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**Results in Mathematics**

# The Summation Formulae of Euler–Maclaurin, Abel–Plana, Poisson, and their Interconnections with the Approximate Sampling Formula of Signal Analysis

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*In Memory of Alexander Ostrowski, 1893–1986*
*Dedicated to Heinrich Wefelscheid to mark his 70th birthday*

**Abstract.** This paper is concerned with the two summation formulae of Euler–Maclaurin (EMSF) and Abel–Plana (APSF) of numerical analysis, that of Poisson (PSF) of Fourier analysis, and the approximate sampling formula (ASF) of signal analysis. It is shown that these four fundamental propositions are all equivalent, in the sense that each is a corollary of any of the others. For this purpose ten of the twelve possible implications are established. Four of these, namely the implications of the grouping APSF $\Leftarrow$ ASF $\Rightarrow$ EMSF $\Leftrightarrow$ PSF are shown here for the first time. The proofs of the others, which are already known and were established by three of the above authors, have been adapted to the present setting. In this unified exposition the use of powerful methods of proof has been avoided as far as possible, in order that the implications may stand in a clear light and not be overwhelmed by external factors. Finally, the four propositions of this paper are brought into connection with four

After this manuscript was accepted for publication the authors learned that special issues were being prepared to honour Heinrich Wefelscheid on the occasion of his 70th anniversary. Since they valued his unstinted service to mathematics as a whole, thus the founding of the present journal and his leading editorship over the years, his publication of the Collected Works, in several volumes, of W. Blaschke and E. Landau, they expressed interest that the present paper, already dedicated to A. Ostrowski, be included in the special issues.

propositions of mathematical analysis for bandlimited functions, including the Whittaker–Kotel'nikov–Shannon sampling theorem. In conclusion, all eight propositions are equivalent to another. Finally, the first three summation formulae are interpreted as quadrature formulae.

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**Keywords.** Euler–Maclaurin summation formula, Abel–Plana summation formula, Poisson summation formula, Whittaker–Kotel'nikov–Shannon sampling theorem, Approximate sampling formula, Bandlimited signals, Quadrature formulae.

## 1. Introduction

First to the Euler–Maclaurin summation formula of the broad area of numerical analysis, a very important one, which has indeed many applications.

**Euler–Maclaurin Summation Formula (EMSF)**

*For $n, r \in \mathbb{N}$ and $f \in C^{(2r)}[0,n]$, we have*

$$\begin{aligned}\sum_{k=0}^{n} f(k) = \int_0^n f(x)\,dx \\ &+\frac{1}{2}\left[f(0)+f(n)\right]+\sum_{k=1}^{r}\frac{B_{2k}}{(2k)!}\left[f^{(2k-1)}(n)-f^{(2k-1)}(0)\right] \\ &+(-1)^r\sum_{k=1}^{\infty}\int_0^n \frac{e^{i2\pi kt}+e^{-i2\pi kt}}{(2\pi k)^{2r}}\,f^{(2r)}(t)\,dt, \qquad (1.1)\end{aligned}$$

*where $B_{2k}$ are the Bernoulli numbers.*

Euler discovered this summation formula in connection with the so-called Basel problem, i.e., with determining $\zeta(2)$ in modern terminology. It can be found without proof in [23] (submitted 1732) and with a complete deduction in [24] (submitted 1735), where he used it to calculate $\zeta(2), \zeta(3), \zeta(4)$, and the Euler constant $\gamma$; for more details see [26]. The formula was found independently by Maclaurin in 1738 [38]; cf. [39] for the differences between Euler's and Maclaurin's approaches. See also the review of Euler's life and works by Gautschi [27], and the overview by Apostol [3]. Of further interest is that Ramanujan [4, Chaps. 6, 8] introduced a method of summation based on EMSF. Candelpergher et al. [17,18] presented a rigorous treatment of the method together with many applications. This formula is also treated in Ostrowski's three-volume treatise on differential and integral calculus [42, vol. II, pp. 287–288], which was very popular in German speaking universities in the 1960s.

Whereas the Euler–Maclaurin summation formula is a standard one in mathematics per se, our second formula, that of Abel–Plana[1] is not. It reads

**Abel–Plana Summation Formula (APSF)**

*Let $f$ be analytic in $\{z \in \mathbb{C} : \Re z \geq 0\}$. Suppose that*

$$\lim_{y\to\infty} |f(x \pm iy)|e^{-2\pi y} = 0 \tag{1.2}$$

*uniformly in $x$ on every finite interval, and that*

$$\int_0^\infty |f(x+iy) - f(x-iy)|e^{-2\pi y}\, dy \tag{1.3}$$

*exists for every $x \geq 0$ and tends to zero when $x \to \infty$. Then*

$$\lim_{N\to\infty} \left[\sum_{k=0}^{N} f(k) - \int_0^{N+1/2} f(x)\, dx\right] = \frac{1}{2} f(0) + i \int_0^\infty \frac{f(iy) - f(-iy)}{e^{2\pi y} - 1}\, dy. \tag{1.4}$$

This formula was obtained by Plana [44] in 1820 and independently by Abel [1, p. 23] in 1823 in the form

$$\sum_{k=0}^{\infty} f(k) - \int_0^\infty f(x)\, dx = \frac{1}{2} f(0) + i \int_0^\infty \frac{f(iy) - f(-iy)}{e^{2\pi y} - 1}\, dy. \tag{1.5}$$

As it was customary at that time, neither author gave hypotheses on $f$. They proceeded via an integral representation of the Bernoulli numbers. Abel [1, p. 27] also established the following interesting "alternating series version":

$$\sum_{k=0}^{\infty} (-1)^k f(k) = \frac{1}{2} f(0) + i \int_0^\infty \frac{f(iy) - f(-iy)}{2 \sinh \pi y}\, dy.$$

Although his proof was different, this formula can be obtained by applying (1.5) to the function $z \mapsto 2f(2z)$ and subtracting the resulting equation from (1.5).

[1] Giovanni (Antonio Amedea) Plana, born 1781 in Voghera, Lombardy (Italy), died 1864 in Turin, studied 1800–1803 at the École Polytéchnique in Paris under Fourier and Lagrange. On the recommendation of Fourier, Plana in 1803 was appointed to the mathematics chair at the Artillery school of Piedmont in Turin. With Lagrange's recommendation he was also appointed to the chair of astronomy at the University of Turin in 1811. He taught both astronomy and mathematics at both institutions for the rest of his life. His teaching was of the highest quality. Plana, one of the major Italian scientists of his time, stood in contact with Francesco Carlini, Cauchy and Babbage. He was elected to the Turin Academy of Sciences in 1841 and to the Académie des Sciences in Paris in 1860. Plana worked in astronomy (on movement of the moon), Eulerian integrals, elliptic functions, heat, electrostatics and geodesy. See the MacTutor history of mathematics archive http://www-history.mcs.st-andrews.ac.uk/.

Some authors, [19, I, p. 258, III, p. 53], [45, Sect. 2], consider (1.5) to require in addition to our assumptions that $\int_0^\infty f(x)\,dx$ exists *and* $f(N) \to 0$ as $N \to \infty$. However, the last assumption is superfluous. Indeed, by (1.4) the existence of the integral implies the convergence of the series which entails that $f(N) \to 0$ as $N \to \infty$.

APSF, as stated by us, is exactly what Henrici proves by contour integration in his book [30, Sect. 4.9]. However, his resulting theorem [30, Theorem 4.9 c] is presented in a somewhat different form.

The Abel–Plana formula, connected with several great names of the past, including Cauchy (1826), Kronecker (1889), Lindelöf (1905) (see his account of the history until 1904 [37, pp. 68–69]), was almost forgotten until Henrici [30, pp. 270–275] brought it up again in his text-book and applied it. One of the most important recent applications of the APSF is to the vacuum expectation values for the physical observables in the Casimir[2] effect; see [21,40,45,50]. For a proof of the functional equation for the Riemann zeta function using the Abel–Plana formula see [51], and for the interconnections of Plana's formula with Ramanujan's integral formula see [55].

The third formula, which is fundamental in a variety of mathematical fields, the Poisson summation formula, will be stated in the following form (see e.g. [11, p. 202]):

**Poisson Summation Formula (PSF)**

*Let* $f \in L^1(\mathbb{R}) \cap AC(\mathbb{R})$. *Then*

$$\sum_{k=-\infty}^{\infty} f(k) = \sqrt{2\pi} \sum_{k=-\infty}^{\infty} \widehat{f}(2\pi k), \tag{1.6}$$

*where the series on the left-hand side converges absolutely.*

Above, $\widehat{f}$ denotes the Fourier transform of $f \in L^1(\mathbb{R})$; it, as well as the class $AC(\mathbb{R})$, is defined in Sect. 2.

Although this formula is named after Poisson, it was given in a general form without proof by Gauß in a note written on the interior of the cover page of a book entitled "Opuscula mathematica 1799–1813"; see [25, p. 88]. Jacobi in his work on the transformation formula for elliptic functions [36, p. 260] attributes it to Poisson. Further remarks concerning the history of the Poisson summation formula can be found in [13, p. 28], [53, pp. 36 f.].

Finally to the approximate sampling formula (ASF) of signal analysis, actually due already to de la Vallée-Poussin (see [20], pp. 65–156 for a reproduction of this paper and pp. 421–453 for a commentary by Butzer-Stens), Weiss [54], Brown [6], and Butzer-Splettstößer [14].

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[2] The Dutch physicist Hendrik Casimir (1909–2000) was awarded an Honorary Doctorate by the RWTH Aachen University in 1966, the Laudation having been held by Professor Josef Meixner.

**Approximate Sampling Formula (ASF)**

*For $f$ belonging to the class $F^2$ (see Sect. 2 for definition), $w > 0$, we have*

$$f(t) = \sum_{k=-\infty}^{\infty} f\left(\frac{k}{w}\right) \operatorname{sinc}(wt - k) + (R_w f)(t) \qquad (t \in \mathbb{R}), \tag{1.7}$$

*where the remainder $R_w f$ is defined by*

$$(R_w f)(t) := \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \left(1 - e^{-i2\pi kwt}\right) \int\limits_{(2k-1)\pi w}^{(2k+1)\pi w} \widehat{f}(v) e^{ivt}\, dv \quad (t \in \mathbb{R}), \tag{1.8}$$

*the* sinc*-function being given by*

$$\operatorname{sinc} z := \begin{cases} \frac{\sin \pi z}{\pi z} & \textit{if } z \in \mathbb{C}\backslash\{0\}, \\ 1 & \textit{if } z = 0. \end{cases}$$

It is the sampling theorem for not necessarily bandlimited functions, generalizing the classical Whittaker–Kotel'nikov–Shannon sampling theorem (CSF) of signal analysis

$$f(t) = \sum_{k=-\infty}^{\infty} f\left(\frac{k}{w}\right) \operatorname{sinc}(wt - k) \qquad (t \in \mathbb{R}) \tag{1.9}$$

for $f \in B^2_{\pi w}$ (for definition see Sect. 2). To proceed from (1.9) to (1.7) one has to add the "error" term $(R_w f)(t)$.

*Remark.* A consequence of the definition (1.8) of $R_w f$ is the inequality

$$\left|(R_w f)(t)\right| \le \sqrt{\frac{2}{\pi}} \int\limits_{|v| \ge \pi w} |\widehat{f}(v)|\, dv, \tag{1.10}$$

yielding

$$\lim_{w\to\infty} \sum_{k=-\infty}^{\infty} f\left(\frac{k}{w}\right) \operatorname{sinc}(wt - k) = f(t)$$

uniformly for $t \in \mathbb{R}$.

The implications to be established in this paper between the assertions of the summation formulae of Euler–Maclaurin (EMSF), Abel–Plana (APSF), Poisson (PSF) and the approximate sampling formula (ASF), are indicated by the arrows of the graphic in Fig. 1. As the reader observes, there are 12 possible implications; of these we shall prove 10 implications, namely the grouping

$$\text{APSF} \Longleftarrow \text{ASF} \Longrightarrow \text{EMSF} \Longleftrightarrow \text{PSF}$$

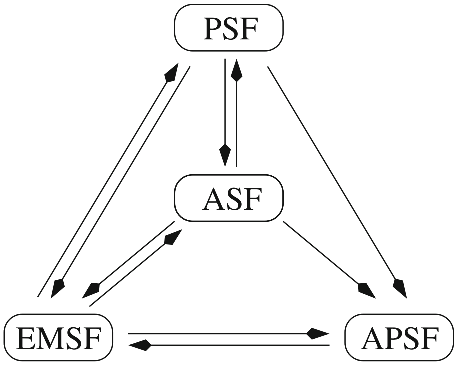


Figure 1. Diagram of the implications to be proven

as well as the grouping

$$\text{ASF} \Longleftrightarrow \text{PSF} \Longrightarrow \text{APSF} \Longleftrightarrow \text{EMSF} \Longrightarrow \text{ASF}.$$

The conclusion: All the formulae are equivalent to each other.

The first grouping consists of four new implications which are established in this paper for the first time. Of the second grouping, the equivalence ASF $\Leftrightarrow$ PSF, was first presented in the proceedings of the Edmonton conference of 1982 [15], the implication EMSF $\Rightarrow$ ASF in the special issues dedicated to the 90th birthday of A. Ostrowski in 1983 [16], and that of PSF $\Rightarrow$ APSF $\Leftrightarrow$ EMSF in memory of U. N. Singh in 1994 [49].

The proofs of five of these six known implications of the second grouping, established by three of the present authors since 1982, are to be found in conference proceedings and special issues not always readily available. For this reason they are presented here with complete proofs, especially to make sure that they do not involve "circular arguments", so important in establishing equivalences between many assertions. Further, they have been adapted to those of the new implications of the first grouping in order to give a unified exposition of the material. So the reader will find full proofs of all of the results discussed in the present paper.

All the proofs use only a handful of modest side results, namely classical Fourier analysis, the Weierstraß theorem for algebraic (not trigonometric) polynomials and Cauchy's theorem of complex function theory. Therefore, if we prove an implication X $\Rightarrow$ Y, then it is fully justified to say that Y is a direct corollary of X. Here "direct" means that the proof of the implication proceeds directly from X to Y, and is not of the form X $\Rightarrow$ Z $\Rightarrow$ Y, where Z is any other of the formulae under consideration.

The implications APSF $\Rightarrow$ ASF and APSF $\Rightarrow$ PSF are left open. The problem is that APSF requires functions which are analytic in a half-plane, whereas ASF and PSF hold for larger function classes. Thus it seems to be appropriate to proceed via a density argument based on an approximation process which could be so cumbersome that one could not speak of "establishing a direct corollary".

For those readers who are primarily interested in the equivalence of the four formulae, the presentation also allows several closed loops of implications, one being PSF $\Rightarrow$ APSF $\Rightarrow$ EMSF $\Rightarrow$ ASF $\Rightarrow$ PSF. In this case only the proofs of four implications are needed.

Concerning contents, Sect. 2 is devoted to notations and side results needed. Section 3 is concerned with the proof of the equivalence of EMSF and APSF, and Sect. 4 with the equivalence of PSF and ASF. Section 5 deals with the PSF $\Leftrightarrow$ EMSF and Sect. 6 with EMSF $\Leftrightarrow$ ASF. Section 7 treats the APSF as a corollary of ASF and PSF.

Section 8 presents a proof of EMSF; thus since one of the formulae has been shown to be actually valid, the four formulae are not only equivalent among themselves, but are all valid too.

Section 9 is concerned with APSF, EMSF and PSF considered as quadrature formulae in case the functions in question are bandlimited. Finally, Sect. 10 compares the results presented here with a similar equivalence grouping for bandlimited functions in [7]. Section 11 contains a short biography of A. Ostrowski.

## 2. Notations and Side Results

In what follows, the Fourier transform of $f \in L^p(\mathbb{R})$, $p = 1, 2$, is defined by

$$\widehat{f}(v) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(u) e^{-ivu}\, du \qquad (v \in \mathbb{R}),$$

the integral being understood as the limit in the $L^2(\mathbb{R})$-norm for $p = 2$. We shall consider the following function spaces:

$$F^p := \left\{ f\colon \mathbb{R} \to \mathbb{C};\, f \in L^p(\mathbb{R}) \cap C(\mathbb{R}), \widehat{f} \in L^1(\mathbb{R}) \right\} \qquad (p = 1, 2),$$

where $C(\mathbb{R})$ denotes the space of all uniformly continuous and bounded functions on $\mathbb{R}$. Since $\widehat{f} \in L^1(\mathbb{R})$ implies that $f$ is bounded, there holds $F^1 \subset F^2$.

We also make use of the space $AC(\mathbb{R}) = AC^{(1)}(\mathbb{R})$. It comprises all functions $f\colon \mathbb{R} \to \mathbb{C}$ being absolutely continuous on $\mathbb{R}$, i.e., those $f$ having the representation

$$f(t) = \int_{-\infty}^{t} f'(u)\, du + c \qquad (t \in \mathbb{R})$$

with $f' \in L^1(\mathbb{R})$. More generally, $AC^{(r+1)}(\mathbb{R})$, $r \in \mathbb{N}$, consists of all functions $f$ with $f' \in AC^{(r)}(\mathbb{R}) \cap L^1(\mathbb{R})$.

The Bernstein spaces $B^p_\sigma$ for $p = 1, 2$ and $\sigma > 0$ are defined in terms of the Fourier transform via

$$B^p_\sigma := \left\{ f \in L^p(\mathbb{R}); \widehat{f}(v) = 0 \ a.e. \,\text{outside}\ [-\pi\sigma, \pi\sigma] \right\}.$$

In view of the Paley–Wiener theorem (see e.g. [56, vol. II, p. 272]) these spaces can be equivalently characterized as the set of all entire functions of exponential type $\sigma$, the restriction to $\mathbb{R}$ of which belongs to $L^p(\mathbb{R})$. For this reason the spaces $B^2_\sigma$ are also known as Paley–Wiener spaces.

Let us list some well-known results from Fourier analysis; see any textbook in the matter, e.g. [11, Sects. 5.1, 5.2].

**Proposition 2.1.** a) *If $f \in L^p(\mathbb{R})$, $p = 1, 2$, then for each $h \in \mathbb{R}$,*

$$[e^{-ih\cdot} f(\cdot)]^\wedge(v) = \widehat{f}(v + h) \tag{2.1}$$

*for all $v \in \mathbb{R}$ in case $p = 1$ and a. e. in case $p = 2$.*

b) *If $f \in L^1(\mathbb{R})$, $g \in L^p(\mathbb{R})$, $p = 1, 2$, then the convolution*

$$(f * g)(t) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(u) g(t - u)\, du \tag{2.2}$$

*belongs to $L^p(\mathbb{R})$, and for the Fourier transform one has the convolution theorem*

$$(f * g)^\wedge(v) = \widehat{f}(v)\,\widehat{g}(v) \tag{2.3}$$

*for all $v \in \mathbb{R}$ in case $p = 1$ and a. e. in case $p = 2$.*

c) *If $f, g \in L^p(\mathbb{R})$, $p = 1, 2$, then there holds the exchange formula*

$$\int_{\mathbb{R}} f(v)\widehat{g}(v)\, dv = \int_{\mathbb{R}} \widehat{f}(v) g(v)\, dv. \tag{2.4}$$

d) *If $f \in L^2(\mathbb{R})$, then $\widehat{f}$ also belongs to $L^2(\mathbb{R})$ and $\|f\|_2 = \|\widehat{f}\|_2$. Furthermore, there holds the inversion formula*

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{f}(v) e^{ivt}\, dv = \widehat{\widehat{f}}(-t) \quad a.\ e., \tag{2.5}$$

*where the integral is again understood as the limit in $L^2(\mathbb{R})$-norm. If $f \in F^p$, $p = 1, 2$, then the integral in (2.5) exists as an ordinary Lebesgue integral and both equalities hold for all $t \in \mathbb{R}$.*

**Lemma 2.2.** *There hold the formulae*

$$\left[ \frac{1}{\sqrt{2\pi}\, w} \operatorname{rect}\left(\frac{\cdot}{w}\right) e^{i\cdot t} \right]^\wedge (v) = \operatorname{sinc} w(v - t) = \operatorname{sinc} w(t - v) \qquad (t \in \mathbb{R}) \tag{2.6}$$

$$[\operatorname{sinc} w(\cdot - t)]^\wedge (v) = \frac{1}{\sqrt{2\pi}\, w} \operatorname{rect}\left(\frac{v}{w}\right) e^{-ivt} \qquad (t \in \mathbb{R}) \tag{2.7}$$

*with the rectangle function*

$$\operatorname{rect}(t) := \begin{cases} 1, & |t| < \pi \\ \frac{1}{2}, & |t| = \pi \\ 0, & |t| > \pi \end{cases}.$$

*Proof.* Equation (2.6) is shown by a simple integration, and (2.7) follows from (2.6) by the Fourier inversion formula. (For an elementary proof of (2.7) without using the inversion formula see [5, p. 13 f].) □

Proposition 2.1 and Lemma 2.2 enable us to prove alternative representations of the sampling series

$$(S_w f)(t) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{w}\right) \operatorname{sinc} w(t-k) \tag{2.8}$$

and the remainder $R_w f$ of (1.8).

**Proposition 2.3.** *For fixed $t \in \mathbb{R}$, $w > 0$ let $e_{t,w}(v)$, $v \in \mathbb{R}$, be the function obtained by restricting $v \mapsto e^{itv}$ to the interval $[-\pi w, \pi w)$, and then extending it to $\mathbb{R}$ by $2\pi w$-periodic continuation. Then for $f \in F^2$,*

$$(S_w f)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{f}(v) e_{t,w}(v)\, dv. \tag{2.9}$$

*Proof.* By a simple calculation $e_{t,w}$ has the Fourier expansion

$$e_{t,w}(v) = \sum_{k=-\infty}^{\infty} \operatorname{sinc}(wt-k) e^{ikv/w}. \tag{2.10}$$

Since $e_{t,w}$ is of bounded variation, its Fourier series converges and (2.10) holds at each point of continuity. Moreover, its partial sums

$$s_n(v) := \sum_{|k|\le n} \operatorname{sinc} w(t-k) e^{ikv/w} \tag{2.11}$$

are uniformly bounded with respect to $v \in \mathbb{R}$ and $n \in \mathbb{N}$, that is

$$|s_n(v)| \le C \qquad (v \in \mathbb{R}; n \in \mathbb{N})$$

for some constant $C > 0$ depending on $t$ only; see [56, p. 90, Theorem 3.7]. Since $\widehat{f} \in L^1(\mathbb{R})$, we see that $\widehat{f} \cdot s_n$ has an absolutely integrable majorant for all $n \in \mathbb{N}$. Therefore, Lebesgue's dominated convergence theorem allows us to conclude that

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \widehat{f}(v)e_{t,w}(v)\,dv = \lim_{n\to\infty}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \widehat{f}(v)s_n(v)\,dv$$
$$= \lim_{n\to\infty}\sum_{|k|\le n}\left\{\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \widehat{f}(v)e^{ikv/w}\,dv\right\}\operatorname{sinc}(wt-k)$$
$$= \lim_{n\to\infty}\sum_{|k|\le n} f\left(\frac{k}{w}\right)\operatorname{sinc} w(t-k),$$

where the Fourier inversion theorem (Proposition 2.1 d) has been used in the last step. □

**Proposition 2.4.** *For $f \in F^2$ and $w > 0$ we have the following two alternative representations of the remainder $R_w f$ of (1.8),*

$$(R_w f)(t) = \sum_{k=-\infty}^{\infty}\left(e^{i2\pi kwt}-1\right)\int_{\mathbb{R}} f(u)e^{-i2\pi kwu}\operatorname{sinc} w(t-u)\,du \tag{2.12}$$
$$= f(t) - \sum_{k=-\infty}^{\infty}\int_{\mathbb{R}} f(u)e^{-i2\pi kwu}\operatorname{sinc} w(t-u)\,du \quad (t\in\mathbb{R}). \tag{2.13}$$

*Proof.* Using (2.4), (2.6) and (2.1) we obtain

$$\int_{(2k-1)\pi w}^{(2k+1)\pi w} \widehat{f}(v)e^{itv}\,dv = e^{i2\pi kwt}\int_{\mathbb{R}} \widehat{f}(v+2\pi kw)\operatorname{rect}\left(\frac{v}{w}\right)e^{itv}\,dv$$
$$= \sqrt{2\pi}\,e^{i2\pi kwt}\int_{\mathbb{R}} f(u)e^{-i2\pi kwu}\operatorname{sinc}(t-u)\,du. \tag{2.14}$$

Inserting this into (1.8) yields (2.12).

Moreover, (1.8) can be rewritten as

$$(R_w f)(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widehat{f}(v)e^{ivt}\,dv$$
$$-\frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{\infty} e^{-i2\pi kwt}\int_{(2k-1)\pi w}^{(2k+1)\pi w}\widehat{f}(v)e^{ivt}\,dv.$$

Noting that the first integral equals $f(t)$ by the Fourier inversion formula (2.5), and inserting (2.14) into the infinite series, we obtain the representation (2.13). □

We also need the following variant of Nikolskiĭ's inequality. Its proof is nearly the same as that of [41, Theorem 3.3.1]. It requires only the mean value theorems of integral calculus as well as Minkowski's inequality for sums and Hölder's inequality for integrals.

**Lemma 2.5.** *For $f \in L^1(\mathbb{R}) \cap AC(\mathbb{R})$ there holds*

$$\sum_{k=-\infty}^{\infty} |f(k)| \leq \|f\|_{L^1(\mathbb{R})} + \|f'\|_{L^1(\mathbb{R})}. \tag{2.15}$$

In the proofs below it will often be convenient to prove the result first for a particular case, and then the general case by a density argument using a suitable approximation. In this respect we will make use of the singular integral of Fejér or Fejér means (cf. (2.2)), namely,

$$(\sigma_\rho f)(t) := (f * \chi_\rho)(t) \quad (t \in \mathbb{R}, \rho > 0), \tag{2.16}$$

$\chi_\rho$ being Fejér's kernel

$$\chi_\rho(u) := \rho\sqrt{2\pi}\,\mathrm{sinc}^2(\rho u) \quad (u \in \mathbb{R}, \rho > 0), \tag{2.17}$$

having Fourier transform

$$\widehat{\chi_\rho}(v) = \left(1 - \frac{|v|}{2\pi\rho}\right)_+ \quad (v \in \mathbb{R}, \rho > 0), \tag{2.18}$$

where $h_+(u) := h(u)$ if $h(u) \geq 0$, and $h_+(u) = 0$ if $h(u) < 0$.

**Proposition 2.6.** a) *If $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$, then for each $r \in \mathbb{N}$,*

$$\sigma_\rho f \in L^p(\mathbb{R}) \cap C^{(r)}(\mathbb{R}) \quad \textit{with} \quad (\sigma_\rho f)^{(r)} \in L^p(\mathbb{R}) \quad (\rho > 0)$$

*and*

$$\lim_{\rho\to\infty} \|\sigma_\rho f - f\|_{L^p(\mathbb{R})} = 0.$$

*Furthermore, there holds $\lim_{\rho\to\infty}(\sigma_\rho f)(t) = f(t)$ at each point $t \in \mathbb{R}$ where $f$ is continuous.*

b) *If $f \in L^p(\mathbb{R}) \cap C^{(s-1)}(\mathbb{R})$ with $f^{(s)} \in L^p(\mathbb{R})$ for some $s \in \mathbb{N}$ and $1 \leq p < \infty$, then*

$$\lim_{\rho\to\infty} \|(\sigma_\rho f)^{(s)} - f^{(s)}\|_{L^p(\mathbb{R})} = 0.$$

c) *If $f \in F^2$, then for each $w > 0$,*

$$\lim_{\rho\to\infty} (S_w(\sigma_\rho f))(t) = (S_w f)(t) \qquad (t \in \mathbb{R}), \tag{2.19}$$

*and*

$$\lim_{\rho\to\infty} (R_w(\sigma_\rho f))(t) = (R_w f)(t) \qquad (t \in \mathbb{R}). \tag{2.20}$$

d) *If $f \in L^1(\mathbb{R}) \cap AC(\mathbb{R})$, then*

$$\lim_{\rho\to\infty} \sum_{k=-\infty}^{\infty} (\sigma_\rho f)(k) = \sum_{k=-\infty}^{\infty} f(k), \tag{2.21}$$

*all series being absolutely convergent.*

*Proof.* Parts a) and b) are well known (see e.g. [11, pp. 122, 134]). As to c), one has by the convolution theorem (2.3) and by (2.9), noting that $|e_{t,w}(v)| = 1$,

$$\big|(S_w(\sigma_\rho f))(t) - (S_w f)(t)\big| \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |\widehat{f}(v)| \left| \left(1 - \frac{|v|}{2\pi\rho}\right)_+ - 1\right| dv.$$

Since $\widehat{f} \in L^1(\mathbb{R})$, the last integral tends to zero for $\rho \to \infty$ by Lebesgue's dominated convergence theorem. This yields (2.19).

Similarly, one has by (1.8),

$$\begin{aligned}\big|(R_w(\sigma_\rho f))(t) - (R_w f)(t)\big| &\leq \sqrt{\frac{2}{\pi}} \sum_{k=-\infty}^{\infty} \int_{(2k-1)\pi w}^{(2k+1)\pi w} |\widehat{f}(v)| \left| \left(1 - \frac{|v|}{2\pi\rho}\right)_+ - 1\right| dv \\ &= \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} |\widehat{f}(v)| \left| \left(1 - \frac{|v|}{2\pi\rho}\right)_+ - 1\right| dv,\end{aligned}$$

which proves (2.20).

d) The proof of the absolute convergence of the series follows by Lemma 2.5, noting that the assumptions upon $f$ imply that $\sigma_\rho f \in L^1(\mathbb{R}) \cap AC(\mathbb{R})$ by part a). Assertion (2.21) is now an easy consequence of Lemma 2.5 and part b), since

$$\sum_{k=-\infty}^{\infty} \big|(\sigma_\rho f)(k) - f(k)\big| \leq \|\sigma_\rho f - f\|_{L^1(\mathbb{R})} + \|(\sigma_\rho f') - f'\|_{L^1(\mathbb{R})}.$$

$\square$

**Lemma 2.7.** a) *Let $g$ be a continuous $\lambda$-periodic function with $\int_0^\lambda g(u)du = 0$. Then*

$$\left| \int_1^\rho \frac{g(u)}{u}\, du \right| \leq 2 \int_0^\lambda |g(u)|\, du$$

*for $\rho \in [1, \infty)$.*

b) *There exists a constant $C \in \mathbb{R}$, such that for all $a, b, t \in \mathbb{R}$ and $k \in \mathbb{Z}$,*

$$\left| \int_a^b \operatorname{sinc}(t-u) e^{i2\pi k u}\, du \right| \leq C.$$

*Proof.* By the second law of the mean for integrals, there exists $\xi \in (1, \rho)$ such that

$$\int_1^\rho \frac{g(u)}{u}\,du = \int_1^\xi g(u)\,du + \frac{1}{\rho}\int_\xi^\rho g(u)\,du.$$

Thus

$$\Big|\int_1^\rho \frac{g(u)}{u}\,du\Big| \leq \Big(1+\frac{1}{\rho}\Big)\int_0^\lambda |g(u)|\,du \leq 2\int_0^\lambda |g(u)|\,du.$$

Part b) follows immediately by choosing $g(u) := \sin\pi(t-u)e^{i2\pi u}$ in a). □

## 3. The Summation Formulae of Euler–Maclaurin and Abel–Plana

The summation formulae of Euler–Maclaurin and Abel–Plana can be interpreted as trapezoidal rules with remainders for quadrature over $[0, n]$ and $[0, \infty]$, respectively; see Sect. 9 below. As such, one may suspect that APSF is somehow the limiting case $n \to \infty$ of EMSF. However, the situation is more intricate since the hypotheses on $f$ and the representations of the remainders in the two formulae are very different. Nevertheless, it was shown in [49] that EMSF and APSF can be deduced from each other.[3] The cited paper established a conjecture raised by one of the authors at a lecture held at the University of Erlangen-Nuremberg in February 1991.

The basic idea is to transform the remainders by using Cauchy's theorem for contour integration of analytic functions along rectangles. However, in the hypotheses of EMSF the function $f$ need not be analytic. Therefore $f$ is approximated by an analytic function $\varphi$ so that APSF becomes applicable. Such a function $\varphi$ is obtained by a process based on the classical approximation theorem of Weierstraß.

Since the paper [49] is not easily available, we borrow details from it and present them here in a somewhat different form.

For our purpose it is convenient to introduce

$$L_j(z) := (-1)^j \sum_{k=1}^{\infty} \frac{e^{-2\pi kz}}{(2\pi k)^j} \quad (j \in \mathbb{N}_0). \tag{3.1}$$

For $j = 0$ and $j = 1$ the series converges in the open right half-plane, where it defines $L_j$ as an analytic function.

Using the formula for the geometric series, we obtain

$$L_0(z) = \frac{1}{e^{2\pi z}-1} \tag{3.2}$$

[3] Hardy [29, Sect. 13.14] only indicated a proof for EMSF ⇒ APSF, but he needed additional assumptions on $f$ for carrying it out.

and may therefore consider $L_0$ a meromorphic function which has simple poles at $z \in i\mathbb{Z}$ and is analytic in $\mathbb{C}\setminus i\mathbb{Z}$.

In the case $j = 1$ the Cauchy-Schwarz inequality implies for $x > 0$ and $y \in \mathbb{R}$ that

$$|L_1(x+iy)| \leq \left(\sum_{k=1}^{\infty} \frac{1}{(2\pi k)^2}\right)^{1/2} \left(\sum_{k=1}^{\infty} e^{-4\pi kx}\right)^{1/2} = \frac{1}{2\sqrt{6}} \left(e^{4\pi x} - 1\right)^{-1/2},$$

and so

$$L_1(x+iy) = \mathcal{O}(x^{-1/2}) \quad (x \to 0+). \tag{3.3}$$

For $j \geq 2$ the functions $L_j$ are defined in the closed right half-plane and analytic in its interior. Two useful obvious properties are

$$L_j(z+i) = L_j(z), \quad L'_{j+1}(z) = L_j(z) \tag{3.4}$$

holding for $\Re z > 0$ and all $j \in \mathbb{N}_0$. We also mention that

$$\frac{B_{2k}}{(2k)!} = (-1)^{k+1} 2L_{2k}(0). \tag{3.5}$$

As in [49], we will consider EMSF for $r = 1$ only, since the more general form (1.1) can be deduced from that special case by repeated integration by parts of the remainder. Employing the functions $L_j$, we may write EMSF, thus (1.1), for $r = 1$, as

$$\sum_{k=0}^{n} f(k) = \frac{1}{2}\left[f(0) + f(n)\right] + \int_0^n f(x)\,dx$$
$$+\frac{1}{12}\left[f'(n) - f'(0)\right] - \int_0^n \left[L_2(it) + L_2(-it)\right] f''(t)\,dt \tag{3.6}$$

while APSF (1.4) takes the form

$$\lim_{N\to\infty}\left\{\sum_{k=0}^{N} f(k) - \int_0^{N+1/2} f(x)\,dx\right\}$$
$$= \frac{1}{2} f(0) + i\int_0^{\infty} L_0(y)\left[f(iy) - f(-iy)\right] dy. \tag{3.7}$$

The crucial link between the two summation formulae (3.6) and (3.7) is contained in the following lemma.

**Lemma 3.1.** *Let $f$ satisfy the hypotheses of APSF. Then the expressions*

$$A_n(f) := i\int_0^{\infty} L_0(t)\left[f(it) - f(-it)\right] dt - i\int_0^{\infty} L_0(t)\left[f(n+it) - f(n-it)\right] dt$$

*and*

$$B_n(f) := \frac{1}{12}\left[f'(n) - f'(0)\right] - \int\limits_0^n \left[L_2(it) + L_2(-it)\right] f''(t)\, dt$$

*are equal.*

*Proof.* For $z_0, z_1 \in \mathbb{C}$, we mean by $\int_{z_0}^{z_1} \dots dz$ the integral along the line segment starting at $z_0$ and ending at $z_1$.

Let $\varepsilon > 0$ and $K > \varepsilon$. Introducing

$$F(z) := L_0(-iz)f(z) \quad \text{and} \quad G(z) := L_0(iz)f(z),$$

we define

$$I_n^+(\varepsilon, K) := \int\limits_{i\varepsilon}^{iK} F(z)\, dz - \int\limits_{n+i\varepsilon}^{n+iK} F(z)\, dz$$

and

$$I_n^-(\varepsilon, K) := \int\limits_{-i\varepsilon}^{-iK} G(z)\, dz - \int\limits_{n-i\varepsilon}^{n-iK} G(z)\, dz.$$

Then the expression $A_n(f)$ can be rewritten as

$$A_n(f) = \lim_{\varepsilon\to 0+} \lim_{K\to+\infty} \left[I_n^+(\varepsilon, K) + I_n^-(\varepsilon, K)\right]. \tag{3.8}$$

Clearly, $F$ is analytic in $\{z \in \mathbb{C} : \Re z \geq 0,\ \Im z \geq \varepsilon\}$. Hence, by Cauchy's theorem, integration of the function $F$ along the rectangle with vertices at $i\varepsilon, n + i\varepsilon, n + iK, iK$ yields

$$I_n^+(\varepsilon, K) = \int\limits_{i\varepsilon}^{n+i\varepsilon} F(z)\, dz - \int\limits_{iK}^{n+iK} F(z)\, dz.$$

As a consequence of (1.2), the last integral vanishes as $K \to \infty$. Analogous considerations hold for $G$ in $\{z \in \mathbb{C} : \Re z \geq 0,\ \Im z \leq -\varepsilon\}$, and so (3.8) reduces to

$$A_n(f) = \lim_{\varepsilon\to 0+} \left[\int\limits_{i\varepsilon}^{n+i\varepsilon} F(z)\, dz + \int\limits_{-i\varepsilon}^{n-i\varepsilon} G(z)\, dz\right]. \tag{3.9}$$

Now integrating by parts twice and using (3.4), we find that

$$\int\limits_{i\varepsilon}^{n+i\varepsilon} F(z)\, dz = \int\limits_0^n L_0(\varepsilon - it) f(t + i\varepsilon)\, dt$$

$$= \left[iL_1(\varepsilon)f(t + i\varepsilon) + L_2(\varepsilon)f'(t + i\varepsilon)\right]_{t=0}^{t=n} - \int\limits_0^n L_2(\varepsilon - it)f''(t+i\varepsilon)\, dt. \tag{3.10}$$

Similarly for the second integral in (3.9),

$$\int_{-i\varepsilon}^{n-i\varepsilon} G(z)\,dz = \int_0^n L_0(\varepsilon+it)f(t-i\varepsilon)\,dt$$

$$= [-iL_1(\varepsilon)f(t-i\varepsilon)+L_2(\varepsilon)f'(t-i\varepsilon)]_{t=0}^{t=n} - \int_0^n L_2(\varepsilon+it)f''(t-i\varepsilon)\,dt. \quad (3.11)$$

But from (3.3) it follows that

$$L_1(\varepsilon)\left[f(t+i\varepsilon)-f(t-i\varepsilon)\right] = \mathcal{O}\left(\varepsilon^{1/2}\right) \to 0 \quad (\varepsilon\to 0+).$$

Hence, combining (3.9)–(3.11) and letting $\varepsilon \to 0+$, we obtain, since $2L_2(0) = B_2/2 = 1/12$,

$$A_n(f) = 2L_2(0)\left[f'(n)-f'(0)\right] - \int_0^n \left[L_2(-it)+L_2(it)\right] f''(t)\,dt = B_n(f)$$

as was to be shown. □

Now we turn to the aforementioned approximation process.

**Lemma 3.2.** *Let $f \in C^{(2)}[0,n]$, where $n\in\mathbb{N}$, and denote by $\|\cdot\|_{C[0,n]}$ the supremum norm on $[0,n]$. Then, given $\varepsilon > 0$, there exists a function $\varphi$ satisfying the hypotheses of APSF such that*

$$\|f-\varphi\|_{C[0,n]} \le \frac{n\varepsilon}{2\,(n^2+1)} \quad (3.12)$$

*and*

$$\|f''-\varphi''\|_{C[0,n]} \le \frac{\varepsilon}{n\,(n^2+1)}. \quad (3.13)$$

*Proof.* By the approximation theorem of Weierstraß, there exists a polynomial $P(x)$ such that

$$\|f''-P\|_{C[0,n]} \le \frac{\varepsilon}{2n(n^2+1)}. \quad (3.14)$$

Defining

$$Q(x) := f(0) + xf'(0) + \int_0^x (x-t)P(t)\,dt,$$

and choosing $\lambda \in (0,2\pi)$ such that

$$\lambda\left(n\,\|Q''\|_{C[0,n]} + 2\,\|Q'\|_{C[0,n]} + 2\pi\,\|Q\|_{C[0,n]}\right) \le \frac{\varepsilon}{2n\,(n^2+1)}, \quad (3.15)$$

we claim that $\varphi(z) := e^{-\lambda z}Q(z)$ has all the desired properties.

Indeed, since $Q$ is a polynomial and $\lambda > 0$, it is easily verified that $\varphi$ satisfies the hypotheses of APSF.

Next, writing $f$ as

$$f(x) \;=\; f(0) + xf'(0) + \int_0^x (x-t)f''(t)\,dt,$$

which is a special case of Taylor's formula, and recalling the definition of $Q$, we find that

$$|f(x) - Q(x)| \leq \left| \int_0^x (x-t)\left(f''(t) - P(t)\right)\,dt \right| \leq \frac{n^2}{2}\,\|\, f'' - P \,\|_{C[0,n]} \tag{3.16}$$

for $x \in [0, n]$. On the same interval,

$$|1 - e^{-\lambda x}| \;\leq\; 1 - e^{-\lambda n} \leq n\lambda. \tag{3.17}$$

Since

$$f(x) - \varphi(x) = f(x) - Q(x) + \left(1 - e^{-\lambda x}\right) Q(x),$$

we may employ (3.16) and (3.17) to deduce that

$$\|\, f - \varphi \,\|_{C[0,n]} \leq \frac{n^2}{2}\,\|\, f'' - P \,\|_{C[0,n]} + n\lambda\,\|\, Q \,\|_{C[0,n]}\,.$$

Now, estimating the right-hand side with the help of (3.14) and (3.15), we see that (3.12) holds.

Finally, since $Q''(x) \equiv P(x)$, we have

$$f''(x) - \varphi''(x) = f''(x) - P(x) \;+\; Q''(x) - \frac{d^2}{dx^2}\left(e^{-\lambda x}Q(x)\right)$$

and deduce that

$$\begin{aligned}\|\, f'' - \varphi'' \,\|_{C[0,n]} &\leq \|\, f'' - P \,\|_{C[0,n]} + n\lambda\,\|\, Q'' \,\|_{C[0,n]} \\ &\qquad + 2\lambda\,\|\, Q' \,\|_{C[0,n]} + \lambda^2\,\|\, Q \,\|_{C[0,n]}\,.\end{aligned}$$

In conjunction with (3.14) and (3.15), this implies (3.13). □

*Proof of APSF ⇒ EMSF.* As mentioned above, it suffices to prove (3.6). If $f$ satisfies the hypotheses of APSF, then so does $f(\cdot + n)$. Applying APSF to these two functions and subtracting the results, we obtain

$$\sum_{k=0}^{n-1} f(k) = \frac{1}{2}\left[f(0) - f(n)\right] + \int_0^n f(x)\,dx + A_n(f)$$

with $A_n(f)$ as defined in Lemma 3.1. The conclusion of Lemma 3.1 gives (3.6) immediately.

Now suppose that $f$ satisfies the hypotheses of EMSF only, i.e. $f \in C^{(2)}[0,n]$. We have to show that

$$E(f) := \frac{1}{2}\,[f(0)+f(n)] + \sum_{k=1}^{n-1} f(k) - \int_0^n f(x)\,dx - \frac{1}{12}\,\big[f'(n)-f'(0)\big]$$
$$- \int_0^n \big[L_2(it) - L_2(-it)\big]\, f''(t)\,dt$$

vanishes. Given any $\varepsilon > 0$, we now employ Lemma 3.2 and define $\psi := f - \varphi$. By what we have proved already, namely that $E(\varphi) = 0$,

$$E(f) = E(\varphi) + E(\psi) = E(\psi).$$

Finally, noting that

$$\big|L_2(it) - L_2(-it)\big| \le \left|\sum_{k=1}^{\infty} \frac{2\cos(2\pi k t)}{(2\pi k)^2}\right| \le \frac{1}{2\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{1}{12}$$

for $t \in \mathbb{R}$, and using (3.12) and (3.13), we conclude that

$$\begin{aligned}|E(\psi)| &\le \|\,\psi\,\|_{C[0,n]} + (n-1)\,\|\,\psi\,\|_{C[0,n]} + n\,\|\,\psi\,\|_{C[0,n]}\\ &\quad + \frac{1}{12}\int_0^n |\psi''(t)|\,dt + \frac{n}{12}\,\|\,\psi''\,\|_{C[0,n]}\\ &\le 2n\,\|\,\psi\,\|_{C[0,n]} + \frac{n}{6}\,\|\,\psi''\,\|_{C[0,n]} < \varepsilon.\end{aligned}$$

This implies that $E(f) = 0$. $\square$

*Proof of EMSF $\Rightarrow$ APSF.* Suppose that $f$ satisfies the hypotheses of APSF and that (3.6) holds. Employing Lemma 3.1, we may rewrite (3.6) as

$$\begin{aligned}\sum_{k=0}^{n} f(k) &= \frac{1}{2}\,[f(0)+f(n)] + \int_0^n f(x)\,dx\\ &\quad + i\int_0^\infty L_0(t)\,[f(it) - f(-it)]\;dt\\ &\quad - i\int_0^\infty L_0(t)\,[f(n+it) - f(n-it)]\,dt.\end{aligned} \tag{3.18}$$

Now let $\varepsilon > 0$ and $K > \varepsilon$. Consider the rectangle with vertices at

$$n + i\varepsilon, \quad n + \frac{1}{2} + i\varepsilon, \quad n + \frac{1}{2} + iK, \quad n + iK$$

and its conjugate in the lower half-plane. By contour integration of the function $L_0(-iz)f(z)$ along the first rectangle and $L_0(iz)f(z)$ along the second, we find, as in the proof of Lemma 3.1, that

$$\begin{aligned}
i\int_0^\infty L_0(t)\,[f(n+it)-f(n-it)]dt\\
\quad -i\int_0^\infty L_0\left(t+\frac{1}{2}i\right)\left[f\left(n+\frac{1}{2}+it\right)-f\left(n+\frac{1}{2}-it\right)\right]dt\\
=\lim_{\varepsilon\to 0+}\int_n^{n+1/2}\left[L_0(\varepsilon-it)f(t+i\varepsilon)+L_0(\varepsilon+it)f(t-i\varepsilon)\right]\,dt =: C_n(f).
\end{aligned}$$

Next we observe that

$$\begin{aligned}
&\lim_{\varepsilon\to 0+}\int_n^{n+1/2} L_0(\varepsilon\mp it)\,[f(t\pm i\varepsilon)-f(t)]\,dt\\
&\quad=\lim_{\varepsilon\to 0+}\int_n^{n+1/2}\frac{\varepsilon}{e^{2\pi(\varepsilon\mp it)}-1}\cdot\frac{f(t\pm i\varepsilon)-f(t)}{\varepsilon}\,dt=0
\end{aligned}$$

since the integrand remains bounded on $[n,n+\frac{1}{2}]$ as $\varepsilon\to 0+$ and approaches zero on $(n,n+\frac{1}{2}]$. Hence $C_n(f)$ can be expressed as

$$\begin{aligned}
C_n(f)&=\lim_{\varepsilon\to 0+}\int_n^{n+1/2}[L_0(\varepsilon-it)+L_0(\varepsilon+it)]\,f(t)\,dt\\
&=\lim_{\varepsilon\to 0+}\int_n^{n+1/2}2\sum_{k=1}^{\infty}e^{-2\pi k\varepsilon}\cos(2\pi kt)f(t)\,dt\\
&=\lim_{\varepsilon\to 0+}\sum_{k=1}^{\infty}e^{-2\pi k\varepsilon}a_k,
\end{aligned}$$

where

$$a_k := 2\int_n^{n+1/2} f(t)\cos(2\pi kt)\,dt.$$

We now interpret $a_k$ as the coefficients of a Fourier series. Indeed, let $\widetilde{g}$ be the 1-periodic continuation of the function

$$g(x) := \frac{1}{2} f(n + |x - n|) - \int_{n}^{n+1/2} f(t)\, dt$$

defined on $[n - \frac{1}{2}, n + \frac{1}{2}]$. Then $\widetilde{g}$ is continuous and its associated Fourier series is

$$\widetilde{g}(x) \sim \sum_{k=1}^{\infty} a_k \cos(2\pi k x).$$

Hence $C_n(f)$ is the Abel–Poisson limit of the Fourier series of $\widetilde{g}$ at $x = 0$; see [11, p. 46, Proposition 1.2.8]. Therefore,

$$C_n(f) = \widetilde{g}(0) = g(n) = \frac{1}{2} f(n) - \int_{n}^{n+1/2} f(t)\, dt.$$

Altogether, we can rewrite (3.18) as

$$\sum_{k=0}^{n} f(k) = \frac{1}{2} f(0) \int_{0}^{n+1/2} f(x)\, dx + i \int_{0}^{\infty} L_0(t) \left[ f(it) - f(-it) \right] dt$$
$$- i \int_{0}^{\infty} L_0 \left( t + \frac{1}{2} i \right) \left[ f\left( n + \frac{1}{2} + it \right) y - f\left( n + \frac{1}{2} - it \right) \right] dt.$$

Since

$$L_0 \left( t + \frac{i}{2} \right) = - \frac{1}{e^{2\pi t} + 1},$$

we see from (1.3) that the last integral vanishes as $n \to \infty$. Thus we arrive at (3.7). $\square$

## 4. Poisson's Summation Formula and the Approximate Sampling Formula

When we want to deduce ASF from one of the other formulae, it suffices to do it for $w = 1$. Indeed, the general formula is obtained from that special case by replacing $f$ by $f(\cdot/w)$, $t$ by $wt$ and noting that

$$\left[ (R_1 f) \left( \frac{\cdot}{w} \right) \right] (wt) = (R_w f)(t).$$

The two proofs of this section are adapted from [15].

*Proof of PSF $\Rightarrow$ ASF.* First we assume that $f \in L^2(\mathbb{R}) \cap AC(\mathbb{R})$. Then, for fixed $t \in \mathbb{R}$, the function $g_t(\cdot) := f(\cdot)\,\mathrm{sinc}(t-\cdot)$ belongs to $L^1(\mathbb{R}) \cap AC(\mathbb{R})$, and PSF (1.6) applies. Noting that

$$\sum_{k=-\infty}^{\infty} g_t(k) = (S_1 f)(t)$$

and

$$\sqrt{2\pi}\,\widehat{g_t}(2\pi k) = \int_{\mathbb{R}} f(u)e^{-i2\pi ku}\,\mathrm{sinc}(t-u)\,du,$$

PSF yields

$$(S_1 f)(t) = \sum_{k=-\infty}^{\infty} \int_{\mathbb{R}} f(u)e^{-i2\pi ku}\,\mathrm{sinc}(t-u)\,du = f(t) - (R_1 f)(t),$$

the representation (2.13) being used for the remainder $R_1 f$. This is ASF for $w = 1$ in that particular case.

In order to prove ASF for $f \in F^2$, we approximate $f$ by the convolution integral of Fejér $\sigma_\rho f$; see (2.16). According to Proposition 2.6 a), $\sigma_\rho f$ satisfies the assumptions of the particular case, hence

$$(S_1(\sigma_\rho f))(t) = (\sigma_\rho f)(t) - (R_1(\sigma_\rho f))(t) \quad (t \in \mathbb{R}, \rho > 0)). \tag{4.1}$$

Letting $\rho \to \infty$, ASF for $w = 1$ follows again by Proposition 2.6. $\square$

Next to the inverse implication.

*Proof of ASF $\Rightarrow$ PSF.* Assume that $f \in L^1(\mathbb{R}) \cap AC(\mathbb{R})$ with $\widehat{f} \in L^1(\mathbb{R})$. Then ASF applies to $f$ and we obtain for $w = 1$,

$$\begin{aligned} f(t) = &\sum_{k=-\infty}^{\infty} f(k)\,\mathrm{sinc}(t-k) \\ &+ \sum_{k=-\infty}^{\infty} \left(e^{i2\pi kt} - 1\right) \int_{\mathbb{R}} f(u)e^{-i2\pi ku}\,\mathrm{sinc}(t-u)\,du \quad (t \in \mathbb{R}), \end{aligned} \tag{4.2}$$

where we have used the remainder in the form (2.12). Integrating this equation term by term formally, one obtains

$$\begin{aligned} \int_{\mathbb{R}} f(t)\,dt = &\sum_{k=-\infty}^{\infty} f(k) \int_{\mathbb{R}} \mathrm{sinc}(t-k)\,dt \\ &+ \sum_{k=-\infty}^{\infty} \int_{\mathbb{R}} f(u)e^{-i2\pi ku} \left\{ \int_{\mathbb{R}} \left(e^{i2\pi kt} - 1\right) \mathrm{sinc}(t-u)\,dt \right\} du. \end{aligned}$$

Now, the integrals in the first series are equal to 1 by (2.7), and, similarly, the integrals in curly brackets are equal to $-1$ for $k \neq 0$ and equal to 0 for $k = 0$, again by (2.7). So one has

$$\int_{\mathbb{R}} f(t)\,dt = \sum_{k=-\infty}^{\infty} f(k) - \sqrt{2\pi} \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \widehat{f}(2\pi k),$$

which is PSF.

To make these considerations rigorous, first assume that $\widehat{f}(v) = 0$ for $|v| > \rho$ for some $\rho > 0$. This implies in particular that $\widehat{f} \in L^1(\mathbb{R})$. Now it follows from (2.14) that the integrals in (4.2) vanish for $|k| > \rho$, i.e., (4.2) can be rewritten as

$$\begin{aligned} f(t) &= \sum_{k=-\infty}^{\infty} f(k)\,\mathrm{sinc}(t-k) \\ &\quad + \sum_{|k|\leq\rho} \left(e^{i2\pi kt} - 1\right) \int_{\mathbb{R}} f(u) e^{-i2\pi ku}\,\mathrm{sinc}(t-u)\,du \quad (t \in \mathbb{R}). \end{aligned} \tag{4.3}$$

Since the infinite series is uniformly convergent with respect to $t$ in view of (2.15), one can integrate (4.3) term by term from $-R$ to $R$ to deduce

$$\begin{aligned} \int_{-R}^{R} f(t)\,dt &= \sum_{k=-\infty}^{\infty} f(k) \int_{-R}^{R} \mathrm{sinc}(t-k)\,dt \\ &\quad + \sum_{|k|\leq\rho} \int_{\mathbb{R}} f(u) e^{-i2\pi ku} \left\{ \int_{-R}^{R} \left(e^{i2\pi kt} - 1\right) \mathrm{sinc}(t-u)\,dt \right\} du, \end{aligned} \tag{4.4}$$

where the interchange of the order of integration is justified by Fubini's theorem.

Now take the limit for $R \to \infty$. As to the infinite series, since the integrals are uniformly bounded with respect to $R > 0$ and $k \in \mathbb{Z}$ by Lemma 2.7 b), one can interchange the limit with summation. Similarly, in the second term on the right-hand side of (4.4), the inner integrals are uniformly bounded with respect to $R > 0$ and $u \in \mathbb{R}$ by Lemma 2.7 b), and hence one can also interchange the limit for $R \to \infty$ with the outer integral. So one obtains from (4.4) for $R \to \infty$,

$$\begin{aligned} \int_{-\infty}^{\infty} f(t)\,dt &= \sum_{k=-\infty}^{\infty} f(k) \int_{-\infty}^{\infty} \mathrm{sinc}(t-k)\,dt \\ &\quad + \sum_{|k|\leq\rho} \int_{\mathbb{R}} f(u) e^{-i2\pi ku} \left\{ \int_{-\infty}^{\infty} \left(e^{i2\pi kt} - 1\right) \mathrm{sinc}(t-u)\,dt \right\} du. \end{aligned}$$

This yields PSF as in the formal proof, noting that by assumption $\widehat{f}(2\pi k) = 0$ for $|k| > \rho$.

It remains to show that the additional assumption $\widehat{f}(v) = 0$ for $|v| > \rho$ can be dropped. To this end, we consider the Fejér means $\sigma_\rho f$ (cf. (2.16)). It follows from (2.3), (2.18) and Proposition 2.6 a) that $\sigma_\rho f$ satisfies the assumptions of the particular case, and hence

$$\sum_{k=-\infty}^{\infty} (\sigma_\rho f)(k) = \sqrt{2\pi} \sum_{|k|\leq\rho} \widehat{f}(2\pi k)\left(1 - \frac{|k|}{\rho}\right).$$

Letting now $\rho \to \infty$ we obtain by Proposition 2.6 d) that

$$\sum_{k=-\infty}^{\infty} f(k) = \lim_{\rho\to\infty} \sqrt{2\pi} \sum_{|k|\leq\rho} \widehat{f}(2\pi k)\left(1 - \frac{|k|}{\rho}\right). \tag{4.5}$$

Finally, since $f \in L^1(\mathbb{R}) \cap AC(\mathbb{R})$ implies that $\widehat{f}(v) = o\left(|v|^{-1}\right)$ for $|v| \to \infty$ (cf. [11, p. 194]), it follows that

$$\lim_{\rho\to\infty} \sum_{|k|\leq\rho} \widehat{f}(2\pi k)\frac{|k|}{\rho} = 0,$$

so that one can replace the right-hand side of (4.5) by $\sqrt{2\pi}\sum_{-\infty}^{\infty} \widehat{f}(2\pi k)$, giving PSF. □

## 5. Poisson's Summation Formula and the Euler–Maclaurin Formula

*Proof of PSF ⇒ EMSF.* Let $f$ satisfy the hypothesis of EMSF for $r = 1$, that is, $f \in C^{(2)}[0, n]$. We may and will assume that $f(0) = f(n) = 0$. Indeed, we simply have to add to $f$ an appropriate linear function and observe that EMSF is trivial for linear functions.

Defining

$$\phi(x) := \begin{cases} f(x), & \text{if } x \in [0, n] \\ 0, & \text{if } x \in \mathbb{R}\backslash(0, n), \end{cases} \tag{5.1}$$

we see that

$$\widehat{\phi}(v) = \frac{1}{\sqrt{2\pi}} \int_0^n f(u)e^{-iuv}\, du.$$

Integrating by parts twice, we find that

$$\widehat{\phi}(v) = \frac{1}{\sqrt{2\pi}v^2} \left[ f'(n)e^{-inv} - f'(0) - \int_0^n f''(u)e^{-iuv}\, du \right] \tag{5.2}$$

for $v \neq 0$. Hence there exist constants $c_1 > 0$ and $c_2 > 0$ such that

$$|\widehat{\phi}(v)| \leq \min\left\{c_1, \frac{c_2}{v^2}\right\} \qquad (v \in \mathbb{R}). \tag{5.3}$$

With this it is readily seen that $\phi$ satisfies the hypothesis of Poisson's summation formula. Hence we obtain

$$\sum_{k=0}^{n} f(k) = \sqrt{2\pi} \sum_{k=-\infty}^{\infty} \widehat{\phi}(2\pi k). \tag{5.4}$$

Finally, calculating $\widehat{\phi}(2\pi k)$ for $k \neq 0$ with the help of (5.2) and using that

$$\sum_{k=1}^{\infty} \frac{2}{(2\pi k)^2} = \frac{B_2}{2!} = \frac{1}{12}, \tag{5.5}$$

we arrive at

$$\sum_{k=0}^{n} f(k) = \int_0^n f(t)\,dt + \frac{1}{12}\left[f'(n) - f'(0)\right] - \sum_{k=1}^{\infty} \int_0^n \frac{e^{i2\pi k u} + e^{-i2\pi k u}}{(2\pi k)^2} f''(u)\,du.$$

This is EMSF for $r = 1$ and functions satisfying $f(0) = f(n) = 0$. $\square$

The implication PSF $\Rightarrow$ EMSF is new. A variant with $n = \infty$ for functions belonging to the Schwarz space $\mathcal{S}$ of rapidly decreasing functions can be found in [22, p. 112 ff.]. It is also proved as a corollary of PSF.

Now to the converse implication which is also new.

*Proof of EMSF $\Rightarrow$ PSF.* Starting with EMSF for $r = 1$, we find for $f \in C^{(2)}[0, n]$ by an integration by parts of (1.1) that

$$\sum_{k=0}^{n} f(k) = \int_0^n f(x)\,dx + \frac{1}{2}\left[f(0) + f(n)\right] - \frac{1}{\pi}\int_0^n \sum_{k=1}^{\infty} \frac{\sin(2\pi k t)}{k} f'(t)\,dt. \tag{5.6}$$

It is known and can be shown in an elementary way that[4]

$$\sum_{k=1}^{\infty} \frac{\sin(2\pi k t)}{k} = -\pi\left(t - \lfloor t \rfloor - \frac{1}{2}\right) \tag{5.7}$$

for $t \in \mathbb{R}\backslash\mathbb{Z}$. By Abelian summation and standard estimates, it can also be shown that the partial sums of the series are uniformly bounded on the whole of $\mathbb{R}$.

Although (5.6) was derived for $f \in C^{(2)}[0, n]$, it is valid for functions that are absolutely continuous on $[0, n]$. This follows by a density argument

[4] This result can be found in many textbooks on Analysis for first year students.

and a limiting process which is easy since in (5.6) we have a finite sum and integrations over a finite interval.

Now assume $f \in L^1(\mathbb{R}) \cap AC(\mathbb{R})$ as in the hypothesis of PSF. From Lemma 2.5 it follows in particular that $f(n) \to 0$ as $n \to \pm\infty$. Hence we may apply (5.6) to $f(x)$ and to $f(-x)$, add the results and let $n$ approach infinity. This way we obtain

$$\sum_{k=-\infty}^{\infty} f(k) = \int_{\mathbb{R}} f(x)\,dx - \frac{1}{\pi}\int_{\mathbb{R}} \sum_{k=1}^{\infty} \frac{\sin(2\pi kt)}{k} f'(t)\,dt.$$

Since the partial sums of the series are uniformly bounded and $f \in AC(\mathbb{R})$, Lebesgue's theorem of dominated convergence allows us to interchange integration and summation. Thus,

$$\sum_{k=-\infty}^{\infty} f(k) = \int_{\mathbb{R}} f(x)\,dx - \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\int_{\mathbb{R}} \sin(2\pi kt) f'(t)\,dt. \tag{5.8}$$

Now, expressing the sine by exponential functions and performing an integration by parts, we find that

$$\begin{aligned}
&-\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\int_{\mathbb{R}} \sin(2\pi kt) f'(t)\,dt \\
&\quad= \frac{1}{2\pi i}\sum_{k=1}^{\infty}\frac{1}{k}\left[\int_{\mathbb{R}} e^{-i2\pi kt} f'(t)\,dt - \int_{\mathbb{R}} e^{i2\pi kt} f'(t)\,dt\right] \\
&\quad= \sum_{k=1}^{\infty}\left[\int_{\mathbb{R}} e^{-i2\pi kt} f(t)\,dt + \int_{\mathbb{R}} e^{i2\pi kt} f(t)\,dt\right] \\
&\quad= \sqrt{2\pi}\sum_{k=1}^{\infty}\left[\widehat{f}(k) + \widehat{f}(-k)\right].
\end{aligned}$$

Substituting this in (5.8), we arrive at PSF. $\square$

## 6. The Euler–Maclaurin Formula and the Approximate Sampling Formula

*Proof of EMSF $\Rightarrow$ ASF.* Let us first consider the particular case that $f \in L^2(\mathbb{R}) \cap AC^{(2)}(\mathbb{R})$ with $f'' \in C(\mathbb{R})$, and define $g_t(\cdot) := f(\cdot)\,\mathrm{sinc}(t - \cdot)$ with $t \in \mathbb{R}$ fixed. Now apply (1.1) for $r = 1$ to $g_t(\cdot)$ and $g_t(-\cdot)$ and add the two formulae. This yields for each $n \in \mathbb{N}$,

$$\sum_{k=-n}^{n} g_t(k) = \int_{-n}^{n} g_t(u)\,du + \frac{1}{2}\left[g_t(n)+g_t(-n)\right] + \frac{1}{12}\left[g_t'(n)-g_t'(-n)\right]$$
$$-\sum_{k=1}^{\infty}\int_{-n}^{n}\frac{e^{i2\pi ku}+e^{-i2\pi ku}}{(2\pi k)^2}\,g_t''(u)\,du.$$

Since $g_t, g_t'' \in L^1(\mathbb{R}) \cap C(\mathbb{R})$ with $\lim_{j\to\pm\infty} g_t(j) = \lim_{j\to\pm\infty} g_t'(j) = 0$, one obtains for $n \to \infty$ that

$$\sum_{k=-\infty}^{\infty} g_t(u)\,du = \int_{\mathbb{R}} g_t(u)\,du - \lim_{n\to\infty}\sum_{k=1}^{\infty}\int_{-n}^{n}\frac{e^{i2\pi ku}+e^{-i2\pi ku}}{(2\pi k)^2}\,g_t''(u)\,du.$$

Now the infinite series on the right-hand side is uniformly convergent with respect to $n \in \mathbb{N}$ and one can interchange the limits with summation to deduce

$$\sum_{k=-\infty}^{\infty} g_t(k) = \int_{\mathbb{R}} g_t(u)\,du - \sum_{k=1}^{\infty}\int_{\mathbb{R}}\frac{e^{i2\pi ku}+e^{-i2\pi ku}}{(2\pi k)^2}\,g_t''(u)\,du$$
$$= \int_{\mathbb{R}} g_t(u)\,du - \sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\frac{1}{(2\pi k)^2}\int_{\mathbb{R}} e^{-i2\pi ku} g_t''(u)\,du.$$

Integrating by parts twice yields

$$\sum_{k=-\infty}^{\infty} g_t(k) = \sum_{k=-\infty}^{\infty}\int_{\mathbb{R}} g_t(u)e^{-i2\pi ku}\,du. \quad (t \in \mathbb{R}). \tag{6.1}$$

Using now (2.13) with $w = 1$, we can rewrite the last integral as

$$\int_{\mathbb{R}} g_t(u)e^{-i2\pi ku}\,du = \int_{\mathbb{R}} f(u)e^{-i2\pi ku}\,\mathrm{sinc}(t-u)\,du = f(t) - (R_1 f)(t).$$

This is ASF for $w = 1$ in that particular instance. The general case now follows as in the proof of PSF $\Rightarrow$ ASF of Sect. 4 by approximating $f$ by its Fejér means $\sigma_\rho f$ (cf. (2.16)). □

The proof of the foregoing implication is adapted from [16] and that of the converse, to follow, is new.

*Proof of ASF $\Rightarrow$ EMSF.* Let $f \in C^{(2)}[0,n]$. Assuming again that $f(0) = f(n) = 0$, we define $\phi$ by (5.1). Then (5.2) and (5.3) hold, and we see that $\phi$ satisfies all the hypotheses of ASF. Therefore, ASF applies to $\phi$, and we obtain

$$\phi(t) - \sum_{k=-\infty}^{\infty}\phi(k)\,\mathrm{sinc}(t-k) = (R\phi)(t), \tag{6.2}$$

where the remainder is now given by

$$(R\phi)(t) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \left(1 - e^{-i2\pi kt}\right) \int_{(2k-1)\pi}^{(2k+1)\pi} \widehat{\phi}(v) e^{itv}\, dv. \tag{6.3}$$

From (5.3) it follows that the series in (6.3) converges absolutely and uniformly.

Now we integrate both sides of (6.2) over $[-\rho, \rho]$ and let $\rho$ approach infinity. On the left-hand side we easily see by (2.7) that

$$\begin{aligned}
&\lim_{\rho\to\infty} \int_{-\rho}^{\rho} \left[\phi(t) - \sum_{k=-\infty}^{\infty} \phi(k)\, \mathrm{sinc}(t-k)\right] dt \\
&\quad = \int_0^n f(t)\, dt - \sum_{k=0}^{n} f(k) \lim_{\rho\to\infty} \int_{-\rho}^{\rho} \mathrm{sinc}(t-k)\, dt \\
&\quad = \int_0^n f(t)\, dt - \sum_{k=0}^{n} f(k).
\end{aligned}$$

The right-hand side needs more care. Using the representation (2.14) of the integrals, we may rewrite (6.3) as

$$(R\phi)(t) = \sum_{k=-\infty}^{\infty} \left(e^{i2\pi kt} - 1\right) \int_0^n f(u) e^{-i2\pi ku}\, \mathrm{sinc}(t-u)\, du.$$

Again the series converges absolutely and uniformly. Hence, when we integrate over the finite interval $[-\rho, \rho]$, we may interchange integration and summation and employ Fubini's theorem. This leads us to

$$\int_{-\rho}^{\rho} (R\phi)(t)\, dt = \sum_{k=-\infty}^{\infty} \int_0^n f(u) a_{k,\rho}(u)\, du - \sum_{k=-\infty}^{\infty} \int_0^n f(u) b_\rho(u) e^{-i2\pi ku}\, du, \tag{6.4}$$

where

$$a_{k,\rho}(u) := \int_{-\rho-u}^{\rho-u} e^{i2\pi kt}\, \mathrm{sinc}\, t\, dt \quad \text{and} \quad b_\rho(u) := \int_{-\rho-u}^{\rho-u} \mathrm{sinc}\, t\, dt.$$

At the end of this proof we will show that the integrals in both series of (6.4) are of order $\mathcal{O}(k^{-2})$ as $k \to \pm\infty$, uniformly with respect to $\rho$. Therefore the limit $\rho \to \infty$ may be taken inside the summation. Since $a_{k,\rho}(u)$ and $b_\rho(u)$ are uniformly continuous as functions of $(u, \rho)$ on $[0, n] \times [0, \infty)$, that limit may even by taken inside the integration. By (2.7), we have

$$\lim_{\rho\to\infty} a_{k,\rho}(u) = \delta_{k,0} \quad \text{and} \quad \lim_{\rho\to\infty} b_\rho(u) = 1,$$

where Kronecker's delta has been used. Hence

$$\lim_{\rho\to\infty} \int_{-\rho}^{\rho} (R\phi)(t)\,dt = -\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty} \int_0^n f(u)e^{-i2\pi ku}\,du.$$

Altogether we have

$$\int_0^n f(t)\,dt - \sum_{k=0}^{n} f(k) = -\sum_{k=1}^{\infty} \int_0^n \left(e^{i2\pi ku} + e^{-i2\pi ku}\right) f(u)\,du. \tag{6.5}$$

Integrating by parts twice on the right-hand side and noting (5.5) yields EMSF (1.1).

It remains to show that the integrals on the right-hand side of (6.4) are of order $\mathcal{O}(k^{-2})$ as $k \to \pm\infty$ uniformly with respect to $\rho$. Indeed, two integrations by parts yield

$$\begin{aligned} a_{k,\rho}(u) = {} & \left[\frac{e^{i2\pi kt}}{i2\pi k}\operatorname{sinc} t + \frac{e^{i2\pi kt}}{(2\pi k)^2}\operatorname{sinc}' t\right]_{t=-\rho-u}^{t=\rho-u} \\ & - \frac{1}{(2\pi k)^2} \int_{-\rho-u}^{\rho-u} e^{i2\pi kt} \operatorname{sinc}'' t\,dt. \end{aligned}$$

This is a decomposition of $a_{k,\rho}(u)$ into five terms. When we multiply by $f(u)$ and integrate over $[0,n]$, we see for the last three terms that they are $\mathcal{O}(k^{-2})$ as $k \to \pm\infty$, uniformly with respect to $\rho$. For the first two terms we observe the same asymptotic behaviour after one integration by parts taking $f(u)\operatorname{sinc}(\pm\rho - u)$ as the term to be differentiated and $e^{i2\pi k(\pm\rho-u)}$ as the one to be integrated. Hence

$$\int_0^n f(u)a_{k,\rho}(u)\,du = \mathcal{O}\left(\frac{1}{k^2}\right)$$

uniformly with respect to $\rho$.

As regards $b_\rho(u)$, we note that it is uniformly bounded with respect to $\rho$ and $u$ by Lemma 2.7 b), and so are its derivatives. Hence two integrations by parts, as in the calculation of $\widehat{\phi}$ in (5.2), show that also

$$\int_0^n f(u)b_\rho(u)e^{-i2\pi ku}\,du = \mathcal{O}\left(\frac{1}{k^2}\right)$$

uniformly with respect to $\rho$. This justifies the above procedure and completes the proof. $\square$

## 7. The Abel–Plana Formula as a Corollary of Poisson's Summation Formula and of the Approximate Sampling Formula

*Proof of PSF ⇒ APSF.* Suppose that $f$ satisfies the hypotheses of APSF. We may and will assume that, in addition, $f(0) = 0$. Indeed, we simply have to replace $f$ by $f - f(0)$ sinc and observe that APSF is trivial for the sinc function.

For $N \in \mathbb{N}$, we now define

$$f_N(x) := \begin{cases} f(x) & \text{if } x \in [0, N + \frac{1}{2}], \\ 2f(N + \frac{1}{2})(N + 1 - x) & \text{if } x \in (N + \frac{1}{2}, N + 1], \\ 0 & \text{if } x \in \mathbb{R}\backslash[0, N + 1]. \end{cases}$$

Then $f_N$ satisfies the hypothesis of PSF, and so

$$\sum_{k=1}^{N} f(k) = \sqrt{2\pi}\left\{ \widehat{f}_N(0) + \sum_{k=1}^{\infty} \left[ \widehat{f}_N(2\pi k) + \widehat{f}_N(-2\pi k) \right] \right\}. \tag{7.1}$$

Obviously, we may write $\sqrt{2\pi}\widehat{f}_N$ as $\varphi_N + \psi_N$, where

$$\varphi_N(v) := \int_0^{N+1/2} e^{-ivu} f(u)\, du$$

and

$$\psi_N(v) := 2f\left(N + \frac{1}{2}\right) \int_{N+1/2}^{N+1} e^{-ivu}(N + 1 - u)\, du.$$

An easy calculation shows that

$$\psi_N(0) + \sum_{k=1}^{\infty} \left[ \psi_N(2\pi k) + \psi_N(-2\pi k) \right] = 0,$$

and therefore (7.1) reduces to

$$\sum_{k=1}^{N} f(k) = \int_0^{N+1/2} f(x)\, dx + \sum_{k=1}^{\infty} \left[ \varphi_N(2\pi k) + \varphi_N(-2\pi k) \right]. \tag{7.2}$$

Now, by contour integration in the lower half-plane along the rectangle with vertices at $0,\ -iT,\ N+\frac{1}{2}-iT,\ N+\frac{1}{2}$ we find that

$$\varphi_N(v) = -i\int_0^T e^{-vt} f(-it)\,dt + \int_0^{N+1/2} e^{-v(T+it)} f(t-iT)\,dt$$
$$+\, i\int_0^T e^{-v(t+iN+i/2)} f\left(N+\frac{1}{2}-it\right)dt.$$

If $v \geq 2\pi$, then, by (1.2), the second integral approaches zero as $T\to\infty$. Hence

$$\varphi_N(2\pi k) = -i\int_0^\infty e^{-2\pi kt} f(-it) + i(-1)^k \int_0^\infty e^{-2\pi kt} f\left(N+\frac{1}{2}-it\right)dt$$

for $k\in\mathbb{N}$. Analogous considerations in the upper half-plane yield

$$\varphi_N(-2\pi k) = i\int_0^\infty e^{-2\pi kt} f(it)\,dt - i(-1)^k \int_0^\infty e^{-2\pi kt} f\left(N+\frac{1}{2}+it\right)dt.$$

Thus (7.2) may be rewritten as

$$\sum_{k=1}^N f(k) - \int_0^{N+1/2} f(x)\,dx$$
$$= i\sum_{k=1}^\infty\Bigg\{\int_0^\infty e^{-2\pi kt}\,[f(it)-f(-it)]dt$$
$$+\,(-1)^{k+1}\int_0^\infty e^{-2\pi kt}\left[f\left(N+\frac{1}{2}+it\right) - f\left(N+\frac{1}{2}-it\right)\right]dt\Bigg\}.$$

Next, we want to show that summation and integration may be interchanged. For $x\in\{0, N+\frac{1}{2}\}$, the hypotheses of APSF guarantee the existence of constants $c_1>0$ and $c_2>0$ such that

$$|f(x+it)-f(x-it)| \leq c_1 t \qquad \text{for } t\in[0,1]$$

and

$$|f(x+it)-f(x-it)| \leq c_2\, e^{2\pi t} \qquad \text{for } t\in[1,\infty).$$

This yields that for $k>1$, we have

$$\int_0^\infty e^{-2\pi kt}|f(x+it)-f(x-it)|\,dt \leq \frac{c_1}{(2\pi k)^2} + \frac{c_2 e^{-2\pi(k-1)}}{2\pi(k-1)} =: M_k.$$

Obviously, $\sum_{k=2}^{\infty} M_k < \infty$, and so an interchange of summation and integration is justified by the theorem of Beppo Levi. Thus, we obtain

$$\sum_{k=1}^{N} f(k) - \int_0^{N+1/2} f(x)\,dx$$

$$= i\int_0^{\infty} \frac{f(it)-f(-it)}{e^{2\pi t}-1}\,dt + i\int_0^{\infty} \frac{f\left(N+\frac{1}{2}+it\right)-f\left(N+\frac{1}{2}-it\right)}{e^{2\pi t}+1}\,dt.$$

By (1.3), the last integral approaches zero as $N \to \infty$, and so we arrive at (1.4) for functions satisfying $f(0) = 0$. $\square$

The above implication was proved in [49] as an application of a different version of Poisson's formula, namely $f \in AC(\mathbb{R})$ being replaced by $f \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$. The following is new.

*Proof of ASF ⇒APSF.* Let $g \in C^{(2)}[0, N+\frac{1}{2}]$ such that $g(0) = g(N+\frac{1}{2}) = 0$. Proceeding as in the proof of (6.5), we can deduce from ASF the analogous equation

$$\int_0^{N+1/2} g(t)\,dt - \sum_{k=0}^{N} g(k) = -\sum_{k=1}^{\infty} \int_0^{N+1/2} \left(e^{i2\pi ku} + e^{-i2\pi ku}\right) g(u)\,du. \tag{7.3}$$

If $f$ satisfies the hypotheses of APSF, then this equation holds for

$$g(t) := f(t) - f(0) - \frac{t}{N+\frac{1}{2}}\left[f\left(N+\frac{1}{2}\right) - f(0)\right].$$

Substituting this in (7.3), we find by a short calculation in which we use the formula

$$\sum_{m=1}^{\infty} \frac{1}{(2m-1)^2} = \frac{\pi^2}{8}$$

that

$$\int_0^{N+1/2} f(t)\,dt - \sum_{k=0}^{N} f(k) + \frac{1}{2} f(0) = -\sum_{k=1}^{\infty} \left[\varphi_N(2\pi k) + \varphi_N(-2\pi k)\right] \tag{7.4}$$

with

$$\varphi_N(v) := \int_0^{N+1/2} e^{-ivu} f(u)\,du.$$

The series on the right-hand side of (7.4) is exactly that in (7.2). Calculating it as in the previous proof and using (1.3), we arrive at (1.4). $\square$

As mentioned in the introduction, the converse of the two implications presented in this section have not been dealt with. The proofs would probably be so involved that one could not really claim that one is a corollary of the other.

## 8. A Proof of the Euler–Maclaurin Formula

Among the four formulae (EMSF, APSF, PSF, ASF) under consideration, it is EMSF which permits the most elementary proof. Variants of the subsequent approach can be found in many textbooks on Analysis.[5] Proving the implications between the expositions does not include proofs of their truth or falsity. Only if the truth value of any one of them is established independently will all of them have the same truth value.

*Proof of EMSF.* Let $f \in C^{(2r)}[0,n]$ and define[6] $p(x) := \frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{12}$. Two integrations by parts yield

$$\int_0^1 p(x) f''(x)\,dx = -\frac{1}{2}\left[f(0)+f(1)\right] + \frac{1}{12}\left[f'(1)-f'(0)\right] + \int_0^1 f(x)\,dx. \tag{8.1}$$

Applying this formula to $f(\,\cdot\, + k)$ for $k = 0, \dots, n-1$ and summing up the results, we obtain

$$\begin{aligned}\int_0^n \widetilde{p}(x) f''(x)\,dx &= -\frac{1}{2}f(0) - \sum_{k=1}^{n-1} f(k) - \frac{1}{2}f(n) \\ &\quad + \frac{1}{12}\left[f'(n)-f'(0)\right] + \int_0^n f(x)\,dx, \end{aligned} \tag{8.2}$$

where $\widetilde{p}$ is the 1-periodic continuation of $p$ from $[0,1]$ to the whole of $\mathbb{R}$. Since $\widetilde{p}$ is an absolutely continuous even function, its Fourier series is an absolutely convergent cosine series that represents $\widetilde{p}$. Therefore

$$\widetilde{p}(x) = \sum_{k=0}^{\infty} a_k \cos(2\pi k x) = \sum_{k=0}^{\infty} a_k \frac{e^{i2\pi kx} + e^{-i2\pi kx}}{2},$$

---

[5] Some authors first establish (5.6) with the trigonometric series replaced by the right-hand side of (5.7). For this, just one integration by parts for a Riemann-Stieltjes integral is required; see e.g., [2, pp. 149–150], [31, pp. 506–507].

[6] The attentive reader will note that $2p(x)$ is the Bernoulli polynomial $B_2(x)$.

where $a_0 = \int_0^1 p(x)\,dx = 0$ and

$$a_k = 2\int_0^1 p(x)\cos(2\pi kx)\,dx = \frac{2}{(2\pi k)^2} \qquad (k \in \mathbb{N}).$$

The last integral has been calculated by using (8.1) for the function $f(x) = -(2\pi k)^{-2}\cos(2\pi kx)$. Substituting the Fourier series of $\widetilde{p}$ in (8.2) and regrouping terms, we obtain EMSF for $r = 1$. The formula for general $r$ is deduced from this special case by repeated integration by parts on the left-hand side of (8.2) such that $\widetilde{p}$ is integrated and $f''$ is differentiated.

We only carry out the case of $r = 2$. Using the notation (3.1) and the relations (3.4)–(3.5), we find by two integrations by parts:

$$\begin{aligned}
\int_0^n \widetilde{p}(x) f''(x)\,dx &= \int_0^n [L_2(ix) + L_2(-ix)]\, f''(x)\,dx \\
&= \left[\frac{1}{i}\,[L_3(ix) - L_3(-ix)]\, f''(x) + [L_4(ix) + L_4(-ix)]\, f'''(x)\right]_{x=0}^{x=n} \\
&\quad - \int_0^n [L_4(ix) + L_4(-ix)]\, f^{(4)}(x)\,dx \\
&= -\frac{B_4}{4!}\,[f'''(n) - f'''(0)] - \sum_{k=1}^{\infty}\int_0^n \frac{e^{i2\pi kx} + e^{-i2\pi kx}}{(2\pi k)^4} f^{(4)}(x)\,dx.
\end{aligned}$$

Substituting this in (8.2), we obtain the formula for $r = 2$. □

## 9. Quadrature Formulae for Bandlimited Functions

Suppose that the integrals under consideration exist. Then, by regrouping terms, the three summation formulae EMSF, APSF and PSF can be interpreted as trapezoidal rules with remainders. In fact, we may write EMSF in the form

$$\int_0^n f(x)\,dx = \frac{1}{2}f(0) + \sum_{k=1}^{n-1} f(k) + \frac{1}{2}f(n) + R_{[0,n]}(f) \tag{9.1}$$

with remainder

$$\begin{aligned}
R_{[0,n]}(f) = &-\sum_{k=1}^{r} \frac{B_{2k}}{(2k)!}\left[f^{(2k-1)}(n) - f^{(2k-1)}(0)\right] \\
&+ (-1)^r \sum_{k=1}^{\infty}\int_0^n \frac{e^{i2\pi kt} + e^{-i2\pi kt}}{(2\pi k)^r}\, f^{(2r)}(t)\,dt,
\end{aligned}$$

APSF as

$$\int_0^\infty f(x)\,dx = \frac{1}{2} f(0) + \sum_{k=1}^{\infty} f(k) + R_{[0,\infty)}(f) \tag{9.2}$$

with

$$R_{[0,\infty)}(f) = -i \int_0^\infty \frac{f(iy) - f(-iy)}{e^{2\pi y} - 1}\,dy \tag{9.3}$$

and PSF as

$$\int_{\mathbb{R}} f(x)\,dx = \sum_{k\in\mathbb{Z}} f(k) + R_{\mathbb{R}}(f) \tag{9.4}$$

with

$$R_{\mathbb{R}}(f) = -\sqrt{2\pi} \sum_{k\in\mathbb{Z}\setminus\{0\}} \widehat{f}(2\pi k).$$

The three remainders are of a very different nature. If $\widehat{f}(v)$ vanishes for $|v| \geq 2\pi$ as it is the case when $f$ belongs to the Bernstein space $B^1_{2\pi}$, then $R_{\mathbb{R}}(f) = 0$. In other words, the trapezoidal rule on $\mathbb{R}$ with nodes at the integers is exact for absolutely integrable functions that are bandlimited to $(-2\pi, 2\pi)$.

In contrast to this observation, ASF needs $f$ to be bandlimited to $(-\pi, \pi)$ in order that its remainder vanishes and the classical sampling formula is obtained. A similar phenomenon occurs for polynomials in connection with Gaussian quadrature. While a polynomial of degree $n - 1$ needs at least $n$ samples for its reconstruction, the Gaussian quadrature formula with $n$ nodes is exact for polynomials up to degree $2n - 1$. Other characterizations of the Gaussian quadrature formula also hold analogously for the trapezoidal rule over $\mathbb{R}$ applied to bandlimited functions; see [46], [13, Sect. 2.11.2]. It is therefore justified to call the trapezoidal rule on $\mathbb{R}$ a *Gaussian quadrature formula for bandlimited functions.*

What happens to the remainders of the other two formulae (9.1) and (9.2) when we apply them to bandlimited functions? In the case of formula (9.2), this question was studied in [48]. The remainder $R_{[0,\infty)}(f)$ does not vanish unless $f$ is an even function. For practical applications, the representation (9.3) has two disadvantages: It involves values of $f$ on the imaginary axis and it does not provide a sequence of gradual approximations. Therefore in [48, Theorem 1] an expansion of $R_{[0,\infty]}(f)$ in terms of derivatives of $f$ at 0 has been established. It also connects APSF with EMSF. The result is as follows.

**Theorem.** *Let $f$ be an entire function of exponential type $\sigma$ less than $2\pi$ and suppose that $\int_0^\infty f(x)\,dx$ exists in the sense of Cauchy. Then*

$$\int_0^\infty f(x)\,dx = \frac{1}{2}f(0) + \sum_{k=1}^{\infty} f(k) + \sum_{j=1}^{\infty} f^{(2j-1)}(0)\frac{B_{2j}}{(2j)!}. \tag{9.5}$$

*The second series converges absolutely.*

If, in addition, $f$ is bounded on the whole real line, then the terms of the second series can be estimated explicitly and it turns out that they decrease like $\mathcal{O}((\sigma/(2\pi))^{2j})$ as $j \to \infty$. Thus (9.5) includes a representation of $R_{[0,\infty)}(f)$ which yields gradual approximations by truncating the second series.

As a generalization of (9.5), it was shown in [48, Theorem 3], again under the additional hypothesis of boundedness on $\mathbb{R}$, that

$$\int_0^\infty f(x)\,dx = \sum_{k=0}^{\infty} f(z+k) + \sum_{j=1}^{\infty} f^{(j-1)}(0)\frac{B_j(z)}{j!} \quad (z \in \mathbb{C}),$$

where $B_j(z)$ are the Bernoulli polynomials. Both series converge uniformly on compact subsets of $\mathbb{C}$. Moreover, the second series converges absolutely.

Connections to (9.1) are easily found by writing

$$\int_0^n f(x)\,dx = \int_0^\infty \left[f(x) - f(x+n)\right]\,dx$$

and applying (9.2) to the right-hand side.

A very general class of quadrature formulae involving derivatives at all the nodes was established in [47]. It contains a generalization of EMSF [47, Theorem 3] and of APSF [47, Corollary 6]. For another quadrature formula over semi-infinite intervals see [52, formula (1.13)].

## 10. Epilogue

In a recent paper [7] the five authors considered, among others, the following four formulae of mathematical analysis for bandlimited functions, i.e. for functions belonging to the Bernstein spaces $B^p_\sigma$, $\sigma > 0$, $p = 1, 2$, (see Sect. 2 for the definition):

**Poisson Summation Formula for Bandlimited Functions (PSFB)**

*For $f \in B^1_\sigma$ with $\sigma > 0$ we have*

$$\int_{\mathbb{R}} f(u)\,du = \frac{2\pi}{\sigma}\sum_{k\in\mathbb{Z}} f\left(\frac{2k\pi}{\sigma}\right).$$

This means that in $B^1_\sigma$ the trapezoidal rule with step size $2\pi/\sigma$ is exact for integration over $\mathbb{R}$.

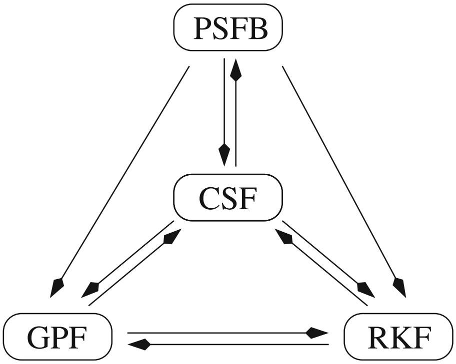


FIGURE 2. The implications proved in [7]

**Classical Sampling Formula (CSF)**

*For $f \in B^2_\sigma$ with $\sigma > 0$ we have*

$$f(z) = \sum_{k\in\mathbb{Z}} f\left(\frac{k\pi}{\sigma}\right) \operatorname{sinc}\left(\frac{\sigma z}{\pi} - k\right) \qquad (z \in \mathbb{C}), \tag{10.1}$$

*the convergence being absolute and uniform in strips of bounded width parallel to the real line, thus in particular, on compact sets.*

**General Parseval Formula (GPF)**

*For $f, g \in B^2_\sigma$ with $\sigma > 0$ we have*

$$\int_{\mathbb{R}} f(u)\overline{g(u)}\, du = \frac{\pi}{\sigma} \sum_{k\in\mathbb{Z}} f\left(\frac{k\pi}{\sigma}\right) \overline{g\left(\frac{k\pi}{\sigma}\right)}.$$

**Reproducing Kernel Formula (RKF)**

*For $f \in B^2_\sigma$ with $\sigma > 0$ we have*

$$f(z) = \frac{\sigma}{\pi} \int_{\mathbb{R}} f(u) \operatorname{sinc} \frac{\sigma}{\pi}(z-u)\, du \quad (z \in \mathbb{C}).$$

In that paper it was shown that these formulae are equivalent. More precisely, the implications indicated in Fig. 2 were established.

Now in [10] it was shown that the classical sampling formula (CSF) for bandlimited functions is equivalent to the approximate sampling formula (ASF) for not necessarily bandlimited functions, those belonging to $F^2$. Thus all the assertions of Fig.1 are equivalent to those of Fig. 2.

Figure 3 shows the implications proved in the present paper together with those established in [7,10].

It should be noted that in [7,10] the Bernstein spaces $B^p_\sigma$ were defined via functions of exponential type rather than in terms of the Fourier transform, as

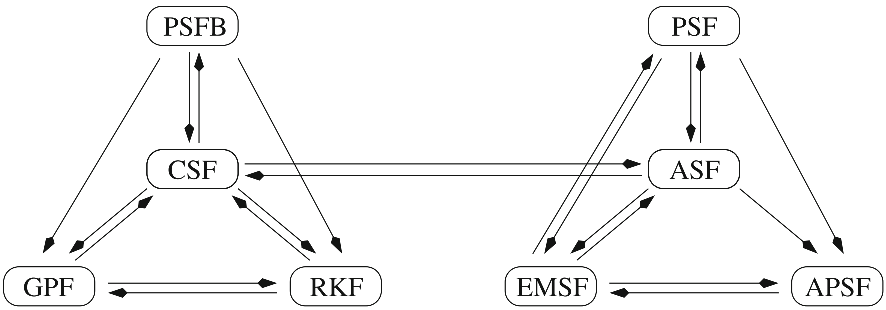


FIGURE 3. The equivalence between the bandlimited and non-bandlimited assertions

we did above in Sect. 2. The powerful Paley–Wiener theorem, however, guarantees that these two definitions are nevertheless equivalent. In view of these different definitions the Paley–Wiener theorem has to be used as a side result in order to establish CSF $\Leftrightarrow$ ASF in Fig. 3.

In earlier papers some of the authors have shown that the four propositions dealing with bandlimited functions are equivalent to Gauß's summation of the hypergeometric function, the generalised Vandermonde–Chou formula, Tschakalov's sampling theorem, Dougall's bilateral sum, to the functional equation of the Riemann zeta function and to the transformation formula for Jacobi's elliptic theta function.

For results of this type showing the equivalence of basic formulae in analysis to those in the diagrams above see, e.g. [8,9,12,32–35,55].

## 11. Short Biography of A. Ostrowski

Alexander (Markowich) Ostrowski (1893–1986) was born in Kiev and was a student of Chebyshev's disciple D. A. Grave at the University there; Grave recommended him to E. Landau and K. Hensel. He then studied under Hensel at Marburg University from 1912 on, moved to Göttingen in 1918 where he received his doctorate under Hilbert and Landau in 1920. The next stop was Hamburg where, as assistant to E. Hecke, he was awarded the Habilitation degree in 1922. He returned to Göttingen as Dozent, spent the year 1925/1926 on a Rockefeller Research Fellowship at Oxford, Cambridge and Edinburgh, and finally was offered the mathematics chair at Basel. He retired in 1958. Ostrowski was the author of some 275 publications, dealing with linear algebra, algebraic equations, estimating their roots, Galois theory, algebraic number theory, differential equations, complex analysis, conformal mappings, numerical analysis, Cauchy functional equation, Cauchy–Frullani integrals, methods for both finding and approximating eigenvalues of linear systems. In fact, his works were published in six volumes [43].

Ostrowski, one of the great mathematicians of the 20th century, was especially known for his desire to unravel the essential features of a problem, for his elegant and succinct proofs, for investigating a topic exhaustively, once having selected it, for his thoroughness based on his deep knowledge of the contemporary literature and especially the original sources. His 16 doctoral students included S. E. Warschawski (1932), who helped build up the University of Minnesota's mathematics department, T. Motzkin (1936), the well-known analyst, E. Batschelet (1944), known for his work in statistics and biomathematics, Werner Gautschi and his twin brother Walter Gautschi (1954), the eminent numerical analyst. For Ostrowski's life and work see, in particular, [28].

The paper on the Euler–Maclaurin formula [16] appeared in the volume dedicated to Ostrowski's 90th birthday. He invited PLB to a colloquium lecture at Basel in 1957 and he participated together with his wife Margaret in the symposium on "Abstract Spaces and Approximation" conducted by PLB, who also had the honour to be invited to his villa at Montagnola in the seventies.

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