraged him to study number theory and analysis. The results of these studies became his Habilitationsschrift *Analytical characteristics of the Riemann function* $\zeta(z)$. In 1922, he became Professor at Sofia University. A second two-year period of study abroad, from 1924 to 1925, found him in Paris, Pisa and finally Naples, where he obtained his doctorate in 1925 with a dissertation on Riccati equations. Tschakaloff is best known for his work in entire and univalent functions, mean value theorems and Gaussian quadrature. His work was characterized by an ability to find original and incisive methods, which were often powerful enough to find wider uses. His scientific creativity was interrupted by two world wars, but apart from these exceptional times he produced a steady stream of scholarly work between 1910 and 1963. He published 112 works during his life, including books on analytic functions and differential equations.
   He was a member of the Bulgarian Academy of Sciences, as well as several foreign Academies. He died in September 1963.
   For further details see http://www.math.bas.bg/~serdica/tschakaloff.html. The authors thank Professor Virginia Kiryakova, Sofia, for supplying biographical information.
2. This result was proved in response to a question raised by one of the authors at the conference SampTA 05, Samsun, Turkey, July 2005, of whether Kluvánek's theorem implies an abstract approximate sampling theorem, in analogy with the classical case treated in [30].
3. Note, however, that an equivalence grouping can be of interest even when the logic value of the propositions is unknown. An example is the collection of propositions equivalent to the Riemann hypothesis.

4. In fact, big $\mathcal{O}$ can be replaced by little $o$ [38, p. 98], but this is much harder to prove and will not be used in the sequel.
5. Let us point out that Riemann's theorem on removable singularities has been used here. The same applies to the function $g$ in connection with Valiron's formula (proof of Theorem 5.1).
6. An elementary proof of this expansion can be found in the appendix. Note that we are not using formula (20) as a side result, since it is a particular case of CSF.
7. This is only used to keep the formulae shorter. We do not apply any results from Hilbert space theory in particular we do not make use of the theory of orthogonal expansions in Hilbert spaces. In every case the inner product notation can be replaced by integrals.
8. The reader will observe that this is Bessel's inequality for the orthonormal system $(\mathrm{sinc}(\cdot - k))_{k\in\mathbb{Z}}$. See also the proof of Theorem 4.10.

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## Appendix

**Proof of Valiron's sampling formula:** Using the estimates (5) we easily conclude with the help of the Cauchy–Schwarz inequality that the series (3) converges absolutely and uniformly on compact subsets of $\mathbb{C}$. It remains to show that it represents $f(z)$.

Now let $N \in \mathbb{N}$, set $N' := N + \frac{1}{2}$ and denote by $\mathcal{S}_N$ the positively oriented square with vertices at $\pm N' \pm iN'$, and set

$$I_N(z) := \frac{z \sin \pi z}{2\pi i} \int_{\mathcal{S}_N} \frac{f(\zeta)}{\zeta(\zeta - z)\sin \pi\zeta}\, \mathrm{d}\zeta.$$

Then, for any $z \in \mathbb{C}\backslash\mathbb{Z}$ inside the square we have, by the residue theorem,

$$I_N(z) = f(z) - f'(0) z \operatorname{sinc} z - f(0) \operatorname{sinc} z - \sum_{k \neq 0} f(k) \frac{z}{k} \operatorname{sinc}(z - k),$$

where we have used that

$$z \sin \pi z \operatorname{Res}\left(\frac{f(\zeta)}{\zeta(\zeta - z)\sin(\pi\zeta)}, \zeta = z\right) = f(z),$$

$$z \sin \pi z \operatorname{Res}\left(\frac{f(\zeta)}{\zeta(\zeta - z)\sin(\pi\zeta)}, \zeta = 0\right) = -f'(0) z \operatorname{sinc} z - f(0) \operatorname{sinc} z,$$

$$z \sin \pi z \operatorname{Res}\left(\frac{f(\zeta)}{\zeta(\zeta - z)\sin(\pi\zeta)}, \zeta = k\right) = -f(k) \frac{z}{k} \operatorname{sinc}(z - k), \quad (k \in \mathbb{Z} \setminus \{0\}).$$

Next, denote by $I^{\pm}_{\mathrm{hor}}$ the contributions to the integral coming from the two horizontal parts of $\mathcal{S}_N$, where $+$ and $-$ refer to the upper and lower line segment, respectively. Similarly, denote by $I^{\pm}_{\mathrm{vert}}$ the contributions coming from the two vertical parts of $\mathcal{S}_N$, where now $+$ and $-$ refer to the right and the left line segment, respectively. Thus

$$I_N(z) = \frac{z \sin(\pi z)}{2\pi i} \left(I^{-}_{\mathrm{hor}} + I^{+}_{\mathrm{vert}} + I^{+}_{\mathrm{hor}} + I^{-}_{\mathrm{vert}}\right), \tag{29}$$

where

$$I^{\pm}_{\mathrm{hor}} = \mp \int_{-N'}^{N'} \frac{f(t \pm iN')}{(t \pm iN')(t \pm iN' - z)\sin(\pi(t \pm iN'))} \mathrm{d}t$$

and

$$I^{\pm}_{\text{vert}} = \pm i \int_{-N'}^{N'} \frac{f(\pm N' + it)}{(\pm N' + it)(\pm N' + it - z)\sin(\pi(\pm N' + it))} \mathrm{d}t.$$

In order to estimate these integrals, we note that

$$|\sin(\pi(t \pm iN'))| \geq \sinh \pi N' = \frac{e^{\pi N'}}{2}\ (1 - e^{-2\pi N'})$$

and

$$|\sin(\pi(\pm N' + it))| = \cosh \pi t \geq \frac{e^{\pi|t|}}{2}.$$

Recalling also $|f(z)| \leq M \exp(\pi|\mathfrak{Im}\, z|)$, we find that

$$|I^{\pm}_{\text{hor}}| \leq \frac{2M}{1 - e^{-2\pi N'}} \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{|t \pm iN'| \cdot |t \pm iN' - z|}$$

and

$$|I^{\pm}_{\text{vert}}| \leq 2M \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{|\pm N' + it| \cdot |\pm N' + it - z|.}$$

Now, for $N' \geq |\mathfrak{Im}\, z| + 1$, we have

$$\phi_N(t) := \frac{1}{|t \pm iN'| \cdot |t \pm iN' - z|} \leq \frac{1}{\sqrt{(t^2 + 1)((t - \mathfrak{Re}\, z)^2 + 1)}} =: \phi(t).$$

Clearly, $\int_{-\infty}^{\infty} \phi(t)\mathrm{d}t < \infty$. Hence, by Lebesgue's theorem of dominated convergence, we have

$$\lim_{N\to\infty} \int_{-\infty}^{\infty} \phi_N(t)\mathrm{d}t = \int_{-\infty}^{\infty} \lim_{N\to\infty} \phi_N(t)\mathrm{d}t = 0,$$

which implies that $\lim_{N\to\infty} |I^{\pm}_{\text{hor}}| = 0$. Similarly, it is seen that $\lim_{N\to\infty} |I^{\pm}_{\text{vert}}| = 0$. Therefore the right-hand side of (29) vanishes as $N \to \infty$. This completes the proof. ■

One could also have established this theorem using a circular contour instead of a square; see e.g. [9].

**Proof of the cotangent expansion:** It is easily seen (e.g. by using l'Hospital's rule twice) that

$$\lim_{z\to n}\left(\pi \cot \pi z - \frac{1}{z - n}\right) = 0$$

exists for each $n \in \mathbb{Z}$. Hence, introducing

$$g(z) := \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{z + n} + \frac{1}{z - n}\right),$$

we find that

$$h(z) := \pi \cot \pi z - g(z)$$

defines a continuous function $h$ on $\mathbb{C}$. We have to show that $h(z) \equiv 0$.

Next, by a simple trigonometric calculation, we verify that

$$f(z) = \frac{1}{2}\left\{f\left(\frac{z}{2}\right) + f\left(\frac{z+1}{2}\right)\right\} \quad (z \in \mathbb{C} \setminus \mathbb{Z}) \tag{30}$$

holds for $f(z) := \pi \cot \pi z$. Setting

$$g_N(z) := \frac{1}{z} + \sum_{n=1}^{N} \left( \frac{1}{z+n} + \frac{1}{z-n} \right),$$

and noting that

$$\frac{1}{2}\left\{ g_N\left(\frac{z}{2}\right) + g_N\left(\frac{z+1}{2}\right) \right\} = g_{2N}(z) + \frac{1}{z+2N+1},$$

we conclude by letting $N \to \infty$ that (30) holds for $f := g$ as well. Hence, in view of the continuity of $h$, we obtain

$$h(z) = \frac{1}{2}\left\{ h\left(\frac{z}{2}\right) + h\left(\frac{z+1}{2}\right) \right\}, \quad (z \in \mathbb{C}). \tag{31}$$

Since $h$ is an odd function, we have $h(0) = 0$. Now, to obtain a contradiction that $h$ is not identically zero. Then there exists an $r > 1$ and a $z_r \in D_r := \{z \in \{\mathbb{C} : |z| \le r\}$ of smallest positive modulus such that

$$|h(z_r)| = \max_{z \in D_r} |h(z)| =: M_r > 0.$$

Obviously, $z_r/2$ and $(z_r + 1)/2$ also belong to $D_r$ and $|z_r/2| < |z_r|$. Hence, by our choice of $z_r$ we have $|h(z_r/2)| < M_r$. Thus, using (31) and the triangular inequality, we obtain

$$M_r = |h(z_r)| \le \frac{1}{2}\left\{ \left| h\left(\frac{z_r}{2}\right) \right| + \left| h\left(\frac{z_r+1}{2}\right) \right| \right\} < M_r,$$

which is a contradiction. ■

The proof given here may be seen as a planar version of the first part of a proof in [44, Chapter 11, Section 2]. Note that we did not need any nontrivial result from complex analysis. Whereas, the proof in [44] needs that $h$ is analytic, here only the continuity of $h$ is used.